Solutions to Homework Set 4

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Exercise 1. Let G be a group, and D be a homomorphism $D : G \to GL(2, \mathbb{R})$, i.e. D is a representation of G, such that

$$D : G \mapsto \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Identify G (i.e. what group is G isomorphic to?). Is D reducible or irreducible?
Suppose that $A \in D$ acts on a 2×1 real vector $(x, y)^T$ in the following way

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

such that

$$a'x'^2 + b'y'^2 = ax^2 + by^2, \;\; a, b, a', b' \in \mathbb{R}. \tag{1}$$

Find the *real* 2×2 matrix M, whose components are *independent* of a, b, a', c' such that

$$\begin{pmatrix} a \\ b \end{pmatrix} = M \begin{pmatrix} a' \\ b' \end{pmatrix}. \tag{2}$$

Finally, show that the map $A \to D(A) = M^{-1}$ is a representation of G.

Solution: *The 2×2 identity matrix corresponds to the element e. Now, observe that for*

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ we have } g^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

I.e. the third element is the square of the second one. Similarly,

$$g^3 = gg^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } g^4 = gg^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So g generates the whole set and thus G must be isomorphic to $\mathbb{Z}_4$.

This is a ***reducible*** *representation. There are many ways of showing this. For example, we know that the number $\mathcal{M}$ of irreducible representations of G satisfies*

$$\sum_{m=1}^{\mathcal{M}} n_m^2 = |G|$$

And since $|G| = 4$ and we have at least one 1-dimensional irreducible representation (the trivial representation), all remaining representations must be of degree 1 ($2^2 + 1 > 4$).

Using character theory, we know that for an irreducible representation, the sum of the squares of the characters of its elements must be equal to the order of the group. In this case, this sum equals 8, which is famously larger than 4. ***There are several more ways of proving this.***

For finding the real 2×2 matrix, we need to find, in terms of $\alpha, \beta, \gamma, \delta$, a matrix which satisfies the condition $a'x'^2 + b'y'^2 = ax^2 + by^2, \quad a, b, a', b' \in \mathbb{R}$. This is done simply by substituting x', y' with x, y via matrix A. If we write the components of M as m_{ij}:

$$a'x'^2 + b'y'^2 = ax^2 + by^2 a'(\alpha x + \beta y)^2 + b'(\gamma x + \delta y)^2 = (m_{11}a' + m_{12}b')x^2 + (m_{21}a' + m_{22}b')y^2$$

expanding the left-hand side:

$$(\alpha^2 a' + \gamma^2 b')x^2 + (\beta^2 a' + \delta^2 b')y^2 + 2xy(a'\alpha\beta + b'\gamma\delta)$$

Notice that the last term vanishes (equals 0) because $A \in D$ so $\alpha = \delta, \beta = -\delta$. Then, comparing with the right-hand side, we get that $m_{11} = \alpha^2$, $m_{12} = \gamma^2$, $M_{21} = \beta^2$ and $m_{22} = \delta^2$.

The map $A \to D(A) = M^{-1}$ is a homomorphism $GL(2, \mathbb{C}) \to GL(2, \mathbb{C})$, and since the composition of homomorphisms is a homomorphism, the map $g \mapsto D(A)$ is a homomorphism from G to $GL(2, \mathbb{C})$ and thus a representation.

Exercise 2. Prove that the number of conjugacy classes of a finite Abelian group G, C is equal to its order, i.e. $\mathcal{C} = |G|$. Hence deduce that all irreps of finite Abelian groups are one-dimensional.

Solution: *This is a classic group theory exercise! If G is Abelian, then all its elements commute. This means that if b is in the conjugacy class of a, $\exists g \in G$ such that $gag^{-1} = b$, so $b = gag^{-1} = gg^{-1}a = ea = a$. I.e for every $a \in G$, the conjugacy class of a is $\{a\}$.*

To see that all irreducible representations of an Abelian group are one dimensional, we use the fact that the number of irreducible representations (up to equivalence) is the same as the number of conjugacy classes. In this case, the number of conjugacy classes is $|G|$, because each class contains exactly one element; i.e. $\mathcal{M} = |G|$. Thus for

$$\sum_{m=1}^{\mathcal{M}} n_m^2 = |G|$$

to hold we require $n_m = 1\ \forall m$. In other words, all irreducible representations must be 1-dimensional.

Exercise 3. Consider the Klein four-group $V_4 = \{1, a, b, c\}$ with group laws $a^2 = b^2 = c^2 = 1$ and $ab = c$. Find all conjugacy classes of V_4. Consider a dimension 3 representation of V_4, $D : V_4 \to GL(3, \mathbb{R})$, with the following matrices for the generators of V_4

$$D(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad D(b) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

Calculate $D(c)$. Decompose D into its irreducible components. How many inequivalent irreps are there for V_4? Find them.

Solution: *We know that $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is abelian, so is V_4, therefore each element is on a conjugacy class of its own (you can simply state that all groups of order four are abelian; there are only two of them: $\mathbb{Z}_4$ and V_4). This means that there are exactly 4 inequivalent irreducible representations, all of which are 1-dimensional.*

Since $ab = c$, and the representation D is a homomorphism, $D(ab) = D(a)D(b) = D(c)$ so

$$D(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Back to representations, other than the trivial one, there are 3 irreps. These will be of the form $g \mapsto \pm 1$ ($\mapsto \pm r \in \mathbb{R} - \{0\}$ up to equivalence). There are exactly 3 choices: $\{1, 1, -1, -1\}$, $\{1, -1, 1, -1\}$, and $\{1, -1, -1, 1\}$.

Exercise 4. For a group G, show that for any $g_1 \in G$ the elements $\{h\}$ such that $hgh^{-1} = g_1$ form a subgroup H_{g1} of G. Show that if $gg_1g^{-1} = g_2$ for some $g \in G$, then H_{g1} is isomorphic to H_{g2}. Show that the conjugacy class of g_1 has $|G|/|H_{g1}|$ elements.

Solution: *Another classic exercise! The set of h is known as the **centraliser** of g_1 in G. It corresponds precisely to the elements in G which commute with g_1. It is then simple to see that the identity commutes with g_1, and if h commutes with g_1 then so does h^{-1} because $h^{-1}g_1 = h^{-1}g_1hh^{-1} = h^{-1}hgh^{-1} = egh^{-1} = gh^{-1}$. Finally, it is clearly closed because if a and b commute with g_1 then $abg_1 = ag_1b = g_1ab$. Since the commutation relation is transitive, if g_1 and g_2 commute, the set of elements that commute with g_1 must be the same as the set of elements which commute with g_2. The last exercise is a trivial application of the orbit-stabiliser theorem under the conjugation action $Conj : G \to G$. If you do not know this theorem, we will have to take the long route: the elements $\tilde{g}$ in the conjugacy class of g_1 are those such that $\exists g_0 \in G$ where $g_0g_1g_0^{-1} = \tilde{g}$. Since all $g \in H_{g1}$ will conjugate into $\tilde{g} = g_1$, only the $g \notin H_{g1}$ will result in an element $\tilde{g} \neq g_1$ in the same conjugacy class as g_1. Moreover, if g_0 and g are in the same centraliser, then they will give the same $\tilde{g}$, therefore the amount of elements will be equivalent to the amount of left cosets of the centraliser. I.e. $|G|/|H_{g1}|$.*

Exercise 5. Some problems on homomorphisms.

(i) Show that there exists a homomorphism from S_3, the permutation/symmetric group of 3 object, to S_2, the permutation group of 2 objects, by explicitly constructing the map. Identify its kernel.

(ii) Show that there exists a homomorphism from S_4, the permutation/symmetric group of 4 objects, to S_3, the permutation group of 3 objects, by explicitly constructing the map. Identify its kernel.

(iii) Prove that there exists no homomorphism from S_5 to S_4.

Solution: *Homomorphisms send conjugacy classes to each other, so we can write the S_n groups in terms of it's conjugacy classes (using cycle notation): S_2 has (e) and (ab), while S_3 has (e), (ab) and (abc), so we construct a homomorphism $\phi : S_3 \to S_2$ with $\ker\phi = (e), (abc)$ and $(ab)_{S3} \mapsto (ab)_{S2}$. Similarly, we can write S_4 in terms of conjugacy classes as having $(e), (ab), (abc), (ab)(cd)$ and $(abcd)$. Then a homomorphism $\phi : S_4 \to S_3$ can be constructed via $\ker\phi = (e0, (abcd)$, $(ab), (ab)(cd) \mapsto (ab)_{S3}$ and $(abc) \mapsto (abc)_{S3}$. Now, notice that S_4 is isomorphic to some subgroup of S_5 (the permutations that keep 1 element fixed). So we can create a homomorphism if we can send everything outside that subgroup to the identity. I.e if $\ker\phi = S_5 - S_4$. In terms of conjugacy classes of S_5, this means that we need to collapse $(abcde)$ and $(abc)(de)$ into the kernel; but this is impossible because the set containing $(abcde)$ and $(abc)(de)$ is not closed under the group operation, and the kernel of a homomorphism must be a subgroup.*

Exercise 6. Find all irreps of the cyclic-3 group, $\mathbb{Z}_3 = \{e, a, a^2\}$ with $a^2 = e$.

Solution: *Since $\mathbb{Z}_3$ is Abelian, we already know there will be exactly three 1-dimensional irreducible representations up to equivalence. These are, over the field $\mathbb{C}$: the trivial one, $\{1, \omega, \omega^2\}$ and $\{1, \omega, \omega^2\}$, where ω is the cube root of 1 ($\omega = e^{2\pi i/3}$) (as a general rule, the representations of the n^{th} order cyclic group will be the n appropriate orderings of the n^{th} roots of unity $e^{2\pi i/n}$).*

Exercise 7. Given a matrix representation for a group A, one can define for *any* non-singular square matrix B a similarity transform

$$A' = BAB^{-1}$$

Prove that the similarity transform forms equivalence classes, i.e. prove that the transformation is reflexive, transitive and symmetric.

Solution: *We say A, A' are **similar** if $\exists B$ an invertible matrix such that $A' = BAB^{-1}$. Clearly, $A = 1A1$ so A is similar to A and thus matrix similarity is reflexive. Now, if A' is similar to A, then $A' = BAB^{-1}$ and so $A = B^{-1}AB$; since B^{-1} is an invertible matrix, A is similar to A'. I.e. similarity is reflexive. Finally, if A'' is similar to A' which is similar to A, then (by substituting) $A'' = CBAB^{-1}C^{-1}$ and since $M = CB$ is an invertible matrix, A'' is similar to A, thus similarity is transitive.*

Exercise 8. Consider an $N = 3$ complex representation of the Dihedral-4 group D_4,

$$T : D_4 \to GL(3, \mathbb{C}).$$

The representation can be generated by the following generators

$$T(R) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad T(m_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Check that this representation is not unitary.

(i) Using the generators, find all the other matrix representations of $T(g)$ for all $g \in D_4$.

(ii) Construct the Hermitian matrix H by

$$H = \sum_{\alpha} T_\alpha T_\alpha^\dagger \tag{3}$$

(iii) Find the eigenvectors and eigenvalues of H. Using these, construct the unitary matrix U which diagonalises H.

(iv) Hence, find $\tilde{T}_\alpha$, the unitary representation equivalent to D_α. Can you find another one?

Deduce that

$$T(g) = F_1(g) \oplus F_2(g) \tag{4}$$

where F_2 is a 2-D irreducible representation of D_4. (You don't have to calculate F_2, but you need to prove that it is irreducible).

Solution *To check if this is unitary, we simply have to check whether* $T(g)T(g)^T = 1_{3\times3}$. *This is not the case:*

$$T(R)T(R)^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

We can find the image of the remaining elements under this representation by matrix multiplication: $m_2 = Rm$, $m_3 = R^2m$, $m_4 = R^3m$. *These correspond to:*

$$m_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \quad m_3 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad m_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$R^2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad R^3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad R^4 = e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We may build H by brute forcing the sum. We can shorten the computation if we notice that $m_2 m_2^T = RR^T$.... Hence $H = 2(ee + RR^T + R^2 R^{2T} + R^3 R^{3T})$. The resulting matrix is

$$H = \begin{pmatrix} 8 & 8 & 0 \\ 8 & 20 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

The eigenvectors are simply the colums of the diagonalising matrix, and the corresponding eigenvalues are $\{24, 12, 4\}$, with eigenvectors $\{1/2, 1, 0\}$, $\{0, 0, 1\}$ and $\{-2, 1, 0\}$. Then, after normalising (to ensure unitarity):

$$U = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \end{pmatrix}$$

This is precisely the unitary representation $\tilde{T}_\alpha$ will be written in terms of the halved (square root) of the diagonalised matrix

$$\lambda = \begin{pmatrix} 24 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

This is simply

$$\lambda^{1/2} = \begin{pmatrix} 2\sqrt{6} & 0 & 0 \\ 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and its inverse

$$\lambda^{-1/2} = \begin{pmatrix} \frac{1}{2\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

So that we build the elements of the new unitary representation by $\tilde{R} = \lambda^{-1/2} U^T R U \lambda^{1/2}$ (and replace R with m_i, R^2... for the remaining elements).

Finally, we use the same method of finding the squares of the characters of the elements in the representation (see exercise 1) and adding them together to see that this is 16. Since the group is of order 8, our representation must be the direct sum of two irreducible representations. Since it is a 3-dimensional representation, these must be a 2-dimensional and 1-dimensional representation ($F1$ and $F2$).

Exercise 9. Consider the Dihedral-5 group, D_5, the symmetry group of the pentagon. Show that there are exactly two inequivalent 1-dimensional representations of D_5. How many irreps are there of D_5? What are their dimensions?

Solution: *The Dihedral group* D_5 *is* $\{e, R, R^2, R^3, R^4, m_1, m_2, m_3, m_4, m_5\}$. *It has order* 10 *and* 4 *conjugacy classes:* $\{e\}$, $\{R, R^4\}$, $\{R^2, R^3\}$, *and* $\{m_1, m_2, m_3, m_4, m_5\}$. *As we know, this means there are* 4 *irreducible representations up to equivalence. One of these is the trivial representation, which sends everything to* 1. *For the three remaining representations* T_1, T_2, T_3, *we can do some dimension counting, because the sum of the squares of the dimensions must add up to* $|D_5| = 10$ *and the dimension of the trivial representation is* 9. *Finally, we can see that the only possible combination is* $1^2 + 2^2 + 2^2 = 9$. *Thus there are exactly two inequivalent 1-dimensional representations, and two 2-dimensional ones; making a total of four irreps.*

Exercise 10. Consider an order 8 group G generated by two elements a and b, with the group laws $a^4 = b^2 = e$ and $ab = ba$.

(i) Calculate all 8 elements of G.

(ii) Consider S, a 2-D complex representation of G, i.e. $S : G \to GL(2, \mathbb{C})$ where the generators are represented by

$$S(a) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad S(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Calculate the other matrix representations for the other elements of the group G. is this a reducible or irreducible representation?

(iii) Consider T, another 2-D complex representation of G, i.e. $T : G \to GL(2, \mathbb{C})$ where the generators are represented by

$$T(a) = \begin{pmatrix} i & 0 \\ 1 & 1 \end{pmatrix}, \quad T(b) = \begin{pmatrix} -1 & 0 \\ i+1 & 1 \end{pmatrix}.$$

Calculate the other matrix representations for the other elements of the group G. Is this a reducible or irreducible representation?

(iv) Calculate the characters for S and T. Using this, state whether S and T are equivalent to each other.

(v) Find all the conjugacy classes of this group, and verify that the characters of the same conjugacy classes are equal.

Solution: *The elements are* $\{a, a^2, a^3, b, b^2, ab, a^2b, a^3b, e\}$. *Onwards to calculate all the remaining elements from the generators... I've written so many matrices in this set of solutions :(:(*

$$S(a^2) = S(a)S(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S(a^3) = S(a)S(a^2) = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

$$S(ab) = S(a)S(b) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad S(a^2b) = S(a^2)S(b) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S(a^3b) = S(a^3)S(b) = \begin{pmatrix} -1 & 0 \\ 0 & -i \end{pmatrix} \quad S(e) = S(b)S(b) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

After a painful amount of /begin{pmatrix} we can see whether this is an irreducible representation. The sum of the squares of the characters of the representation is 16, *which I have been told is greater than* 8, *thus the representation is not irreducible.*

Oh great, now I get to do this all over again, I can't wait to typeset 6 more matrix operations... For the T representation, we get the remaining elements via multiplication (as usual):

$$T(a^2) = T(a)T(a) = \begin{pmatrix} - & 0 \\ 1+i & 1 \end{pmatrix} \quad T(a^3) = T(a)T(a^2) = \begin{pmatrix} -i & 0 \\ i & 1 \end{pmatrix}$$

$$T(ab) = T(a)T(b) = \begin{pmatrix} -i & 0 \\ i & 1 \end{pmatrix} \quad T(a^2b) = T(a^2)T(b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T(a^3b) = T(a^3)T(b) = \begin{pmatrix} i & 0 \\ 1 & 1 \end{pmatrix} \quad T(e) = T(b)T(b) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We do not need to compute the characters to see that this is in fact not irreducible (each matrix appears exactly twice...), but if we add the squares of the characters (this is just the square of the traces of the matrices, where the trace of a matrix is the sum along its diagonal) we get 16*, which after much deliberation I have concluded is larger than* 8 *and you know the drill. Finally, these representations are not equivalent because even though the square of their character sums are the same, the individual elements have different characters (the* (ab) *elements gain a minus sign).*

Now, characters are ***class functions****, which means that they are constant within conjugacy classes (if two elements are in the same conjugacy class, then their character under any given representation will be the same). We are being asked to verify this. Go on.*