Candidate number:	Desk number:

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B.Sc. EXAMINATION

5CCP2332 Symmetry in Physics

Examiner: Dr Eugene A Lim
Spring 2014

Time allowed: TWO hours

Candidates may answer as many parts as they wish from SECTION A, but the total mark for this section will be capped at 20.

Candidates should answer no more than ONE question from SECTION B.
No credit will be be given for answering a further question from this section.

You may use a College-approved calculator for this paper.
The approved calculators are the Casio fx83 and Casio fx85.

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TURN OVER WHEN INSTRUCTED

SECTION A

Answer SECTION A on the question paper in the space below each question. If you require more space, use an answer book.

Answer as many parts of this section as you wish. The final mark for this section will be capped at 20.

1.1 Let S be a set, with $p, q \in S$ are elements **related** by relation $\bowtie$ and we write $p \bowtie q$. We call $\bowtie$ *a relation on* S. Describe *reflexive*, *transitive* and *symmetric* relations between objects in set S.

[*3* marks]

Solution
Bookwork

1.2 Define what is meant by an *isomorphism* and a *homomorphism* between two groups A and B.

Consider the set H of all possible homomorphisms between A and B. If $f : A \to B$ is an isomorphism, is $f \in H$?

[*3* marks]

Solution

Bookwork

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1.3 Let $S = \mathbb{Z}$, the set of all integers, and $\star$ be a binary operator between elements of S, such that

$$a \star b = a + b + 1 \ , \ a, b \in S.$$

Determine whether
(i) $\star$ is commutative,
(ii) $\star$ is associative,
(iii) an identity exists (find it if it does), and
(iv) an inverse exists (find it if it does).

[*5* marks]

Solution

(a) Commutative $a \star b = a + b + 1 = b \star a = b + a + 1$

(b) Associative $a \star (b \star c) = a + b + 2 = (a \star b) \star c$

(c) Identity $\exists b$ s.t. $a \star b = b \star a = a$. Easy to show $a \star b = a = a + b + 1$, i.e. $b = -1$ is the identity.

(d) Inverse $\exists b$ s.t. $a \star b = b \star a = -1$. Easy to show $a + b + 1 = -1$, i.e. $b = -2 - a$. So the inverse for a is $-2 - a$.

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1.4 Let G and H be finite groups, and $G \times H$ be the product group of these two groups. Prove that $G \times H$ is abelian if and only if G and H are both abelian.

[*4* marks]

Solution

Let $g_i \in G$ and $h_i \in H$, so $(g_i, h_i) \in G \times H$. For $G \times H$ to be abelian, the following

$$(g_1, h_1)(g_2, h_2) = (g_2, h_2)(g_1, h_1) \ \forall g_i, h_i \tag{1}$$

must be true. Calculate LHS $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$, while RHS $(g_2, h_2)(g_1, h_1) = (g_2 g_1, h_2 h_1)$. For LHS = RHS, $g_1 g_2 = g_2 g_1$ and $h_1 h_2 = h_2 h_1$ $\forall g_1, h_1$.

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1.5 Let G be a group, and $a \in G$. Let $C_G(a)$ be the set of all elements of G which commute with the element a, i.e.

$$C_G(a) = \{g \in G : ag = ga\}.$$

Prove that C_G is a subgroup of G.

[*5* marks]

Solution

Let $c_1, c_2 \in C_g$, then

- Closure : $c_1 a = a c_1$, multiply from the left with c_2, we get $c_2 c_1 a = c_2 a c_1$, but using commutative property on the RHS, this is $(c_2 c_1) a = a(c_2 c_1)$, so $c_3 = c_1 c_2 \in C_g$.
- Associativity is inherited from G.
- Identity : since e commute with everything, $e \in C_g$.
- Inverse : let g be the inverse of c_1. Now $c_1 a = a c_1$, multiply from right with g, we get $c_1 a g = a c_1 g = a$ since $c g_1 = e$. Multiply from the left with g, we get $g c_1 a g = g a \Rightarrow ag = ga$, hence $g \in C_g$.

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1.6 Consider the map $f : \mathbb{R} \to \mathbb{R}^*$ such that $f(x) = 2^x$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$ is the set of all reals excluding the zero. Show that if we use the additive group law for $\mathbb{R}$, and the multiplicative group law for $\mathbb{R}^*$, then f is a homomorphism. Find its kernel $\ker(f)$.

Is f an isomorphism? Justify your answer.

[*5* marks]

Solution

It is easy to show $f(x + y) = 2^{x+y}$, and $f(x)f(y) = 2^x 2^y = 2^{x+y}$. To find the kernel of $f(x)$, we note that the identity on $\mathbb{R}^*$ is 1, so we want to find all $x \in \mathbb{R}$ s.t.

$$2^x = 1 \tag{2}$$

with the only solution being $x = 0$, so $\ker(f) = \{0\}$.

No it is not an isomorphism since $\mathrm{cod}(f) \subset \mathbb{R}^*$ (i.e. it must map to the entire $\mathbb{R}^*$. However $2^x > 0 \ \forall \ x \in \mathbb{R}$, which is just a subset of $\mathbb{R}^*$.

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1.7 Consider the following differential equation

$$x^2 \frac{dy}{dx} + x^3 y^2 - \frac{1}{x} = 0.$$

Suppose the coordinate x undergoes a dilatation $x \to ax'$. Find the corresponding transformation for y which leaves the equation invariant.

[*5* marks]

Solution

Let $y \to by'$, and then substitute this into the differential equation to find

$$(ab)x'^2 \frac{dy'}{dx'} + (a^3 b^2) x'^3 y'^2 - a^{-1} \frac{1}{x'} = 0 \tag{3}$$

which means that $b = a^{-2}$ to keep the ODE invariant.

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SECTION B - Answer ONE question

Answer section B in an answer book

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(i) State Lagrange's Theorem. Prove that the theorem implies that finite groups of prime order possess no proper subgroup.

[*4* marks]

(ii) Consider the following set of matrices defined by

$$\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Big| a, b, c \in \mathbb{R}, ac \neq 0 \right\}.$$

(a) Prove that $\mathcal{M}$ forms a group under matrix multiplication. Is this group abelian or non-abelian? Justify your answer.

(b) Consider a 2-dimensional target space $\mathbb{R}^2$, spanned by coordinates (x, y) where $x, y \in \mathbb{R}$. Each element $M \in \mathcal{M}$ acts *linearly* on $\mathbb{R}^2$ in the following way

$$M(x, y) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the codomain of this action. Find the subset of $\mathbb{R}^2$ which remains invariant under this transformation.

(c) Consider $\mathcal{M}'$, which is a subset of $\mathcal{M}$ determined by the condition $c = 1$. Show that $\mathcal{M}'$ is a subgroup of $\mathcal{M}$. Consider the action of $\mathcal{M}'$ on $\mathbb{R}^2$. What are the conditions on a, b such that the transformation is a *dilatation* on x?

(d) Finally consider D, the set of 3×3 matrices defined by

$$D = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \Big| a, b, c \in \mathbb{R}, ac \neq 0 \text{ and } 0 \leq \theta < 2\pi \right\}. \tag{4}$$

D is a matrix group under matrix multiplication. Prove that it is isomorphic to the product group $\mathcal{M} \times U(1)$. Is D abelian? Justify your answers.

[*18* marks]

(iii) For the following question, illustrate your answers.

(a) The Methane molecule CH_4 consists of 4 Hydrogen H atoms with a central Carbon C atom. Describe the symmetry group A exhibited by CH_4.

(b) A chemical process replaced one of the H atom with a Chlorine atom Cl, forming a Chloromethane molecule CH_3Cl. Describe the the symmetry group B exhibited by CH_3Cl.

(c) Is B a subgroup of A? Justify your answer.

[*8* marks]

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Solution

(i) Bookwork.

(ii) (a) M is a group.

- Closure:

$$M_1 M_2 = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 c_2 \\ 0 & c_1 c_2 \end{pmatrix} \in M \tag{5}$$

as $a_1 a_2 c_1 c_2 \neq 0$ as $a_1 c_1 \neq 0$ and $a_2 c_2 \neq 0$. It is not abelian, since

$$M_2 M_1 = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & a_2 b_1 + b_2 c_1 \\ 0 & c_1 c_2 \end{pmatrix} \neq M_1 M2 \tag{6}$$

- Identity is $\mathbb{I}$.

- Inverse exists because $ac \neq 0$, hence $\det M \neq 0$.

- Associative inherited.

(b) The action maps $x' \to ax + b$ and $y' \to cy$, which spans the entire $\mathbb{R}^2$ since $a, b, c \in \mathbb{R}$. So the $\text{cod}(M) = \mathbb{R}^2$, i.e. the action is *onto*. The subset of $\mathbb{R}^2$ left invariant under this map $\forall a, b, c$ is the $y = 0$ line since it gets map to $y' = 0$ regardless of the values of a, b, c.

(c) M' is a subgroup of M because $c = 1 \subset \mathbb{R}$. The action of M' on (x, y) leaves $x \to ax + b$ and $y \to y'$. For the transformation to a dilatation, $b = 0$.

(d) The matrix group is $D = M \oplus U(1)$. This means that it is a product group $D = M \times U(1)$. We can rewrite the group elements as

$$D(a, b, c, \theta) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, e^{i\theta} \right\} \tag{7}$$

which then it is easy to show that the multiplication of $D_1 D_2$ leads to

$$D_1 D_2 = \left\{ \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 c_2 \\ 0 & c_1 c_2 \end{pmatrix}, e^{i\theta_1 + i\theta_2} \right\} \tag{8}$$

which is the same if we multiply the 3×3 parameterization together. Since M is not abelian, D is not abelian.

(iii) (a) CH_4 has tetrahedral symmetry. This means that any interchange, or permutation, of its 4 vertices will leave the molecule invariant, hence the symmetry group is S_4 (the group of permutations for 4 objects).

(b) CH_3Cl, one of the H has been swapped out for a Cl molecule, with the 3 remaining H atoms forming an equilateral triangle. The symmetry is now reduced to the discrete rotations around $Cl - C$ axis, where a rotation of 120 degrees will keep the C and Cl atom in the same spot, while interchanging the remaining H atoms. We can also consider reflections around the $C - Cl$ axis. Hence the symmetry group is D_3 (Dihedral-3) group.

(c) Since S_3 is a subgroup of S_4 and $D_3 \cong S_4$, B is a subgroup of A. To justify the answer, one can explicitly find the elements of S_4 that forms D_3 (long way). It is easy to show this geometrically – by fixing the C and one of the H of the group A, the remaining group elements form B.

3.

(i) Let Z_4 be the order 4 cyclic group generated by a.

(a) Define what is a *conjugacy class*. Find all conjugacy classes of Z_4.

(b) State what are *reducible* and *irreducible* representations. Find all inequivalent irreducible representations of Z_4.

(c) Let $D_5 : Z_4 \to GL(4, \mathbb{C})$ be a 3×3 representation of Z_4, and given

$$D_5(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \ D_5(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix}$$

find $D_5(a^2)$ and $D_5(a^3)$. Decompose them into irreducible representations by writing D_5 as a direct sum of irreducible representations you found in part (b).

(d) State the Orthogonality Theorem for Characters. Construct the character table for the D_1, D_2, D_3, D_4, D_5 representations of Z_4 and verify that it satisfies the Orthogonality Theorem.

[*15* marks]

(ii) Consider a *rational fractional transformation* parameterized by (a, b, c, d) which map points on the real line to the real line in the following way

$$x \to x' = \frac{ax + b}{cx + d} \ , \ a, b, c, d \in \mathbb{R}.$$

(a) Show that the maps (a, b, c, d) and $(\lambda a, \lambda b, \lambda c, \lambda d)$ for $\lambda \neq 0$ generate identical mappings.

(b) Compose two successive maps parameterized by (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2) and show that this also a *rational fractional transformation*.

(c) Find the set of transformations which maps x to itself (i.e. $x \to x' = x$).

(d) Finally, show that the inverse transformation for (a, b, c, d) is given by $(\alpha d, -\alpha b, -\alpha c, \alpha a)$ where $\alpha \neq 0$, and that this transformation exists only if $ad - bc \neq 0$.

(e) Given your calculations above, comment on whether rational fractional transformations form a Lie Group. Explain your answer. You may assume that the map is associative.

[*15* marks]

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Solution

(i)

(a) Conjugacy Class (Bookwork). Since Z_4 is abelian (easily proven), it has as many conjugacy classes as its order,i.e. $\{e\}, \{a\}, \{a^2\}, \{a^3\}$.

(b) Reducible/Irrep (Bookwork). Using Burnside's theorem, we know that there exist 4 irreps (since there are 4 classes). Since Z_4 is abelian, all its irreps must be single dimensional. In the complex representation, the group structure means that the elements must be the 4-th root of unity $z^4 = 1$. This have solutions

$$\begin{aligned} D_1 &= \{1, 1, 1, 1\} \\ D_2 &= \{1, -1, 1, -1\} \\ D_3 &= \{1, i, -1, -i\} \\ D_4 &= \{1, -i, -1, i\} \end{aligned} \tag{9}$$

(c) Can solve by brute force calculation or by realizing that $D_5 = D_1 \oplus D_2 \oplus D_3$, i.e.

$$D_5(a^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ , \ D(a^3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -i \end{pmatrix}. \tag{10}$$

(d) Orthogonality Theorem for characters (Bookwork) $\sum_\alpha (\chi_\alpha^m)^* (\chi_\alpha^{m'}) = |G|\delta_{mm'}$. The character table is

	e	a	a^2	a^3
$\chi(D_1)$	1	1	1	1
$\chi(D_2)$	1	-1	1	-1
$\chi(D_3)$	1	i	-1	$-i$
$\chi(D_4)$	1	$-i$	-1	i
$\chi(D_5)$	3	i	1	$-i$

(11)

Since the theorem applies to only irreps, ignoring D_5, it's clear that the theorem is obeyed.

(ii) (a) Plugging in the transform $(\lambda a, \lambda b, \lambda c, \lambda d)$ yields

$$x' = \frac{ax + b}{cx + d} \tag{12}$$

which is the same transform as (a, b, c, d) as long as $\lambda \neq 0$.

(b) Successive transform yields

$$x' = \frac{a_1 x + b_1}{c_1 x + d_1}\ ,\ x'' = \frac{a_2 x' + b_2}{c_2 x' + d_2} = \frac{(a_1 a_2 + b_2 c_1)x + (a_2 b_1 + b_2 d_1)}{(a_2 c_2 + c_1 d_2)x + (b_1 c_2 + d_1 d_2)} \tag{13}$$

which is also a rational fractional transform.

(c) We want $x' = x$, so

$$\frac{ax + b}{cx + d} = x \Rightarrow cx^2 + (d - a)x + b = 0 \tag{14}$$

and it is clear that for this to be solution $c = 0, b = 0$ and $a = d$. So the identity map is $(a, 0, 0, a)$.

(d) To find the inverse transform, we want to map x' back to x, so using the ansatz

$$x'' = \frac{\alpha(dx' - b)}{\alpha(-cx' + a)} = \frac{x(ad - bc)}{(ad - bc)} \tag{15}$$

so as long as $(ad - bc) \neq 0$, the map is the inverse.

(e) We have found the identity, the inverse and proved closure. Assuming associativity, the map forms a group. Since $x' = (a, b, c, d, x)$ is analytic in all its 4 parameters, it forms a Lie Group (it is really the real projective group $\mathbb{RP}^1$.)

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