Differential Equations from $\begin{cases}\text{null vectors}\\ \text{singular vectors}\end{cases}$

Kac determinant $= 0$ $\Rightarrow$ null states orthogonal to all states

$\Rightarrow$ decouple from correlation functions

$|\psi\rangle$ null state

$$\Rightarrow \langle \phi_1 | \phi_2(1)\, \phi_3(z)\, |\psi\rangle = 0$$

Simplest case: $M_1 = 2h$, $\quad h = h_{11} = 0 \Rightarrow L_{-1}|0\rangle = 0$

$$\langle \phi_1 | \phi_2(1)\, \phi_3(z)\, L_{-1}|0\rangle = 0 \quad \Rightarrow \text{ translation invariance}$$

Next case: $M_2 = A_2 \cdot (h - h_{11})(h - h_{12})(h - h_{21})$

$h_{12}(t) = h_{21}(1/t)$ so only one case.

Take $h_{21}(t) = \frac{3}{4t} - \frac{1}{2}$

$$|\psi\rangle = \left(L_{-2} + \alpha L_{-1} L_{-1}\right)|h\rangle = 0$$

In fact $\left.\begin{array}{l} L_1|\psi\rangle = 0 \quad \text{(quasiprimary)} \\ L_2|\psi\rangle = 0 \end{array}\right\}$ primary/h.w. state.

Check: $\left.\begin{array}{l} L_1|\psi\rangle = 0 \Rightarrow \alpha = -\dfrac{3}{2(2h+1)} \\ L_2|\psi\rangle = 0 \Rightarrow \alpha = -\dfrac{(8h+c)}{12h} \end{array}\right\}$ $\det M_2 = 0$

$$\alpha = -t$$

$$|\psi\rangle = \left(L_{-2} - t L_{-1}^2\right)|h_{21}\rangle.$$

General expressions known but cumbersome.

Consider first four point fn

$$F = \langle h | \phi_h(1)\, \phi_h(z) | \ell \rangle \qquad \ell = h_{21}$$

$$\langle h | \phi_h(1)\, \phi_h(z)\, | \psi \rangle = 0 \qquad |\psi\rangle = (L_{-2} - t L_{-1}^2) | \ell \rangle$$

Same method:

$$L_{-2} = L_{-2} - L_0 + L_0$$

$$\langle h | \phi_h(1)\, \phi_h(z)\, (L_{-2}) | h \rangle = \langle h | \phi_h(1)\, \phi_h(z)\, (L_{-2} - L_0 + h) | \ell \rangle$$

$$= \left[h\left(3 + \frac{1}{z^2}\right) + \left(z - \frac{1}{z}\right)\partial \right] F$$

$$\langle h | \phi_h(1)\, \phi_h(z)\, L_{-1} L_{-1} | \ell \rangle$$

$$= \left[2h + 1 + (z-1)\frac{\partial}{\partial z} \right]\left[2h + (z-1)\frac{\partial}{\partial z} \right] F$$

If $\quad F = z^{-2h} (1-z)^{-2h} \cdot G$

$$\Rightarrow \quad (1-z) z\, G'' + \frac{2}{3}(2z-1)(4h-1)\, G' + \frac{4}{3} h (4h-1)\, G = 0$$

Hypergeometric eqn with $\quad a = 2 - 3t, \quad b = 1 - t, \quad c = 2 - 2t$

You know the solutions.

If $t = 4/3$, $h = 1/16$, get two solutions:

$$\langle h | \phi_h(1)\, \phi_h(z) | h \rangle = \begin{cases} (z(1-z))^{-1/8} \sqrt{\frac{1+\sqrt{1-z}}{2}} \\ (z(1-z))^{-1/8} \sqrt{2\left(1-\sqrt{1-z}\right)} \end{cases}$$

[Solve Ising model in § 9]

If $t = 3/4$, $h = 1/2$

$$F = \begin{cases} \frac{1}{1-z} + \frac{1}{z} - 1 \\ \underbrace{\frac{1}{z(1-z)} \cdot z^{5/3} \cdot F\left(-1/3, 4/3, 5/3, z\right)}_{\text{Spurious sol}^n.} \end{cases}$$

$$h_{21}(3/4) = \underline{h_{13}(3/4)}$$

Independent singular vector

Independent 3rd order diff eqn

Eliminates the spurious solution.