Candidate number:
Desk number:

King's College London

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

5CCP2332 Symmetry in Physics

Examiner: Dr. Eugene A. Lim

Examination Period 2
(Summer 2017)

Time allowed: TWO hours

Candidates may answer as many parts as they wish from SECTION A, but the total mark for this section will be capped at 20 out of a total of 50 for the whole paper.

Candidates should also answer ONE question from SECTION B. No credit will be given for answering a further question from this section.
The approximate mark out of 30 for each part of a question is indicated in square brackets.

Calculators may be used. The following models are permitted: Casio fx83 and Casio fx85.

DO NOT REMOVE THIS EXAM PAPER
FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	$\epsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}
Speed of light in free space	c $= 2.998 \times 10^8$ m s^{-1}
Gravitational constant	G $= 6.673 \times 10^{-11}$ N m^2 kg^{-2}
Elementary charge	e $= 1.602 \times 10^{-19}$ C
Electron rest mass	$m_\mathrm{e} = 9.109 \times 10^{-31}$ kg
Unified atomic mass unit	$m_\mathrm{u} = 1.661 \times 10^{-27}$ kg $= 931.494$ MeV c^{-2}
Proton rest mass	$m_\mathrm{p} = 1.673 \times 10^{-27}$ kg
Neutron rest mass	$m_\mathrm{n} = 1.675 \times 10^{-27}$ kg
Planck constant	$h = 6.626 \times 10^{-34}$ J s
Boltzmann constant	$k_\mathrm{B} = 1.381 \times 10^{-23}$ J K^{-1} $= 8.617 \times 10^{-11}$ MeV K^{-1}
Stefan-Boltzmann constant	$\sigma = 5.670 \times 10^{-8}$ W m^{-2} K^{-4}
Gas constant	$R = 8.314$ J mol^{-1} K^{-1}
Avogadro constant	$N_\mathrm{A} = 6.022 \times 10^{23}$ mol^{-1}
Molar volume of ideal gas at STP	$= 2.241 \times 10^{-2}$ m^3
One standard atmosphere	$P_0 = 1.013 \times 10^5$ N m^{-2}

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SECTION A

Answer SECTION A in an answer book. Answer as many parts of this section as you wish. Your total mark for this section will be capped at 20.

1.1 Let S be a set. Define what is meant by a *partition* of S.

[*3* marks]

1.2 Suppose f is a map from set A to set B, i.e. $f : A \to B$, i.e., A is the domain of f, dom(f) and B is the codomain of f, cod(f). What do you call the map if (separate cases)
(i) $\text{Im}(f) = \text{cod}(f)$
(ii) $a, a' \in A$ and if $a \neq a'$ then $f(a) \neq f(a')$

[*3* marks]

1.3 Let Z_2 and Z_3 be the order 2 and order 3 cyclic groups respectively. Construct the multiplication (or Cayley) table for $Z_2 \times Z_3$.

[*5* marks]

1.4 Let Z_n be a cyclic group and $D : Z_n \to GL(2, \mathbb{R})$ be a 2×2 *faithful* representation of Z_n. Let $a \in Z_n$, and

$$D(a) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Determine the *minimum* order of Z_n.

[*4* marks]

QUESTION CONTINUES ON NEXT PAGE

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1.5 Let $\mathbb{Z}$ be the set of all integers, and $\star$ be a binary operator between elements of $\mathbb{Z}$, such that

$$a \star b = a + b - a \cdot b,\ \ a, b \in \mathbb{Z},$$

where the $+$, $-$ and $\cdot$ operators denote usual addition, subtraction and multiplication.

Determine whether
(i) $\star$ is commutative,
(ii) $\star$ is associative,
(iii) an identity exists (find it if it does).
(iv) an inverse exists (find it if it does).

[5 marks]

1.6 Let G be a group, and $a \in G$. Let $C_G(a)$ be the set of all elements of G which commute with the element a, i.e.

$$C_G(a) = \{g \in G : ag = ga\}.$$

Prove that C_G is a group. [5 marks]

1.7 Consider the following differential equation

$$x\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + \frac{x}{y} = 0.$$

Suppose the coordinate x undergoes a dilatation $x \to ax'$ for $a \neq 0$. Find the corresponding transformation for y which leaves the equation invariant.

[5 marks]

Solution 1.1

partition of S is a collection C of subsets of S such that (a)$X \neq \emptyset$ whenever $X \in C$, (b) if $X, Y \in C$ and $X \neq Y$ then $X \cap Y = \emptyset$, and (c) the union of *all* of the elements of the partition is S.

Solution 1.2 (i) surjective/onto (ii) injective/into

Solution 1.3

Let $Z_1 = \{(e, a)\}$ and $Z_2 = \{e, b, b^2\}$ and $Z_2 \times Z_3 = \{(e, e), (e, b), (e, b^2), (a, e), (a, b), (a, b^2)\}$.

(e,e)	(e,b)	(e,b^2)	(a,e)	(a,b)	(a,b^2)
(e,b)	(e,b^2)	(e,e)	(a,b)	(a,b^2)	(a,e)
(e,b^2)	(e,e)	(e,b)	(a,b^2)	(a,e)	(a,b)
(a,e)	(a,b)	(a,b^2)	(e,e)	(e,b)	(e,b^2)
(a,b)	(a,b^2)	(a,e)	(e,b)	(e,b^2)	(e,e)
(a,b^2)	(a,e)	(a,b)	(e,b^2)	(e,e)	(e,b)

(1)

Solution 1.4

Easy to compute

$$D(a)D(a) = D(a^2) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} , \quad (2)$$

$$D(a)D(a)D(a) = D(a^3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D(e) , \quad (3)$$

$$D(a)D(a)D(a)D(a) = D(a^4) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = D(a) , \quad (4)$$

So $n = 3$.

Solution 1.5

(i) $a \star b = a + b - ab = b \star a$ (commutative)

(ii) $a \star (b \star c) = a + b + c - ab - ac - cb + abc = (a \star b) \star c$ (associative)

(iii) $a \star e = a$ so $a + e - ae = a$ or $e(1 - a) = 0$ so $e = 0$ satisfies the condition for all a. (identity exists)

(iv) $a \star b = e = 0$ so $a + b - ab = 0$ or $b = a/(a - 1)$. But since $b \notin \mathbb{Z}$ necesarily (e.g. $a = 5$) inverse does not exist.

Solution 1.6 Let $c_1, c_2 \in C_g$, then

- Closure : $c_1 a = ac_1$, multiply from the left with c_2, we get $c_2 c_1 a = c_2 a c_1$, but using commutative property on the RHS, this is $(c_2 c_1)a = a(c_2 c_1)$, so $c_3 = c_1 c_2 \in C_g$.
- Associativity is inherited from G.
- Identity : since e commute with everything, $e \in C_g$.
- Inverse : let g be the inverse of c_1. Now $c_1 a = ac_1$, multiply from right with g, we get $c_1 a g = a c_1 g = a$ since $c g_1 = e$. Multiply from the left with g, we get $g c_1 a g = ga \Rightarrow ag = ga$, hence $g \in C_g$.

Solution 1.7

Easy to show $y \to ay'$ will leave the equation invariant.

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SECTION B - Answer ONE question

Answer section B in an answer book

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(i) Consider two Lie groups, $SO(3,\mathbb{R})$ and $SL(2,\mathbb{R})$. Let $R \in SO(3,\mathbb{R})$ and $L \in SL(2,\mathbb{R})$ be in the 3×3 and the standard 2×2 matrix representations of these two groups respectively. Furthermore, let T be a 3×2 matrix with real coefficients and $0_{2\times3}$ be a 2×3 matrix with *zero* coefficients. We can then construct the set of 5×5 matrices, $\mathcal{M}$, such that

$$M = \begin{pmatrix} R & T \\ 0_{2\times3} & L \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_{11} & t_{12} \\ r_{21} & r_{22} & r_{23} & t_{21} & t_{22} \\ r_{31} & r_{32} & r_{33} & t_{31} & t_{32} \\ 0 & 0 & 0 & l_{11} & l_{12} \\ 0 & 0 & 0 & l_{21} & l_{22} \end{pmatrix} \ , \ M \in \mathcal{M}$$

where r_{ij}, l_{ij} and t_{ij} are the coefficients for the R, L and T matrices respectively.

(a) Define what it means for a matrix to be *block-diagonal*. What are the coefficients of the matrix T such that the matrix M is block-diagonal?

[*4* marks]

(b) Show that the set $\mathcal{M}$ is closed under matrix multiplication, i.e.

$$M_1 M_2 = \begin{pmatrix} R_1 & T_1 \\ 0_{2\times3} & L_1 \end{pmatrix} \begin{pmatrix} R_2 & T_2 \\ 0_{2\times3} & L_2 \end{pmatrix} \in \mathcal{M}$$

[*5* marks]

(c) Let L^{-1} and R^{-1} be inverses for L and R respectively. Find A such that the following matrix

$$M^{-1} = \begin{pmatrix} R^{-1} & A \\ 0_{2\times3} & L^{-1} \end{pmatrix}$$

is the inverse for M, i.e. $MM^{-1} = M^{-1}M = e$. And hence argue that M forms a group under matrix multiplcation.

[*5* marks]

QUESTION CONTINUES ON NEXT PAGE

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(ii) Consider the following real vector

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

An *affine* transformation of $\mathbf{x}$ is defined to be the following operation

$$\mathbf{x}' = A\mathbf{x} + B$$

where $A \in GL(2, \mathbb{R})$ and B is a 2×1 column matrix with real coefficients,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

(a) Describe what it means for $f(x)$ to be an *analytic function in its domain*?

[*2* marks]

(b) Prove that the affine transformation described in the preamble forms a group. This group is called the *affine group* Aff(2).

[*4* marks]

(c) By considering the analyticity of the transformations, show that the affine group Aff(2) is a Lie group. What is the dimensionality of this Lie group?

[*2* marks]

(d) The elements of Aff(2) can be represented as a 3×3 matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ 0 & 0 & 1 \end{pmatrix} \in \text{Aff}(2).$$

Verify that matrix multiplication reproduces the group composition laws you computed in (b). Identify the identity element and hence calculate the generators for Aff(2) in this representation.

[*8* marks]

Solution

Q2.

(i)

(a) Block-diagonal means that a matrix can be decomposed into the following form

$$\begin{pmatrix} A & & & \\ & B & & \\ & & C & \\ & & & \dots \end{pmatrix}$$

where A, B, C etc are square matrices. Equation (5) is block diagonal if T has zero coefficients.

(b) Multiplying M_1 and M_2 we get

$$M_1 M_2 = \begin{pmatrix} R_1 & T_1 \\ 0_{2\times 3} & L_1 \end{pmatrix} \begin{pmatrix} R_2 & T_2 \\ 0_{2\times 3} & L_2 \end{pmatrix} = \begin{pmatrix} R_1 R_2 & R_1 T_2 + T_1 L_2 \\ 0_{2\times 3} & L_1 L_2 \end{pmatrix} \in \mathcal{M}$$

so it is closed.

(c) Solution follows from plugging in M^{-1} and setting top right element equal to 1.

(ii) (a) An analytic function is an infinitely differentiable (C_∞) function whose Taylor series around a point x_0 converges to $f(x_0)$ for x_0 within its domain.

(b) (Closure) Consider two successive transformations $\mathbf{x}' = A_1\mathbf{x} + B_1$ and $\mathbf{x}'' = A_2\mathbf{x}' + B_2$, so

$$\begin{aligned} \mathbf{x}'' &= A_2(A_1\mathbf{x} + B_1) + B_2 \\ &= A_2 A_1 \mathbf{x} + (A_2 B_1 + B_2). \end{aligned} \tag{5}$$

But since $A_2 A_1 \in GL(2, R)$ and $A_2 B_1 + B_2$ is a 2×1 column matrix with real coefficients, the transformation is closed.

(Identity) Given by $A = I$ and $B = 0$.

(Inverse) To find the inverse, we set $\mathbf{x}'' \to \mathbf{x}$ in equation (6), to get

$$\mathbf{x} = A_2 A_1 \mathbf{x} + (A_2 B_1 + B_2), \tag{6}$$

and the inverse is found when $A_2 = A_1^{-1}$ (which exists since $A \in GL(2, R)$) and $A_2 B_1 + B_2 = 0$, or $B_2 = -A_1^{-1} B_1$, which we can always find a solution for B_2 since A_1^{-1} exists. So an inverse exist.

(Associativity) Inherited from matrix multiplication.

(c) The coordinate transforms for x and y can be written as

$$x' = a_{11}x + a_{12}y + b_1 \ , \ y' = a_{21}x + a_{22}y + b_2 \tag{7}$$

both equations which are clearly analytic in a_{ij}. Since there are six unconstrained parameters, the dimensionality of Aff(2) is 6.

(d) Group multiplication law is given by matrix multiplication

$$M_2 M_1 = \begin{pmatrix} A_2 & B_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_2 A_1 & A_2 B_1 + B_2 \\ 0 & 1 \end{pmatrix} \tag{8}$$

which reproduces (b). Note that we have use left multiplcation rules (not $M_1 M_2$).

The identity element is $a_{11} = a_{22} = 1$ and everything else zero. The generators can be easily found by taylor expanding around the identity, whose first terms for each generator are

$$X_i = \left.\frac{\partial M}{\partial a_i}\right|_I \tag{9}$$

The 6 generators are

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;,\; X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;,\; X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10}$$

$$X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;,\; X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;,\; X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{11}$$

QUESTION CONTINUES ON NEXT PAGE

3.

(i) Let G be a finite group, and $D : G \to GL(2, \mathbb{C})$ be a 2×2 *faithful* representation of the group. G is generated by two generators, g_1 and g_2, which in this representation are given by

$$D(g_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ D(g_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} .$$

(a) Define what is meant by the *generators* of a group?

[*2* marks]

(b) Determine the *orders* of the generators g_1 and g_2. Hence, compute all the other elements of G. What is the order of G?

[*10* marks]

(c) Calculate the characters for the representation $D(g)$. Is this a reducible or irreducible representation? Justify your conclusions.

[*4* marks]

QUESTION CONTINUES ON NEXT PAGE

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(ii) Let G be a finite group, and H be a subgroup of G.

(a) Define what is meant by the *left coset space* G/H.

[*2* marks]

(b) State what is a *normal subgroup* H of G.

[*2* marks]

(c) Hence, prove that a subgroup H with *index* 2, i.e. $|G/H| = 2$, is a normal subgroup. *Hint*: Recall that cosets *partition* G.

[*10* marks]

Solution Q3

(i)

(a) A generator $g \in G$ element of G where by repeated application of the group composition with itself or other generators of G, makes (or generates) other elements of G.

(b) By bruteforce computing

$$D(g_1)D(g_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D(e) \tag{12}$$

and

$$D(g_2)D(g_2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = D(g_4) \tag{13}$$

$$D(g_2)D(g_2)D(g_2) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = D(g_6) \tag{14}$$

$$D(g_2)D(g_2)D(g_2)D(g_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D(e) \tag{15}$$

so g_1 is order 2 while g_2 is order 4.

The other elements of G can be computed easily by bruteforce

$$D(g_1)D(g_2) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \equiv D(g_3) \tag{16}$$

$$D(g_2)D(g_3) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \equiv D(g_5) \tag{17}$$

$$D(g_2)D(g_5) = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \equiv D(g_7) \tag{18}$$

and other combinations will result in existing elements, so $|G| = 8$.

(c) The characters are just traces, so $\chi(g_1) = \chi(g_2) = \chi(g_6) = \chi(g_5) = 0$ and $\chi(e) = 2$, $\chi(g_3) = 2i$, $\chi(g_4) = -2$, $\chi(g_7) = -2i$.

Since $\sum |\chi|^2 = 16 > |G|$, this is a *reducible* representation.

(ii)

(a) A left coset space is the set of all possible left cosets gH, where $g \in G$.

(b) A normal subgroup H of G is one such that $gH = Hg$ for all $g \in G$.

(c) If $|G/H| = 2$, then there are two left cosets. Using the fact that cosets partition G, we can consider two possible cases.

Case 1: If $g \in H$ then $gH = H = Hg$.

Case 2: If $g \notin H$, then $gH = G - H \equiv \bar{G}$ *necessarily*, i.e. in the complement of G of H. But the *right* coset $Hg = G - H$ for the same reason, so $gH = Hg$. It is necessary because there are only 2 cosets – if the index is 3 or more then this proof fails.

We have thus shown that for all g, $gH = Hg$ which means that H is a normal subgroup.

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