

Fusion 2-Categories and Fully Extended Framed 4-dimensional TQFTs

Thibault D. Décoppet

University of Oxford

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Cobordism Hypothesis

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Example

In the symmetric monoidal 1-category of vector spaces, the fully dualizable objects are precisely the finite dimensional vector spaces.

Fusion 1-Categories

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- ▶ The 1-category $\text{Rep}(G)$ of finite dimensional G -representations, with G a finite group,
- ▶ The 1-category $\text{Rep}(\mathcal{V})$ of modules over a nice rational VOA \mathcal{V} .

Remark

The last two examples are braided.

Known Full Dualizability Results

Theorem [Douglas, Schommer-Pries, Snyder 2013]

Fusion 1-categories are fully dualizable objects in the symmetric monoidal 3-category of monoidal 1-categories, bimodules 1-categories, etc.

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Consequences

- ▶ Associated to every fusion 1-category, there is a fully extended framed 3-dimensional TQFT.
- ▶ Associated to every braided fusion 1-category, there is a fully extended framed 4-dimensional TQFT.

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Chronology

- ▶ Step 2 was established by Müger in 2001 (in full generality by Etingof, Nikshych, and Ostrik in 2005).
- ▶ Step 3 was established by Etingof, Nikshych, and Ostrik in 2005.
- ▶ Step 1 was established by Douglas, Schommer-Pries, and Snyder in 2013.

Fusion 2-Categories

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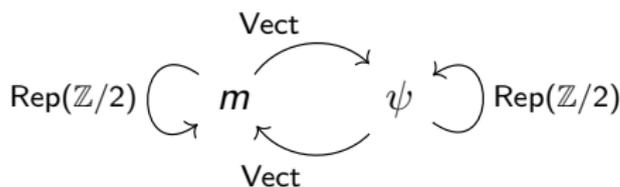
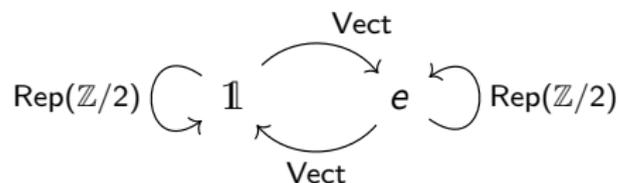
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- ▶ Given \mathcal{B} a braided fusion 1-category, the 2-category $\mathbf{Mod}(\mathcal{B})$ of finite semisimple \mathcal{B} -module 1-categories,
- ▶ In particular, $2\mathbf{Rep}(\mathcal{G})$, the 2-category of 2-representations of a finite 2-group \mathcal{G} .

One Detailed Example

Defects in the (3+1)-dimensional toric code model [Kong, Tian, Zhang 2020]



$\boxtimes \text{TC}_3$	$\mathbb{1}$	e	m	ψ
$\mathbb{1}$	$\mathbb{1}$	e	m	ψ
e	e	$2e$	ψ	2ψ
m	m	ψ	$\mathbb{1}$	e
ψ	ψ	2ψ	e	$2e$

Further Examples

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There are more exotic algebraic examples, but they are harder to define. See Douglas, Reutter 2018; **TD** 2022 (to appear).

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- ▶ The duality and triality defects of Choi, Cordova, Hsin, Lam, and Shao may be used to construct interesting fusion 2-categories.

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Physical Examples (To be investigated!)

- ▶ The duality and triality defects of Choi, Cordova, Hsin, Lam, and Shao may be used to construct interesting fusion 2-categories.
- ▶ The 4-dimensional non-invertible categorical symmetries of Bhardwaj, Bottini, Schafer-Nameki, and Tiwari may be used to construct interesting fusion 2-categories.

Separable Fusion 2-Categories

Definition

A fusion 2-category \mathfrak{C} is separable if its Drinfel'd center $\mathcal{Z}(\mathfrak{C})$ is a finite semisimple 2-category.

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This is precisely the algebraic condition corresponding to [1](#).

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Theorem [Johnson-Freyd 2020, TD 2022 (to appear)]

A fusion 2-category \mathfrak{C} is a fully dualizable object in the symmetric monoidal 4-category of monoidal 2-categories, etc if and only if it is separable.

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Consequence

Associated to every separable fusion 2-category, there is a framed fully extended 4-dimensional TQFT.

Further Remarks

Theorem [TD 2022, TD 2022 (to appear)]

To any fusion 2-category \mathcal{C} , one can associated a scalar $\text{Dim}(\mathcal{C})$ measuring the failure of $\mathcal{Z}(\mathcal{C})$ to be finite semisimple.

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This corresponds precisely to 2 above.

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Remark

This corresponds precisely to 2 above.

Consequence (using [TD 2022])

All of the examples of fusion 2-categories previously mentioned are separable.

Remark

Taking $\mathfrak{C} = \text{Mod}(\mathcal{B})$, with \mathcal{B} a braided fusion 1-category, recovers the theorem of Brochier, Jordan, and Snyder. But there are many more separable fusion 2-categories!

Final Comments

Remark

It has been conjectured by Douglas and Reutter that every fusion 2-category is separable. This is work in progress with David Reutter. This corresponds precisely to 3 above, and would show that every fusion 2-category yields an associated framed fully extended 4-dimensional TQFT.

Thank you!