Candidate number:	Desk number:

King's College London

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B.Sc. EXAMINATION

5CCP2332 Symmetry in Physics

Examiner: Dr Eugene A Lim
Spring 2016

Time allowed: TWO hours

Candidates may answer as many parts as they wish from SECTION A, but the total mark for this section will be capped at 20.

Candidates should answer no more than ONE question from SECTION B.
No credit will be be given for answering a further question from this section.

The approximate mark for each part of a question is indicated in square brackets.

Calculators may be used. The following models are permitted: Casio fx83 and Casio fx85.

DO NOT REMOVE THIS EXAM PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

For examiner's use only

1.1	
1.2	
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SECTION A

Answer SECTION A in an answer book. Answer as many parts of this section as you wish. Your total mark for this section will be capped at 40.

1.1 If S is a set, and $p, q \in S$ are **related** by relation $\bowtie$ and we write $p \bowtie q$. Describe *reflexive*, *transitive* and *symmetric* relations between objects in the set S.

[*3* marks]B

1.2 Let H be a subgroup of group G. Define the *index* of H in G.

[*3* marks]B

1.3 Consider the Klein four-group V_4 with elements $\{e, a, b, c\}$ where e is the identity, and the group laws are given by $ab = c$, $a^2 = b^2 = c^2 = e$. Construct the multiplication table for V_4.

How many proper subgroups are there in V_4?

[*5* marks]B

1.4 The Methane molecule CH_4 is made out of 4 hydrogen atoms H bonded to a central carbon C atom. Due to inter-atomic forces, the hydrogen atoms are equidistant from each other, and individually equidistant from the central carbon atom. The symmetry group of Methane is the set of operations which leave the molecular configuration invariant. What is the symmetry group of Methane? Explain your answer with the help of an illustration.

[*5* marks]UP

QUESTION CONTINUES ON NEXT PAGE

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1.5 Let $\mathbb{Z}$ be the set of all integers, and $\star$ be a binary operator between elements of $\mathbb{Z}$, such that

$$a \star b = 2 \cdot a \cdot b \ , \ a, b \in \mathbb{Z},$$

where the dot operator $\cdot$ denotes usual multiplication.

Determine whether
(i) $\star$ is commutative,
(ii) $\star$ is associative,
(iii) an identity exists (find it if it does).

[*4* marks]UP

1.6 The symmetry group of the pentagon is described by the Dihedral group D_5. This group is generated by two generators, a rotation operator R such that $R^5 = e$ and an operator m. Let U be a 2×2 representation for D_5, with

$$U(R) = \begin{pmatrix} e^{2\pi i/5} & 0 \\ 0 & e^{-2\pi i/5} \end{pmatrix} \ , \ U(m) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{1}$$

Show that the representations for the reflection operators m_j are given by

$$U(m_j) = \begin{pmatrix} 0 & e^{2(j-1)\pi i/5} \\ e^{-2(j-1)\pi i/5} & 0 \end{pmatrix} \tag{2}$$

where $m_1 = m$.

[*5* marks]UP

1.7 Consider the following differential equation

$$x^5 \frac{d^2y}{dx^2} - x^4 \frac{dy}{dx} + \frac{1}{2x^3 y} = 0$$

Suppose the coordinate x undergoes a dilatation $x \to ax'$ for $a \neq 0$. Find the corresponding transformation for y which leaves the equation invariant.

[*5* marks]P

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Solution 1.1 Let S be a set, and $a, b, c \in S$

- **Reflexive**: if $a \bowtie a$ for every $a \in S$
- **Symmetric**: if $a \bowtie b$ then $b \bowtie a$ for $a, b \in S$
- **Transitive**: if $a \bowtie b$, and $b \bowtie c$ then $a \bowtie c$ for $a, b, c \in S$.

Solution 1.2 The index of H in G is the number of distinct left cosets $gH \forall g \in G$.

Solution 1.3

$$\begin{array}{c|ccc} e & a & b & c \\ \hline a & e & c & b \\ b & c & e & a \\ c & b & a & e \end{array} \tag{3}$$

There are 3 order 2 proper subgroups $\{e, a\}$, $\{e, b\}$ and $\{e, c\}$.

Solution 1.4

Methane looks like a tetrahedral where the hydrogens are at the vertices, and the carbon is in the centroid. The molecule is symmetric under any permutations of the hydrogen atoms, and hence is S_4 (i.e. permutation group of 4 elements).

Solution 1.5

(i) $a \star b = 2ab = 2ba = b \star a$, commutative

(ii) $(a \star b) \star c = 4abc = a \star (b \star c)$, associative

(iii) Suppose e is the identity, then $e \star a = a$. But $e \star a = 2ea = a$ and hence $e = 1/2 \notin \mathbb{Z}$. Identity does not exist.

Solution 1.6

Recall that the reflection operators are generated by $m_j = R^{j-1}m$ (taught in class, but a clever student can figure this out), and since $U(R)$ is a diagonal matrix, taking the power of the matrices simply mean taking the power of the diagonal elements. Matrix multiplication yields the required answer.

Solution 1.7

Let $y = by'$, we get

$$ba^3 x'^5 \frac{d^2 y'}{dx'^2} - ba^3 x'^4 \frac{dy'}{dx'} + 2(ba^3)^{-1} \frac{1}{x^3 y} = 0$$

hence $b = a^{-3}$.

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SECTION B - Answer ONE question

Answer section B in an answer book

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(i)

Consider a matrix group G whose elements $M \in G$ are given by

$$M(a,b,c) = \begin{pmatrix} 1+a & b \\ c & (1+bc)/(1+a) \end{pmatrix} \qquad (4)$$

where $a, b, c \in \mathbb{R}$ are the parameters of the group.

(a) Show that G is the Lie group $SL(2,\mathbb{R})$.

[*2* marks]B

(b) Describe what it means for $f(x)$ to be an *analytic function in its domain*?

[*2* marks]B

(c) By expanding around the identity, show that the elements of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ are given by

$$X_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ X_b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ X_c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \qquad (5)$$

[*6* marks]U

(d) Find the structure constant(s) for this algebra.

[*6* marks]U

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(ii)

The motion of a point particle of mass m traveling under the influence of a potential $V(x)$ which depends on the position x (in 1 dimension) is described by Newton's equation of motion

$$m\frac{d^2x}{dt^2} = -\frac{dV}{dx}. \tag{6}$$

Consider the following transformations on both the time and space coordinates

$$x \to \beta x' \ , \ t \to \gamma t' \tag{7}$$

where $\beta, \gamma \in \mathbb{R}$ and $\beta \neq 0, \gamma \neq 0$.

Under these coordinate transformations, the potential transforms as

$$V(x) \to V(\beta x') \equiv \beta^k V(x'), \tag{8}$$

where k is a real constant which depends on the exact form of the potential.

(a) Show that Newton's equation of motion is *invariant* under these transformations if $\beta^{2-k}\gamma^{-2} = 1$.

[*3* marks]U

(b) Consider the momentum of the particle given by $p = m\dot{x}$. Determine how $V(x)$ would transform (by finding k) such that the momentum of the particle is a *constant of motion*.

[*3* marks]U

(c) Consider the special case of a particle falling under a local gravitational field

$$V(x) = mgx \tag{9}$$

where g is the gravitational constant. What is k for this potential? Argue that under this potential, the distance traveled by a particle is proportional to the square of the time elapsed.

[*4* marks]U

QUESTION CONTINUES ON NEXT PAGE

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(d) Finally, consider the special case of a particle in a quadratic potential (i.e. a Harmonic oscillator)

$$V(x) = \frac{1}{2}qx^2 \tag{10}$$

where q is a constant. What is k for this potential? Use this result to argue *Hooke's Law*, i.e. the oscillation period ω is independent of the physical size of the oscillation.

[*4* marks]U

Solution

(i)

(a) Since $a, b, c \in \mathbb{R}$ and the matrix is 2×2, it is a linear group. It's easy to show that $det(M) = 1$, and hence the group is $SL(2, R)$.

(b) An analytic function is an infinitely differentiable (C_∞) function whose Taylor series around a point x_0 converges to $f(x_0)$ for x_0 within its domain.

(c) The identity is given by parameters $a = 0, b = 0, c = 0$. By expanding around the identity, we get

$$X_a = \left.\frac{\partial M}{\partial a}\right|_{a=0,b=0,c=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

$$X_b = \left.\frac{\partial M}{\partial b}\right|_{a=0,b=0,c=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{12}$$

$$X_c = \left.\frac{\partial M}{\partial c}\right|_{a=0,b=0,c=0} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{13}$$

(d) We can compute (some tedious time consuming algebra for the student without mathematica help, but this is compensated by the almost algebra-less part (ii), pun unintended)

$$[X_a, X_b] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \ , \ [X_a, X_c] = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} \ , \ [X_b, X_c] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}$$

Then using $[X_i, X_j] = C_{ij}^k X_k$ (summing over k), we find that $C_{ab}^b = 2$,$C_{ac}^c = -2$, $C_{bc}^a = 1$ and everything else is zero.

(ii)

(a) Simple substitution gets us $\beta\gamma^{-2}d^2x'/dt'^2 = -\beta^{k-1}dV/dx'$, hence $\beta^{2-k}\gamma^{-2} = 1$ is the condition to leave the equation invariant.

(b) Taking time derivative $dp/dt = m\ddot{x} = -dV/dx$. If $dp/dt = 0$ this means that $dV/dx = 0$. Straight integration tells us that $V =$ constant for p to be a constant of motion, hence $k = 0$.

(c) $V(\beta x') = \beta mgx'$, hence $k = 1$. This gives us $\beta = \gamma^2$. This means that the distance traveled distance (scaling like β) is proportional to the square of the time elasped which scales as γ^2.

(d) Here $k = 2$. This gives us $\gamma^2 = \beta^0$, i.e. the time does not scale with physical distanace, hence the since frequency $\omega \propto 1/t$, and this means that the oscillation period is independent of time.

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3.

(i)

Let D be a 2×2 representation of the order 3 cyclic group Z_3 generated by a such that

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ D(a) = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & x \end{pmatrix} \ , \ D(a^2) = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & y \end{pmatrix} \tag{15}$$

where $x, y \in \mathbb{R}$.

(a) Find x and y.

[*2* marks]UP

(b) Calculate the characters χ of this representation. Prove that the representation above is *reducible*. You may use any theorems without proof.

[*4* marks]UP

(c) Show that Z_3 is abelian, and hence deduce its total number of irreps and their dimensionalities. You may use any theorems without proof.

[*4* marks]UP

(d) Let $T : Z_3 \to \mathbb{C}$ and $U : Z_3 \to \mathbb{C}$ be two *inequivalent and non-trivial* 1-dimensional representations of Z_3. Find T and U.

(Hint: You may find the following relations useful

$$e^{i\theta} = \cos\theta + i \sin\theta \ , \ 1 = e^{2n\pi i} \tag{16}$$

where n is an integer.)

[*4* marks]U

(e) Show that $D = U \oplus T$.

(Hint: Any reducible representation can be decomposed into its irreps by using the formula

$$a_m = \frac{1}{|G|} \sum_i \chi_i^{m*} \chi_i \tag{17}$$

where χ_i^m are the characters of the m-th irrep.)

[*4* marks]U

QUESTION CONTINUES ON NEXT PAGE

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(ii)

Consider a finite order group G whose group order $|G| > 2$. Let $g \in G$, and suppose $g^k = e$, then the *minimum* k is called the *order* of g and denoted $|g| = k$.

(a) Prove that an element, *other than the identity*, whose inverse is itself (i.e. $g = g^{-1}$) must have order 2.

[*2* marks]UP

(b) Prove that the order of any element $g \in G$ is also the order of its inverse g^{-1}. I.e. prove that $|g| = |g^{-1}|$.

[*5* marks]U

(c) Given (a) and (b) above, prove that all *even* order groups G (i.e. $|G| = 2n$ where n is an integer) must possess an order 2 element.

[*5* marks]U

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Solution

(i)

(a) Using $D(a^3) = D(e) = D(a)D(a^2) = 1$, we get $x = y = -1$.

(b) The characters are $\chi(D(e)) = 2$, $\chi(D(a)) = \chi(D(a^2)) = -1$ (simply take traces). By using the theorem that $\sum_i \chi_i^2 = |G|$ for irreps, we see that here $\sum_i \chi_i^2 = 2^2 + (-1)^2 + (-1)^2 = 6 > 3$ hence D is a reducible rep.

(c) It is easy show that $ab = ba$ by direct calculation. Since it is abelian, the number of conjugacy classes is equal the order of the group 3, and hence the number of irreps must be equal to 3 (Burnside). Using $\sum_m n_m^2 = |G|$ (where n_m is the dim of irrep m) the only possible solution is that all the three irreps are dimension 1.

(d) Using $U(a^3) = U(a)U(a)U(a) = U(e) = 1$, and setting $z \equiv U(a)$, we have $z^3 = 1$ and hence $z = e^{2n\pi i/3}$. The two inequivalent representations are given by $n = 1$ and $n = -1$. Thus

$$T = \{1, e^{2\pi i/3}, e^{4\pi i/3}\} \ , \ U = \{1, e^{-2\pi i/3}, e^{-4\pi i/3}\} \tag{18}$$

(e) Since U and T (and the trivial rep $V = \{1, 1, 1\}$) are 1-D irreps, the elements are their own characters. Straightforward plugging yields

$$a_V = \frac{1}{3}(2 - 1 - 1) = 0 \tag{19}$$

$$a_T = \frac{1}{3}(2 - e^{-2\pi i/3} - e^{-4\pi i/3}) = 1 \tag{20}$$

$$a_U = \frac{1}{3}(2 - e^{2\pi i/3} - e^{4\pi i/3}) = 1 \tag{21}$$

and thus $D = a_v \cdot V \oplus a_U \cdot U \oplus a_T \cdot T = U \oplus T$.

(ii)

(a) If $g = g^{-1}$ then $g^2 = e$, and hence it is order 2.

(b) First note the following equalities: $(g^k)^{-1} = g^{-k} = (g^{-1})^k$. Now let k be the order of g, i.e. $g^k = e$, so by taking the inverse of both sides we get$(g^{-1})^k = e^{-1} = e$. Hence $|g^{-1}| \leq |g|$. Note the inequality because g^{-1} can have a lower order than g (the importance of *minimum* in th definition of order) since the last equality could also imply that $|g| = n|g^{-1}|$ where n is an integer. To complete that proof, we go "the other way", i.e. $(g^{-1})^k = e$ and hence taking the inverse of both sides again $((g^{-1})^k)^{-1} = e^{-1}$, or $g^k = e^{-1} = e$, implying $|g| \leq |g^{-1}|$. Putting both iequalities together, we get $|g| = |g^{-1}|$.

(c) The identity is its own inverse, so that leaves an *odd* number of elements left to be accounted for. Now, using (b) above, for $|g| > 2$ we can pair off with its inverse $|g^{-1}| > 2$. This leaves one element left, and the group axioms imply that it must has an inverse, so it must be its own inverse. Using (a) above, this last element is order 2.

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