$d=2$

Global transformations $\quad z \to \frac{az+b}{cz+d} \qquad \delta z = a + bz + cz^2$

Extend to $\quad z \to w(z) \qquad \delta z = \alpha(z)$

$$\delta\varphi = h\,\partial\alpha\,\varphi + \alpha\,\partial\varphi$$

$\alpha = z^{m+1}$: $\quad \delta\varphi = h(m+1)z^m\varphi + z^{m+1}\partial\varphi$

$$[L_m, \varphi] = z^{m+1}\partial\varphi + h(m+1)z^m\varphi$$

$$[L_m, L_n] = (m-n)L_{m+n} + \underbrace{\frac{c}{12}\, m(m^2-1)\,\delta_{m+n}}_{\text{vanishes on } sl(2)}$$

As before, $\lim_{z\to 0} \varphi(z)|0\rangle = |\varphi\rangle$

$$\Rightarrow \left.\begin{array}{ll} L_m|\varphi\rangle = 0 & m>0 \\ L_0|\varphi\rangle = h|\varphi\rangle & \end{array}\right\} \text{ h.w. state } |h\rangle$$

Representation determined by (h, c).

Spanned by $\{ L_{-p}^{m_p} \cdots L_{-1}^{m_1}|h\rangle \}$

$$[L_0, L_m] = -mL_m \;, \quad L_0|h\rangle = h|h\rangle$$

$$\Rightarrow L_0\left(L_{-p}^{m_p}\cdots L_{-1}^{m_1}|h\rangle\right) = (\underbrace{p\cdot m_p + \cdots + m_1}_{\text{“level”}} + h)\left(L_{-p}^{m_p}\cdots L_{-1}^{m_1}|h\rangle\right)$$

level	States	No.
0	$\|h\rangle$	1
1	$L_{-1}\|h\rangle$	1
2	$\left\{\begin{array}{l} L_{-1}^2\|h\rangle \\ L_{-2}\|h\rangle \end{array}\right.$	2
3	$\left\{\begin{array}{l} L_{-1}^3\|h\rangle \\ L_{-2}L_{-1}\|h\rangle \\ L_{-3}\|h\rangle \end{array}\right.$	3
n	$\left\{\begin{array}{l} L_{-1}^n\|h\rangle \\ \vdots \\ L_{-n}\|h\rangle \end{array}\right.$	$p(n)$ = No. of partitions of n

Field – state correspondence

$$|\varphi\rangle \longleftrightarrow \varphi(z)$$

$$[L_{-1},\varphi] = \partial\varphi : \qquad L_{-1}|\varphi\rangle \longleftrightarrow \partial\varphi$$

$$L_{-1}^n|\varphi\rangle \longleftrightarrow \partial^n\varphi$$

L_m are modes of $T(z) = \frac{1}{2\pi} T_{zz} = \sum L_m z^{-m-2}$

$$L_{-2}|\varphi\rangle \longleftrightarrow (T\varphi)$$

$$(T\varphi) = \Big(\sum_{m\leq -2} L_m z^{-m-2}\Big)\varphi + \varphi\Big(\sum_{m\geq -1} L_m z^{-m-2}\Big)$$

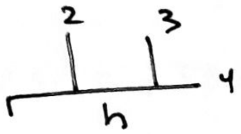
$$L_{-2}|\varphi\rangle = \lim_{z\to 0} (T\varphi)(z)|0\rangle$$

In case of $h=0$, $|0\rangle =$ vacuum state

$$L_{-1}|0\rangle = 0$$

$$L_{-2}|0\rangle \longleftrightarrow T(z)$$

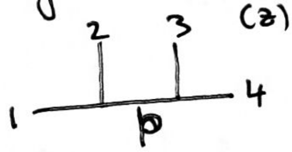
Virasoro chiral blocks defined as $su(1,1)$ blocks.



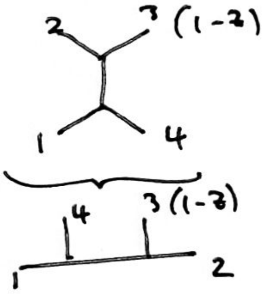
$$= \langle \varphi_1 | \varphi_2(1) \underset{\substack{\text{projector onto} \\ \text{rep}^{\underline{n}}\ h}}{\uparrow} \varphi_3(z) | \varphi_4 \rangle$$

normalised: $= z^{h-h_3-h_4} \{ \underbrace{1 + \dots}_{\text{power series in } z} \}$

Crossing relation:



$$= \sum_q F_{pq} \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$



<u>Example</u> Free fermion $\langle \psi(z)\psi(w)\rangle = \frac{1}{z-w}$

$$\langle \psi(u)\psi(v)\psi(z)\psi(w)\rangle = \frac{1}{u-v}\frac{1}{z-w} + \frac{1}{v-z}\frac{1}{u-w} - \frac{1}{v-w}\frac{1}{u-z}$$

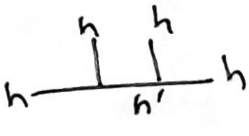
$h = \frac{1}{2}, \bar{h} = 0$

$$\langle \psi | \psi(1)\psi(z)|\psi\rangle = \lim_{\substack{v\to 1 \\ w \to 0 \\ u \to 0}} u^{2\cdot\frac{1}{2}} \langle \psi(u)\psi(v)\psi(z)\psi(w)\rangle$$

$$= \frac{1}{1-z} + \frac{1}{z} - 1$$

$$\frac{1}{1-z} + \frac{1}{z} - 1 = \frac{1}{z} + z + z^2 + z^3 + \dots$$

Unique expression in terms of <u>global</u> blocks



(global) $= z^{h'-2h} \; F(h', h'; 2h'; z)$

$$F(h', h'; 2h'; z) = 1 + \frac{h'\,h'}{2h'} z + \frac{h'(h'+1)\, h'(h'+1)}{2h'(2h'+1)\, 2!} z^2 + \dots$$

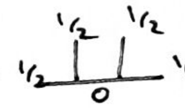
$$F(2,2;4;z) = 1 + z + \frac{9}{10} z^2 + \frac{4}{5} z^3 + \dots$$

$$F(4,4;8;z) = 1 + 2z + \frac{25}{9} z^2 + \frac{10}{3} z^3 + \dots$$

<u>Special case</u>

(global) $= z^{-2h}$

$$\frac{1}{1-z} + \frac{1}{z} - 1 = 1 \cdot$$



(global)

$+ \; 1 \cdot$

(global)

$+ \; \frac{1}{10} \cdot$

(global)

$+ \; \frac{1}{126} \cdot$

(global)

$+ \; \dots$

Free fermion: naturally a CFT with $c = 1/2$

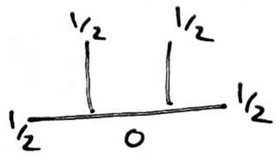
$$T = \frac{1}{2}\psi'\psi$$

$$\psi = \sum_{m \in \mathbb{Z}+\frac{1}{2}} \psi_m z^{-m-1/2}$$

$$L_{-2}|0\rangle = \lim_{z\to 0} T(z)|0\rangle = \frac{1}{2}\psi_{-3/2}\psi_{-1/2}|0\rangle$$

Only allowed values of h are $0, \frac{1}{2}, \frac{1}{16}$

$$\frac{1}{1-z} + \frac{1}{z} - 1 =$$

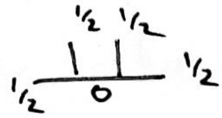


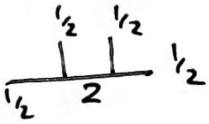
A single block (at $c = 1/2$).

This is by itself crossing symmetric. $\left(F_{00}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = 1 \right)$

Virasoro blocks depend on h and c

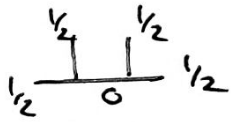
$$\langle \psi | \psi(1) \psi(z) | \psi \rangle = \frac{1}{1-z} + \frac{1}{z} - 1$$

$= 1 \cdot$ 

$+ \left(\frac{c - 1/2}{c} \right)$ 

$+ \frac{1}{10} \frac{(c - 1/2)(c + 39/10)}{(c + \frac{22}{5})(c + \frac{44}{5})}$ 

$+ \cdots$

At $c = 1/2$, only one term: $\langle \psi | \psi(1) \psi(z) | \psi \rangle =$ 

This is itself crossing symmetric.

Values of c in factors are special: All of the form

$$c = 13 - 6t - \frac{6}{t} \qquad t \text{ rational.}$$

$$\frac{1}{2} = c(3/4) \qquad -39/10 = c(5/12)$$

$$-\frac{22}{5} = c(2/5) \qquad -\frac{44}{5} = c(3/10) \qquad 0 = c(2/3)$$

Factor in denominator : null states that decouple

Factor in numerator : truncation of spectrum (values of h in the model)

Next 4 sections cover 4 methods of calculating the Virasoro conformal blocks

5. From definition (brute force)
6. Null states give differential equations
7. Recursion relations from analytic considerations
8. Exact formula from AGT correspondence