Candidate number:
Desk number:

King's College London

B.Sc. EXAMINATION

6CCP3212 Statistical Mechanics

Examiner: Dr Eugene Lim

Examination Period 2
(Summer 2019)

Time allowed: THREE hours

Candidates may answer as many parts as they wish from SECTION A, but the total mark for this section will be capped at 40.

Candidates should also answer TWO questions from SECTION B. No credit will be given for answering a further question from this section. The approximate mark for each part of a question is indicated in square brackets.

Calculators may be used. The following models are permitted: Casio fx83 and Casio fx85.

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Physical Constants

Permittivity of free space	$\epsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}
Speed of light in free space	c = 2.998×10^{8} m s^{-1}
Gravitational constant	G = 6.673×10^{-11} N m^2 kg^{-2}
Elementary charge	e = 1.602×10^{-19} C
Electron rest mass	$m_\mathrm{e} = 9.109 \times 10^{-31}$ kg
Unified atomic mass unit	$m_\mathrm{u} = 1.661 \times 10^{-27}$ kg = 931.494 MeV c^{-2}
Proton rest mass	$m_\mathrm{p} = 1.673 \times 10^{-27}$ kg
Neutron rest mass	$m_\mathrm{n} = 1.675 \times 10^{-27}$ kg
Planck constant	$h = 6.626 \times 10^{-34}$ J s
Boltzmann constant	$k_\mathrm{B} = 1.381 \times 10^{-23}$ J K^{-1} = 8.617 $\times 10^{-11}$ MeV K^{-1}
Stefan-Boltzmann constant	$\sigma = 5.670 \times 10^{-8}$ W m^{-2} K^{-4}
Gas constant	$R = 8.314$ J mol^{-1} K^{-1}
Avogadro constant	$N_\mathrm{A} = 6.022 \times 10^{23}$ mol^{-1}
Molar volume of ideal gas at STP	= 2.241×10^{-2} m^3
One standard atmosphere	$P_0 = 1.013 \times 10^{5}$ N m^{-2}

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Useful Information

Maxwell Relations

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T \ , \ \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V ,$$

$$\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P \ , \ \left(\frac{\partial V}{\partial S}\right)_P = \left(\frac{\partial T}{\partial P}\right)_S .$$

Fundamental Equation of Thermodynamics

$$dE = TdS - PdV + \mu dN$$

Thermodynamic Potentials

$$F = E - TS \ , \ \Phi = E - TS + PV \ , \ H = E + PV \ .$$

with differentials

$$dF = -SdT - PdV + \mu dN \ , \ d\Phi = -SdT + VdP + \mu dN \ , \ dH = TdS + VdP + \mu dN \ .$$

Heat Capacities

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V \ , \ C_P = T\left(\frac{\partial S}{\partial T}\right)_P = \left(\frac{\partial H}{\partial T}\right)_P \ .$$

Microcanonical Ensemble Entropy

$$S = k_b \ln \Omega$$

Canonical Partition Function and formulas

$$Z = \sum_r e^{-\beta E_r} \ , \ P_r = \frac{1}{Z} e^{-\beta E_r} \ , \ \langle X \rangle = \sum_r P_r X_r \ ,$$

$$F = -k_b T \ln Z \ , \ S = k_b \frac{\partial}{\partial T}(T \ln Z) \ , \ \text{Mean Energy } \langle E \rangle = -\left(\frac{\partial \ln Z}{\partial \beta}\right)$$

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Grand Canonical Ensemble Partition Function

$$\mathcal{Z} = \sum_r e^{-\beta(E_r - \mu N_r)} \ ,$$

Mean Energy $\langle E \rangle = -\left(\frac{\partial \ln \mathcal{Z}}{\partial \beta}\right) + \mu \langle N \rangle$, Mean Particle Number $\langle N \rangle = \frac{1}{\beta}\left(\frac{\partial \ln \mathcal{Z}}{\partial \mu}\right)$.

Fermi-Dirac Distribution

$$\langle N_{\mathbf{n}} \rangle = \frac{1}{e^{\beta(E_{\mathbf{n}} - \mu)} + 1} \ .$$

Bose-Einstein Distribution

$$\langle N_{\mathbf{n}} \rangle = \frac{1}{e^{\beta(E_{\mathbf{n}} - \mu)} - 1} \ .$$

Thermal de Broglie wavelength

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{mk_b T}} \ .$$

Stirling's Formula

$$\ln N! = N \ln N - N \ .$$

Polylog integrals

$$\int_0^\infty \frac{x^{n-1}}{e^x + 1} dx = (1 - 2^{1-n})\Gamma(n)\zeta(n) \ , \ (n > 0),$$

and

$$\int_0^\infty \frac{x^{n-1}}{e^x - 1} dx = \Gamma(n)\zeta(n) \ , \ (n > 1),$$

with Riemann Zeta function

$$\zeta(p) \equiv \sum_{n=1}^{\infty} \frac{1}{n^p} \ ,$$

and the Gamma function

$$\Gamma(n) \equiv \int_0^\infty x^{n-1} e^{-x} dx \ .$$

The Gamma function for $n > 0$ integers is

$$\Gamma(n) = (n-1)! \ , \ n \in \mathcal{N} - \{0\} \ .$$

Common values for half-integer Gamma functions

$$\Gamma(1/2) = \sqrt{\pi}\ ,\ \Gamma(3/2) = \frac{\sqrt{\pi}}{2}\ ,\ \Gamma(5/2) = \frac{3\sqrt{\pi}}{4}\ ,\ \Gamma(7/2) = \frac{15\sqrt{\pi}}{8}\ .$$

and Zeta functions

$$\zeta(3/2) = 2.612\ ,\ \zeta(2) = \frac{\pi^2}{6}\ ,\ \zeta(5/2) = 1.341\ ,\ \zeta(3) = 1.202\ ,\ \zeta(7/2) = 1.127\ .$$

Gaussian Integral

$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}\ .$$

Geometric Sum

$$\sum_{n=0}^{n=\infty} x^n = \frac{1}{1-x}\ ,\ |x| < 1\ .$$

A derivative identity between x, y and z with a single constraint $x(y,z)$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

Differential transform from $f(x,y) \to f(x,z)$ for a function $f(x,y)$ with a constraint $x = x(y,z)$

$$\left(\frac{\partial f}{\partial x}\right)_z = \left(\frac{\partial f}{\partial x}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z\ .$$

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SECTION A

Answer SECTION A in an answer book.
Answer as many parts of this section as you wish. Your total mark for this section will be capped at 40.

1.1 State the Zeroth and 1st laws of Classical Thermodynamics.

[3 marks]

1.2 What is the *statistical weight* of an ensemble?

[3 marks]

1.3 State which of the following are exact differentials. Integrate the equation if it is exact.

(i) $dG(x, y, z) = yzdx + zxdy + xydz$.

(ii) $dG(x, y) = 2xe^{-y}dx - x^2e^{-y}dy$.

(iii) $dG(x, y) = (3x^2 + y)dx + 2ydy$.

[9 marks]

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1.4 Suppose X and Y are *extensive* variables. Show that this implies that X/Y is an *intensive* variable.

[6 marks]

1.5 The free energy of blackbody radiation is given by

$$F = -\frac{V\pi^2(k_bT)^4}{45(\hbar c)^3} \ ,$$

where V and T are the volume and temperature. Prove that the energy density $\rho \propto T^4$.

[8 marks]

1.6 Consider a thermodynamic system with state variables P, T and V, related by some equation of state $P(V,T)$. By performing a coordinate transform from $S(P,T) \to S(T,V)$, show that

$$\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial S}{\partial P}\right)_T + \left(\frac{\partial S}{\partial T}\right)_P \ .$$

You may use any formula provided.

[6 marks]

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1.7 A lattice has 5 possible sites, each with a magnetic dipole with two possible spin states, + and -. An energy of a + state has energy ϵ while the energy of a - state has energy $-\epsilon$. The total energy of the system is held by an external magnetic field such that $E = \epsilon$. What kind of statistical ensemble does this represent? Calculate the statistical weight of this system.

[8 marks]

1.8 The Maxwell-Boltzmann distribution of a gas at temperature T is given by

$$f(v)e^{-mv^2/2k_bT} = \sqrt{\frac{2}{\pi}}\left(\frac{m}{k_bT}\right)^{3/2} v^2 e^{-mv^2/2k_bT} \ ,$$

where the absolute velocity is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \ .$$

Suppose the particles of the gas is *non-relativistic*, argue that the distribution in terms of energy E is

$$f(E)e^{-E/k_bT} = \sqrt{\frac{8}{\pi}}\left(\frac{m}{k_bT}\right)^{3/2} \frac{E}{m} e^{-E/k_bT} \ .$$

Prove that the most likely energy state of each gas particle is $E_{max} = k_bT$.

[8 marks]

1.9 The Landau functional for a first order phase transition is given by

$$F = F_0(T) = a(T)m^2 + jm^3 + bm^4 \ ,$$

with

$$a(T) = \alpha(T) = \alpha(T - T_c) + \frac{9j^2}{32b} \tag{1}$$

with $\alpha > 0$, $b > 0$ and $j < 0$ being real constants. Find the equilibrium points of F as a function of the order parameter m. Show that when $T > T_c$, there are only one equilibrium point. Is this equilibrium point stable or unstable?

[9 marks]B

Solutions

1.1 bookwork

1.2 bookwork

1.3 (i) Exact $G(x,y,z) = xyz$. (ii) Exact $G(x,y) = x^2 e^{-y}$. (iii) Inexact

1.4 An extensive variable scales as $X(aP, aV, \dots) = aX(P, V, \dots)$. So if X and Y are extensive, they both scales $X/Y \to aX/aY = X/Y$, and hence X/Y does not scale as a, and thus is intensive.

1.5 bookwork

1.6 Homework 1 Q2.

1.7 A system with fixed energy $E = \epsilon$ represents a micro-canonical ensemble. This means that the total energies of the 5 lattice sides must add up to $E = \epsilon$. This means that there must be a total of 3 + states and 2 - states. The total number of combinations is then ${}^5C_2 = 5!/(2!3!) = 10$.

1.8 To get from the v distribution to the E distribution, simply substitute the non-relativistic kinetic energy $E = mv^2/2$. To find the most likely energy state, find the maxima point via

$$\frac{\partial}{\partial E}\left(f(E)e^{-E/k_bT}\right) = 0 \tag{2}$$

and some algeba yields $E_{max} = k_bT$.

1.9 bookwork.

SECTION B - Answer TWO questions
Answer SECTION B in an answer book

2) Consider an analog of the Van der Waals equation of state for a *diatomic* gas as follows

$$P = \frac{Nk_bT}{V-b} + \frac{N^2}{V^2}a \tag{3}$$

where $a > 0$ and $b > 0$ are constants. The $+$ sign for the a term in the above equation means that the gas has *long range repulsive forces* instead of long range attractive forces for the standard Van der Waals gas.

(a) State the conditions for a system to be in a *critical point*.

[2 marks]

(b) Does the gas described by the equation of state above possess a critical point? If so, find T_c, P_c and V_c as functions of N, a and b. If not, prove it.

[6 marks]

(c) The energy E of any system is given by

$$\left(\frac{\partial E}{\partial V}\right)_T = -P + T\left(\frac{\partial P}{\partial T}\right)_V .$$

In the limit of very low densities and room temperature, it is found experimentally that the heat capacity of this gas is $C_V = 5/2Nk_b$. Show that the energy E of the system is given by

$$E(T,V) = \frac{5}{2}Nk_bT + \frac{aN^2}{V} .$$

[10 marks]

(d) For a gas of fixed N, and using the fundamental equation of thermodynamics, show that

$$dS = \frac{5}{2}Nk_b\frac{dT}{T} + \frac{Nk_b}{V-b}dV .$$

[6 marks]

QUESTION CONTINUES ON NEXT PAGE

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(e) The isothermal compressibility of a gas is given by

$$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \ .$$

Calculate K_T for this gas. Show that as $V \to \infty$, $\lim_{V\to\infty} K_T \to 1/P$.

[6 marks]

Solution 2)

(a) The critical conditions are $(\partial P/\partial V)_T = 0$ and $(\partial^2 P/\partial V^2)_T = 0$.

(b) From the first condition

$$\left(\frac{\partial P}{\partial V}\right)_T = -\left(\frac{Nk_bT}{(V-b)^2} + \frac{2N^2}{V^3}a\right), \tag{4}$$

but for $(V-b) > 0$, there is no solution for $(\partial P/\partial V)_T = 0$ since the RHS is always negative definite (compare to the standard Van der Waals case the sign of $a \to -a$), so there is no critical point.

(c) (This is almost identical to Homework 5 Q2ii, with $a \to -a$ and $3/2 \to 5/2$.) Using the equation of state, we can compute

$$\begin{aligned}\left(\frac{\partial E}{\partial V}\right)_T &= -P + T\left(\frac{\partial P}{\partial T}\right)_V \\ &= -P + \frac{Nk_bT}{V-Nb} = -\frac{N^2}{V^2}a \end{aligned}\tag{5}$$

Now expressing $E(T,V)$, we have

$$dE = \left(\frac{\partial E}{\partial V}\right)_T dV + \left(\frac{\partial E}{\partial T}\right)_V dT \tag{6}$$

Integrating this equation over dV from some constant V_0 to V, we get

$$\begin{aligned} E(T,V) &= E(T,V_0) + \int_{V_0}^{V}\left(\frac{\partial E}{\partial V}\right)_T dV\ . \\ &= E(T,V_0) - \int_{V_0}^{V}\frac{N^2a}{V^2}dV \end{aligned}\tag{7}$$

We have the freedom to choose V_0. Since we are given that at $V \to \infty$, $C_V = 5/2Nk_b$, we can choose $V_0 = \infty$, the integral then becomes

$$\begin{aligned} E(T,V) &= E(T,V_0) - \int_{V_0}^{V}\frac{N^2a}{V^2}dV \\ &= E(T,V_0\to\infty) - \int_{\infty}^{V}\frac{N^2a}{V^2}dV \\ &= E^{\mathrm{id}}(T) + \frac{N^2a}{V} \end{aligned}\tag{8}$$

But we are given that $C_V^{\mathrm{id}} = (5/2)Nk_b$, thus $E^{\mathrm{id}} = (5/2)Nk_bT$, and we get the required result

$$E = \frac{5}{2}Nk_bT + \frac{aN^2}{V}\ . \tag{9}$$

(d) Easy to use $TdS = dE + PdV = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV + PdV$, and some algebra yields the final answer.

(e) From

$$\left(\frac{\partial P}{\partial V}\right)_T = -\left(\frac{Nk_bT}{(V-b)^2} + \frac{2N^2}{V^3}a\right) \tag{10}$$

so

$$K_T = \left(\frac{Nk_bTV}{(V-b)^2} + \frac{2N^2}{V^2}a\right)^{-1} \tag{11}$$

In the limit of $V \to \infty$, the 1st term above scales as $1/V$ while the 2nd term as $1/V^2$, so the 1st term dominates and we can neglect the 2nd term. But in this limit $\lim_{V\to\infty}(V-b) \to V$ and we get $K_T \to V/(Nk_bT) = 1/P$.

3) Consider a gas of ideal non-relativistic fermoins of spin-1/2 and mass m_F, trapped on a 2 dimensional surface of area A. For these quantum particles, the energy is given by

$$E_{\mathbf{n}} = \frac{\hbar^2 k^2}{2m_F} = \frac{4\pi^2\hbar^2}{2m_F a^2}(n_x^2 + n_y^2) \ , \ k = \sqrt{k_x^2 + k_y^2} \ ,$$

where $n_x, n_y = 0, 1, 2, 3, \ldots$ label the possible quantum numbers, and $n_i = ak_i/(2\pi)$.

(a) By converting the sum into an integral

$$\sum_{\mathbf{n}} \to \int dn_x dn_y \ ,$$

show that the density of states $g(E)$ is given by

$$g(E) = \tilde{g}\frac{2\pi m_F A}{(2\pi\hbar)^2} \ ,$$

where $\tilde{g} = 2$ for spin-1/2 fermions.

[6 marks]

(b) The *fugacity* of a species of particle with chemical potential μ is defined to be

$$z \equiv e^{\beta\mu} \ .$$

Show that the number density n_F of the fermions on this two dimensional plane is given by

$$n_F \equiv \frac{\langle N \rangle}{A} = \tilde{g}\lambda_F^{-2}\ln(1+z) \ ,$$

where $\lambda = \sqrt{2\pi\hbar^2/m_F k_b T}$ is the thermal de Broglie wavelength.

(*Hint*: You may find the following integral

$$\int_0^\infty \frac{dx}{\alpha^{-1}e^x \pm 1} = \pm\ln(1 \pm \alpha) \ ,$$

for any real constant a useful.)

[6 marks]

QUESTION CONTINUES ON NEXT PAGE

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(c) Repeat the calculation for part (b), but this time for a gas of spin-0 *bosons* of mass m_B. Show that the number density of bosons is given by

$$n_B\lambda_B^2 = -\ln(1 - z_B) \ , \tag{12}$$

where z_B is the fugacity of the boson, and $\lambda_B = \sqrt{2\pi\hbar^2/m_B k_b T}$.

[2 marks]

(d) At some sufficiently low temperature T_*, the fermions are non-interacting with the exception that opposite spin fermions can pair up to form spin-0 bosons of mass $m_B = 2m_F$, with interaction energy Δ, such that the energy of each boson is given by

$$E_B = -\Delta + \frac{\hbar^2 k^2}{2m_B} \ ,$$

with chemical potential $\mu_B \equiv 2\mu$. At equilibrium with temperature T, due to the reactions n_F and n_B are not conserved, but the *total number density* $n = n_F + 2n_B$ is conserved. Show that

$$n\lambda_F^2 = 2\ln(1 + z) - 4\ln(1 - z^2 e^{\beta\Delta}) \ . \tag{13}$$

[8 marks]

(e) Prove that in the limit $n\lambda_F^2 \gg 1$, almost all the fermions are paired up into bosons, i.e.

$$n_B \approx \frac{1}{2}n \ .$$

[8 marks]

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Solution 3)

(a) Homework problem Q5a

(b) First calculate N using the Fermi-Dirac distribution,

$$\langle N\rangle = \int_0^\infty \frac{g(E)}{e^{\beta E}z^{-1}+1}\, dE = \frac{g(E)}{\beta}\ln(1+z) \tag{14}$$

where we have used the formula provided with $x = \beta E$ and $\alpha = z$. Some algebra gives us the number density n_F as required.

(c) For bosons, we use the Bose-Einstein distribution, with $\tilde{g} = 1$ for spin-0,

$$\langle N_B\rangle = \int_0^\infty \frac{g(E)}{e^{\beta E}z^{-1}-1}\, dE = -\frac{g(E)}{\beta}\ln(1-z)\ . \tag{15}$$

and using the appropriate mass $m \to m_B$ and some algebra we get the final answer for n_B.

(d) First we want to calculate the fugacity for the excited boson with energy $E_B = -\Delta + (\hbar^2k^2)/2m_B$. This energy, when compared with the "free" energy $E_B = (\hbar^2k^2)/2m_B$, means that the number density of bosons in such a system would be,

$$\begin{aligned}\langle N_B\rangle &= \int_0^\infty \frac{g(E)}{e^{\beta E_B-\beta\mu_B}-1}\, dE \\ &= \int_0^\infty \frac{g(E)}{e^{\beta E}z_B^{-1}-1}\, dE\end{aligned} \tag{16}$$

where the fugacity would be $z_B = e^{\beta\Delta}e^{\beta\mu_B}$. Given that $\mu_B = 2\mu$, this is then $z_B = e^{\beta\Delta}z^2$, and hence we get

$$\langle N_B\rangle = \int_0^\infty \frac{g(E)}{e^{\beta E}z_B^{-1}-1}\, dE = -\frac{g(E)}{\beta}\ln(1-z_B)\ , \tag{17}$$

and hence $n_B\lambda_B^2 = -\ln(1-z_B)$. From $\sqrt{2}\lambda_B = \lambda_F$, and $n = n_F + 2n_B$, we finally get the required result

$$n\lambda_F^2 = 2\ln(1+z) - 4\ln(1-z^2e^{\beta\Delta})\ . \tag{18}$$

(e) At equilibrium at fixed T, we can see from the equation in part (d) that for the RHS to become very big, the 2nd term must become positive (it is much harder for the first term to dominate – z has to be exponentially big since there is a log term in front), so the log must be the log of a very small number that is much less than 1 (but still positive). This means that $1 - z_B \to 0_+$ where the subscript + means we approach 0 from the positive side. Or $z_B \to 1_-$, and hence $z = e^{-\Delta/2k_bT}$. To leading order this means that the n_B term will dominate so $n_B = n/2$ as requested. (Not examined : Note that this is 'pairing' of 2 fermions into a boson is called a "Cooper pair", which is the fundamental mechanism for how superconductivity arises.)

4) A system consists of two identical, non-interacting, spin-less particles. The system has only 3 single particle states, labeled 1, 2 and 3, with energies $\epsilon_1 = 0 < \epsilon_2 < \epsilon_3$ respectively.

(a) Define what is meant by a *microcanonical ensemble* and a *canonical ensemble.*

[4 marks]

(b) Write down all the possible states and their total energies for the above system if the particles are (i) fermions and (ii) bosons.

[8 marks]

(c) Write down the partition function for a canonical ensemble at temperature $T = 1/k_b\beta$, for both the fermionic and bosonic cases of part (b).

[4 marks]

(d) In the low temperature limit $\beta\epsilon_3 \gg 1$, we can neglect all but the first two leading order terms of the partition functions derived in part (c). Compute the mean energy densities of the fermionic system $\langle E_F \rangle$ and of the bosonic system $\langle E_B \rangle$ respectively.

[8 marks]

(e) What are the mean energies of these systems in the limit as $T \to 0$? Discuss this results in the context of the symmetric/anti-symmetric properties of bosonic/fermionic wavefunctions.

[6 marks]

FINAL PAGE

Solution 4)

(a) Bookwork

(b) Suppose (n_1, n_2, n_3, E) denotes the occupation of states 1, 2 and 3 respectively, with energy E. Then

(i) For fermions Pauli exclusion means that only one particle can occupy each state, we can have $(1, 1, 0, \epsilon_2)$, $(1, 0, 1, \epsilon_3)$, $(0, 1, 1, \epsilon_2 + \epsilon_3)$.

(ii) For bosons, more than one particle can occupy each state, so we have $(2, 0, 0, 0)$, $(0, 2, 0, 2\epsilon_2)$, $(0, 0, 2, 2\epsilon_3)$, $(1, 1, 0, \epsilon_2)$, $(1, 0, 1, \epsilon_3)$, $(0, 1, 1, \epsilon_2 + \epsilon_3)$.

(c) For fermions,

$$Z_F = e^{-\beta\epsilon_2} + e^{-\beta\epsilon_3} + e^{-\beta(\epsilon_2+\epsilon_3)} \tag{19}$$

while for bosons

$$Z_B = 1 + e^{-\beta\epsilon_2} + e^{-\beta\epsilon_3} + e^{-\beta(\epsilon_2+\epsilon_3)} + e^{-2\beta\epsilon_2} + e^{-2\beta\epsilon_3} \ . \tag{20}$$

(d) Since $\epsilon_2 < \epsilon_3$, then the 2 leading order terms for fermions and bosons are

$$Z_F \approx e^{-\beta\epsilon_2} + e^{-\beta\epsilon_3} \tag{21}$$

and

$$Z_B \approx 1 + e^{-\beta\epsilon_2} \tag{22}$$

The energy is then using $\langle E\rangle = -\partial \ln Z/\partial\beta$, we have

$$\langle E_F\rangle = \frac{\epsilon_2 e^{-\beta\epsilon_2} + \epsilon_3 e^{-\beta\epsilon_3}}{Z_F} \tag{23}$$

and

$$\langle E_B\rangle = \frac{\epsilon_2 e^{-\beta\epsilon_2}}{Z_B} \tag{24}$$

(e) In the limit of $T \to 0$, $e^{\beta\epsilon_i \to}0$, but since $\epsilon_2 < \epsilon_3$, $e^{-\beta\epsilon_3}$ will go to zero faster than $e^{-\beta\epsilon_2}$, so for fermions this lead us to

$$\lim_{T\to 0}\langle E_F\rangle \to \frac{\epsilon_2 e^{-\beta\epsilon_2}}{e^{-\beta\epsilon_2}} = \epsilon_2 \tag{25}$$

while for bosons

$$\lim_{T\to 0}\langle E_B\rangle \to \frac{\epsilon_2 e^{-\beta\epsilon_2}}{1 + e^{-\beta\epsilon_2}} = 0 \ . \tag{26}$$

The anti-symmetry of fermionic wavefunctions mean that the 2 particles must repel each other, so they cannot be in the same zero energy state. Thus the mean ground state for the 2 fermion state is ϵ_2 which is non-zero. On the other hand, the symmetry of the bosonic wavefunctions mean that both particles can be in the zero energy state, hence the ground state has zero mean energy.