**Symmetry in Physics 2015 Class Test Feb 27 Lecturer : Dr. Eugene A. Lim**

This is a class test for the Symmety in Physics CP2332 course. It will account for 10% of the total class mark. You have 1 hour to complete the test. This is a closed-book test and you are not allowed to use any calculators. [X] indicates the total number of marks for that question.

**Question 1**

State Lagrange's Theorem. [1 marks]

**Question 2**

Define *conjugacy class*. Let $C$ be a conjugacy class of a group $G$. Is $C$ necessary a subgroup of $G$ ? Justify your answer. [3 marks]

**Question 3**

What is a *generating set* of a group $G$? Consider the Dihedral-4 group, $D_4$. Find all generating sets for $D_4$. Generalize your answer to $D_N$ groups for $N > 4$. [6 marks]

Solutions

1. Bookwork [1 mark]

2. Conjugacy class (bookwork) [1 mark]. $C$ is not necessary a subgroup. [1 mark]. There are many ways to justify this. The easiest is to argue that conjugacy classes partition the group, and the identity can only appear once in one particular conjugacy class, so not all conjugacy classes are subgroups. [1 mark].

3. Generating set (bookwork) [2 mark]. Let $D_4 = \{e, R, R^2, R^3, m_i\}$ for $i = 1, 2, 3, 4$. The generating set is a *minimal* set, so we need a rotation and a reflection to make the set. In class, we showed that one such set is $\{R, m_1\}$. By symmetry, we can consider the inverse of $R$ as the generator of rotations, so $\{R^3, m_1\}$ is also a set. Using the formula $m_i = R^{i-1}m_1$, we can replace the $m_1$ with any other $m_i \; \forall \; i$. Furthermore, we can rewrite the formula as $R^{i-1} = m_i m_1$ (using $m_1^{-1} = m_1$), and hence we can generate $R$ and $R^3$ with $m_1 m_4$ and $m_1 m_2$ respectively. Thus all the generating sets of $D_4$ are $\{R, m_i\}$, $\{R^3, m_i\}$, $\{m_1, m_2\}$ and $\{m_1, m_4\}$. Note that $\{R^2, m_i\}$ (and hence $\{m_3, m_1\}$)) are not generating sets since we cannot generate all the rotations with $R^2$. [3 mark]

[This is the hard bonus question part.] To generalize to $D_N$, we know that one of the generator has to be a reflection operator $m_i$, and our problem reduces to which of the rotation operators can generate all the rotations. Such an operator must *cycle* through all possible rotations when operated repeatedly on itself, i.e. if $X$ is such a rotation operator, then $X^2$, $X^3$, $X^4$, $\dots$, $X^{N-1}$, $X^N = e$ must generate *different* rotation operators. In other words the *order* of the element $X$ must be $N$. Let $X = R^M$, then for $X$ to be order $N$, then the greatest common divisor of $M$ and $N$ must be 1, i.e. they must be *co-primes.* As an example, for $D_{12}$, $R, R^5, R^7, R^{11}$ and any $m_i$ will form a generating set. [1 mark].