Candidate number: ....

Desk number: ....

King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

5CCP2332 Symmetry in Physics

Examiner: Dr Eugene Lim

May 2013

Time allowed: TWO hours

Candidates may answer as many parts as they wish from SECTION A, but the total mark for this section will be capped at 20.

Candidates should answer no more than ONE question from SECTION B.
No credit will be given for answering a further question from this section.

The approximate mark for each part of a question is indicated in square brackets.

You may use a College-approved calculator for this paper. The approved calculators are the Casio fx83 and Casio fx85.

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For examiner's use only

TURN OVER WHEN INSTRUCTED

SECTION A

Answer SECTION A on the question paper in the space below each question. If you require more space, use an answer book.

Answer as many parts of this section as you wish. The final mark for this section will be capped at 20.

1.1) Define what is meant by an *isomorphism* and a *homomorphism* between two groups A and B.

Consider the set H of all possible homomorphisms between A and B. If $f : A \to B$ is an isomorphism, is $f \in H$?

[3 marks]

1.2) State Schur's First and Second Lemmas, and the Orthogonality Theorem for different irreducible representations.

[3 marks]

1.3) If H and K are subgroups of G, prove that the intersection $H \cap K$ is also a group.

[5 marks]

1.4) Consider the Dihedral-3 group D_3. Identify its elements. How many order 2 subgroups are there? Find them. Show by explicit construction that the union of two order 2 subgroups do not form a group.

[5 marks]

1.5) Let $G = \{-i, i, 1, -1\}$ be a set, where i is the unit imaginary number. Prove that G is a group under the usual rule of multiplication.

[Hint: It may be useful to rewrite the group elements using the formula $a + ib = r \exp i\theta$ where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1} b/a$, and a, b are real numbers.]

[4 marks]

1.6) Let $SO(N)$ be a set of $N \times N$ matrices R such that $\det R = 1$, and the condition $R^T R = e$, where R^T is the **Transpose** the matrix R. Prove that $SO(N)$ is a group under the usual matrix multiplication rule.

The Coulomb Potential of a point electric charge e, located at the origin in Cartesian coordinates, is given by

$$V(x, y, z) = \frac{e}{4\pi\epsilon} \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

where ϵ is the electric constant. What symmetry group is this potential invariant under? Explain the physical significance of this symmetry.

[5 marks]

1.7) Consider the 3-dimensional real matrix representation D of the order 4 cyclic group $Z_4 = \{e, a, a^2, a^3\}$ given by

$$D(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}$$

What are the conditions on the real constants b and c such that D is a (i) *faithful* and (ii) *unfaithful* representation?

[5 marks]

SECTION B – Answer ONE question

Answer SECTION B in an answer book.

2) (i) Define what is meant by a *conjugacy class*, and what is meant by a *normal* proper subgroup.

[2 marks]

Consider the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with the group composition laws

$$-1 = i^2 = j^2 = k^2 = ijk\ .$$

Find all 5 conjugacy classes of Q_8. Hence deduce the number of irreducible representations of Q_8 and state their dimensions, stating any theorems you used.

[8 marks]

(ii) What is the order of the symmetry group associated with a rectangle that is not a square? Identify the elements and explain their geometrical action on the rectangle, and then construct the group table. Find all proper subgroups. Determine its conjugacy classes.

Describe in physical terms why the symmetry group of the rectangle describes the symmetries of a water molecule.

Prove that this group is isomorphic to the Klein four-group.

[10 marks]

Consider now the Dihedral D_n group, the symmetry group of an n-gon for $n \geq 3$. Prove that when n is even, there exists only one order 2 normal subgroup while when n is odd, there exists no order 2 normal subgroup.

[10 marks]

3) (i) Let $S_3 = \{e, x, y, y^2, xy, xy^2\}$ be the $n = 3$ symmetric group of three objects. The group composition laws are given by

$$x^2 = y^3 = e \ , \ yx = xy^2 \ , \ y^2x = xy$$

Given the following 2×2 representations of the generators for S_3, where $z = e^{2\pi i/3}$,

$$R(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ R(y) = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}$$

calculate the representations for $R(xy), R(y^2), R(xy^2)$. Show that this representation is *irreducible*.

[5 marks]

Given that the trivial 1-dimensional complex representation of S_3, $T(s) = 1$ where $s \in S_3$, find the non-trivial 1-dimensional complex representation of S_3, $U : S_3 \to \mathbb{C}$. What is $\mathrm{Ker}(U)$? Is U a faithful representation?

Prove that there exist no other inequivalent irreducible representations for S_3. You may state any theorems you use without proof.

[10 marks]

(ii) Consider the following matrix representation for some continuous group, parameterized by θ

$$M_\theta = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

M_θ acts from the left on a complex vector $\mathbb{V}$ space.

Show that M_θ has two eigenvalues ± 1, and find their corresponding *normalized* eigenvectors $v_\pm \in \mathbb{V}$ in terms of θ. Show that $v_\pm$ are orthogonal to each other.

[10 marks]

Consider the general state vector,

$$V_0 = \alpha v_+ + \beta v_-$$

where α, β are complex numbers. Calculate $v_\pm$ for the case when $\theta = \pi$. Suppose $\mathbb{V}$ describes a quantum mechanical Hilbert Space. Describe physically why V_0 can represent the spin state of an electron.

[5 marks]