$$x^\mu \to x'^\mu$$

$$g'_{\mu\nu} \to \left(\frac{\partial x^\rho}{\partial x'^\mu}\right)\left(\frac{\partial x^\sigma}{\partial x'^\nu}\right) g_{\rho\sigma} = \Omega^2 g_{\mu\nu}$$

$$x'^\mu = x^\mu + \alpha^\mu$$

$$\Rightarrow \partial_\mu \alpha_\nu + \partial_\nu \alpha_\mu = f \cdot g_{\mu\nu} = \frac{2}{d}(\partial \cdot \alpha)\, g_{\mu\nu}$$

$$\Rightarrow \Box \alpha_\nu = \frac{2-d}{d}\, \partial_\nu (\partial \cdot \alpha)$$

$$\Rightarrow \frac{2}{d}(1-d)\, \Box(\partial \cdot \alpha) = 0 \qquad \Box(\partial\cdot\alpha) = 0 \quad (d \neq 1)$$

$$(2-d)\,\partial_\mu \partial_\nu \partial_\tau \alpha_\rho = \frac{\Box(\partial\cdot\alpha)}{d}\left\{ g_{\mu\nu} g_{\tau\rho} - g_{\mu\rho} g_{\nu\tau} + g_{\mu\tau} g_{\rho\nu} \right)$$

$$\Rightarrow \partial_\mu \partial_\nu \partial_\tau \alpha_\rho = 0 \qquad (d \neq 1, 2)$$

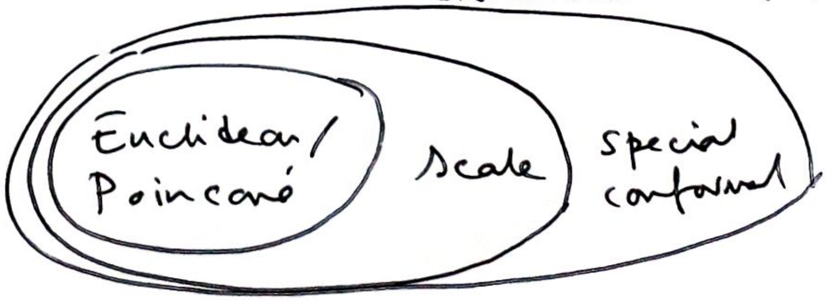
Soln: $$\alpha^\mu = \underbrace{a^\mu}_{\text{translation}} + \underbrace{\omega^\mu{}_\nu x^\nu}_{\text{rotation/ Lorentz}} + \underbrace{\lambda x^\mu}_{\text{scale}} + \underbrace{b^\mu x^2 - 2x^\mu (b\cdot x)}_{\text{special conformal trans.}}$$

| translation | rotation/ Lorentz | scale | special conformal trans. |
|---|---|---|---|
| $d$ | $\frac{1}{2}d(d-1)$ | $1$ | $d$ |

Total: $\frac{1}{2}(d+1)(d+2)$ — Euclidean: $so(1, d+1)$

Lorentzian: $so(2, d)$

n.b



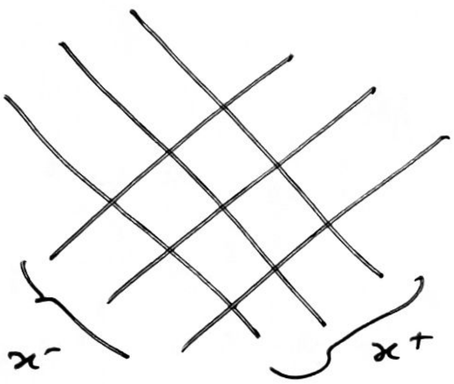
$d=2$: $\qquad \partial_\mu \alpha_\nu + \partial_\nu \alpha_\mu = g_{\mu\nu} (\partial \cdot \alpha)$

Lorentzian: light cone coordinates $\qquad x^\pm = t \pm x$

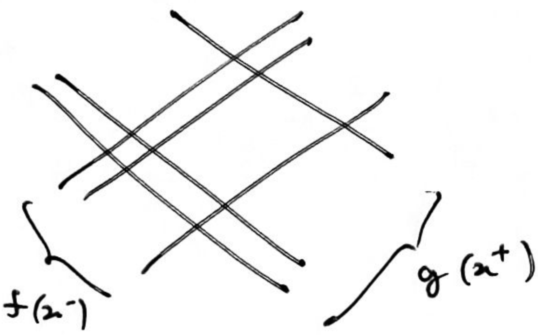
$$\Rightarrow \quad \boxed{\partial_+ \alpha^- = \partial_- \alpha^+ = 0}$$

Independent reparametrisations of the light cones

$$x^+ \to f(x^+) \qquad x^- \to g(x^-)$$




$\to$




Euclidean: $\qquad z = x + iy$

$$\partial_z \alpha^{\bar z} = \partial_{\bar z} \alpha^z = 0$$

$$\boxed{z \to f(z), \quad \bar z \to g(\bar z)}$$

Scale transformation $\quad x \to \lambda x$

Scaling field: $\quad \varphi(x) \to \lambda^{\Delta} \varphi(x')$

$x'^{\mu} = x^{\mu} + \alpha x^{\mu} \qquad \delta\varphi = \alpha\left(\Delta\varphi + x\cdot\partial\varphi\right)$

$$\varphi(x) \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta/D} \varphi(x')$$

$$\delta\varphi = \frac{\Delta}{D}(\partial\cdot\alpha)\varphi + \alpha\cdot\partial\varphi$$

$$\left[\ \Box\varphi \to \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta+2}{D}} \Box'\varphi + \dots + \left(\Box\left(\frac{\partial x'}{\partial x}\right)^{\frac{\Delta}{D}}\right)\varphi\ \right]$$

$\underline{D=2}$: $\quad z \to w(z), \quad \bar{z} \to \bar{w}(\bar{z})$

$$\varphi(z,\bar{z}) \to \left(\frac{\partial w}{\partial z}\right)^{h}\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \varphi(w,\bar{w}) \qquad (*)$$

Scale: $w = \lambda z$, $\bar{w} = \lambda\bar{z}$ $\qquad \varphi(z,\bar{z}) \to \lambda^{h+\bar{h}} \varphi(w,\bar{w})$ $\qquad \Delta = h+\bar{h}$ scale dim.

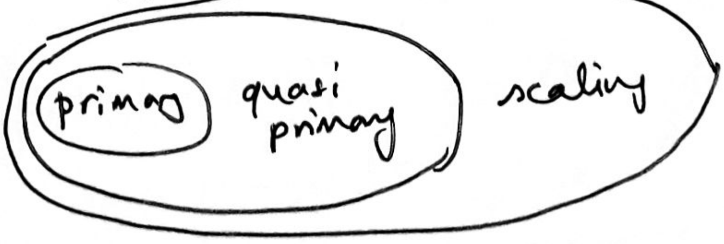
Rot2: $w = e^{i\theta} z$, $\bar{w} = e^{-i\theta}\bar{z}$ $\qquad \varphi(z,\bar{z}) \to e^{i\theta(h-\bar{h})} \varphi(w,\bar{w})$ $\qquad s = h-\bar{h}$ "spin"

If $(*)$ for scale trans. $\Rightarrow$ scaling field

$(*)$ for global conf trans $\Rightarrow$ quasiprimary field

$$z \to \frac{az+b}{cz+d}$$

If $(*)$ for all $w(z)$, $\bar{w}(\bar{z})$ $\Rightarrow$ primary field



$$\langle \varphi(x)\ \varphi(y) \rangle$$

· translate by $(-y)$

· Rotate so $(x-y)^\mu \to |x-y|(1,0\ldots\ldots 0)$

$$\langle \varphi(x)\ \varphi(y) \rangle = \frac{1}{|x-y|^{2\Delta}} \langle \varphi(1,0..)\ \varphi(0) \rangle$$

Define $|\varphi\rangle = \lim\limits_{x\to 0} \varphi(x)|0\rangle$, $\quad \langle\varphi| = \lim\limits_{x\to\infty} |x|^{2\Delta} \langle 0|\varphi(x)$

$$\langle \varphi(x)\ \varphi(y) \rangle = \frac{\langle \varphi | \varphi \rangle}{|x-y|^{2\Delta}}$$

$$\langle \varphi_1(x)\ \varphi_2(y)\ \varphi_3(z) \rangle$$

$$= \underbrace{\langle \varphi_1 |\ \varphi_2(1,0..)\ | \varphi_3 \rangle}_{C_{123}} \cdot |x-y|^{\Delta_1-\Delta_2-\Delta_3} \cdot |x-z|^{\Delta_2-\Delta_1-\Delta_3} \cdot |y-z|^{\Delta_1-\Delta_2-\Delta_3}$$

$$\langle \varphi_1(x_1)\ \varphi_2(x_2)\ \varphi_3(x_3)\ \varphi_4(x_4) \rangle$$

$$=$$

$x_1 \to \infty$

$x_4 \to 0$

scale and rotate $\quad x_2 \to (1, 0, \dots)$

rotate $x_3$ into 1-2 plane $\quad x_3 \to (x, y, \dots)$

Result:

$$\langle \varphi_1(x_1)\, \varphi_2(x_2)\, \varphi_3(x_3)\, \varphi_4(x_4) \rangle$$

$$= \Big(\text{prefactor}\Big) \cdot \langle \varphi_1 |\, \varphi_2(1,0..)\, \varphi_3(x,y..)\, | \varphi_4 \rangle$$

Now insert complete set of states

$$\langle \varphi_1 | \varphi_2(1,0) \sum |i\rangle\langle i|\, \varphi_3(x,y..)\, |\varphi_4\rangle$$

Can group by representations of the conformal group

$$= \sum_\rho \langle \varphi_1 | \varphi_2(1,0) \sum_{i \in \rho} |i\rangle\langle i|\, \varphi_3(x,y)\, |\varphi_4\rangle$$

$$= \sum_\rho C_{12\rho} C_{\rho 34} \cdot \underbrace{F_\rho(x,y)}$$

{ Conformal partial wave
{ Conformal block

$$\langle \varphi_1 | \varphi_2\, \varphi_3 | \varphi_4 \rangle = \sum_\rho C_{12\rho} C_{\rho 34}$$ 


$$= \sum_\sigma C_{23\sigma} C_{\sigma 14}$$