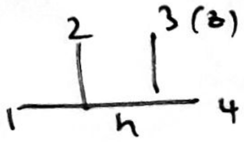
Brute Force method: use the definition



$$= \langle \varphi_1 | \varphi_2(1) \sum_{\substack{\text{orthonormal}\\ \text{basis}}} |\psi\rangle\langle\psi| \, \varphi_3(z) \, | \varphi_4\rangle$$

Not easy to write an orthonormal basis

Use a set of states at level n – $\{ |i,n\rangle \}$

$$\sum_n \langle \varphi_1 | \varphi_2(1) \sum_{i,j} |i,n\rangle M^{-1}_{ij} \langle j,n | \varphi_3(z) | \varphi_4 \rangle$$

$$= \sum_n z^{n+h-h_3-h_4} \sum_{ij} \langle \varphi_1 | \varphi_2(1) | i,n\rangle \, M^{-1}_{ij} \, \langle j,n | \varphi_3(1) | 4 \rangle$$

Basis, level by level:

0: $|h\rangle$

1: $L_{-1}|h\rangle$

2: $L_{-1}L_{-1}|h\rangle$, $L_{-2}|h\rangle$

Need 1. Matrix of inner products $M_{ij} = \langle i,n | j,n\rangle$

2. 3 point functions $\langle \varphi_1 | \varphi_2(1) | i,n\rangle$

1. Inner products:

Use $L_m|h\rangle = 0 \quad m>0 \qquad \Rightarrow \langle h|L_m = 0 \quad m<0$

$L_0|h\rangle = h|h\rangle \qquad \Rightarrow \langle h|L_0 = h\langle h|$

And $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$

Level 0:

$$\langle h|h\rangle = 1$$

Level 1:

$$\langle h|L_1L_{-1}|h\rangle = 2h\langle h|h\rangle = 2h$$

Level 2:

$$\langle h|L_1L_1L_{-1}L_{-1}|h\rangle = 2\cdot 2h(2h+1)$$

$$\langle h|L_2L_{-2}|h\rangle = \langle h|\underbrace{[L_2,L_{-2}]}_{\frac{c}{2}+4L_0} + \cancelto{0}{L_{-2}L_2}|h\rangle$$

$$= \frac{c}{2} + 4h$$

$$\langle h|L_1L_1L_{-2}|h\rangle = 6h$$

$$M = \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & \frac{c}{2}+4h \end{pmatrix}$$

$$M^{-1} = \frac{1}{\det M}\begin{pmatrix} \frac{c}{2}+4h & -6h \\ -6h & 4h(2h+1) \end{pmatrix} \qquad \det M = 2h(2h+1)(c+8h) - 36h^2$$

Clearly problems when $\det M = 0$.

det M is the Kac determinant

$$\det M_n = A_n \cdot \prod_{rs<n} (h - h_{rs}(n))^{p(n-rs)}$$ Number of partitions

$h_{r,s}(c)$ defined implicitly by

$$c = 13 - 6t - 6/t$$

$$h_{r,s} = \frac{r^2-1}{4t} + \frac{(s^2-1)t}{4} - \frac{rs-1}{2}$$

Eg $$\det M_2 = A_2 \cdot \prod_{rs=1} (h-h_{rs})^{p(1)} \cdot \prod_{rs=2} (h-h_{rs})^{p(0)}$$

$$= A_2 \cdot (h-h_{11})(h-h_{12})(h-h_{21})$$

2. 3 pt fns as in sl(2) case.

$$[L_m, \varphi(z)] = z^{m+1}\partial\varphi + h(m+1)z^m\varphi$$

$$[L_m - L_0, \varphi(1)] = m h \varphi$$

$$\langle\varphi_1|\varphi_2(1)\, L_{-1}^2|h\rangle = (h+h_2-h_1+1)(h+h_2-h_1)\,\langle\varphi_1|\varphi_2|h\rangle$$

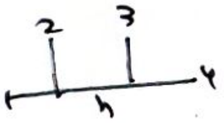
$$\langle\varphi_1|\varphi_2(1)\, L_{-2}|h\rangle = \langle\varphi_1|\varphi_2(1)\,(L_2 - L_0 + h)\,|h\rangle$$

$$= \langle\varphi_1|\,(L_2 - L_0 + h)\,\varphi_2(1) - [L_2 - L_0, \varphi_2(1)]\;|h\rangle$$

$$= (h - h_1 + 2h_2)\,\langle\varphi_1|\varphi_2|h\rangle .$$

Result:

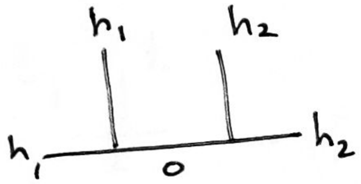
$\boxed{h_{ij} = h_i - h_j}$



$$= z^{h-h_3-h_4}\left\{ 1 + \frac{(h-h_{12})(h+h_{34})}{2h}\, z + z^2 \cdot \frac{\begin{pmatrix}(h+h_2-h_1)_2 & (h+2h_2-h_1)\end{pmatrix}\cdot\begin{pmatrix}\frac{c}{2}+4h & -6h \\ -6h & 4h(2h+1)\end{pmatrix}\begin{pmatrix}(h+h_3-h_4)_2 \\ (h+2h_3-h_4)\end{pmatrix}}{2h(2h+1)(c+8h) - 36h^2} + \cdots \right\}$$

Vacuum Sector: $L_{-1}|0\rangle = 0$

level	States
0	$\|0\rangle$
1	———
2	$L_{-2}\|0\rangle$
3	$L_{-3}\|0\rangle$
4	$L_{-2}L_{-2}\|0\rangle,\ L_{-4}\|0\rangle$



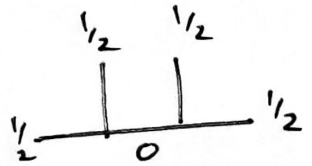
$$= z^{-2h_2}\Big\{ 1 + \frac{2h_1h_2}{c} z^2 + \frac{2h_1h_2}{c} z^3 + \frac{h_1h_2\left(9c + 2(20 + h_1 + h_2 + 5h_1h_2)\right)}{c\,(22+5c)} z^4 + \dots$$

$h_{14} = 0$ for $c = -22/5$.

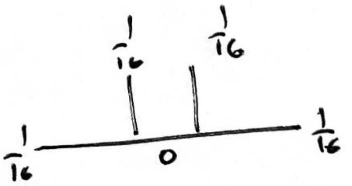
Useful examples

$c = 1/2$ "allowed" h-values

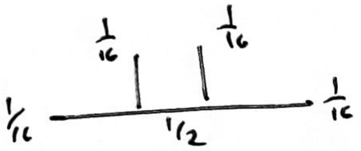
$r \uparrow$	$1/2$	$1/16$	0
	0	$1/16$	$1/2$
	$\rightarrow s$		



$$= \frac{1}{1-z} + \frac{1}{z} - 1$$



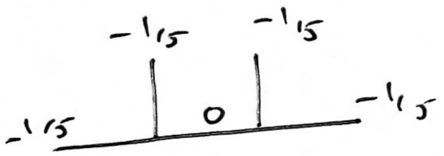
$$= x^{-1/8}(1-x)^{-1/8}\sqrt{\frac{1+\sqrt{1-x}}{2}}$$



$$= x^{3/8}(1-x)^{-1/8}\sqrt{2\left(1-\sqrt{1-x}\right)}$$

Lee Yang, $c = -22/5$

$r \uparrow$	0	$-1/5$	$-1/5$	0
	$\rightarrow s$			



$$= x^{2/5}(1-x)^{2/5}\, F\left(\tfrac{3}{5}, \tfrac{4}{5}; \tfrac{6}{5}; x\right)$$

$$= x^{1/5}(1-x)^{2/5}\, F\left(3/5, 2/5; 4/5; x\right)$$

$c = 1$, free boson, h irrational

$$= \left(\frac{x}{(1-x)}\right)^{2h}$$