Candidate number:	Desk number:

King's College London

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B.Sc. EXAMINATION

5CCP2332 Symmetry in Physics

Examiner: Dr Eugene A Lim
Spring 2015

Time allowed: TWO hours

Candidates may answer as many parts as they wish from SECTION A, but the total mark for this section will be capped at 20.

Candidates should answer no more than ONE question from SECTION B.
No credit will be be given for answering a further question from this section.

The approximate mark for each part of a question is indicated in square brackets.

Calculators may be used. The following models are permitted: Casio fx83 and Casio fx85.

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TURN OVER WHEN INSTRUCTED

For examiner's use only

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1.2	
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Total	

SECTION A

Answer SECTION A in an answer book. Answer as many parts of this section as you wish. Your total mark for this section will be capped at 40.

1.1 Define what is meant by the permutation group $Perm(S)$ for a finite set S. State *Cayley's Theorem* for finite groups.

[*3* marks]

1.2 Let S be a set. Define what is meant by a *partition* of S.

[*3* marks]

1.3 Construct the multiplication table for Z_4, the cyclic group with 4 elements.

Prove that Z_4 is abelian.

[*5* marks]

1.4 Consider a 2 dimensional rectangle that is *not* a square. Find all its symmetry operations. You may find it useful to draw a diagram to illustrate the operations.

[*4* marks]

1.5 Let $\mathbb{Z}$ be the set of all integers, and $\star$ be a binary operator between elements of $\mathbb{Z}$, such that

$$a \star b = a + b + 2 \ , \ a, b \in \mathbb{Z}.$$

Determine whether
(i) $\star$ is commutative,
(ii) $\star$ is associative,
(iii) an identity exists (find it if it does),
(iv) an inverse exists (find it if it does).

[*5* marks]

1.6 Consider the Klein four-group $V_4 = \{e, a, b, c\}$ with the group law $ab = c$. A 2×2 representation of V_4 is given by

$$D(a) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \ , \ D(b) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

where $x \neq 0 \ , \ y \neq 0$ and $x \neq y$.

Find the conditions on x and y such that the representation D is *unfaithful.*

[*5* marks]

1.7 Consider the following differential equation

$$x^3 \frac{d^2 y}{dx^2} + 3x^2 y^2 - \frac{1}{xy} = 0.$$

Suppose the coordinate x undergoes a dilatation $x \to ax'$ for $a \neq 0$. Find the corresponding transformation for y which leaves the equation invariant.

[*5* marks]

Solution 1.1

$Perm(S)$ is the group of all permutations on the finite set S.

Any finite group G is isomorphic to a subgroup of Permutation Group $Perm(S)$ for some choice of S.

(Bookwork)

Solution 1.2

A **partition** of S is a collection C of subsets of S such that (a)$X \neq \emptyset$ whenever $X \in C$, (b) if $X, Y \in C$ and $X \neq Y$ then $X \cap Y = \emptyset$, and (c) the union of *all* of the elements of the partition is S.

(Bookwork).

Solution 1.3

Let $Z_4 = \{e, a, a^2, a^3\}$, then the multiplication table is easily constructed as

$$\begin{array}{c|ccc} e & a^2 & a & a^3 \\ \hline a^2 & e & a^3 & a \\ a & a^3 & a^2 & e \\ a^3 & a & e & a^2 \end{array} \tag{1}$$

Solution 1.4 The symmetries are (1) identity (2) reflection on horizontal (3) reflection on vertical (4) rotation by 180 degrees.

Solution 1.5

(a) Commutative $a \star b = a + b + 2 = b \star a = b + a + 2$

(b) Associative $a \star (b \star c) = a + b + c + 4 = (a \star b) \star c$

(c) Identity $\exists b$ s.t. $a \star b = b \star a = a$. Easy to show $a \star b = a = a + b + 2$, i.e. $b = -2$ is the identity.

(d) Inverse $\exists b$ s.t. $a \star b = b \star a = -2$. Easy to show $a + b + 2 = -2$, i.e. $b = -4 - a$. So the inverse for a is $-4 - a$.

Solution 1.6 It is easy to calculate

$$D(c) = D(a)D(b) = \begin{pmatrix} 1 & x + y \\ 0 & 1 \end{pmatrix}$$

while $D(e)$ is just the identity matrix.

For the representation to be *unfaithful*, we can set $D(c) = D(e)$, or $x = -y$.

Solution 1.7

Let $y \to by'$, and then substitute this into the differential equation to find

$$(ab)x'^2 \frac{d^2y'}{dx'^2} + 3(a^2b^2)x'^2y'^2 - a^{-1}b^{-1}\frac{1}{x'y'} = 0 \tag{2}$$

which means that $b = a^{-1}$ to keep the ODE invariant.

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SECTION B - Answer ONE question
Answer section B in an answer book

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(i) Consider the set of all square matrices of the form

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where a, b, c are integers obeying addition modulo 5 (i.e. $(3 + 6) \bmod 5 = 4$ and $(1 + 4) \bmod 5 = 0$ etc).

(a) Prove that for $a \neq 0$ and $c \neq 0$, the set of all possible A's forms a group G under matrix multiplication.

How many elements are there in this group?

[*4* marks]

(b) Let G be a finite group and H be a subgroup. Define what is meant by the *left coset* of an element $g \in G$ of H.

[*2* marks]

(c) Consider the subset of the group G given by the condition $b = a - c$.

Show that this subset forms an abelian subgroup H of G.

What is $|H|$?

[*4* marks]

(d) How many distinct left cosets of H are in G? State any theorem(s) that you may use.

[*2* marks]

(e) Find all elements of G whose square is the identity. Prove that this subset cannot be a subgroup of G. State any theorem(s) you use.

[*8* marks]

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(ii) Consider three coloured balls and three coloured boxes, both having the same set of colours, red (1), green (2) and blue (3). The balls are first put in matching colour boxes (i.e. red ball in red box, green ball in green box etc.).

(a) We now swap the contents of the red and green boxes, and *then* swap the contents of the red and blue boxes. Write these operations as permutation operators of the permutation group of 3 objects S_3.

State what are the colours of the balls in each coloured box at the end of these two operations.

[*3* marks]

(b) Now, suppose you are *allowed to make a pair-wise swap of the contents of two boxes only once*, showed that given the remaining possible permutations you cannot restore the balls back into their original boxes (i.e. red in red, blue in blue etc.).

[*3* marks]

(c) Suppose at this stage, additionally you are given an orange box with a orange ball inside and a yellow box with a yellow ball inside. Suppose you are still only allowed to make a pair-wise swap of the contents of each pair of boxes at most once, show given these two additional balls and boxes, that you can restore all five balls to the boxes of their respective colours.

Hint: Note that the addition of 2 more boxes and balls means that we are now considering permutations of 5 objects, thus the initial permutation P is given by

$$P^* = \begin{pmatrix} 1 & 2 & 3 & x & y \\ 2 & 3 & 1 & x & y \end{pmatrix}$$

where x labels orange and y labels yellow. Then find any sequence of pair-wise permutations which, when composed, give the identity.

[*4* marks]

Solution

(i)

(a) The left coset of $g \in G$ of H is the set constructed by acting on all the elements of $H = \{e, h_1, h_2, \ldots\}$ from the left with g, i.e.

$$L_g(H) = \{ge, gh_1, gh_2, \ldots\} \equiv gH.$$

(b) Associativity is inherited. Inverses exists because $\det(A) = ac \neq 0$ since $a \neq 0$ and $c \neq 0$. Id is simply identity matrix. Finally, matrix multiplication imply

$$A_1A_2 = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1c_2 \\ 0 & c_1c_2 \end{pmatrix}$$

and since a_1a_2, $a_1b_2 + b_1 + c_2$ and c_1c_2 modulo 5 are $\in \{0, 1, 2, 3, 4\}$, closure is proven.

Since $a \neq 0$, $c \neq 0$ but there are no conditions on b, the total number of elements are $4 \times 4 \times 5 = 80$.

(c) To show that H is abelian, calculate

$$A_1A_2 = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1c_2 \\ 0 & c_1c_2 \end{pmatrix}, \quad A_2A_1 = \begin{pmatrix} a_2a_1 & a_2b_1 + b_2c_1 \\ 0 & c_2c_1 \end{pmatrix}.$$

Then it is easy to show that

$$a_1b_2 + b_1c_2 = a_1(a_2 - c_2) + (a_1 - c_1)c_2 = a_1a_2 - c_1c_2$$

and

$$a_2b_1 + b_2c_1 = a_2(a_1 - c_1) + (a_2 - c_2)c_1 = a_1a_2 - c_1c_2,$$

thus $A_1A_2 = A_2A_1$, i.e. H is an abelian group.

Since b is no longer an independent variable, $|H| = 4 \times 4 = 16$.

(d) Let G/H be the set of all left cosets of H of G, and $|G/H|$ be the index (i.e. the total number of left cosets). Then Lagrange Theorem states that

$$|G| = |G/H||H|.$$

Thus using this theorem it is easy to show that $|G/H| = 80/16 = 5$, i.e. there are 5 distinct left cosets of H of G.

(e) First calculate

$$AA = \begin{pmatrix} a^2 & ab + bc \\ 0 & c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

giving the conditions $a^2 = 1$,$c^2 = 1$ and $b(a + c) = 0$. Now $a^2 = 1$ implies that $a = \{1, 4\}$, since $1^2 = 1$ and $4^2 \bmod 5 = 16 \bmod 5 = 1$. For b, we go through all the possibilities:

- $a = 1$,$c = 1$: $a + c = 2$ so $b = 0$.

- $a = 1, c = 4$: $a + c = 0$ so $b = \{0, 1, 2, 3, 4\}$.
- $a = 4, c = 1$: $a + c = 0$ so $b = \{0, 1, 2, 3, 4\}$.
- $a = 4, c = 4$: $a + c = 3$ so $b = 0$.

This gives a total number of 12 elements. However, $|G| = 80$ is not divisible by 12, and hence Lagrange's Theorem states that this subset cannot be a subgroup of G.

(ii)

(a) The permutation operators are

$$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

(b) To restore P to the identity, we need to find its inverse P^{-1}

$$P^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

To construct P^{-1} out of pair-wise permutations require

$$P^{-1} = (P_a P_b)^{-1} = P_b^{-1} P_a^{-1}$$

but we know that the inverse of any pair-wise permutation is itself, i.e. $P_a^2 = e$ implying $P_a^{-1} = P_a$ and similarly for P_b. Since we are only allowed a single use of each permutation (by the statement of the problem), this is impossible.

(c) While there are no mechanical way of finding the answer except by brute-force, this problem is not as hard as it looks as there are more than one solution. Two possible sequence of permutations using cyclic notation are

$$\sigma = (xy)(x1)(x2)(y3)(x3)(y1)$$

or

$$\sigma = (xy)(x1)(y2)(y3)(x2)(y1)$$

both which gives

$$P^* \sigma = \begin{pmatrix} 1 & 2 & 3 & x & y \\ 1 & 2 & 3 & x & y \end{pmatrix}.$$

This problem is a toy version of the so-called *Futurama Theorem*. Google it!

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3.

(i) Consider the matrix group $SO(2)$. The elements $M \in SO(2)$ whose elements are real obey the condition $M^T M = e$, with $\det M = 1$.

(a) Let x be the parameter of the group $SO(2)$. By considering the following 2×2 matrix,

$$M = \begin{pmatrix} a & x \\ b & c \end{pmatrix},$$

show that every matrix in $M(x) \in SO(2)$ can be written in the form

$$M_\pm(x) = \begin{pmatrix} \pm\sqrt{1-x^2} & x \\ -x & \pm\sqrt{1-x^2} \end{pmatrix}.$$

[*4* marks]

(b) $SO(2)$ is the rotation group on a 2-dimensional plane. Argue that the elements of M_+ and M_- cover distinct halves of the group respectively, and hence prove that

$$M_+ \cup M_- = SO(2).$$

[*4* marks]

(ii) Consider the Lie group $HT(1,1)$ whose elements are represented by the set of 2×2 real matrices

$$L(a,b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a \neq 0$.

(a) By acting $L(a,b)$ on the vector space parameterized by

$$V(x) = \begin{pmatrix} x \\ 1 \end{pmatrix},$$

show that $M(a,b)$ generates the transformation $x \to ax + b$.

[*2* marks]

QUESTION CONTINUES ON NEXT PAGE

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3.

(b) Show, by expanding around the identity of the group, that the generators for the Lie Algebra $\mathfrak{ht}(1,1)$ are given by

$$X_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \; X_b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

[*6* marks]

(c) Find the structure constant(s) for the algebra $\mathfrak{ht}(1,1)$.

[*4* marks]

(iii) Consider the Dihedral-4 group D_4.

(a) Define what is meant by the *order of an element* of a group.
Find the order of all the elements of D_4.

[*5* marks]

(b) Prove that D_4 is non-abelian, and find the dimensions of all the irreducible representations of D_4. State any theorem(s) that you use.

[*5* marks]

Solution

(i)

(a) The conditions on M mean that $a^2+x^2=1$, $b^2+c^2=1$ and $ab+cx=0$, while $\det M=1$ gives the condition $ac-bx=1$. The first condition means that $a=\pm\sqrt{1-x^2}$. The 3rd condition gives $ab=-cx$, and now using the unitary determinant condition we get

$$1 = ac - bx = ac - (-cx/a)x = (c/a)(a^2+x^2) = \frac{c}{a} \tag{3}$$

which can be fulfilled if $b=-x$ and $a=c$, giving the required form.

(b) Since the $SO(2)$ is the rotation group, we can parameterize it in the usual way using the angular parameter $0<\theta\leq 2\pi$

$$M(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} .$$

By comparing the θ parameterization to the x parameterization, it is clear that since $\sqrt{1-x^2}>0$ by construction while $\cos\theta>0$ for $0<\theta\leq\pi$ and

$\cos\theta < 0$ for $\pi < \theta \leq 2\pi$, we can see that M_- and M_+ cover distinct halves of the rotation group.

(ii)

(a) It is easy to show that

$$L(a,b)V(x) = \begin{pmatrix} ax+b \\ 1 \end{pmatrix} \equiv \begin{pmatrix} x' \\ 1 \end{pmatrix}$$

hence it maps $x \to ax+b$.

(b) Noting that $L(1,0)$ is the identity, we can then Taylor expand around it

$$L(a,b) = L(1,0) + \left.\frac{\partial L}{\partial a}\right|_{a=1,b=0} \delta a + \left.\frac{\partial L}{\partial b}\right|_{a=1,b=0} \delta b$$

where the generators are

$$X_a = \left.\frac{\partial L}{\partial a}\right|_{a=1,b=0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \ X_b = \left.\frac{\partial L}{\partial b}\right|_{a=1,b=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

(c) The structure constants can be computed by taking the commutator

$$[X_a, X_b] = X_a \ , \ [X_b, X_a] = -X_a$$

so $C^a_{ab} = 1$, $C^a_{ba} = -1$ and the rest zeroes.

(iii)

(a) The order n of an element of a group $a \in G$ is such that $a^n = e$. D_4 has 4 mirror reflections m_i for $i = 1,2,3,4$ where $m_i^2 = e$, a rotation by 90°, R where $R^4 = e$, a rotation by 180°, R^2 where $(R^2)^2 = e$, and a rotation by 270°, R^3 where $(R^3)^3 = e$. So D_4 has 5 elements of order 2 (m_i and R^2), 1 element of order 4 (R), 1 element of order 3 (R^3) and the identity.

(b) It is easy to show that D_4 is non-abelian by finding a counter example, e.g. $Rm_1 \neq m_1 R$.

To find the number and dimensionality of the representations, we first note that there must exist the trivial representation of dimension one. Now using Burnside's theorem, we know that the sum of the square of the dimensions of the representations n_i must be equal to $|G|$, i.e.

$$\sum n_i^2 = |G| = 8$$

.

The only possibility that this could occur is $1 + 1 + 1 + 1 + 2^2 = 8$, i.e. there must exist 4 1-d irrep. and 1 2-d irrep.

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