Symmetry in Physics Homework 3

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1. Let C_g be the conjugacy class of g of the group G, and Let H be a *normal* subgroup of G. For each $h \in H$, we can form the conjugacy class C_h of h of the group G. Prove that H is a union of C_h.

2. Let G_1 and G_2 be groups and f be a group homomorphism $f : G_1 \to G_2$. Let H_1 be a *normal* subgroup of G_1. Prove that if f is *onto*, then $f(H_1)$ is a normal subgroup in G_2.

3. Find all the proper subgroups of D_4. Which of these proper subgroups are *normal*? One of the proper subgroup is $Z_2 = \{e, R^2\}$. Calculate the quotient group D_4/Z_2 and construct its multiplication table.

4. Prove the trace identity. For any two square matrices A, B,

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \tag{1}$$

5. Consider the matrix group $SU(2)$. Let $M \in SU(2)$ be a 2×2 matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a, b, c, d \in \mathbb{C}.$$

Show that the group can be represented as

$$M = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

with the *constraint* $\mathrm{Re}(a)^2 + \mathrm{Im}(a)^2 + \mathrm{Re}(b)^2 + \mathrm{Im}(b)^2 = 1$. Show geometrically that this describes a 3-sphere S_3 embedded in a 4-dimensional cartesian space $\mathbb{R}^4$, i.e. it is a 3-dimensional sphere in a 4-dimensional space.

6. Consider the equilateral triangle, or the 3-gon. Let R be the symmetry operation which rotate the triangle clockwise by 120° and m be the reflection around the vertical axis through the center. Construct the multiplication table for D_3 by using these two generators. What is the order of the Group? How many proper subgroups are there? What are the conjugacy classes of D_3? How many of these classes are also subgroups (hence is a normal subgroup)? Construct a

(i) regular representation

(ii) 3×3 faithful representation

(iii) 2×2 faithful representation.

7. Consider the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$$

where a, b, c are integers mod 4.
(i) Prove that the set of of all possible $\mathcal{M} = \{M\}$ forms a finite group under matrix multiplication, and that $|\mathcal{M}| = 64$. Is this group abelian or non-abelian?
(ii) Consider a subgroup H of M where $a = c$. What is $|H|$? Is this group abelian or non-abelian?

8. Prove that $GL(n, \mathbb{R})$ is a group. Prove that $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$ by explicitly constructing an isomorphism $SL(n, \mathbb{R})$ and a subset of $GL(n, \mathbb{R})$ which is also a group.

9. (**Mobius Transformation**) Consider now the *Mobius Transform*, which is a map of the *extended complex plane* $\tilde{\mathbb{C}} = \mathbb{C} \cup \infty$ back to itself, i.e. let $z \in \tilde{\mathbb{C}}$ be an element of this set, the transform is a map of $\tilde{\mathbb{C}}$ back to itself in the following way

$$f : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}; f(z) = \frac{az+b}{cz+d}\ ,\ a,b,c,d \in \mathbb{C}.$$

Let $\mathcal{M}$ be the set of all possible f.
(i) What are the conditions on a,b,c,d such that $f(z)$ is a (a) translation (b) rotation around the origin (c) contraction/expansion (or *dilations*) in distance from the origin, of the point z?
(ii) Show that the set of all possible f forms a group under the group composition law $f_1 \circ f_2(z) = f_1(f_2(z))$.
(iii) Let the matrix group $SL(2,\mathbb{C})$, with $A \in SL(2,\mathbb{C})$ described by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose μ maps the elements of $SL(2,\mathbb{C})$ to $\mathcal{M}$, show that this map is a homomorphism and surjective.
(iv) Find $\mathrm{Ker}(\mu)$ and show that $\mathrm{Ker}(\mu)$ forms the group Z_2.
(v) Hence argue that the *Mobius Group* is the quotient group $\mathcal{M} = SL(2,\mathbb{C})/Z_2$.

10. Suppose that G is a group and the set $\{D(g)\}$ is a matrix representation of the elements $g \in G$. Let B be a non-singular matrix that executes a linear transformation on the basis vectors for $D(g)$. Show that the set $\{BD(g)B^{-1}\}$ forms a new matrix representation for G by proving that it obeys all the group axioms. (In class we argue that $\{D(g)\} \sim \{BD(g)B^{-1}\}$.)

11. In class, we showed that $U(1)$ is the group of planar rotations around the origin. This rotations "traces out" a circle S_1 drawn on $\mathbb{R}^2$ – in other words the symmetry group of the circle is $U(1)$. Now, consider a donut or a "beigel/bagel (if you are American)". The donut traces out a *compact surface* on $\mathbb{R}^3$ – topologically speaking a T_2. Show that the symmetry group of T_2 is $U(1) \times U(1)$. Construct a $N = 2$ Group representation of T_2 acting on a vector space (θ, ϕ) where $0 \le \theta < 2\pi$ and $0 \le \phi < 2\pi$ describe points on T_2.

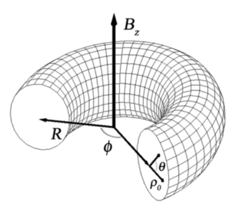

Figure 1: A 2-Torus T_2.

12. Let G be a group, and $D_1 : G \to GL(N,\mathbb{C})$ be a homomorphism, and hence $D_1(g)$ is a matrix representation of $g \in G$. Suppose we define the set

$$D_2(g) = [D_1(g^{-1})]^\dagger$$

where † is the Hermitian conjugate. Prove that the set $D_2(g)$ is also a representation of G.