Solutions to Homework Set 5

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April 19, 2022

Exercise 1. Let G be a group and the homomorphisms $D : G \to GL(N_1, \mathbb{C})$ and $T : G \to GL(N_2, \mathbb{C})$ be two matrix representations of G. Consider the reducible representations $M_1 = D \oplus T$ and $M_2 = T \oplus D$. Show that the characters are the same for both reducible representations, and hence prove that there exists a similarity transform B that transforms M_1 to M_2.

Solution: *Direct sums of matrix representations act as block-diagonal compositions (hence why the $\oplus$ symbol is sometimes called **outer sum**). We can write*

$$M_1 = D \oplus T = \begin{pmatrix} D_{N_1 \times N_1} & 0 \\ 0 & T_{N_2 \times N_2} \end{pmatrix}$$
$$M_2 = T \oplus D = \begin{pmatrix} T_{N_2 \times N_2} & 0 \\ 0 & D_{N_1 \times N_1} \end{pmatrix}$$

The characters are the traces of M_1 and M_2, which will be the same because multiplication in C is commutative and their diagonals are the same up to rearranging. Moreover, since the rows and columns are also the same under different orderings, their eigenvalues and eigenvectors must be the same, which means that they are similar. In fact, the similarity transform is quite easy to derive: we just need a matrix that will swap the two matrix blocks:

$$B = \begin{pmatrix} 0 & \mathbb{I}_{N_2 \times N_2} \\ \mathbb{I}_{N_1 \times N_1} & 0 \end{pmatrix} \quad \textit{so that} \quad BM_2 = M_1 B$$

Exercise 2. Consider the matrix group $SO(3)$, acting on $\mathbb{R}^3$, i.e. the usual 3 dimensional Euclidean space with *Cartesian* coordinates labeled by (x, y, z). Show that this action is analytic, and hence $SO(3)$ forms a Lie group.

Solution: *An action is analytic if it can be described in terms of analytic functions over the acted field. In this case, if we parameterise $SO(3)$ in terms of 3 rotation angles about the three Cartesian axes ($SO(3)$ are 3D rotations so they can be described by rotations along the axes, but there are many more ways of parametersing this group. For example, (the aforementioned) Tait-Bryant or Euler angles, or points over the solid 3D unit ball), we may write any $SO(3)$ as*

$$\mathbf{x'} = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where a general $SO(3)$ transformation will be the product of the three rotation matrices by the corresponding angles $\theta_1, \theta_2, \theta_3$. In this case, we need to show that

$$\begin{aligned} x_1' &= f_1(x_1, x_2, x_3, \theta_1, \theta_2, \theta_3) \\ x_2' &= f_2(x_1, x_2, x_3, \theta_1, \theta_2, \theta_3) \\ x_3' &= f_3(x_1, x_2, x_3, \theta_1, \theta_2, \theta_3) \end{aligned}$$

for f_1, f_2, f_3 analytic functions over $\mathbb{C}$. Luckily, we know from the expression above that all these functions will be sums and products of cosines of real numbers with the components of $\mathbf{x}$. We also know that the sin *and* cos *functions are analytic in $\mathbb{C}$, and the sums, products and compositions of analytic functions are analytic too, so our whole functions and thus our action must be analytic -which makes $SO(3)$ a Lie group.*

Exercise 3. Consider the circle, S_1, parameterised by the coordinate θ such that $0 \leq \theta \leq 2\pi$. Consider a transformation T which maps the circle to itself as follows

$$T : \theta \to \theta'; \theta' = \theta + k + f(\theta)\ ,\ 0 \leq k \leq 2\pi \tag{1}$$

(i) Argue that since $0 \leq \theta' \leq 2\pi$, $f(\theta)$ is periodic, i.e. $f(\theta + 2\pi) = f(\theta)$.

(ii) Show that, insisting that T is one-to-one requires an additional condition $f(\theta) : df(\theta)/d\theta > -1$ to be imposed everywhere on S_1.

(iii) Show that the set of transformations $T(\theta)$ forms a group.

(iv) Is this a Lie Group? Justify your answers

Solution: *To show that $f(\theta)$ is periodic, we simply look at the definition of $\theta' = \theta' \pm 2\pi = \theta + k + f(\theta)$. Treating $\theta' \pm 2\pi \equiv \theta'_0$ as the new output angle we may write $\theta + k + f(\theta) = \theta' = \theta' \pm 2\pi = \theta'_0 = \theta_0 + k + f(\theta_0) = (\theta \pm 2\pi) + k + f(\theta \pm 2\pi)$ but since θ is an angle, $\theta \pm 2\pi = \theta$ so we can replace the last expression with $\theta + k + f(\theta \pm 2\pi)$. Subtracting by θ and k on both sides yields $f(\theta) = f(\theta_0) = f(\theta \pm 2\pi)$.*

T will be one-to-one only if $d\theta'/d\theta > 0$ everywhere (if it is equal to zero then we would have $g(\theta + d\theta) = g(\theta)$ which is not one-to-one, and if it is < 0 then it must have changed sign at some point, and since the derivative is continuous, it must have gone through zero. This statement is known as the ***intermediate value theorem (ITV)*** *and it appears extremely often in the branch of mathematics known as Analysis. Incidentally, ITV also stands for Spain's version of the MOT test). Now, by differentiating the definition fo T, and requiring that $d\theta'/d\theta > 0$ we get $0 < d\theta'/d\theta = 1 + df/d\theta$ i.e. $df/d\theta > -1$.*

To show the space of transformations T forms a group (under composition), we simply go through the group axioms. You should be able to do this by now. Associative: composition of maps is associative. Identity: when $f(\theta) = -k$. Inverses: Since T is one-to-one, we will have inverses in the space of continuous monotonically increasing functions $f(\theta)$. Closure: the composition of periodic functions with the same period is periodic and the composition of monotonically increasing functions is monotonically increasing.

It is a Lie group only if we restrict $f(\theta)$ to continuous and analytic functions over $0 \leq \theta \leq 2\pi$ (i.e. $f \in \mathcal{C}^2([0, 2\pi])$).

Exercise 4. In an $n-$dimensional linear vector space, two coordinate systems x^i and y^i are related by a *linear basis transformation*

$$y^j = M_i^j x^i \tag{2}$$

where M_i^j can be represented by an $n \times n$ square matrix. Show that the derivatives are related by the same transformation M_i^j, i.e.

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} = M_i^j \frac{\partial}{\partial y^j} \tag{3}$$

(Don't forget to use the Einstein summation convention to sum over indices. In this problem, the location of the indices (superscript/subscript) does not matter).

Solution: *From the algebra of partial derivatives, we know that*

$$dy^j = \frac{\partial y^j}{\partial x^1}dx^1 + \frac{\partial y^j}{\partial x^2}dx^2 + \quad ... \quad + \frac{\partial y^j}{\partial x^n}dx^n$$

so we can write in Einstein notation:

$$dy^j = \frac{\partial y^j}{\partial x^i}dx^i$$

But taking differentials in equation 2*, we can identify these partials with the coefficients of the matrix:*

$$\frac{\partial y^j}{\partial x^i} = M_i^j$$

Then, using the chain rule:

$$\frac{\partial}{\partial x^i} = \frac{\partial y^1}{\partial x^i}\frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial x^i}\frac{\partial}{\partial y^2} + \quad ... \quad \frac{\partial y^n}{\partial x^i}\frac{\partial}{\partial y^n} = \frac{\partial y^j}{\partial x^i}\frac{\partial}{y^j} = M_i^j \frac{\partial}{\partial y^j}$$

as required.

Exercise 5. In class, we claim that the exponentiation of a square matrix A is defined by

$$e^A \equiv \sum_{k=0}^{\infty} \frac{1}{k!} A^k \tag{4}$$

where $A^0 = \mathbb{I}$. By expanding the RHS of the following equation

$$e^A = \lim_{m\to\infty} \left(\mathbb{I} + \frac{1}{m}A\right)^m \tag{5}$$

show that it is equivalent to the first definition of e^A we discussed in class. Prove that

$$\det(e^A) = \lim_{m\to\infty} \left[\det\left(\mathbb{I} + \frac{1}{m}A\right)\right]^m. \tag{6}$$

Now show that,

$$\det\left(\mathbb{I} + \frac{1}{m}A\right) = 1 + \frac{1}{m}\mathrm{Tr}(A) + \mathcal{O}(1/m^2) + ... \tag{7}$$

where ... indicate terms at higher orders in $1/m$. Substituting this result in equation 7 into equation 6 derive the identity

$$\det(e^A) = e^{\mathrm{Tr}(A)}. \tag{8}$$

Solution: *To show that these definitions are equivalent, we simply (Taylor) expand the exponential and let $m \to \infty$. I.e.*

$$e^A = \lim_{m\to\infty} \left(\mathbb{I} + \frac{1}{m}A\right)^m = \lim_{m\to\infty} \left(\mathbb{I} + m\frac{1}{m}A + \frac{m(m-1)}{2!}\left(\frac{A}{m}\right)^2 + \ ...\right)$$

noticing that $m(m-1)/m^2 \to 1$ as $m \to \infty$, and substituting for the limit, we get

$$e^A = \mathbb{I} + A + \frac{1}{2!}A^2 + \ \ ... \ = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

as required.

In the case of the determinant, we use the fact that $\det(AB) - \det A \det B$ *so that* $\det(A^n) = (\det A)^n$. *Then we simply take the determinant of both sides in* 5 *and push the determinant inside the power m.*

For the next part, simply expand in matrix form (the hint is that we have to go from determinants to traces, so clearly we'll have to do something involving the coefficients of the matrix (I + 1/m A). Indeed,

$$\det\left(\mathbb{I} + \frac{1}{m}A\right) = \det\begin{pmatrix} 1 + A_{11}/m & A_{12}/m & \cdots \\ A_{21}/m & 1 + A_{22}/m & \\ \vdots & & \ddots \end{pmatrix},$$

and expanding the determinant and collecting terms along the diagonal:

$$= \left(1 + \frac{A_{11}}{m}\right)\left(1 + \frac{A_{22}}{m}\right)\left(\dots\right) + \mathcal{O}\left(\frac{1}{m^2}\right) = 1 + \frac{1}{m}(A_{11} + A_{22} + \dots) + \mathcal{O}\left(\frac{1}{m^2}\right)$$

Where $(A_{11} + A_{22} + \dots)$ *is simply* $\mathrm{Tr}(A)$ *so the expression in* 6 *becomes*

$$\det(e^A) = \lim_{m\to\infty} \left[1 + \frac{1}{m}\mathrm{Tr}(A) + \mathcal{O}\left(\frac{1}{m^2}\right)\right]^m$$

Again, expanding the power of m:

$$\det(e^A) = \lim_{m\to\infty} \left(1 + \mathrm{Tr}(A) + \frac{m(m-1)}{2!}\left(\frac{\mathrm{Tr}(A)}{m}\right)^2 + \dots\right)$$

And taking the limit as $m \to \infty$*:*

$$\det(e^A) = 1 + \mathrm{Tr}(A) + \frac{1}{2!}[\mathrm{Tr}(A)]^2 + \dots = e^{\mathrm{Tr}(A)}$$

as required.

Exercise 6. In class, we showed that the 2-D representation of $SO(2)$ can be obtained by exponentiating the generator

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{9}$$

Show that

$$X^{2n} = (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

Using this result, prove that the exponentiation can be broken into even and odd powers as follows

$$\exp(\theta X) = \mathbb{I} + \sum_{n=1}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} \mathbb{I} + \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n-1)!} X, \tag{11}$$

and hence show that the exponentiation recovers the matrix operator for $SO(2)$

$$\exp(\theta X) = \cos(\theta)\mathbb{I} + \sin(\theta) X \tag{12}$$

Solution: *This is simply a matrix algebra exercise:*

$$X^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (-1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore

$$X^{2n} = (X^2)^n = (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^n$$

We can now use the results from exercise 5 *and expand the matrix exponential to get the desired result:*

$$\begin{aligned} \exp(\theta X) &= \mathbb{I} + \theta X + \frac{(\theta X)^2}{2!} + \frac{(\theta X)^3}{3!} + \\ &= \mathbb{I} + \sum_{k \text{ odd}} \frac{\theta^k X^k}{k!} + \sum_{k \text{ even}} \frac{\theta^k X^k}{k!} \\ &= \mathbb{I} + \sum_{n=0}^{\infty} \frac{\theta^{2n+1} X^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{\theta^{2n} X^{2n}}{(2n)!} \end{aligned}$$

Now if we substitute X^m with the appropriate $(-1)^m X$ or $(-1)^m I$ we can see that the remaining series in θ correspond precisely to the Maclaurin series of $\sin\theta$ and $\cos\theta$, so we can substitute in to get the desired result (Search Maclaurin -or Taylor at zero- expansions of sin *and* cos *if you are having difficulties doing this. The idea is that since we are working with series definitions of functions, it only makes sense to keep using that tool to arrive at a sin and a cosine).*

Exercise 7. Prove that the commutator $[A, B] = AB - BA$ obeys the Jacobi Identity

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0. \tag{13}$$

Solution: *The Jacobi Identity is a property of bilinear forms, in particular of Lie brackets. This is done by straight computation and using the associative properties of Lie algebras:*

$$[[A, B], C] = [A, B]C - C[A, B] = ABC - BAC - CAB + CBA \tag{1}$$

$$[[B, C], A] = [B, C]A - A[B, C] = BCA - CBA - ABC + ACB \tag{2}$$

$$[[C, A], B] = [C, A]B - B[C, A] = CAB - ACB - BCA + BAC \tag{3}$$

Observe that $(1) + (2) + (3) = 0$ *as required.*

Exercise 8. Consider the differential equation

$$y\frac{dy}{dx} + x^{\alpha}y - xy^{\beta} = 0 \tag{14}$$

where α, β are integers. Consider the transformations $x \to ax$, $y \to a^2y$ where $a \in \mathbb{R} - \{0\}$. What are the values of α, β for which this transformation leaves the above equation invariant?

Solution: *Another purely computational exercise. We apply the transformation and solve for α and β so that the new a terms vanish and keep the equation the same:*

$$a^{-2}y\frac{a^{-2}}{a}\frac{dy}{dx} + a^{\alpha}x^{\alpha}\alpha^{-2}y - axa^{-2\beta}y^{\beta} = 0$$

$$a^{-3}y\frac{dy}{dx} + a^{\alpha-2}x^{\alpha}y - \alpha^{1-2\beta}xy^{\beta} = 0$$

$$y\frac{dy}{dx} + a^{\alpha+1}x^{\alpha}y - a^{4-2\beta}xy^{\beta} = 0$$

$$\Rightarrow \alpha = -1 \quad \& \quad \beta = 2$$

Exercise 9. Let K_i be the Lie Algebra of $SO(3)$. Show by computing the structure constants that the $l = 1$ **triplet** representation of K_i can be represented by the following matrices

$$K_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad , \quad K_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad K_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} . \tag{15}$$

Solution: *For $l = 1$, we have $m \in \{-1, 0, 1\}$ so we can label the sates as*

$$v_{1,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_{1,0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_{1,-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for which K_3 is then

$$K_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(you can check this easily).

Using that $K_+ v_{1,1} = 0$ and $K_+ v_{1,-1} = \sqrt{1(1+1) - (-1)(-1+1)}v_{1,0} = \sqrt{2}v_{1,0}$ and $K_+ v_{1,0} = \sqrt{2}v_{1,1}$ we can derive the whole form of K_+ (each product with these vectors corresponds to the column number for where the v-vector si non-zero. I.e. the columns of K_+ will be the RHS of these three expressions):

$$K_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly, from $K_- v_{1,-1} = 0$, $K_- v_{1,1} = \sqrt{2}v_{1,0}$ and $K_- v_{1,0} = \sqrt{2}v_{1,-1}$ we can derive

$$K_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

as required.

Exercise 10. There is no exercise 10 :D

Solution: *The empty set* $\emptyset \equiv \{\}$

Exercise 11. (Hard.) As discussed in class, $SL(2, \mathbb{C}$ are complex 2×2 matrices with determinant $= 1$. Matrices of this group M have structure

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad , \quad \det(M) = \alpha\delta - \beta\gamma = 1. \tag{16}$$

Consider a matrix X parameterised by

$$X(x, y, z, t) = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}. \tag{17}$$

(i) Show that, if we define $\mathbf{x} \equiv (x, y, z)$ as the usual 3-vector , and the dot product $\cdot$ ad the usual 3-D vector dot product, we can express X as

$$X(x, y, z, t) = t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sigma \cdot \mathbf{x} \tag{18}$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the usual **Pauli spin Matrices**

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{19}$$

and t is the time coordinate.

(ii) Show that X is Hermitian.

(iii) Show that the most general 2×2 Hermitian matrix can be written in the form of the decomposition equation 17.

(iv) If $M \in SL(2, \mathbb{C})$, consider the transformation $X(x', y', z', t') = M^\dagger X(x, y, z, t) M$, i.e. a transform X by any element of $SL(2, \mathbb{C})$ leaves X in the form of equation 18. Show that this transformation leaves the metric $(t^2) - x^2 - y^2 - z^2$ invariant. *(Hint: Consider the determinants.)*

(v) How do the new space-time coordinates (x', y', z', t') relate to the original coordinates (x, y, z, t)? I.e. calculate x', y', z', t' as functions of (x, y, z, t) and $(\alpha, \beta, \gamma, \delta)$, using the condition $\alpha\delta - \beta\gamma = 1$.

(vi) Find the subgroup of $SL(2, \mathbb{C}$ which leaves $t' = t$ in the transformation defined in (iv). Show that this subgroup is $SU(2)$. (*Hint: Consider the conditions on* $(\alpha\beta\gamma\delta)$ *such that* $t' = t$.)

(vii) (Optional: Lorentz Transformation) Let $H \in SU(2)$, show that H can be represented by the exponentiation of the Pauli matrices

$$H = \exp\left(\frac{i}{2}\sigma \cdot \theta\right) \tag{20}$$

where $\theta = (\theta_x, \theta_y, \theta_z)$ is a set of rotation angles along the x, y, z axes respectively. COnsider the Hermitian matrix K, i.e. $K^\dagger = K$, given by

$$K = \exp\left(\frac{1}{2}\sigma \cdot \mathbf{b}\right) \tag{21}$$

where $\mathbf{b} = (b_x, b_y, b_z)$ is a real vector in 3-D Euclidean space. Show that

$$M = KH \tag{22}$$

i.e. elements of $SL(2, \mathbb{C})$ can be "factored" into a unitary matrix H (which is a subgroup of $SU(2)$ and a Hermitian matrix K). (Note: The form $M = KH$ means that the set of all possible K forms a *coset* space $K \in SL(2, \mathbb{C})/SU(2)$. Is this a group?) *(Andres' hint: This is a group **if and only if** $SU(2)$ is **normal** in $SL(2, \mathbb{C})$. Cool, huh?).* Consider the case $b_x = b_y = 0$. Calculate the transformation

$$K^\dagger X(x, y, z, t) K = X(x', y', z', t') \tag{23}$$

and show that it is the Lorentz transformation law for a *boost* along the z direction.

Solution: *(i) Separate the components of X in terms of a sum:*

$$X(x, y, z, t) = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is precisely what we need ($\sigma \cdot \mathrm{x} = \sigma_x x + \sigma_y y + \sigma_z z$).

(ii) A matrix M is Hermitian if $M = M^\dagger$. We can use the decomposition from (i) to easily check this; the sum of Hermitian matrices is Hermitian, and X is precisely that.

(iii) Let H be a general 2×2 matrix. Then

$$H \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies H^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

For H to be Hermitian, we require $a = a^*$, $d = d^*$ (i.e. the diagonal entries are real) and $b = c^*$, $c = b^*$ (i.e. the off-diagonal entries have the same real part and opposite imaginary parts. We thus write $b = x - iy$ and $c = x + iy$). Any pair of real values (a, d) can be obtained from $(t + z, t - z)$ for appropriate choices of t and z. To have this expression in the diagonal, we may write

$$t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For the off-diagonal terms, we simply need a similar construction with appropriate real and imaginary parts:

$$x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

and thus we have the required decomposition.

(iv) We know that $\det M = \det M^\dagger = 1$. *Then* $\det X(x', y', z', t') = \det(M^\dagger X M) = \det M^\dagger \det X \det M$. *Expanding both sides we get, from the LHS:* $(t'+z')(t'-z') - (x'-iy')(x'+iy') = t'^2 - z'^2 - x'^2 - y'2$. *From the RHS:* $\det M^\dagger \det X \det M = \det X = t^2 - z^2 - x^2 - y^2$ *as required.* ***Andres' note: I want you to think back to your relativity and QM courses. What is the metric of spacetime? It's called the Minkowsky metric (see Hermann Minkowsky) and it tells us the separation between points in 3+1 Dimensional space-time. Its form is*** $\boldsymbol{c^2t^2 - x^2 - y^2 - z^2}$ ***where t is the time coordinate,*** $\boldsymbol{(x, y, z)}$ ***are space coordinates, and c is the speed of light in vacuum. Now you can glimpse why the language of fundamental physics like QM is in terms of complex Hermitian matrices: they encompass precisely those space/time transformations that leave spacetime intervals invariant.***

(v) This is a lot of algebraic rearranging: we simply have to expand out the transformation and see what we get $X(t', x', y', z') = M^\dagger X M =$

$$\begin{pmatrix} |\alpha|^2(t+z)+|\alpha|^2(t-z)+\alpha^*\gamma(x-iy)+\alpha\gamma^*(x+iy) & \beta[\gamma^*(x+iy)+\alpha^*(t+z)]+\delta[\alpha^*(x-iy)+\gamma^*(t-z)] \\ \beta^*[\gamma(x-iy)+\alpha(t+z)]+\delta^*[\alpha(x+iy)+\gamma(t-z)] & |\beta|^2(t+z)+|\delta|^2(t-z)+\beta^*\delta(x-iy)+\beta\delta^*(x+iy) \end{pmatrix} =$$

$$= \begin{pmatrix} t'+z' & x'-iy' \\ x'+iy' & t'-z' \end{pmatrix}$$

We can simplify this mess by noticing that $t' + z' + (t' - z') = 2t'$ *so that we find* $2t' = (|\alpha|^2 + |\beta|^2)(t+z) + (|\gamma|^2 + |\delta|^2)(t-z) + (\alpha^*\gamma + \beta^*\delta)(x-iy) + (\alpha\gamma^* + \beta\delta^*)(x+iy)$ *and we can do the same for* $2z'$*...*

(vi) This corresponds to choosing values of $\alpha, \beta, \gamma, \delta$ *so that the expression we found in (v) simplifies to* $t' = t$. *I.e. we need* $\alpha^*\gamma + \beta^*\delta = 0$, $\alpha\gamma^{*+\beta\delta^*=0}$ *and* $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2$. *As I alluded to in (iv), this corresponds to* $SU(2)$ *matrices (check the conditions for* $M^\dagger M = \mathbb{I}_{2\times 2}$ *and you will see that it precisely corresponds to the 3 conditions derived above).*

(vii) Consider a matrix of the form $H = e^{i/2\sigma\cdot\theta}$. *We first show that* $H \in SU(2)$*: we know that* $\det H = \det e^{i/2\sigma\cdot\theta} = e^{\mathrm{Tr}(i/2\sigma\cdot\theta)}$ *and, using the solution to exercise 3:*

$$\mathrm{Tr}\left(\frac{i}{2}\sigma\cdot\theta\right) = \mathrm{Tr}\left[\begin{pmatrix}\frac{i}{2}\theta_z & 0\\ 0 & -\frac{i}{2}\theta_z\end{pmatrix} + \frac{i}{2}\begin{pmatrix}0 & -iy\\ iy & 0\end{pmatrix} + \frac{i}{2}\begin{pmatrix}0 & x\\ x & 0\end{pmatrix}\right] = 0$$

Hence $e^{Tr(i/2\sigma\cdot\theta)} = 1$. *To show unitarity, just compute* $H^\dagger H$ *by brute force. For the second part, we consider a Hermitian matrix* $K = K^\dagger$ *with* $K = e^{1/2(\sigma\cdot\vec{b})}$. *We may write*

$M = KH = e^{1/2\sigma \cot(\vec{b}+i\theta)}$. Here M is clearly still special (see (iv)) but no longer unitary ($M^\dagger M \neq 1$) hence $M \in SL(2,\mathbb{C})$.

*Any normal subgroup of $SL(2,\mathbb{C})$ must be discrete because $SL(2,\mathbb{C})$ is a simple Lie Algebra. $SU(2)$ is clearly non-discrete and thus cannot be a **normal** subgroup. Therefore the quotient $SL(2,\mathbb{C})/SU(2)$ is **not** a group.*

Let $b_x = b_y = 0$. Then we have

$$K = e^{\frac{1}{2}\sigma_z b_z} = e^{\frac{1}{2}\begin{pmatrix} b_z & 0 \\ 0 & -b_z \end{pmatrix}}$$

using $e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$ we get

$$e^{\frac{1}{2}\begin{pmatrix} b_z & 0 \\ 0 & -b_z \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} b_z/2 & 0 \\ 0 & b_z/2 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} b_z^2/4 & 0 \\ 0 & b_z^2/4 \end{pmatrix} + \dots$$

Since $K^\dagger = K$, we have $X' = K^\dagger X K = KXK$, which we compute:

$$X' = \begin{pmatrix} t' + z' & x' - iy' \\ x' + iy' & t' - z' \end{pmatrix} = \begin{pmatrix} e^{b_z}(t+z) & x - iy \\ t + iy & e^{-b_z}(t-z) \end{pmatrix}$$

so $x \to x'$ and $y \to y'$. For t' and z', we have:

$$\begin{aligned} t' =& \frac{t}{2}\left(e^{b_z} + e^{-b_z}\right) + \frac{z}{2}\left(e^{b_z} - e^{-b_z}\right) \\ z' =& \frac{t}{2}\left(e^{b_z} - e^{-b_z}\right) + \frac{z}{2}\left(e^{b_z} + e^{-b_z}\right) \end{aligned}$$

which is precisely the form of a Lorentz boost along the z direction.