1 Solution for 2013 Symmetry in Physics exam

1.1 Part A

1.1.1 Problem 1.1

Isomorphism/Homomorphism : bookwork. (lecture notes)

The set of isomorphisms is a subset of all homomorphisms, so $f \in H$.

1.1.2 Problem 1.2

Bookwork.

1.1.3 Problem 1.3

(Identity) Since both H and K are subgroups of G, both must contain the unit element e, and hence $e \in H \cap K$.

(Closure) Let $u_1, u_2 \in H \cap K$. Now since $u_1, u_2 \in H$, this means that $u_1 u_2 \in H$ via closure for subgroup H. But since u_1, u_2 are also elements of the subgroup K, then $u_1 u_2 \in K$ via closure for subgroup K, hence $u_1 u_2 \in H \cap K$.

(Inverse) Let $u \in H \cap K$, then $u \in H$ and $u \in K$, and since H and K are subgroups, $u^{-1} \in H$ and $u^{-1} \in K$ and hence $u^{-1} \in H \cap K$.

(Associativity) Inherited from G.

1.1.4 Problem 1.4

D_3 is the symmetry group of the equilateral triangle – this is bookwork. (lecture notes)

Elements of $D_3 = \{e, R, R^2, m_1, m_2, m_3\}$ where (students can simply draw an appropriate diagram).

- e : the identity
- R : Rotation around centroid by 60 degrees clockwise
- R^2 : Rotation around centroid by 180 degrees clockwise
- m_1 : Reflection about the altitude through vertex 1
- m_2 : Reflection about the altitude through vertex 2
- m_3 : Reflection about the altitude through vertex 3

The following is not bookwork.

There are 3 order 2 subgroups in D_3 which are the reflections : $G_1 = \{e, m_1\}$, $G_2 = \{e, m_2\}$, $G_3 = \{e, m_3\}$ since $m_i^2 = e$.

Consider $G_1 \cup G_2 = \{e, m_1, m_2\}$, then $m_1 m_2 = R^2 \notin G_1 \cup G_2$ hence it is not a subgroup.

1.1.5 Problem 1.5

The student can either construct a multiplication table, or show via bruteforce that G is a group. The following is the slickest way, using the hint.

We can write $1 = e^{0\times\pi i}$,$-1 = e^{\pi i}$,$i = e^{\pi i/2}$,$-i = e^{3\pi i/2}$, or

$$G = \{g_n = e^{in\pi/2} | n \in \mathbb{N} \bmod 4\} \tag{1}$$

so under multiplication

$$g_n g_{n'} = g_{n+n'} \in G \tag{2}$$

is *closed*.

The *identity* is clerly g_0, or $e^{0\pi i}$. The *inverse* for each element is $g_n = g_{n+4 \bmod 4}$, and finally *associativity* is inherited from addition operations of integers.

1.1.6 Problem 1.6

This is bookwork (in the lecture notes).

1.1.7 Problem 1.7

Using the $D(a)$ representation, and realizing that it generates the entire group, one can use group homomorphism to generate the other representations easily by matrix multiplication

$$D(a^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & bc & 0 \\ 0 & 0 & bc \end{pmatrix} \ , \ D(a^3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & b^2c \\ 0 & bc^2 & 0 \end{pmatrix} \ , \ D(a^4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b^2c^2 & 0 \\ 0 & 0 & b^2c^2 \end{pmatrix}. \tag{3}$$

Now $D(a^4) = D(e)$ implies $b^2c^2 = 1$, so $bc = \pm 1$. If (i) D is faithful, then $bc = -1$.If (ii) D is unfaithful, then $bc = 1$.

2 Part B

Part B questions are all in two parts. They are designed using inverted "60-40" rule, i.e. 40% effort to get the first 60% of the marks, and then invert. So both question has a "this is hard" bit that challenges the students. I expect an average of 18/30.

2.0.8 Problem 1.8

(i) Quartenion Q_8

Definition of conjugacy classes is bookwork.

The conjugacy classes are (by brute force calculation) $\{1\}$,$\{-1\}$,$\{\pm i\}$,$\{\pm j\}$,$\{\pm k\}$. (Just need to show that, $g(\pm 1)g^{-1} = L_g(\pm 1) = \pm 1$ and $L_g(\pm i) = \{\pm i\}$, and by symmetry j and k.)

Second part is direct application of Burnside's theorem – taught in detailed in class.

Since there are 5 conjugacy classes, there must be 5 irreducible representations of Q_8. The sum of the dimensions squared of these 5 irreducible representations must equal the order of the group, i.e. 8, and hence their dimensions must be 1,1,1,1,2. (The only possible solution for a sum of 5 integer square that gives 8).

(ii) Dihedral Groups and its subgroups

(a) The elements of the group that preserves rectangular but non-square symmetry are (students can just draw an appropriate diagram)

- e : identity
- m_1 : reflection about the horizontal axis
- m_2 : reflection about the vertical axis
- R : rotation by 180 degrees

The order of the group is 4. The proper subgroups are $\{e, m_1\}$,$\{e, m_2\}$ and $\{e, R\}$.

$$\begin{array}{c|cccc} & e & m_1 & m_2 & R \\ \hline e & e & m_1 & m_2 & R \\ m_1 & m_1 & e & R & m_2 \\ m_2 & m_2 & R & e & m_1 \\ R & R & m_2 & m_1 & e \end{array} \tag{4}$$

To describe the symmetry of the water molecule, student can draw a diagram. Can also identify that the group above is the Dihedral-2 group D_2. To prove that the D_2 group is $Z_2 \times Z_2$, student can explicitly find an isomorphism, or simply remark that the above group is generated by 2 elements $\{R, m_1\}$, while the only *other* order 4 group other than the four-group is Z_4 which is generated by a single element. Hence $D_2 \cong Z_2 \times Z_2$.

(b) This is the hardest problem in the exam, as it requires actual thinking instead of crank-turning, so I expect only the best students to solve it. It is also the prettiest problem. The Dihedral-N group is studied in detailed in class and the lecture notes, so all the following notation is familiar to the students. They should be able to get the answer if they systematically think through the possibilities.

In an order 2 normal subgroup, one of the element must be the identity. So we must find an element $h \in H$ such that $ghg^{-1} = h^{-1}$ for all $g \in G$. Let R^i be the set of all rotations, and m_i be the set of all reflections. Consider reflections $h = m_i$. All reflections are their own inverse and since $m_i m_j$ do not commute for all i, j and $= R^k$ if $i + j$ is odd, there exist no order 2 normal subgroups of $\{e, m_i\}$.

Consider now rotations $h = R^j$. Since the inverse of a rotation is just another rotation, and all rotations commute, $R^i R^j R^{-i} = R^j$.This leaves $m_i R^j m_i^{-1} = R^{-j}$. Since the only rotation that is its own inverse is rotation by 180 degrees, there exist only one such element if n is even, no such element if n is odd.

2.0.9 Problem 1.9

(i) Symmetric/Permutation Group S_3

(a) Using the group laws, and the generators, it is easy to do some matrix algebra to calculate the other representations.

Verify by explicit calculation that, with $z = \exp(2\pi i/3)$,

$$R(y^2) = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}, \quad R(xy) = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}, \quad R(xy^2) = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix},$$

To show that it is irreducible, first note that it is not a trivial representation. So the sum of the characters must be zero $\sum_i \chi(R_i) = 0$, and it's easy to calculate $\chi(R(e)) = 2$, $\chi(R(x)) = 0$, $\chi(R(y)) = z + z^2$, $\chi(R(xy)) = 0$, $\chi(R(y^2)) = z^2 + z$, and $\chi(R(xy^2)) = 0$, and $2(z + z^2) = 2(\exp(2\pi i/3 + \exp 4\pi i/3) = -2$, so the sum is indeed zero.

(b) This part is the second hardest part of the exam which I expect only the best students to solve.

From the group law $U(x^2) = U(x)U(x) = U(e) = 1$, hence $U(x) = \pm 1$. Also, $U(y^3) = U(y)U(y)U(y) = 1$, then $U(y) = 1, e^{i2\pi/3}, e^{i4\pi/3}$, but since $U(yx) = U(xy^2) \to U(y)U(x) = U(x)U(y^2)$, hence $U(y) = U(y)^2$ thus the only consistent representation for y is $U(y) = 1$. If we now choose $U(x) = 1$, we find that $U(xy^2) = U(xy) = 1$, i.e. this is the trivial representation T. The only other choice is then $U(x) = -1$. Thus the representation U is given by

$$U(e) = U(y) = U(y^2) = 1 \ , \ U(x) = U(xy) = U(xy^2) = -1$$

The kernel is then $\{e, y, y^2\}$. Since the kernel is not trivial, this representation is *not* faithful.

Finally, we already know that there are 2 1-D representations and from the previous section, 1 2-D representation, hence $1+1+2^2 = 6 = |S_3|$ which shows that there are no more irreps.

(ii) Rotation operators

The eigenvalues can be calculated with $\det(M - I\lambda) = 0$, and $\lambda = \pm 1$. The normalized eigenvectors are

$$v_+ = \begin{pmatrix} \sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \quad , \quad v_- = \begin{pmatrix} -\cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \tag{5}$$

It is easy to show orthogonality by $v_- \cdot v_+ = 0$.

For $\theta = \pi$, the eigenvectors are $v_+ = (1, 0)$ and $v_- = (0, 1)$ so a general quantum mechanical state is

$$V_0 = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

This is a general quantum mechanical state vector for a quantum state with two-states. An electron spin can be up or down, and hence is a 2-state system, hence V_0 can represent the spin state of an electron.