6CCP3212 Statistical Mechanics Homework 4

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1)(**Ultra-relativistic degenerate fermion gas**). In the lectures, we have derived the equation of state for the non-relativistic degenerate fermion gas and showed that it behaves like $P \propto \rho^{5/3}$. In this problem, we will derive the equation of state for the ultra-relativistic case. Assume that fermion has degeneracy parameter $\tilde{g}$.

(i) The dispersion relation for a relativistic particle is given by

$$E^2 = \hbar^2 k^2 c^2 + m^2 c^4 \ , \ p = \hbar k \ . \tag{1}$$

In Homework 3, you have shown that in ultra-relativistic case, $p \gg mc$ such that $E = pc$. Show that the density of states for this case is then

$$g(E)dE = \tilde{g}\frac{4\pi V}{(2\pi\hbar)^3}\frac{E^2}{c^3}dE \ , \tag{2}$$

i.e. it's the same as the massless case.

(ii) Assuming that the $\langle N \rangle = 1$ for $E < E_F$ and $\langle N \rangle = 0$ for $E > E_F$ (i.e. fully degenerate gas) where E_F is the Fermi energy, calculate the mean particle number and mean energy of the system and show that they are

$$N = \tilde{g}\frac{4\pi V}{(2\pi\hbar c)^3}\frac{E_F^3}{3} \ , \tag{3}$$

and

$$E = \tilde{g}\frac{\pi V}{(2\pi\hbar c)^3}E_F^4 \ . \tag{4}$$

(iii) Equivalently with the non-relativistic Fermi momentum, the relativistic Fermi momentum is given by $p_F = E_F/c$. Show that, in terms of density, this is

$$p_F = \left(\frac{3(2\pi\hbar)^3}{4\pi\tilde{g}}\right)^{1/3}\left(\frac{N}{V}\right)^{1/3} \ . \tag{5}$$

(iv) Calculate the equation of state of the ultra-relativistic degenerate fermion gas and show that it scales like $P \propto (N/V)^{4/3}$.

2)(**Rotation modes of quantum diatomic gas**.) In your quantum mechanics class, you learned that the energy spectrum for the angular momentum modes are labeled by the spherical harmonic parameter l. For each l, there are $m = 0 - l, -l+1, \ldots, -1, 0, 1, \ldots, l-1, l$ magnetic angular momentum modes. The energy for each l mode is given by

$$E_l = \frac{\hbar^2}{2I}l(l+1) \ . \tag{6}$$

(i) Write down the partition function for the angular mometum modes, and show that it is

$$Z_{\rm rot} = \sum_{l=0}^{l=\infty}(2l+1)e^{-\beta\hbar^2 l(l+1)/2I} \ . \tag{7}$$

(ii) Consider the classical limit where temperatures are high $T \gg \hbar^2/2Ik_b$. Which l modes will contribute to the partition function in this limit? Use this fact to show that the partition function is approximated by

$$Z_{\rm rot} \approx \frac{2I}{\beta\hbar^2} \ . \tag{8}$$

(*Hint*: You can approximate a sum with an integral in the limit of high l.)
(iii) What is $Z_{\rm rot}$ in the limit of very low temperatures? Show that the modes are effectively "frozen" out by computing the partition function in this limit.
(iv) The natural frequency of the oxygen molecule is $\omega = 4.6 \times 10^{13}$ Hz, while its moment of inertia is $I = 2 \times 10^{-39}$ kg m^2. Calculate the ratio of the "activation" temperatures $T_{\rm vib}/T_{\rm rot}$. Which degrees of freedom will be activated first as a function of temperature?

3) Consider a non-relativistic boson gas with a degeneracy of $\tilde{g} = 1$ in the classical limit $e^{\beta\mu} \ll 1$. (You might find the discussion on non-relativistic fermion gas in the lectures useful.)
(i) Show that the mean particle number and mean energy are given by (where $x \equiv \beta E$),

$$\langle N \rangle = \int_0^\infty \frac{4\sqrt{2}\pi V}{(2\pi\hbar)^3} \frac{m^{3/2}}{\beta^{3/2}} \frac{x^{1/2}}{e^{x-\beta\mu} - 1} dx \ , \tag{9}$$

and

$$\langle E \rangle = \int_0^\infty \frac{4\sqrt{2}\pi V}{(2\pi\hbar)^3} \frac{m^{3/2}}{\beta^{5/2}} \frac{x^{3/2}}{e^{x-\beta\mu} - 1} dx \ . \tag{10}$$

(ii) By Taylor expanding around small $e^{\beta\mu}$, show that the integral

$$I = \int_0^\infty \frac{x^p}{e^{x-\beta\mu} - 1} dx = e^{\beta\mu} \left[\Gamma(p+1) + \frac{1}{2^{p+1}} \Gamma(p+1) e^{\beta\mu} + \dots \right] \ , \tag{11}$$

and hence show that to 2nd order in $e^{\beta\mu}$,

$$\langle E \rangle \approx \tilde{g} V \frac{3}{2\beta} \frac{e^{\beta\mu}}{\lambda^3} \left(1 + \frac{1}{2^{5/2}} e^{\beta\mu} \right) \ , \ \langle N \rangle \approx \tilde{g} V \frac{e^{\beta\mu}}{\lambda^3} \left(1 + \frac{1}{2^{3/2}} e^{\beta\mu} \right) \ . \tag{12}$$

(Note that $\Gamma(3/2) = \sqrt{\pi}/2$, and $\Gamma(5/2) = 3\sqrt{\pi}/4$.)
(iii) Hence, derive the equation of state to 2nd order, i.e.

$$PV = N k_b T \left[1 - 2^{-5/2} \frac{N/\tilde{g}}{V} \lambda^3 + \dots \right] \ . \tag{13}$$

4) Complete the calculation of the energy for the slightly non-degenerate fermion gas we covered in class (section 4.3.4 of the lecture notes) and show the steps to obtain

$$\langle E \rangle = \langle E_0 \rangle \left[1 + 0.27 \left(\frac{m k_b T}{\hbar^2} \right)^2 \left(\frac{V}{N} \right)^{4/3} + \mathcal{O}(T^4) \right] \ . \tag{14}$$

5) Consider a gas of electrons trapped on a 2 dimensional plane. The energy spectrum of a non-relativistic electron with mass m in 2 dimensions is given by

$$E_{\mathbf{n}} = \frac{\hbar^2 k^2}{2m} = \frac{4\pi^2\hbar^2}{2ma^2} (n_x^2 + n_y^2) \ , \ k = \sqrt{k_x^2 + k_y^2} \ . \tag{15}$$

(i) Calculate the density of states $g(E)$ (recall electrons are spin-1/2 particles) and show that $g(E)$ is independent of E.
(ii) Write down the expressions for the $\langle N \rangle$ and $\langle E \rangle$ in terms of its temperature T and chemical potential μ.
(iii) Calculate the Fermi energy E_F in terms of N.
(iv) Consider the low temperature limit $k_b T \ll E_F$, and assuming that μ does not change with T, compute the heat capacity for the gas of fermions in 2 dimensions.

6) Consider a system with 4 *distinguishable* and non-interacting particles, localised on 4 fixed lattice sites. The energies for each particle is quantised, and they are restricted to the discrete values 0, ϵ, 2ϵ, 3ϵ, 4ϵ,

$5\epsilon, \ldots$. The particles can occupy the same energy state (e.g. particles can possess the same energy). This system is divided into two sub-systems, A and B, each with 2 particles. We can represent the system notationally as ${}_A(a,b|c,d)_B$ where $a,b,c,d = 0,1,2,3,\ldots$ represent the energy state of the particles, e.g. ${}_A(0,3|2,2)_B$ means that system A has a particle with 0 energy, and a particle with 3ϵ, while system B has 2 particles with 2ϵ, such that $E_A = 3\epsilon$ and $E_B = 4\epsilon$.
(i) The two sub-systems are initially thermally insulated from one another, such that $E_A = 5\epsilon$ and $E_B = \epsilon$. How many possible microstates are in this system? State them in the ${}_A(a,b|c,d)_B$ notation.
(ii) The thermal insulation is now removed, and the two sub-systeams are allowed to interact with one another. At equilibrium, prove that the statistical weight of the system is 84. Show your working and arguments clearly.

Hint : It might be useful to consider to break the problem down into smaller parts, e.g. there are 4 *microstates with 1 particle with* 6ϵ *and 3 particlces with* 0 *energy etc.*
(iii) Using your results in (i) and (ii), calculate the probability of the system A to possess $E_A = 5\epsilon$ at equilibrium.

7) For an ideal Fermi gas, the mean particle number (or mean occupation number) of state $\mathbf{n}$ is $N_{\mathbf{n}}$, where we have used the shorthand $N_{\mathbf{n}} = \langle N_{\mathbf{n}} \rangle$. Show that the entropy of the system

$$S = \frac{\partial}{\partial T}\left(k_b T \ln \mathcal{Z}\right)_{V,\mu} \tag{16}$$

is given by

$$S = -k_b \sum_{\mathbf{n}} \left[(1-N_{\mathbf{n}})\ln(1-N_{\mathbf{n}}) + N_{\mathbf{n}} \ln N_{\mathbf{n}}\right] \ . \tag{17}$$

8) (**Chandrasekhar Limit**). A white dwarf is the end state of a low mass star. It is a star made mostly out of baryons (i.e. normal matter), but supported by *electron degeneracy pressure*, i.e. the free electrons that are contained in the star are so tightly packed together that they are in the degeneracy limit. We will consider white dwarfs in the *ultra-relativistic* limit, so you may use the results of Q1 in this problem. ($M_\odot = 2\times 10^{33}$ g, $m_{\text{neutron}} \approx m_{\text{proton}} = m_p = 1.68\times 10^{-24}$ g, and $m_{\text{electron}} \approx 9.1\times 10^{-28}$ g.)
(i) The total kinetic energy of the relativistic electrons is given by

$$E_K = N E_F \ , \tag{18}$$

while the gravitational energy of a spherically symmetric object with mass M with radius R is given by

$$E_G = -\frac{3}{5}\frac{GM^2}{R} \ , \tag{19}$$

giving the total energy of the system to be $E_T = E_G + E_K$. Show that $E_T \propto 1/R$.
(ii) When $E_T < 0$, the gravitational energy dominates, and hence the star collapses into a neutron star or a black hole. Show that this occurs when

$$M_C = 5\sqrt{\frac{5\pi}{6\tilde{g}}}\left(\frac{\hbar c}{G}\right)^{3/2} m_p^{-2} \ , \tag{20}$$

What is M_C in terms of $M_\odot$? A more careful calculation will show that $M_C \approx 1.4 M_\odot$. This is the famous Chandrasekhar limit.