Symmetry in Physics 2016 Class Test Feb 25 Lecturer : Dr. Eugene A. Lim

This is a class test for the Symmety in Physics CP2332 course. It will account for 10% of the total class mark. You have 1 hour to complete the test. This is a closed-book test and you are not allowed to use any calculators. [X] indicates the total number of marks for that question.

Question

The Klein Four-group $V_4 = \{e, a, b, c\}$ has the group laws $a^2 = b^2 = c^2 = e$ and $ab = c$. As we have shown in class it can be constructed out of the product group $Z_2 \times Z_2$ where $Z_2 = \{e, \mu\}$ is the order 2 cyclic group (or parity group) with group law $\mu^2 = e$.

(i) Show that V_4 is the group that describe the symmetry of the rectangle which is not the square. [3]

(ii) Construct the product group $V_4 \times Z_2$, illustrating it with a Cayley Table. What is the order of this group? [5]

(iii) From class, we learn that V_4 is abelian. Is $V_4 \times Z_2$ abelian or non-abelian? Prove your assertion. [2]

Solution

(i) This can be easily proven by drawing a rectangle. Then one can show that a denotes reflection along horizontal axis, b denotes reflection on the vertical axis and c denotes rotation by 180 degrees (clockwise or counterclockwise).

(ii) The group elements for $V_4 \times Z_2 = \{(e, e), (e, \mu), (a, e), (a, \mu), (b, e), (b, \mu), (c, e), (c, \mu)\}$, and the cayley table has 64 entries which can computed by brute force (easy marks here). The order of the group is 8.

(iii) Since Z_2 is also abelian, the product group $V_4 \times Z_2$ is a product of two abelian groups, which is also abelian. Pf : Let $g_1 \in G_1$ and $g_2 \in G_2$, where G_1 and G_2 are abelian groups. Then an element of of $G_1 \times G_2$ is (g_1, g_2). But $(g_1, g_2) \cdot (g'_1, g'_2) = (g_1 g'_1, g_2 g'_2)$, and $(g'_1, g'_2) \cdot (g_1, g_2) = (g'_1 g_1, g'_2 g_2)$ but since $g_1 g'_1 = g'_1 g_1$ and similarly for $g_2 g'_2$ as both groups are abelian, $(g_1, g_2) \cdot (g'_1, g'_2) = (g'_1, g'_2) \cdot (g_1, g_2)$, and we are done.