Fusion 2-Categories and Fully Extended Framed 4-dimensional TQFTs

Thibault D. Décoppet

University of Oxford

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Cobordism Hypothesis

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Fully extended n-dimensional framed topological field theories with target a symmetric monoidal n-category $\mathscr{C}$ are in one-to-one correspondence with the fully dualizable objects of $\mathscr{C}$.

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Example

In the symmetric monoidal 1-category of vector spaces, the fully dualizable objects are precisely the finite dimensional vector spaces.

Fusion 1-Categories

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- The 1-category $\mathrm{Rep}(G)$ of finite dimensional G-representations, with G a finite group,
- The 1-category $\mathrm{Rep}(\mathcal{V})$ of modules over a nice rational VOA $\mathcal{V}$.

Remark

The last two examples are braided.

Known Full Dualizability Results

Theorem [Douglas, Schommer-Pries, Snyder 2013]

Fusion 1-categories are fully dualizable objects in the symmetric monoidal 3-category of monoidal 1-categories, bimodules 1-categories, etc.

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Braided fusion 1-categories are fully dualizable objects in the symmetric monoidal 4-category of braided monoidal 1-categories, etc.

Consequences

- Associated to every fusion 1-category, there is a fully extended framed 3-dimensional TQFT.
- Associated to every braided fusion 1-category, there is a fully extended framed 4-dimensional TQFT.

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Chronology

- Step 2 was established by Müger in 2001 (in full generality by Etingof, Nikshych, and Ostrik in 2005).
- Step 3 was established by Etingof, Nikshych, and Ostrik in 2005.
- Step 1 was established by Douglas, Schommer-Pries, and Snyder in 2013.

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- In particular, $2\mathrm{Rep}(\mathcal{G})$, the 2-category of 2-representations of a finite 2-group $\mathcal{G}$.

One Detailed Example

Defects in the (3+1)-dimensional toric code model [Kong, Tian, Zhang 2020]

$$
\begin{tikzcd}
\mathbb{1} \arrow[loop left, "\mathrm{Rep}(\mathbb{Z}/2)"] \arrow[r, bend left, "\mathrm{Vect}"] & e \arrow[l, bend left, "\mathrm{Vect}"] \arrow[loop right, "\mathrm{Rep}(\mathbb{Z}/2)"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
m \arrow[loop left, "\mathrm{Rep}(\mathbb{Z}/2)"] \arrow[r, bend left, "\mathrm{Vect}"] & \psi \arrow[l, bend left, "\mathrm{Vect}"] \arrow[loop right, "\mathrm{Rep}(\mathbb{Z}/2)"]
\end{tikzcd}
$$

$\boxtimes_{\mathrm{TC}_3}$	$\mathbb{1}$	e	m	ψ
$\mathbb{1}$	$\mathbb{1}$	e	m	ψ
e	e	$2e$	ψ	2ψ
m	m	ψ	$\mathbb{1}$	e
ψ	ψ	2ψ	e	$2e$

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There are more exotic algebraic examples, but they are harder to define. See Douglas, Reutter 2018; **TD** 2022 (to appear).

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Physical Examples (To be investigated!)

- The duality and triality defects of Choi, Cordova, Hsin, Lam, and Shao may be used to construct interesting fusion 2-categories.
- The 4-dimensional non-invertible categorical symmetries of Bhardwaj, Bottini, Schafer-Nameki, and Tiwari may be used to construct interesting fusion 2-categories.

Separable Fusion 2-Categories

Definition

A fusion 2-category $\mathfrak{C}$ is separable if its Drinfel'd center $\mathscr{Z}(\mathfrak{C})$ is a finite semisimple 2-category.

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This is precisely the algebraic condition corresponding to 1.

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Theorem [Johnson-Freyd 2020, **TD** 2022 (to appear)]

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Consequence

Associated to every separable fusion 2-category, there is a framed fully extended 4-dimensional TQFT.

Further Remarks

Theorem [**TD** 2022, **TD** 2022 (to appear)]

To any fusion 2-category $\mathfrak{C}$, one can associated a scalar $\mathrm{Dim}(\mathfrak{C})$ measuring the failure of $\mathscr{Z}(\mathfrak{C})$ to be finite semisimple.

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This corresponds precisely to 2 above.

Consequence (using [**TD** 2022])

All of the examples of fusion 2-categories previously mentioned are separable.

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Remark

This corresponds precisely to 2 above.

Consequence (using [TD 2022])

All of the examples of fusion 2-categories previously mentioned are separable.

Remark

Taking $\mathfrak{C} = \mathsf{Mod}(\mathcal{B})$, with $\mathcal{B}$ a braided fusion 1-category, recovers the theorem of Brochier, Jordan, and Snyder. But there are many more separable fusion 2-categories!

Final Comments

Remark

It has been conjectured by Douglas and Reutter that every fusion 2-category is separable. This is work in progress with David Reutter. This corresponds precisely to 3 above, and would show that every fusion 2-category yields an associated framed fully extended 4-dimensional TQFT.

Thank you!