Symmetry in Physics Homework 5

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1. Let G be a group and the homomorphisms $D : G \to GL(N_1, \mathbb{C})$ and $T : G \to GL(N_2, \mathbb{C})$ be two matrix representations of G. Consider the reducible representations $M_1 = D \oplus T$ an $M_2 = T \oplus D$. Show that the characters are the same for both reducible representations, and hence prove that there exists a similarity transform B that transforms M_1 to M_2.

2. Consider the matrix group $SO(3)$, acting on $\mathbb{R}^3$, i.e. the usual 3 dimensional Euclidean space with *Cartesian* coordinates labeled by (x, y, z). Show that this action is analytic, and hence $SO(3)$ forms a Lie Group.

3. Consider the circle, S_1, parameterized by the coordinate θ such that $0 \leq \theta < 2\pi$. Consider a transformation T which maps the circle to itself as follows

$$T : \theta \to \theta' ; \theta' = \theta + k + f(\theta) \ , \ 0 \leq k < 2\pi \ . \tag{1}$$

(i) Argue that since $0 \leq \theta' < 2\pi$, $f(\theta)$ is periodic, i.e. $f(\theta + 2\pi) = f(\theta)$.
(ii) Show that, insisting that T is one-to-one requires an additional condition $f(\theta) : df(\theta)/d\theta > -1$ to be imposed everywhere on S_1.
(iii) Show that the set of transformations $T(\theta)$ forms a group.
(iv) Is this a Lie Group? Justify your answers.

4. In an n-dimensional linear vector space, two coodinate systems x^i and y^i are related by a *linear basis transformation*

$$y^j = M_i{}^j x^i \tag{2}$$

where $M_i{}^j$ can be represented by an $n \times n$ square matrix. Show that the derivatives are related by the same transformation $M_i{}^j$, i.e.

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} = M_i{}^j \frac{\partial}{\partial y^i}. \tag{3}$$

(Don't forget to use the Einstein summation convention to sum over indices. In this problem, the location of the indices (superscript/subscript) do not matter.)

5. In class, we claim that the exponentiation of a square matrix A is defined by

$$e^A \equiv \sum_{k=0}^{\infty} \frac{1}{k!} A^k \tag{4}$$

where $A^0 = \mathbb{I}$. By expanding the RHS of the following equation

$$e^A = \lim_{m\to\infty} \left(\mathbb{I} + \frac{1}{m} A\right)^m \tag{5}$$

show that it is equivalent to the first definition of e^A we discussed in class. Prove that

$$\det(e^A) = \lim_{m\to\infty} \left[\det\left(\mathbb{I} + \frac{1}{m} A\right)\right]^m . \tag{6}$$

Now show that,

$$\det\left(\mathbb{I} + \frac{1}{m} A\right) = 1 + \frac{1}{m} Tr(A) + \mathcal{O}(1/m^2) + \ldots \tag{7}$$

where ... indicate terms at higher orders in $1/m$. Substituting this result equation (7) into equation (6), derive the identity

$$\det(e^{A}) = e^{Tr(A)}. \tag{8}$$

6. In class, we show that the 2-D representation for $SO(2)$ can be obtained by exponentiating the generator

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{9}$$

Show that

$$X^{2n} = (-1)^{n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

Using this result, prove that the exponentiation can be broken into even and odd powers as follows

$$\exp(\theta X) = \mathbb{I} + \sum_{n=1}^{\infty} \frac{(-1)^{n}\theta^{2n}}{(2n)!}\mathbb{I} + \sum_{n=0}^{\infty} \frac{(-1)^{n}\theta^{2n+1}}{(2n+1)!}X, \tag{11}$$

and hence show that the exponentiation recovers the matrix operator for $SO(2)$

$$\exp(\theta X) = \cos(\theta)\mathbb{I} + \sin(\theta)X. \tag{12}$$

7. Prove that the commutator $[A, B] = AB - BA$ obeys the Jacobi Identity

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0. \tag{13}$$

8. Consider the differential equation

$$y\frac{dy}{dx} + x^{\alpha}y - xy^{\beta} = 0 \tag{14}$$

where α, β are integers. Consider the transformations $x \to ax$, $y \to a^{-2}y$ where $a \in \mathbb{R} - \{0\}$. What are the values of α, β for which this transformation leaves the above equation invariant?

9. Let K_i be the Lie Algebra of $SO(3)$. Show by computing the structure constants that the $l = 1$ **triplet** representation of K_i can be represented by the following matrices

$$K_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \; K_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} , \; K_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \tag{15}$$

11. (Hard.) As discussed in class, $SL(2, \mathbb{C})$ are complex 2×2 matrices with determinant $+1$. Matrices of this group M have structure

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} , \; \det(M) = \alpha\delta - \beta\gamma = 1 \tag{16}$$

Consider a matrix X paramaterized by

$$X(x, y, z, t) = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}. \tag{17}$$

(i) Show that, if we define $\mathbf{x} = (x, y, z)$ as the usual 3-vector, and the dot product $\cdot$ as the usual 3-D vector dot product, we can express X as

$$X(x, y, z, t) = t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sigma \cdot \mathbf{x} \tag{18}$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the usual **Pauli spin Matrices**

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{19}$$

and t is the time coordinate.
(ii) Show that X is Hermitian.
(iii) Show that the most general 2×2 hermitian matrix can be written in the form of the decomposition equation (17).
(iv) If $M \in SL(2, \mathbb{C})$, consider the transformation $X(x', y', z', t') = M^\dagger X(x, y, z, t)M$, i.e. a transform X by an element of $SL(2, \mathbb{C})$ leaves X in the form equation (18). Show that this transformation leaves the metric $(t^2) - x^2 - y^2 - z^2$ invariant. (*Hint : Consider the determinants.*)
(v) How are the new space-time coordinates (x', y', z', t') related to the original coordinates (x, y, z, t)? I.e. calculate x', y', z', t' as functions of (x, y, z, t) and $(\alpha, \beta, \gamma, \delta)$, using the condition $\alpha\delta - \beta\gamma = 1$.
(vi) Find the subgroup of $SL(2, \mathbb{C})$ which leaves $t' = t$ in the transformation defined in (iv). Show that this subgroup is $SU(2)$. (*Hint : Consider the conditions on* $(\alpha, \beta, \gamma, \delta)$ *such that* $t' = t$.)
(vii) (Optional : Lorentz Transformation) Let $H \in SU(2)$, show that H can be represented by the exponentiation of the Pauli matrices

$$H = \exp\left(\frac{i}{2}\sigma \cdot \theta\right) \tag{20}$$

where $\theta = (\theta_x, \theta_y, \theta_z)$ is rotation angles along the x, y, z axes respectively. Consider the hermitian matrix K, i.e. $K^\dagger = K$, given by

$$K = \exp\left(\frac{1}{2}\sigma \cdot \mathbf{b}\right) \tag{21}$$

where $\mathbf{b} = (b_x, b_y, b_z)$ is a real vector in 3-D Euclidean space. Show that

$$M = KH \tag{22}$$

i.e. elements of $SL(2, \mathbb{C})$ can be "factored" into a unitary matrix H (which is a subgroup of $SU(2)$ and a hermitian matrix K). (Note : The form $M = KH$ means that the set of all possible K forms a *coset* space $K \in SL(2, \mathbb{C})/SU(2)$. Is this a group?)

Consider the case $b_x = b_y = 0$. Calculate the transformation

$$K^\dagger X(x, y, z, t)K = X(x', y', z', t') \tag{23}$$

and show that it is the Lorentz transformation law for a *boost* along the z direction.