Solutions to Homework Set 3

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Exercise 1. Let C_g be the conjugacy class of $g \in G$, and let H be a *normal* subgroup of G. For each $h \in H$, we can form the conjugacy class C_h of $h \in G$. Prove that H is a union of the conjugacy classes of its elements C_h.

Solution: *C_g contains g $\forall g \in G$ because $ege^{-1} = ege = g$, so the union of all C_h, $h \in H$ contains H (because it contains all h). It remains to show that the union of conjugacy classes is contained in H (if two sets contain each other then they are the same set). Now, the union is contained in H iff every element in the union is contained in H. Have $h_0 \in H$ and consider C_{h_0}: every element is of the form gh_0g^{-1} for some $g \in G$, so it is contained in $(gH)g^{-1}$. Since H is normal, this us the same as $(Hg)g^{-1} = H(gg^{-1} = H$. Thus every element of C_{h_0} ins in H so every C_h is contained in H ($\forall h \in H$) and so their union is contained in H.*

Exercise 2. Let G_1 and G_2 be groups and f be a group homomorphism $f : G_1 \to G_2$. Let H_1 be a *normal* subgroup of G_1. Prove that if f is *onto*, then $f(H_1)$ is a normal subgroup in G_2.

Solution: *Onto means surjective. Suppose H_1 is normal. Then $gH_1g^{-1} = H_1 \; \forall g \in G_1$. Since f is a group homomorpism, $H_2 = f(H_1) = (gH_1g^{-1}) = f(g)f(H_1)f(g^{-1}) = f(g)H_2f(g)^{-1}$. Thus we have shown that $aH_2a^{-1} = H_2$ for every a that can be written as $a = f(g)$ for some $g \in G_1$. Since f is surjective, every element of G_2 is of this form, so our condition holds for every element in G_2. I.e. H_2 is normal in G_2.*

Exercise 3. Find all the proper subgroups of D_4. Which of these proper subgroups are *normal*? One of the proper subgroups is $Z_2 = \{e, R^2\}$. Calculate the quotient group D_4/Z_2 and construct its multiplication table.

Solution: *You can solve this exercise in any presentation of* D_4 *(note: presentation* $\neq$ *representation). As someone who studied group theory in a Mathematics degree, I am used to the cycle presentation (elements of* D_4 *are of the form* $(abcd)$*). The wording of the exercise makes me think it wants us to do it using the rotation + mirror (mirror = rotation) presentation (elements are* R^i*,* m_j*). The proper subgroups are, firstly, the ones generated by its generators:* R *gives* $\{e, R, R^2, R^3\}$ *(isomorphic to* Z_4*) ans* m_i *gives* $\{e, m_1\}, ..., \{e, m_4\}$*. Since* R^2 *is self-inverse, we have* $\{e, R^2\}$ *(these are all isomorphic to* Z_2*). The remaining subgroups contain both rotations and mirrors. Any such subgroup containing* R *or* R^3 *must contain* Z_4*, so if in addition it contains a mirror by Lagrange's theorem it must be the entire group (there are no divisors of 8 larger than 4). Therefore we are only missing subgroups containing mirrors and* R^2*. These are* $\{e, m_1, R^2, m_3\}$ *and* $\{e, m_2, R^2, m_4\}$ *(because* $m_3 = R^2 m_1$ *and* $m_4 = R^2 m_2$*), which are isomorphic to the Klein 4-group* V_4*.*

$\{e, R, R^2, R^3\}$ *is normal because conjugating by mirrors sends rotations to each other.* $\{e, R^2\}$ *is the centre of* D_4 *so it is clearly normal. The remaining copies of* Z_2 *are not normal because mirrors do not commute with all rotations. The copies of* V_4 *are normal because mirrors anti-commute with rotations and* (R^1, R^3) *are inverses of each other, so* $R^j m_i R^{j-1} = R^2 m_i = m_{i+2}$*, and mirrors commute.*

The quotient group D_4/Z_2 *is the set of left cosets* $\{gZ_2 | g \in D_4\}$*. The exercise asks us to look at the quotient with a particular copy of* Z_2 *(the centre* $\{e, R^2\}$*). The distinct elements are* $\boldsymbol{\{e, R^2\}}$ *(*eZ_2*),* $\boldsymbol{\{R, R^3\}}$ *(*$RZ_2 = R^3 Z_2$*),* $\boldsymbol{\{m_1, m_3\}}$ *(*$m_1 Z_2 = m_3 Z_2$*) and* $\boldsymbol{\{m_2, m_4\}}$ *(*$m_2 Z_2 = m_4 Z_2$*). This is the Klein 4-group with* $\{e, R^2\}$ *as the identity; i.e. every element is self-inverse and the product of two non-identity elements gives the third non-identity element (this should be enough for you to fill in the Cayley table).*

Exercise 4. Prove the trace identity. For any two square matrices A,B,

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \tag{1}$$

Solution: *The trace of a matrix is* $\mathrm{Tr}(M_{N\times N}) = \sum_{i=1}^{N} M_{ii}$*. If* M *is a product of matrices, we can substitute* M_{ii} *with the appropriate sum of products:* $\mathrm{Tr}(AB_{N\times N}) = \sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}B_{ji}$*. Assuming these are matrices over either the reals or the complex numbers, the entries are elements of a commutative field, thus the order of the summation and the ordering of the product are up to choice. Moreover, since the summations are both over the same range, the indices (and summations) can be swapped, so that we end up with* $\sum_{i=1}^{N}\sum_{j=1}^{N} B_{ij}A_{ji} = \mathrm{Tr}(BA)$

Exercise 5. Consider the matrix group $SU(2)$. Let $M \in SU(2)$ be a 2×2 matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}$$

Show that The group can be represented as

$$M = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

with the *constraint* $\mathrm{Re}(a)^2 + \mathrm{Im}(a)^2 + \mathrm{Re}(b)^2 + \mathrm{Im}(b)^2 = 1$. Show geometrically that this describes a 3-sphere S_3 embedded in a 4-dimensional Cartesian space $\mathbb{R}^4$.

Solution: *$SU(2)$ is the group of 2×2 unitary matrices over $\mathbb{C}$. In particular, since its matrices are unitary, $MM^\dagger = 1$ and $\mathrm{Det}(M) = 1$ $\forall M \in SU(2)$ (this constraint on the determinant already hints towards a geometric object with radius $r = 1$!). Expanding the unitarity constraint on some generic $M \in SU(2)$:*

$$MM^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{2\times 2}$$

This implies that $a^2 + b^2 = 1$ and $ac^* = -bd^*$, which in particular implies that $Re(a)^2 + Im(a)^2 + Re(b)^2 + Im(b)^2 = 1$ as required. Moreover, since M is unitary it must have orthonormal rows/columns. Pick a and b; then the remaining row is uniquely defined by a vector normal to (a, b). This means that, given a and b, any solution to $ac^* = -bd^*$ must be the unique solution. Since $d^* = a$ and $c^* = -b$ is a solution, it must be the only one.

Thus we can write our matrix as

$$M = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

for some a and b fulfilling $Re(a)^2 + Im(a)^2 + Re(b)^2 + Im(b)^2 = 1$.

Finally, writing $a = x + iy$ and $b = z + iw$ yields $x^2 + y^2 + z^2 + w^2 = 1$ for $x, y, z, w \in \mathbb{R}$; the equation of a 3-sphere (there are 3 degrees of freedom because the 4th coordinate is uniquely determined by the other three $1 - (x^2 + y^2 + z^2) = w$) embedded in $\mathbb{R}^4$.

Exercise 6. Consider the equilateral triangle, or he 3-gon. Let R be the symmetry operation which rotates the triangle clockwise by $120\,^\circ$ and m be the reflection around the vertical axis through the center. Construct the multiplication table for D_3 by using these two generators. What is the order of the Group? How many proper subgroups are there? What are the conjugacy classes of D_3? How many of these classes are also subgroups (hence normal subgroups)? Construct the following:

(i) A regular representation.

(ii) A faithful 3×3 representation.

(iii) A faithful 2×2 representation.

Solution: *D_3 is a group of order 6, which in this presentation can be written as $\{e, R, R^2, m_1, m_2, m_3\}$ where $m_2 = Rm_1$ and $m_3 = R^2 m_1$. Naturally, rotations are cyclic and reflections (mirrors) are self-inverse. Thus the table is:*

e	R	R^2	m_1	m_2	m_3
R	R^2	e	m_2	m_3	m_1
R^2	e	R	m_3	m_1	m_2
m_1	m_3	m_2	e	R^2	R
m_2	m_1	m_3	R	e	R^2
m_3	m_2	m_1	R^2	R	e

Its proper subgroups, following the same argument as in exercise 3, are $\{e, R, R^2\}$, $\{e, m_1\}$, $\{e, m_2\}$, $\{e, m_3\}$. The conjugacy classes are $\{e\}$ (no more single-element classes because the centre of D_3 is trivial), the rotations $\{R, R^2\}$ and the mirrors $\{m_1, m_2, m_3\}$ (you can figure out this from the Cayley table). Again, following the same argument as in exercise 3, the only normal subgroup is $\{e, R, R^2\}$.

A (left) regular representation is constructed by writing the generators in terms of 6×6 matrices (with the remaining elements coming from matrix multiplication of the generators) given by left translation. In this case:

$$U(e) = 1_{6\times 6} \qquad U(R) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad U(m_1) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

A 3×3 faithful representation can be famously constructed through the action of D_3 on the vertices of an equilateral triangle. You can check this action is indeed faithful, *and it only remains to write the explicit matrix form. The identity is clearly* $1_{3\times3}$*. If the corners are* A, B, C*. Then R is such that it sends (A,B,C) to (C,A,B) and* m_1 *is such that it sends (A,B,C) to (A,C,B). This gives:*

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where the remaining elements are derived through matrix multiplication.

A faithful 2×2 *representation may be had by looking at the action of* D_3 *on the coordinates of the vertices. This can be done with any triangle but for simplicity let us have one centered at* $(0, 0)$ *with vertices* $(0, 1)$, $(-1, -1)$ *and* $(1, -1)$*. Then e is the identity matrix, R is a rotation by* $120\,^\circ$ *and the mirror is a flipping of the x-coordinate: (A,C,B). This gives:*

$$R = \begin{pmatrix} cos(120) & sin(120) \\ -sin(120) & cos(120) \end{pmatrix} \qquad m_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where again, the remaining elements are uniquely defined trough matrix multiplication of the generators.

Exercise 7. Consider the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$$

where a, b, c are integers mod 4.

(i) Prove that the set of all possible $\mathcal{M} = \{M\}$ forms a finite group under matrix multiplication, and that $|\mathcal{M}| = 64$. Is this group abelian or non-abelian?

(ii) Consider a subgroup H of M where $a = c$. What is $|H|$? Is this group abelian or non-abelian?

Solution *Since the only possible values of a, b, c, d are $\{0, 1, 2, 3\}$, there are $4^3 = 64$ distinct matrices which can be built through these construction, hence $|M| = 64$. The determinant of a lower-triangular matrix is always its trace, which in this case is 1, so we have good reason to think that inverses exist. The set is clearly closed and inverses are given by $a \to -a$, $c \to -c$, $b \to ac - b$. Associativity is inherited from matrix multiplication and the identity is clearly in the group. The group cannot be abelian because the lower-left element inherits the full expression of matrix multiplication, which is famously non-commutative.*

If $a = c$ then the set is uniquely defined by choosing a and b, so there are only $4^2 = 16$ elements. I.e $|H| = 16$. Under this condition the group does become abelian because the product in the lower-left element becomes symmetric in its inputs.

Exercise 8. Prove that $GL(n, \mathbb{R})$ is a group. Prove that $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$ by explicitly constructing an isomorphism between $SL(n, \mathbb{R})$ and a subset of $GL(n, \mathbb{R})$ which is also a group.

Solution: *$GL(n, \mathbb{R})$ is the set of invertible $n \times n$ matrices over $\mathbb{R}$ under matrix multiplication. By definition, every element has an inverse, and associativity and the identity are inherited from matrix multiplication. For closure, since real matrices are closed, we just have to make sure that the product of two invertible matrices is invertible. This is clearly true as $(AB)^{-1} = B{-}1A^{-1}$ so $GL(n, \mathbb{R})$ is a group.*

$SL(n, \mathbb{R})$ is the subset of $GL(n, \mathbb{R})$ with matrices of unit determinant. Honestly, I don't really understand the isomorphism part because $SL(n, \mathbb{R})$ is set theoretically contained in $GL(n, \mathbb{R})$. To show it is a subgroup, we simply need to show it is closed, contains inverses and contains the identity. $1_{n\times n}$ has determinant 1, and since the product of the determinants is the determinant of the product, $SL(n, \mathbb{R})$ is closed in $GL(n, \mathbb{R})$. Every element in $GL(n, \mathbb{R})$ has an inverse and the determinant of the inverse is the multiplicative inverse of the determinant so all inverses of elements in $SL(n, \mathbb{R})$ have determinant 1 and thus are in $SL(n, \mathbb{R})$.

Exercise 9. (Mobius Transformation) Consider now the *Mobius Transform*, which is a map of the *extended complex plane* $\tilde{\mathbb{C}} = \mathbb{C} \cup \infty$ back to itself, i.e let $z \in \tilde{\mathbb{C}}$ be an element of this set, the transform is a map of $\tilde{\mathbb{C}}$ back to itself in the following way

$$f : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}};\ f(z) = \frac{az+b}{cz+d}\ ,\quad a,b,c,d \in \mathbb{C}$$

Let $\mathcal{M}$ be the set of all possible f.

(i) What are the conditions on a, b, c, d such that $f(z)$ is a (a) translation (b) rotation around the origin (c) contraction/expansion (or *dilations*) in distance from the origin, of the point z?

(ii) Show that the set of all possible f forms a group under the group composition law $f_1 \circ f_2(z) = f_1(f_2(z))$.

(iii) Let the matrix group $SL(2,\mathbb{C})$ with $A \in SL(2,\mathbb{C})$ be described by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose μ maps the elements of $SL(2,\mathbb{C})$ to $\mathcal{M}$, show that this map is a homomorphism and surjective.

(iv) Find $\mathrm{Ker}(\mu)$ and show that $\mathrm{Ker}(\mu)$ forms the group Z_2.

(v) Hence argue that the *Mobius Group* is the quotient group $\mathcal{M} = SL(2,\mathbb{C})/Z_2$.

Solution: *(i)For a rotation, we need $c = b = 0$, $|a/d| = 1$ so that $f(z) = e^{i\phi}z$. For a translation, $c = 0, a = d$ so that $f(z) = z + b$. For a dilation, $c = b = 0$, $a, d \in R^*$ so that $f(z) = Az$ where $A \in \mathbb{R}$.*

(ii) The identity is given by $a = d = 1$, $b = c = 0$. For closure and identities, just brute force the composition law by explicitly calculating $f_1(f_2(z))$. Associativity comes from composition of maps.

*(iii) You need to verify that for $A_1, A_2 \in SL(2,\mathbb{C})$ the entries of the product $A_3 = A_1A_2$ correspond to the four numbers defining the map $f3 = f_1 \circ f_2$ (just some straightforward rearranging). This is enough to show that μ is a group homomorphism. It is clearly surjective by construction of this presentation (every matrix is written in terms of a, b, c, d from the definition of some f). Note that μ is **not** injective, because it sends f_1 defined through a, b, c, d and f_2 defined through $2a, 2b, 2c, 2d$ to the same matrix.*

(iv) The kernel of a map is the preimage of the identity. This is clearly the set of f such that $a = d$ and $b = c = 0$ where $az/d = a$. I.e. the identity and f_{-1} defined through

$a = d = -1$ and $b = c = 0$. The kernel of a homomorphism is always a group (this is a famous theorem) and so it must be isomorphic to the only group with 2 elements: Z_2

(v) The First Isomorphism Theorem (damn this is one beautiful theorem) says that if $\mu : X \to Y$ is a homomorphism then the map $\tilde{\mu} : X/\ker(\mu) \to \mathrm{Im}(\mu)$, $\tilde{\mu}(a\ker(\mu)) = \mu(a)$, $\forall a \in A$ is an isomorphism. Here $X/\ker(\mu)$ is $SL(2,\mathbb{C})/Z_2$ and the image of μ is $\mathcal{M}$; so $\mathcal{M}$ is isomorphic to $SL(2,\mathbb{C})/Z_2$.

Exercise 10. Suppose that G is a group and the set $\{D(g)\}$ is a matrix representation of the elements $g \in G$. Let B be a non-singular matrix that executes a linear transformation on the basis vectors for $D(g)$. Show that the set $\{BD(g)B^{-1}\}$ forms a new matrix representation for G by proving that it obeys all the group axioms.

Solution: *Consider the set $\{BD(g)B^{-1}\}$ under matrix multiplication. Associativity is inherited from the operation, and the identity is as required ($BD(e)B^{-1} = BIB^{-1} = BB^{-1} = 1$). Same goes for inverses: $BD(g)D(g^{-1})B^{-1} = 1 = BD(g)B^{-1}BD(g^{-1})B^{-1}$. Closure is inherited from $\{D(g)\}$ by the exact same argument as inverses.*

Exercise 11. In class, we showed that $U(1)$ is the group of planar rotations around the origin. These rotation "trace out" a circle S_1 embedded on $\mathbb{R}^2$ - in other words, the symmetry group of the circle is $U(1)$. Now consider a torus T_2. Show that the symmetry group of T_2 is $U(1) \times U(1)$. Construct an $N = 2$ group representation fo T_2 acting on a vector space (θ, ϕ) where $0 \leq \theta \leq 2\pi$ and $\leq \phi \leq 2\pi$ describe points on T_2.

Solution: *Note: T_2 is a topological object called a* ***compact surface****. We know a whole lot about these kinds of objects and one of the first key results you will learn in an introduction to topology course is that, using a tool called* ***edge words****, which constructs homeomorphisms (the topological equivalent of an isomorphism) between compact surfaces and simplicial complexes (a glorified version of polygons), you can find a* ***complete classification of compact surfaces****. The original proof was given by Henry Poincare over 150 years ago and was extremely complex, but the new proof is visually so simple it can be explained to a child. The result is that any compact surface (any finite 2D surface which does not have an end (i.e. the surface of a sphere, but not a square) is either a glue-ing of spheres, a glue-ing of torii (plural of torus) or a glue-ing of real projective planes $\mathbb{R}P2$ (like a disk where every time you go to the edge you reappear at the other side). Cool!!*

From your geometrical intuition, you might be able to figure out that a torus is just a circle translated along another circle, hence $S_1 \times S_1$. We can formalise this by considering the representation on the vector space generated by (θ, ϕ): If $e^{i\theta}, e^{i\phi}$ is a point in T_2 (the first coordinate is the angle along the cross-sectional circle and the second coordinate is the angle along the principal-plane circle). Note that this is, in terms of set, showing that $T_2 = S_1 \times S_1$. But since the Cartesian product respect group structure, the symmetry group of T_2 must be precisely $S_1 \times S_1$. We can write the action of $U(1) \times U(1)$ on our vector space via:

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} e^{i\theta} \\ e^{i\phi} \end{pmatrix} = \begin{pmatrix} e^{i(\theta+\alpha)} \\ e^{i(\phi+\beta)} \end{pmatrix}$$

Exercise 12. Let G be a group, and $D_1 : G \to GL(N, \mathbb{C})$ be a homomorphism, and hence $D_1(g)$ a matrix representation of g. Suppose we define the set

$$D_2(g) = [D_1(g^{-1})]^\dagger$$

where † denotes the Hermitian conjugate (i.e. conjugate-transpose). Prove that the set $D_2(g)$ is also a representation of G.

Solution: *This is a repetition of question 7, where instead of using the algebraic properties of matrix multiplication you use the algebraic properties of Hermitian conjugates. Come see me in my Wednesday office hours if you cannot figure this out from the solutions of question 7.*