$d=2$. $\quad so(3,1) \simeq sl(2) \oplus sl(2)$

$z = x+iy, \quad \bar{z} = x-iy$

Translation: $\frac{\partial}{\partial x} = \partial + \bar{\partial}, \quad \frac{\partial}{\partial y} = i(\partial - \bar{\partial})$

Rotation: $\frac{\partial}{\partial \theta} = x\partial_y - y\partial_x = i(z\partial - \bar{z}\bar{\partial})$

Scale : $r\frac{\partial}{\partial r} = x\partial_x + y\partial_y = z\partial + \bar{z}\bar{\partial}$

Special conformal: $z^2\partial + \bar{z}^2\bar{\partial}, \quad i(z^2\partial - \bar{z}^2\bar{\partial})$

2 commuting sets: $\underbrace{\{\partial, z\partial, z^2\partial\}}_{sl(2)} \oplus \underbrace{\{\bar{\partial}, \bar{z}\bar{\partial}, \bar{z}^2\bar{\partial}\}}_{sl(2)}$

choose $L_{-1} : \partial \qquad [L_1, L_{-1}] = 2L_0$

$L_0 : z\partial \qquad [L_0, L_1] = -L_1$

$L_1 : z^2\partial \qquad [L_0, L_{-1}] = L_{-1}$

Transformations of fields:

$z \to z + \alpha(z) \qquad \delta\varphi = h\,\partial\alpha\,\varphi + \alpha\partial\varphi$

$\delta z = a, \quad \delta\varphi = a\partial\varphi, \quad [L_{-1}, \varphi] = \partial\varphi$

$\delta z = bz, \quad \delta\varphi = hb\varphi + bz\partial\varphi$

$[L_0, \varphi] = h\varphi + z\partial\varphi$

$\delta z = cz^2 \quad \delta\varphi = 2hcz\varphi + cz^2\varphi$

$[L_1, \varphi] = 2hz\varphi + z^2\partial\varphi$

$$\boxed{[L_m, \varphi] = z^{m+1}\partial\varphi + h(m+1)z^m\varphi}$$

If φ quasiprimary,

$$|\varphi\rangle = \lim_{z\to 0} \varphi(z)|0\rangle$$

$$\Rightarrow \quad \left.\begin{array}{l} L_1|\varphi\rangle = 0 \\ L_0|\varphi\rangle = h|\varphi\rangle \end{array}\right\} \text{ defn of a highest weight state.}$$

Space of states spanned by $\{ L_{-1}^n |\varphi\rangle \}$

Conjugation: $(L_m)^\dagger = L_{-m}$

Projector is $\displaystyle\sum_n \frac{L_{-1}^n |\varphi\rangle\langle\varphi| L_1^n}{\langle\varphi| L_1^n L_{-1}^n |\varphi\rangle}$

Recursion: $L_1 L_{-1}^n |\varphi\rangle = n(2h+n-1) L_{-1}^{n-1} |\varphi\rangle$

$$\Rightarrow \langle\varphi| L_1^n L_{-1}^n |\varphi\rangle = n!\, \underbrace{(2h+n-1)\cdots(2h)}_{(2h)_n} \langle\varphi|\varphi\rangle$$

[n.b. notation is not universal.]

Need $\quad \langle \varphi_1 | \varphi_2(1) L_{-1}^n | h \rangle$

Consider $L_{-1} - L_0$

$$[L_{-1} - L_0, \varphi_h(z)] = \partial\varphi_h(z) - (z\partial\varphi_h(z) + h\varphi_h)$$
$$= (1-z)\,\partial\varphi_h - h\varphi_h$$
$$[L_{-1} - L_0, \varphi(1)] = -h\varphi$$

Also $\quad \langle \varphi_1 | L_1 = 0 \qquad \langle \varphi_1 | L_0 = h_1 \langle \varphi_1 |$

$$L_{-1}(L_{-1})^m |\varphi_h\rangle = (L_{-1} - L_0 + L_0) L_{-1}^m |\varphi_h\rangle$$
$$= (L_{-1} - L_0 + m + h) L_{-1}^m |\varphi_h\rangle$$

$$\Rightarrow \quad \langle \varphi_1 | \varphi_2(1) L_{-1}^n | \varphi_h \rangle$$
$$= (h + n - 1 + h_2 - h_1) \langle \varphi_1 | \varphi_2(1) L_{-1}^{n-1} | \varphi_h \rangle$$
$$= (h + n - 1 + h_2 - h_1)(h + n - 2 + h_2 - h_1) \cdots (h + h_2 - h_1) \langle \varphi_1 | \varphi_2(1) | \varphi_h \rangle$$
$$= (h + h_2 - h_1)_n \underbrace{\langle \varphi_1 | \varphi_2(1) | \varphi_h \rangle}_{\text{const.}}$$

Likewise, $\quad \langle \varphi_h | L_1^n \varphi_3(z) | \varphi_4 \rangle$

$$= (h + h_3 - h_4)_n \, z^n \underbrace{\langle \varphi_h | \varphi_3(z) | \varphi_4 \rangle}_{z^{h - h_3 - h_4} \cdot \text{const}}$$

$$\langle \varphi_1 | \, \varphi_2(1) \sum_n \frac{L_{-1}^n |\varphi_h\rangle\langle \varphi_h| L_1^n}{\langle \varphi_h | L_1^n \, L_{-1}^n | \varphi_h \rangle} \varphi_3(z) \,|\, \varphi_4 \rangle$$

$$= \sum_n \cdot \left[(h + h_2 - h_1)_n \cdot \text{const} \right] \cdot \frac{1}{n! \, (2h)_n} \cdot \left[z^{n + h - h_3 - h_4} (h + h_3 - h_4)_n \cdot \text{const} \right]$$

$$= \text{const.} \; z^{h - h_3 - h_4} \sum_n z^n \frac{(h + h_2 - h_1)_n (h + h_3 - h_4)_n}{n! \; (2h)_n}$$

$$= \text{const.} \; z^{h - h_3 - h_4} {}_2F_1 (h + h_2 - h_1, h + h_3 - h_4; \; 2h; \; z)$$

$$\text{Conformal block} = \begin{cases} z^{h - h_3 - h_4} F(\ldots) \\ z^h F(\ldots) \\ F(\ldots) \end{cases}$$

<u>Note</u> $F(a, b; c; z) = \sum_n \frac{(a)_n (b)_n}{(c)_n n!} z^n$

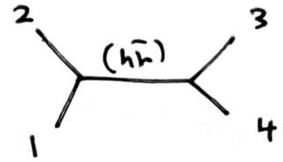
satisfies $z(z-1)F'' + [(a+b+1)z - c]F' + abF = 0$

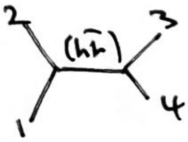
2nd solution $z^{1-c} F(a - c + 1, \; b - c + 1; \; 2 - c; \; z)$

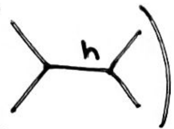
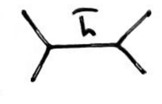
Full result:

$$\langle \varphi_1 | \varphi_2(1) \quad \varphi_3(z,\bar{z}) \quad | \varphi_4 \rangle$$

$$= \sum_{(h,\bar{h})} C_{12}^{(h\bar{h})} \cdot C_{34}^{(h\bar{h})} \cdot$$

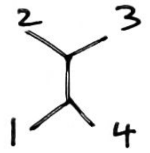




$= \Big($  $\Big)(z) \cdot \Big($  $\Big)(\bar{z})$

conformal block in $d=2$

Crossing symmetry.



$$z \mapsto 1-z$$
$$0 \mapsto 1$$
$$1 \mapsto 0$$
$$\infty \mapsto \infty$$

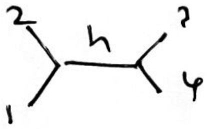
$$\langle \varphi_1 | \varphi_2(1) \, \varphi_3(z) | \varphi_4 \rangle$$
$$= (\text{phase}) \cdot \langle \varphi_1 | \varphi_3(1) \, \varphi_2(1-z) | \varphi_4 \rangle$$

Known relation

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z)$$
$$+ \quad (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b; c-a-b+1; 1-z)$$

[can also relate to $F\left(\frac{z}{z-1}\right)$, $F\left(\frac{1}{1-z}\right)$, $F\left(\frac{1}{z}\right)$]

Note that

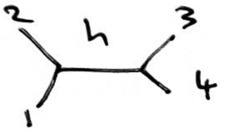


$$= z^{h-h_3-h_4}\left(1 + \frac{(h-h_1+h_2)(h+h_3-h_4)}{2h} z + \dots\right)$$

Reason: only consistent if $\langle \varphi_1 \; \varphi_2 \rangle \neq 0$

Requires $h_1 = h_2$

If $h_1 = h_2$, $h_3 = h_4$,



$$= z^{h-2h_3}\left(1 + \frac{h}{2} z + \dots\right)$$

$$= z^{h-2h_3} F(h, h; 2h; z)$$