Generalized Symmetries in QFT and in String Theory

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work in collaboration over the last 2a with Fabio Apruzzi, Lakshya Bhardwaj, Federico Bonetti, Lea Bottini, Michele del Zotto, Inaki Garcia-Etxebarria, Dewi Gould, Sagar Hosseini, Apoorv Tiwari

Generalized Global Symmetries

Consider any topological operators in a QFT as generators of symmetries.

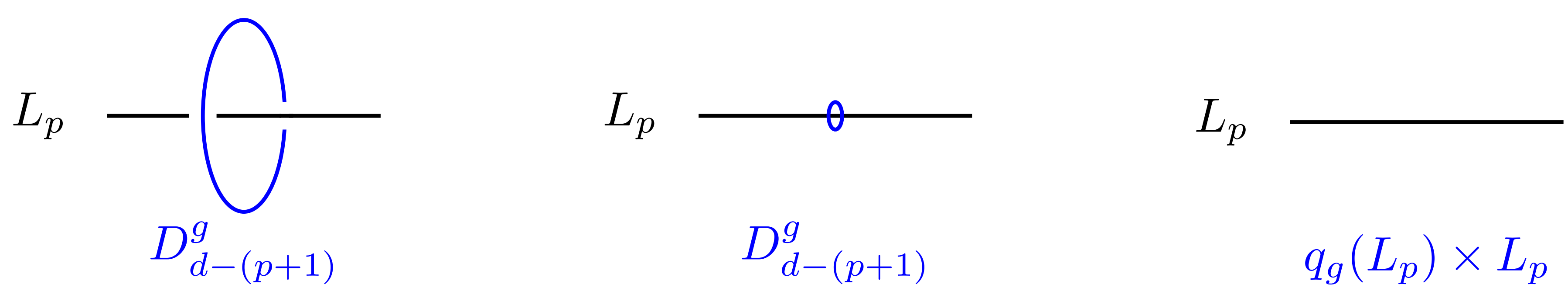
1. Higher-form symmetries $\Gamma^{(p)}$:
   charged objects are p-dimensional defects, charge measured by topological operators $D^g_{d-(p+1)}$.
2. Higher-group symmetries:
   p-form symmetries might not form product groups
3. Non-invertible symmetries:
   relax group law $\Rightarrow$ fusion algebra
4. Higher-categorical symmetries:
   topological operators of dimensions $0, \cdots, d-1$, with non-invertible fusion

Higher-Form Symmetries $\Gamma^{(p)}$

A $\Gamma^{(p)}$-form symmetry is generated by topological operators of codimension $p+1$, $D^{g}_{d-(p+1)}$, $g \in \Gamma^{(p)}$ satisfying an (abelian) group law

$$D^{g}_{d-(p+1)} \otimes D^{h}_{d-(p+1)} = D^{gh}_{d-(p+1)}\,, \qquad g, h \in \Gamma^{(p)}$$

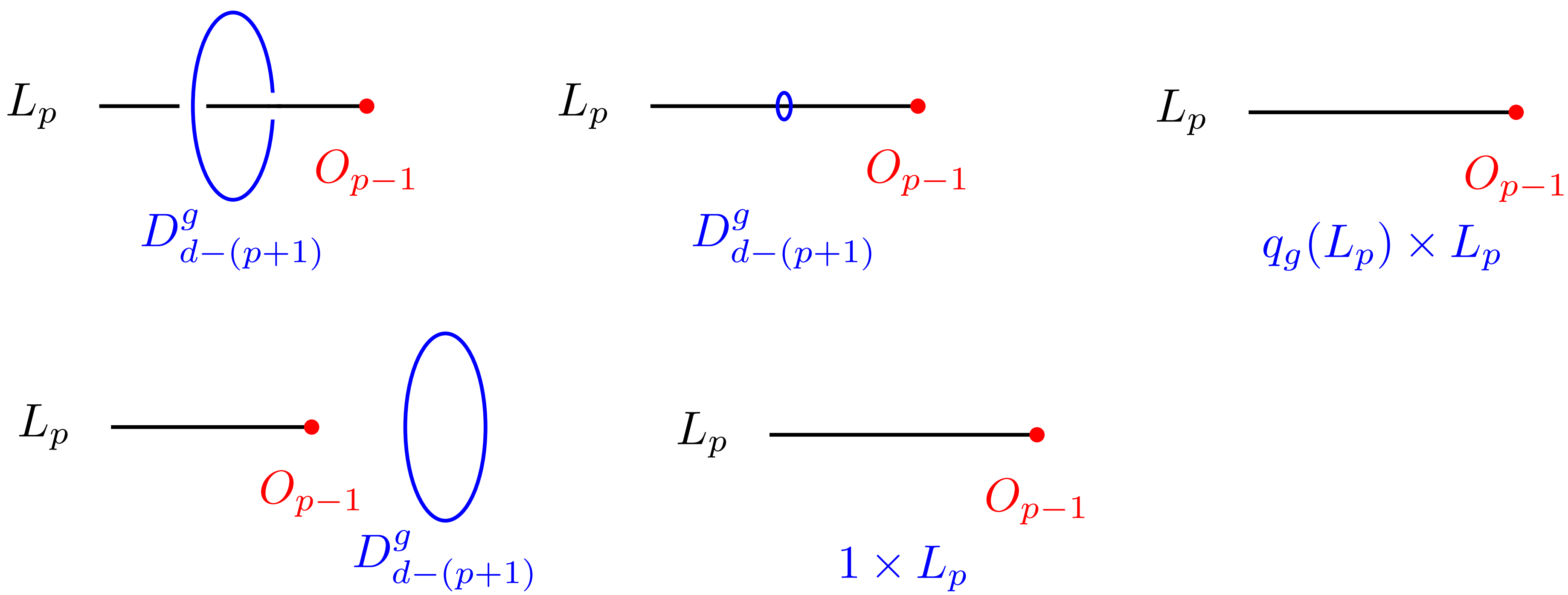
p-dim extended operators links with these, and can be charged under $\Gamma^{(p)}$:



- Background field: $B_{p+1} \in H^{p+1}(M, \Gamma^{(p)})$
- Gauging: summing over all such B.

Screening of Higher-Form Symmetries

p-form symmetries can be screened by $p-1$ dimensional operators: ″endable″



Perspective from Charged Defects

Let $\mathcal{L}_p$ be the dim p defects.

There is an equivalence relation on $\mathcal{L}_p$:

$$L_p^{(1)} \sim L_p^{(2)} \quad \Leftrightarrow \quad \exists\, O_{p-1} \text{ at the junction between } L_p^{(1)} \text{ and } L_p^{(2)}$$

Define the quotient:

$$\widehat{\Gamma^{(p)}} := \mathcal{L}_p / \sim$$

This is the Pontryagin dual group of $\Gamma^{(p)}$

$$\widehat{\Gamma^{(p)}} = \mathrm{Hom}(\Gamma^{(p)}, U(1))$$

Line Operators and 1-form symmetry

<u>Example:</u>
$p = 1$: line operators, with junctions formed by local operators.

$$L_1^{(1)} \sim L_1^{(2)} \quad \Longleftrightarrow \quad \text{there exists } \; L_1^{(1)} \;\underset{O_0^{(12)} \neq 0}{\text{———•———}}\; L_1^{(2)}$$

E.g. in a pure G (simply-connected) Yang Mills theory, we have fundamental Wilson lines. The only local operators are in the adjoint, so

$$\Gamma^{(1)} = Z_G = \text{Center}(G)$$

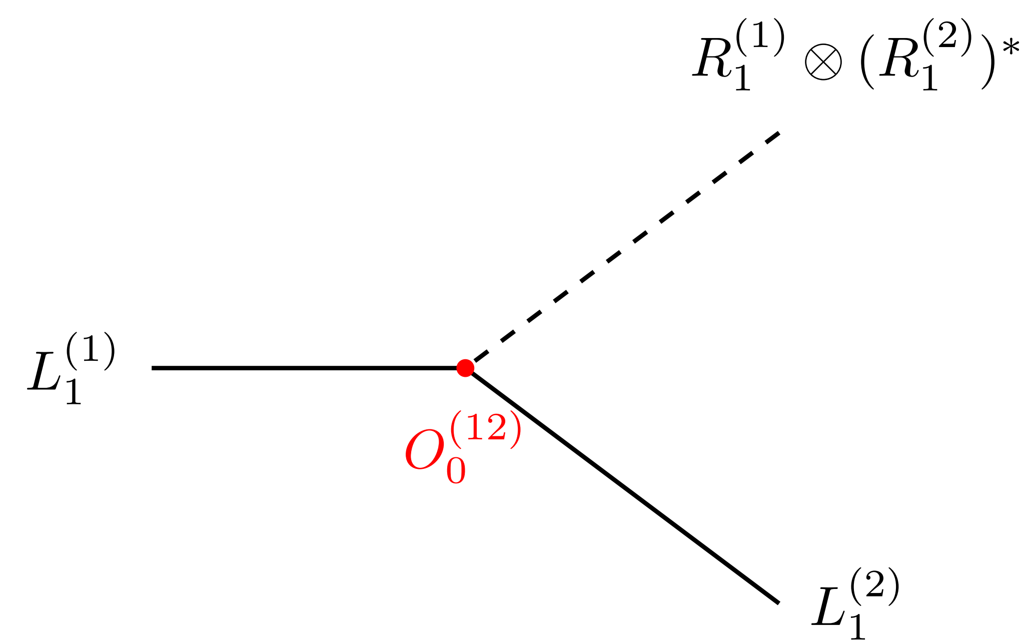
Screening by matter, depends on charge of reps under center.

Higher-Group Symmetries

2-group symmetries:
in this screening picture, the local operators could be charged under a 0-form (flavor) symmetry $\mathcal{F} = F/C$ (F simply connected)

Refine the equivalence relation on the set of genuine line operators and flavor Wilson lines (L, R):

$$\left(L_1^{(1)}, R_1^{(1)}\right) \sim \left(L_1^{(2)}, R_1^{(2)}\right) \quad \Longleftrightarrow \quad \exists$$



Define: $\quad \widehat{\mathcal{E}} = \{(L, R)\}/\sim$

$$\widehat{\mathcal{E}} = \{(L,R)\}/\sim$$

- $\widehat{\mathcal{E}} \twoheadrightarrow \widehat{\Gamma^{(1)}}$ by forgetting the flavor Wilson lines.
- $\widehat{C} \hookrightarrow \widehat{\mathcal{E}}$ by taking (id, R).
  $\widehat{C}$ which is the "center-symmetry" of the flavor symmetry F.

$$0 \to \Gamma^{(1)} \to \mathcal{E} \to C \to 0$$

If this sequence does not split then there is a non-trivial Bockstein homomorphism

$$\text{Bock} : H^2(-,C) \to H^3(-,\Gamma^{(1)})$$

and there is a 2-group if

$$\delta B_2 = \Theta \in H^3(B\mathcal{F},\Gamma^{(1)})$$

is non-zero. Θ is called the Postnikov class.

The 1-form symmetry background is not closed, and depends on the 0-form symmetry background.

Example

Consider a QFT with 0-form symmetry and 1-form symmetry satisfying the following conditions:

- $F = SU(2)$ $\mathcal{F} = SO(3) = \frac{SU(2)}{\mathbb{Z}_2}$, $C = \mathbb{Z}_2$. And let

  $$w_2 \in H^2(BSO(3), \mathbb{Z}_2)$$

  be the class obstructing lifting the flavor group to $SU(2)$.

- $\Gamma^{(1)} = \mathbb{Z}_2$

- From the spectrum of local operators, the charge under gauge and flavor symmetries implies that

  $$0 \to \mathbb{Z}_2^{(1)} \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$$

  I.e. $\mathcal{E} = \mathbb{Z}_4$

Then

$$\delta B_2 = \text{Bock}(w_2) \in H^3(BSO(3), \mathbb{Z}_2^{(1)})$$

Ubiquity of Higher-Group Symmetries

In physics:

[Sharpe][Tachikawa][Benini, Cordova, Hsin][Cordova, Dumitrescu, Intriligator]
6d SCFT (full classification) [Apruzzi, Bhardwaj, Gould, SSN] 5d SCFTs [Apruzzi, Bhardwaj, Oh, SSN][Del Zotto, Heckman, Meynet, Moscrop, Zhang],
4d class S [Bhardwaj] 3d/4d: [Hsin, Lam][Lee, Ohmori, Tachikawa][Apruzzi, Bhardwaj, Gould, SSN][Bhardwaj, Bullimore, Ferrari, SSN].

Higher-groups are at least as ubiquitous as (non-anomalous) $\Gamma^{(p)}$ with mixed anomalies. Gauging $\delta B_2 = \mathrm{Bock}(w_2)$, with a non-anomalous 1-form symmetry yields a mixed anomaly

$$\int B_{d-2} \cup \mathrm{Bock}(w_2)$$

between the $\Gamma^{(d-3)}$ and 0-form symmetry. Other mixed anomalies yielding 2-groups

$$\int A_1 \cup B_2 \cup C_2$$

which are dual to 2-groups after gauging C_2 or B_2.

Non-Invertible and Higher-categorical Symmetries

So far we assumed that the topological operators obey a group like fusion:

$$D^{g}_{d-(p+1)} \otimes D^{h}_{d-(p+1)} = D^{gh}_{d-(p+1)}\,, \qquad g, h \in \Gamma^{(p)}$$

Well-known . in $d = 2, 3$ that topological lines can obey non-trivial fusion algebra relations:

$$D_1^{\alpha} \otimes D_1^{\beta} = \bigoplus_{\gamma} N_{\gamma}^{\alpha,\beta} D_1^{\gamma}$$

with $N \in \mathbb{Z}_{>0}$.

In $d = 2, 3$: modular tensor categories (lines can fuse and braid).

Until last year there were no examples known of higher dimensional – $d \geq 4$ – non-invertible symmetries of QFTs: [Heidenreich, McNamara, Montero,Reece, Rudelius, Valenzuela] [Kaidi, Ohmori, Zheng][Choi,Cordova, Hsin, Lam, Shao][Bhardwaj, Bottini, SSN, Tiwari]

Non-Invertible Higher-Categorical Symmetries

Higher fusion category in terms of the topological operators for d$\geq$ 3 QFTs.
See also $\Rightarrow$ Lakshya Bahrdwaj's talk

Simple example: 4d Spin$(4N)$ Yang-Mills [Bhardwaj, Bottini, SSN, Tiwari]

$$\Gamma^{(1)} = \mathbb{Z}_2^S \times \mathbb{Z}_2^C \,.$$

The diagonal $\mathbb{Z}_2^V$. The topological operators generating $\Gamma^{(1)}$ are

$$\mathcal{C}^{\text{objects}}_{\text{Spin}(4N)} = \left\{ D_2^{(\text{id})}, D_2^{(S)}, D_2^{(C)}, D_2^{(V)} \right\}$$

There is on each topological surface defect $D_2^{(g)}$ an endomorphism

$$\mathcal{C}^{\text{1-endo}}_{\text{Spin}(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\}$$

All these satisfy the group law

$$D_i^{(g)} \otimes D_i^{(h)} = D_i^{(gh)}$$

There is an outer automorphism

$$\Gamma^{(0)} = \mathbb{Z}_2$$

which acts

$$\Gamma^{(0)}: \qquad D_i^{(S)} \leftrightarrow D_i^{(C)}$$

Gauging $\Gamma^{(0)}$ results in the $\mathrm{Pin}^+(4N)$ gauge theory.

The invariant topological operators, i.e. objects

$$\mathcal{C}_{\mathrm{Pin}^+(4N)}^{\text{objects}} = \left\{D_2^{(\mathrm{id})}, D_2^{(SC)}, D_2^{(V)}\right\}, \qquad D_2^{(SC)} = \left(D_2^{(S)} \oplus D_2^{(C)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}}$$

Trivial fusion of invariant objects:

$$D_2^{(\mathrm{id})} \otimes D_2^{(V)} = D_2^{(V)}, \qquad D_2^{(V)} \otimes D_2^{(V)} = D_2^{(\mathrm{id})}$$

But the non-invertible fusion of quantum dim 2 objects:

$$D_2^{(SC)} \otimes D_2^{(SC)} = D_2^{(\mathrm{id})} \oplus D_2^{(V)}$$

1-morphisms are the invariant ones but also the dual to the 0-form symmetry: 2-form symmetry, generated by a topological line

$$\Gamma^{(0)} = \mathbb{Z}_2 \ \text{gauged} \quad \Rightarrow \quad \Gamma^{(2)} = \mathbb{Z}_2 : \quad \text{generated by } D_1^{(-)}$$

Similarly: 1-morphism on the invariant $D_2^{(V)}$, in addition to $D_1^{(V)}$: $D_1^{(V_-)}$.

The morphisms of the new category are:

$$\mathcal{C}^{\text{1-endo}}_{\text{Pin}^+(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)} \right\}$$

with the non-invertible fusion

$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\text{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)}$$

There are also 2-morphisms, which are point-like local operators on the topological defects.

$\Rightarrow$ Generically the symmetries of 4d QFTs form 2-categories, with non-invertible fusion algebras

Generalized Symmetries in String Theory

Physically there are many motivations for studying these generalized symmetries (strong coupling, anomalies, vacuum structure etc.). To utilize e.g. powerful constraints arising from symmetries for strongly-coupled theories such as 5d and 6d SCFTs, we need to understand how they are realized within string theory.

How do we determine the generalized global symmetries
(and anomalies) of QFTs realized in string theory?

To be precise: lets focus on M-theory on canonical Calabi-Yau three-fold singularities, constructing 5d $\mathcal{N} = 1$ SCFTs.

5d $\mathcal{N}=1$ SCFTs – from Geometry

$\mathcal{T}$= M-theory
X =canonical Calabi-Yau 3-fold singularities
realizes $\mathcal{T}(\mathbf{X})$ =5d SCFTs

Dictionary:

$$\begin{aligned}\text{Canonical singularity} &\longleftrightarrow \text{SCFT}\\ \text{Kähler cone} &\longleftrightarrow \text{Coulomb Branch: vev of vector-multiplet scalars}\\ \text{Complex deformations} &\longleftrightarrow \text{Higgs Branch: vev of hyper-multiplet scalars}\end{aligned}$$

On the Coulomb branch:

1. Gauge theory descriptions arise from introducing compact divisors, with gauge coupling $1/g^2$ =vol.
2. Non-compact divisors realize flavor (0-form) symmetries.
3. Matter: wrapped M2-brane states on compact curves.

Relative Homology cycles to Higher-form Symmetry

Recall equivalence relation definition of $\Gamma^{(1)}$:

$$L_1^{(1)} \sim L_1^{(2)} \quad \Longleftrightarrow \quad \text{there exists } \; L_1^{(1)} \xrightarrow[O_0^{(12)} \neq 0]{\qquad\qquad} L_1^{(2)}$$

[Morrison, SSN, Willett][Albertini, del Zotto, Garcia Etxebarria, Hosseini]

- # M2-branes on compact 2-cycles: $H_2(\mathbf{X})$: mass $m < \infty$ particles
- # M2-brane on non-compact 2-cycle: $H_2(\mathbf{X}, \partial\mathbf{X})$
  infinite massive particle, worldline defines line operator.

$$\Rightarrow \qquad \widehat{\Gamma}^{(1)} = \frac{H_2(\mathbf{X}, \partial\mathbf{X})}{H_2(\mathbf{X})}$$

We can also map this problem to the boundary of $\mathbf{X}$:

$$\cdots \to H_2(\mathbf{X}) \xrightarrow{f_2} H_2(\mathbf{X}, \partial\mathbf{X}) \xrightarrow{g_2} H_1(\partial\mathbf{X}) \xrightarrow{h_1} H_1(\mathbf{X}) \to \cdots$$

$$\Rightarrow \; \widehat{\Gamma^{(1)}} = \mathrm{im}(g_2) = \ker(h_1) \subseteq H_1(\partial\mathbf{X})$$

1-Form Symmetry: Examples

For $SU(2)_0$, **X** is the cone over Hirzebruch surface $\mathbb{F}_2$:

$$\Gamma^{(1)} = \mathbb{Z}_2$$

as expected for a pure gauge theory.

The link is ∂**X** particularly simple for **X** toric: Sasaki-Einstein manifolds. E.g.

$$\partial \mathbf{X} = Y^{p,q}\,, \qquad H_1(\partial \mathbf{X}) = \mathbb{Z}_{\mathrm{gcd}(p,q)}\,.$$

UV-fixed points with IR description $SU(p)_q$ (q is here the Chern-Simons Coupling in 5d)

2-group Symmetries

To compute the higher-form symmetries we need to determine the 0-form symmetry group, as well as $\Gamma^{(1)}$.

There are two approaches we have considered for the 2-groups:

1. $\mathcal{E}$ from the spectrum of local operators, i.e. curves in **X**, including flavor group $\mathcal{F}$

   [Apruzzi, Bhardwaj, Oh, SSN]

   Example: $SU(2)_0$ realized by the cone over $\mathbb{F}_2$ have $\mathcal{F} = SO(3)$ flavor and a non-trivial 2-group: with extension and Postnikov class

   $$0 \to \mathbb{Z}_2^{(1)} \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$$

   $$\delta B_2 = \mathrm{Bock}(w_2) \in H^3(BSO(3), \mathbb{Z}_2^{(1)})$$

   where $w_2 \in H^2(BSO(3), \mathbb{Z}_2)$

2. From the boundary ∂**X**

   [del Zotto, Garcia Etxebarria, SSN]

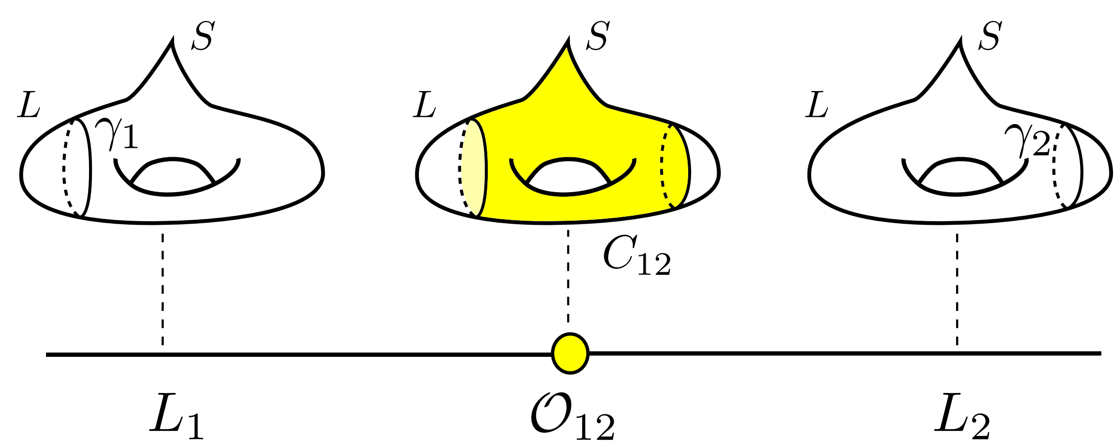
Two-groups from the Link

[del Zotto, Garcia Etxebarria, SSN]

Let $L = \partial \mathbf{X}$.
Let $\mathcal{S}$ = singular locus modeling the flavor symmetry intersected with L (i.e. intersection of non-compact divisors with the boundary)

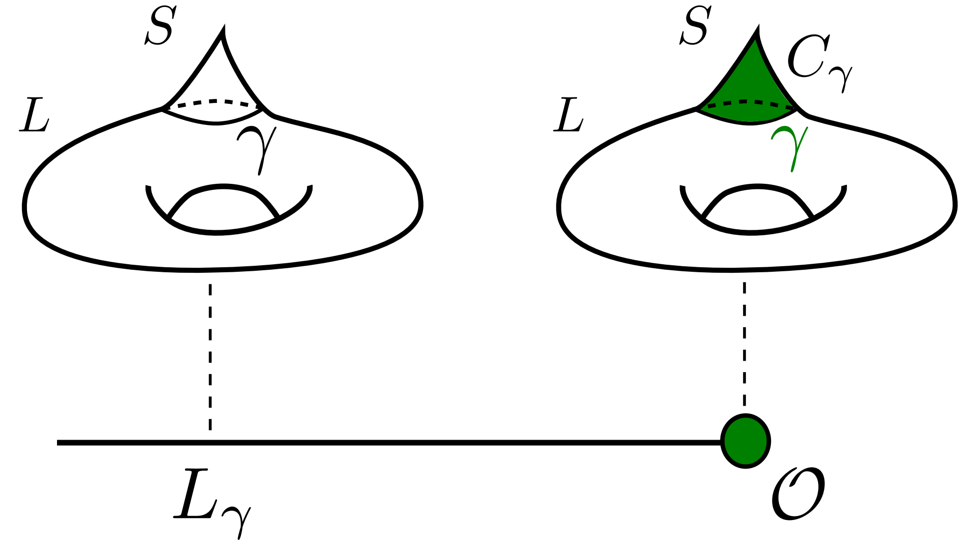
On the boundary the relation on line operators $\sim$ has the following depiction:

$\gamma_i = \partial C_i$, $C_i \in H_2(\mathbf{X}, L)$.
If there is a 2-chain C_{12} with $\partial C_{12} = \gamma_1 \cup \gamma_2$ then M2s on C_{12} give rise to $\mathcal{O}_{12}$:



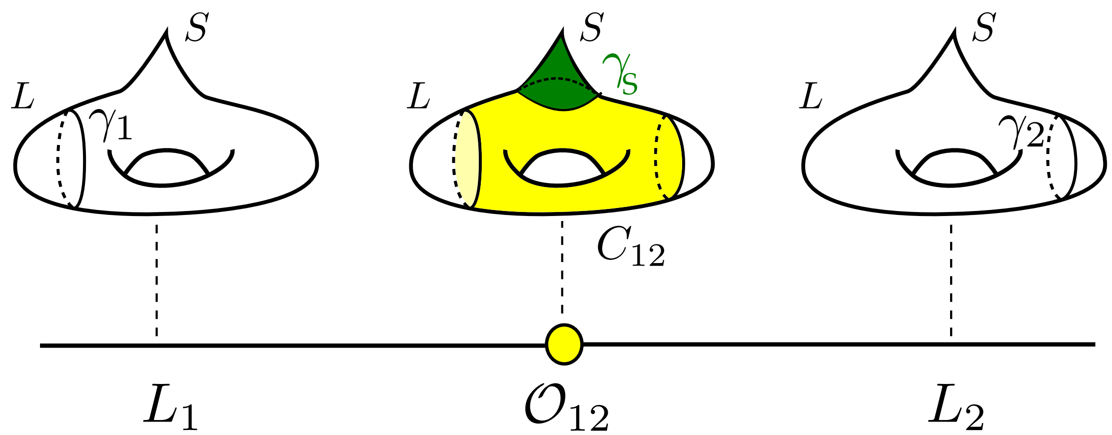
$\mathcal{E}$ from Link Geometry

We can also model geometrically the lines that are pure "flavor Wilson lines" charged under C consider a line that is trivial in $\widehat{\Gamma}^{(1)}$, i.e. γ is trivial in $H_1(L)$. This is charged under the flavor center symmetry C, if $\gamma \in \mathrm{Tor}\, H_1(\partial\mathcal{S})$ is nontrivial.



The relation $\sim'$ defining the 2-group can now be fully geometrized:

$\mathcal{E}$ from Link Geometry



Consider $L - \mathcal{S}$. The chain (yellow) acquire an extra boundary γ_S. Wrapped M2-branes now charged also under the flavor symmetry.

The homology relations now come from chains, that are in $L - \mathcal{S}$, which give rise to lines uncharged under C and

$$\widehat{\mathcal{E}} = H_1(L - \mathcal{S}).$$

Overall we find the geometrized version of the extension sequence:

$$0 \to \text{Tor}\, H_1(\partial \mathcal{S}) \to H_1(\mathbf{L} - \mathcal{S}) \to H_1(\mathbf{L}) \to 0$$

$\Rightarrow$ Methods allows determining 2-group symmetries for 5d and 6d SCFTs.

Anomaly theories from the Boundary

On several occasions we have seen now that the boundary of **X** provides an elegant description of symmetries.

What are the 't Hooft anomalies for the generalized symmetries?

Conjecture [Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, SSN]:
Reducing the 10d (11d) string/M-theory on the boundary $\partial\mathbf{X}$ results in a $D+1$ topological field theory, the Symmetry TFT, which after imposing boundary conditions, results in the anomaly theory of $\mathcal{T}(\mathbf{X})$
⇒ See Saghar Hosseini's talk Comments:

1. The Symmetry TFT is independent of the resolution/singular structure: "Renormalization invariance"

2. The reduction involves a generalized version of differential forms: differential cohomology, which allows reduction on torsion cycles of the supergravity theories

3. The anomaly theory $\mathcal{A}_{d+1}$ is an absolute theory, the Symmetry TFT is a relative theory (see [Freed])

’t Hooft Anomalies for 5d SCFTs

Using this conjecture we derived anomalies for 1-form and 0-form symmetries of 5d SCFTs: these are 6d theories, that depend on the background fields of the global symmetries.

E.g. for $SU(p)_q$ theories, where $\partial\mathbf{X} = Y^{p,q}$ we find

$$\mathcal{A}_6^{(B^3)} = \frac{qp(p-1)(p-2)}{6\,\mathrm{gcd}(p,q)} B_2^3$$

$$\mathcal{A}_6^{(FB^2)} = \frac{p(p-1)}{2\,\mathrm{gcd}(p,q)^2} F_I B_2^2$$

For B_2 the background of $\Gamma^{(1)} = \mathbb{Z}_{\mathrm{gcd}(p,q)}$ and F_I is the background for the $U(1)_I$ flavor symmetry.

This corroborates QFT arguments [Gukov, Pei, Hsin][Benetti Genolini, Tizzano], that the strongly coupled SCFT should have these anomalies as well.

Summary

It is fair to say that mathematical physics is undergoing a

"Symmetry Revolution":

Starting with Noether's notion of symmetries and conserved charges, we can now vastly generalize this to higher-form, higher-group and higher-categorical symmetries.

The main immediate open questions are:

- what is the precise structure of higher-fusion categories that we construct from a physics perspective as symmetry categories of QFTs?
- Physical implications of non-invertible symmetries in QFTs?
- String theoretic realization of non-invertible symmetries in QFTs: e.g. using mixed anomalies or outer automorphipsm gauging