Symmetry in Physics Homework 2

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`https://nms.kcl.ac.uk/eugene.lim/teach/symmetry/symroot.html`

1. (**Cosets**). Suppose H is a subset of G. Consider $b \in G$ but $b \notin H$ then does the set of all left cosets $\{bH\}$ forms a Group? If so, prove it. If not, explain why. Suppose now H is a *normal subgroup* of G, does the left Coset Space form a group? If so, prove it.

2. Show that there exist a bijection between the left coset space and the right coset space of the normal subgroup H in G.

3. Suppose G is a group, $g \in G$ is an element of G whose order is n. Show that n divides $|G|$. (Hint : show that g generates a cyclic subgroup of G, and then use Lagrange's theorem).

4. Let X and Y be discrete and finite sets, and Y is a proper subset of X, $Y \subset X$. Consider the following map

$$j : \text{Perm(Y)} \to \text{Perm(X)}; (j(f))(x) = \begin{cases} f(x)\ ,\ \text{if } x \in Y \\ j(f))(x) = x\ ,\ \text{otherwise} \end{cases} \tag{1}$$

Show that j is an isomorphism of $\text{Perm}(Y)$ into a subgroup of $\text{Perm}(X)$.

5. Consider the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with the group composition laws

$$-1 = i^2 = j^2 = k^2 = ijk. \tag{2}$$

Show that $i = jk = -kj$, $j = ki = -ik$, $k = ij = -ji$. What is the order of this Group? Prove that it is not isomorphic to the Dihedral group D_4. Construct the product group $Z_2 \times V_4$. Is $Q_8 \cong Z_2 \times V_4$? If so prove it, if not explain.

6. Suppose X and Y are subgroups of the finite group G. Consider the set $XY = \{xy | x \in X, y \in Y\}$.
(a) Suppose that $a, b, c, d \in G$ and $ab = cd$. Prove that there is a unique $h \in G$ such that $c = ah$ and $d = h^{-1}b$.
(b) Suppose that $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Show that $x_1y_1 = x_2y_2$, iff $\exists\ a \in X \cap Y$ such that $x_2 = x_1 a$ and $y_2 = a^{-1}y_1$.
(c) Show that $X \cap Y$ is a subgroup of G.
(d) Show that

$$|XY| = \frac{|X||Y|}{|X \cap Y|} \tag{3}$$

(e) Now let $|G| = 256$. Suppose that $|X| = |Y| = 32$. Show that $|X \cap Y| \geq 4$, and that there are at most four possible values of $|X \cap Y|$. You may state any theorems you use in this calculation.

7. Prove that every subgroup of a finite cyclic subgroup Z_n is also a cyclic subgroup.

8. In class, we showed that the set of integers $\mathbb{Z}$ forms a group under the additive operation. Consider the map

$$\tau : \mathbb{Z} \to \mathbb{Z}; \tau(x) = -x\ \forall\ x \in \mathbb{Z}. \tag{4}$$

Show that τ is an isomorphism from the additive group of $\mathbb{Z}$ to itself.

9. (**Logarithms**). Let T be the set of all integer powers of 10, i.e. $T = \{10^z | z \in \mathbb{Z}\}$. Show that T forms a group under the usual multiplicative operator. Show that this group is isomorphic to the additive group of $\mathbb{Z}$, by finding the isomorphism ϕ. Show that the inverse map ϕ^{-1} is the logarithm $\log_{10}$.

10. Consider the Tetrahedron. Explain why the symmetry group of the tetrahedron is the symmetric group S_4, i.e. it is the permutation group of 4 objects. Geometrically describe all symmetry operations. Is D_3 a subgroup of S_4? Find the isomorphic map if it is. Explain if it is not.

11. Prove the theorem that all prime ordered finite groups are cyclic groups.

12. Let H_1, H_2 be *normal* subgroups of G. Prove that the intersection $H_1 \cap H_2$ is also a *normal* subgroup of G. (*Hint:* First prove that the intersection of two subgroups is also a subgroup. Then prove that the normality property gets inherited.)