Symmetry in Physics Homework 1

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2022 Year 2 Semester 2 Theoretical Physics
Week 1+2
`https://nms.kcl.ac.uk/eugene.lim/teach/symmetry/symroot.html`

1. Consider the following maps. State which of these maps are surjective, injective, both or none?

- $A = \{1,2,3\}$, $B = \{l,j,k\}$, and $f : A \to B; f(1) = l, f(2) = j, f(3) = k$.
- $A = \{1,2,3,4,5\}$, $B = \{2,4,6,8,10,12\}$, and $f : A \to B; f : x \mapsto 2x \ \forall \ x \in A\}$.
- $A = \mathbb{R} - \{0\}$, $B = \mathbb{R} - \{0\}$, and $f : A \to B;\ f : x \mapsto x + 1/x \ \forall \ x \in A$.

2. Let $G = \{-1, 1\}$ be a set. Show that G is a group under the usual rule of multiplication. Is it a group under the usual rule for addition?

3. Consider the set of complex numbers $\mathbb{C}$. Let it be a Field – describe the action and properties of the two binary operators required.

4. The group of integers $N_n = \{0, 1, 2, 3, \ldots, n-1\}$ is equipped with the composition law given by the usual "addition modulo n". Show that this forms a group. Find the identity. Find the inverses for each element.

5. Let S is a finite set of complex numbers without 0, i.e. $S \in \mathbb{C} - \{0\}$. The algebra is specified such that for two elements $z, w \in S$, $zw = wz \in S$. Prove that the modulus of all the elements in S is 1. Show that an identity element in S *may* not exist. We now impose the condition that it must contain an identity. What is the identity? Show that S forms a group. Given that $|S| = n$, prove that it is isomorphic to Z_n by finding the appropriate bijective map.

6. Construct the product group $Z_2 \times Z_4$ by finding all the group elements and the group composition laws. You might find that explicitly building a Cayley Table helpful. What is $|Z_2 \times Z_4|$? Is it isomorphic to Z_6? If so, prove it. If not, explain why. Which element has the largest order, and what is the order?

7. Find all the proper subgroups of the Klein four-group V_4. Which if any is a normal proper subgroup?

8. Let $S = \mathbb{N}$ equipped with the binary operators $\star$ such that $m \star n = \max(m, n)$. State whether (a) $\star$ is associative (b) $\star$ is commutative (c) an identity exist (d) inverses exist.

9. Let $A = \{1,2\}$, $B = \{3,4\}$, $C = \{5,6\}$. Calculate the following Cartesian products (a) $A \times B \times C$ (b) $A \times B$ (c) $B \times A$ (d) $(A \times B) \times C$ (e) $A \times (B \times C)$. Is $A \times B = B \times A$? Is $(A \times B) \times C = A \times (B \times C)$?