Symmetry in Physics Homework 4

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1. Let G be a group, and D be a homomorphism $D : G \to GL(2,\mathbb{R})$, i.e. D is a representation of G, such that

$$D : G \mapsto \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \tag{1}$$

Identify G (i.e. what group is G isomorphic to?) Is D reducible or irreducible?

Suppose that $A \in D$ acts on a 2×1 real vector (x, y) in the following way

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \ , \text{ where } A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} , \tag{2}$$

such that

$$a'x'^2 + b'y'^2 = ax^2 + by^2 \ , \ a, b, a', b' \in \mathrm{R}. \tag{3}$$

Find the *real* 2×2 matrix M, whose components are *independent* of a, b, a', c' such that

$$\begin{pmatrix} a \\ b \end{pmatrix} = M \begin{pmatrix} a' \\ b' \end{pmatrix} . \tag{4}$$

Finally, show that the map $A \to D(A) = M^{-1}$ is a representation of G.

2. Prove that the number of conjugacy classes of a finite Abelian group G, $\mathcal{C}$ is equal to its order, i.e. $\mathcal{C} = |G|$. Hence deduce that all irreps of finite Abelian groups are one-dimensional.

3. Consider the Klein four-group $V_4 = \{1, a, b, c\}$ with group laws $a^2 = b^2 = c^2 = 1$ and $ab = c$. Find all conjugacy classes of V_4. Consider a dimension 3 representation of V_4, $D : V_4 \to GL(3,\mathbb{R})$, with the following matrices for the generators of V_4

$$D(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ , \ D(b) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5}$$

Calculate $D(c)$. Decompose D into its irreducible components. How many inequivalent irreps are there for V_4? Find them.

4. For a group G, show that for any $g_1 \in G$ the elements $\{h\}$ such that $hg_1h^{-1} = g_1$ form a subgroup H_{g_1} of G. Show that if $gg_1g^{-1} = g_2$ for some $g \in G$, then H_{g_1} is isomorphic to H_{g_2}. Show that the conjugacy class of g_1 has $|G|/|H_{g_1}|$ elements.

5. Some problems on homomorphisms.
(i) Show that there exist a homomorphism from S_3, the permutation/symmetric group of 3 objects, to S_2, the permutation group of 2 objects, by explicitly constructing the map. Identify its kernel.
(ii) Show that there exist a homomorphism from S_4, the permutation/symmetric group of 4 objects, to S_3, the permutation group of 3 objects, by explicitly constructing the map. Identify its kernel.
(iii) Prove that there exist no homomorphism from S_5 to S_4.

6. Find all irreps of the cyclic-3 group, $Z_3 = \{e, a, a^2\}$ with $a^3 = e$.

7. Given a matrix representation of a group A, one can define for *any* non-singular square matrix B a similarity transform

$$A' = BAB^{-1} \tag{6}$$

Prove that the similarity transform form an equivalence class, i.e. prove that the transformation is reflexive, transitive and symmetric.

8. Consider an $N = 3$ complex representation of the Dihedral-4 group D_4,

$$T : D_4 \to GL(3, \mathbb{C}). \tag{7}$$

The representation can be generated by the following generators

$$T(R) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} , \ T(m_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{8}$$

Check that these representations are not unitary.

(i) Using the generators, find all the other matrix representations of $T(g)$ for all $g \in D_4$.

(ii) Construct the Hermitian matrix H by

$$H = \sum_\alpha T_\alpha T_\alpha^\dagger \tag{9}$$

(iii) Find the eigenvectors and eigenvalues of H. Using these, construct the unitary matrix U which diagonalizes H.

(iv) Hence, find $\tilde{T}_\alpha$, the unitary representation equivalent to D_α.

From this information, find $F_1(g)$, a 1-D irreducible representation of D_4. Can you find another one? Deduce that

$$T(g) = F_1(g) \oplus F_2(g) \tag{10}$$

where F_2 is a 2-D irreducible representation of D_4. (You don't have to calculate F_2, but you need to prove that it is irreducible.)

9. Consider the Diheral-5 group, D_5, the symmetry group of the pentagon. Show that there are exactly two inequivalent 1-dimensional represntations of D_5. How many irreps are there of D_5? What are their dimensions?

10. Consider an order 8 group G generated by two elements a and b, with the group laws $a^4 = b^2 = e$ and $ab = ba$.

(i) Calculate all 8 elements of G.

(ii) Consider S, a 2-D complex representation of G, i.e. $S : G \to GL(2, \mathbb{C})$ where the generators are represented by

$$S(a) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} , \ S(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{11}$$

Calculate the other matrix representations for the other elements of the group G. Is this a reducible or irreducible representation?

(iii) Consider T, another 2-D complex representation of G, i.e. $T : G \to GL(2, \mathbb{C})$ where the generators are represented by

$$T(a) = \begin{pmatrix} i & 0 \\ 1 & 1 \end{pmatrix} , \ S(b) = \begin{pmatrix} -1 & 0 \\ i+1 & 1 \end{pmatrix} . \tag{12}$$

Calculate the other matrix representations for the other elements of the group G. Is this a reducible or irreduicble representation?

(iv) Calculate the characters for S and T. Using this, state whether S and T are equivalent to each other.

(v) Find all the conjugacy classes of this group, and verify that the characters of the same conjugacy classes are equal.