6CCP3212 Statistical Mechanics Solutions 4

Lecturer: Prof. Eugene A. Lim
https://nms.kcl.ac.uk/eugene.lim/teach/statmech/sm.html

1) (i) Bookwork.
(ii) The mean particle number is

$$\begin{aligned}\langle N\rangle &= \int_0^\infty g(E)\frac{1}{e^{\beta(E-\mu)}+1}dE \\ &= \int_0^{E_F} g(E)dE \\ &= \int_0^{E_F}\tilde{g}\frac{4\pi V}{(2\pi\hbar)^3}\frac{E^2}{c^3}dE \\ &= \tilde{g}\frac{4\pi V}{(2\pi\hbar c)^3}\frac{E_F^3}{3} \end{aligned} \tag{1}$$

and the mean energy is

$$\begin{aligned}\langle E\rangle &= \int_0^\infty g(E)\frac{E}{e^{\beta(E-\mu)}+1}dE \\ &= \int_0^{E_F} Eg(E)dE \\ &= \int_0^{E_F}\tilde{g}\frac{4\pi V}{(2\pi\hbar)^3}\frac{E^3}{c^3}dE \\ &= \tilde{g}\frac{\pi V}{(2\pi\hbar c)^3}E_F^4\ . \end{aligned} \tag{2}$$

(iii) We can solve for E_F as a function of N using (ii) to get

$$E_F = \left(\frac{3N(2\pi\hbar c)^3}{\tilde{g}4\pi V}\right)^{1/3} \tag{3}$$

and then using $p_F = E_F/c$ we get the answer.
(iv) From (ii),

$$E_F = \left(\frac{N}{V}\right)^{1/3}\left(\frac{3(\pi\hbar c)^3}{4\tilde{g}\pi}\right)^{1/3} \tag{4}$$

and now using $PV = (1/3)E$ (which you can derive from the same trick as in the lecture notes section 4.3.2, or use the knowledge that for ultra-relativistic fermions, we can ignore the mass and consider it as a massless particle, so the standard relation from the Stefan-Boltzmann relation $P = \rho/3 = 1/3(E/V)$), we get

$$\begin{aligned} P &= \frac{1}{3}\tilde{g}\frac{\pi}{(2\pi\hbar c)^3}E_F^4 \\ &= \frac{1}{3}\tilde{g}\frac{\pi}{(2\pi\hbar c)^3}\left(\frac{3(2\pi\hbar c)^3}{4\tilde{g}\pi}\right)^{4/3}\left(\frac{N}{V}\right)^{4/3} \\ &= \frac{1}{3}\left(\frac{3}{4}\right)^{4/3}\left(\frac{(2\pi\hbar c)^3}{\tilde{g}\pi}\right)^{1/3}\left(\frac{N}{V}\right)^{4/3} \end{aligned} \tag{5}$$

as required.

2)

(i) The partition function can be written down immediately

$$Z_{\rm rot} = \underbrace{\sum_{l=0}^{l=\infty}}_{\text{all } l} \underbrace{(2l+1)}_{\text{degeneracy for every l}} e^{-\beta\hbar^2 l(l+1)/2I} \ . \tag{6}$$

(ii) From (i), and using $k_b T = \beta^{-1}$, we see from the partition function in (i), that in the high temperature limit $\beta\hbar^2/2I \ll 1$, so only the high l limit will contribute. This means that we can express the partition function as an integral over l

$$\begin{aligned} Z_{\rm rot} &= \int_0^\infty (2l+1)e^{-\beta\hbar^2 l(l+1)/2I} dl \\ &\approx \int_0^\infty 2l e^{-\beta\hbar^2 l^2/2I} dl \\ &= \frac{2I}{\beta\hbar^2} \end{aligned} \tag{7}$$

where we have used the fact that $l \gg 1$ in the 2nd line to simply the integral.

(iii) At the limit of low temperature, $\beta\hbar^2/2I \gg 1$, so all the high l modes get suppressed in the partition function since $e^{-\beta\hbar^2 l(l+1)/2I} \ll 1$ when l is big. Thus

$$Z_{\rm rot} \to 1 \ . \tag{8}$$

(iv) From the lecture notes $T_{\rm vib} = \hbar\omega/k_b$, and here $T_{\rm rot} = \hbar^2/2Ik_b$

$$\frac{T_{\rm vib}}{T_{\rm rot}} = \frac{2I\omega}{\hbar} = 1.7 \times 10^9 \tag{9}$$

and hence the rotation modes will be activated before the vibration modes.

3) This problem is a straightforward copy-pasta of the Lecture notes Section 4.3.2, with slight modification of signs – replace the Fermi-Dirac distribution $1/(e^{\beta(E-\mu)}+1)$ with the Bose-Einstein distribution $1/(e^{\beta(E-\mu)}-1)$.

(i) Section 4.3.2.

(ii) Section 4.3.2., but replacing the expansion $1/(1+\epsilon) = 1-\epsilon+\ldots$ with $1/(1-\epsilon) = 1+\epsilon+\ldots$. To keep the terms to second order in $e^{\beta\mu}$, we see that all we need to do is to replace

$$\Gamma(p+1) \to \Gamma(p+1)\left(1+\frac{1}{2^{p+1}}e^{\beta\mu}\right) \tag{10}$$

in the respective integrals to get

$$\langle E\rangle \approx \tilde{g}V\frac{3}{2\beta}\frac{e^{\beta\mu}}{\lambda^3}\left(1+\frac{1}{2^{5/2}}e^{\beta\mu}\right) \ , \ \langle N\rangle \approx \tilde{g}V\frac{e^{\beta\mu}}{\lambda^3}\left(1+\frac{1}{2^{3/2}}e^{\beta\mu}\right) \ . \tag{11}$$

(iii) The equation of state can then be easily derived by using $PV = (2/3)E$,

$$\begin{aligned} PV &= \frac{2}{3}\frac{3}{2\beta}N\left(1+\frac{1}{2^{5/2}}e^{\beta\mu}\right)\left(1+\frac{1}{2^{3/2}}e^{\beta\mu}\right)^{-1} \\ &= Nk_bT\left(1+\frac{1}{2^{5/2}}e^{\beta\mu}\right)\left(1-\frac{1}{2^{3/2}}e^{\beta\mu}+\ldots\right) \\ &= Nk_bT\left(1-\frac{1}{2^{5/2}}e^{\beta\mu}+\ldots\right) \\ &= Nk_bT\left(1-\frac{1}{2^{5/2}}\frac{N/\tilde{g}}{V}\lambda^3+\ldots\right) \ , \end{aligned} \tag{12}$$

and done.

4) The energy density up to first order is given by (from lecture notes)

$$E = A\left[\int_0^{E_F} f(E)dE + \frac{\pi^2}{6}(k_bT)^2 f'(E_F)\right] \tag{13}$$

where A is just some constants such that $A\int_0^{E_F} f(E)dE = E_0$ is the fully degenerate case. We don't have to keep track of A fortunately (else it is a big mess). Now using $f(E) = E^{3/2}$ for energy, and $f'(E_F) = 3/2E_F^{1/2}$, we have

$$E = \frac{2A}{3}E_F^{3/2}\left[1 + \frac{3}{2}\frac{\pi^2}{6}(k_bT)^2 E_F^{-2}\right] . \tag{14}$$

The Fermi energy in terms of N and V is

$$E_F = \frac{1}{2m}(2\pi\hbar)^2\left(\frac{4\pi}{3}\right)^{-2/3}\left(\frac{N}{\tilde{g}V}\right)^{2/3} \tag{15}$$

which we can plug in to get

$$E = E_0\left[1 + \left(\frac{mk_bT}{\hbar^2}\right)^2 \frac{9}{4}\frac{\pi^2}{6}\left(\frac{4\pi}{3}\right)^{4/3}\frac{1}{(2\pi)^4}\tilde{g}^{-4/3}\left(\frac{V}{N}\right)^{4/3}\right] \tag{16}$$

and plugging in the numbers with $\tilde{g} = 2$ (for fermions) gets us the required answer.

5)
(i) In two dimensions, the sum over all microstates is a sum over all possible n_x and n_y, which we can convert into an integral

$$\sum_{\mathbf{n}} \rightarrow \int dn_x dn_y . \tag{17}$$

And now using $dE = \hbar^2 k/mdk$, and $dn = a/(2\pi)dk$ we have

$$\begin{aligned}\int dn_x dn_y &= \tilde{g}\frac{A}{(2\pi)^2}\int d^2k \\ &= \tilde{g}\frac{A}{(2\pi)^2}\int 2\pi k dk \\ &= \tilde{g}\frac{A}{(2\pi)^2}\int 2\pi\frac{m}{\hbar^2}dE \\ &= \tilde{g}\frac{A}{(2\pi\hbar)^2}\int 2\pi m dE\end{aligned} \tag{18}$$

where in the 2nd line we have used the fact that $\int dk^2 \rightarrow \int_0^{2\pi} d\theta \int_0^k kdk$, and hence

$$g(E) = \tilde{g}\frac{2\pi mA}{(2\pi\hbar)^2} \tag{19}$$

which is independent of E.
(ii)

$$\langle N\rangle = \int \tilde{g}\frac{2\pi mA}{(2\pi\hbar)^2}\frac{1}{e^{\beta E-\beta\mu}+1}dE \tag{20}$$

and

$$\langle E\rangle = \int \tilde{g}\frac{2\pi mA}{(2\pi\hbar)^2}\frac{E}{e^{\beta E-\beta\mu}+1}dE . \tag{21}$$

(iii) We defined the Fermi energy as the surface energy of the Fermi sea of N fermions, i.e.

$$\langle N\rangle = \int_0^{E_F} \tilde{g}\frac{2\pi mA}{(2\pi\hbar)^2}dE \tag{22}$$

which can be easily integrated to yield

$$E_F = \frac{N}{A}\frac{(2\pi\hbar)^2}{2\pi\tilde{g}m} \tag{23}$$

(iv) The heat capacity is $C_V = (\partial E/\partial T)_V$, but in the low temperature limit, we are in the degenerate limit, and hence

$$\langle E\rangle = \int \tilde{g}\frac{2\pi mA}{(2\pi\hbar)^2}\frac{E}{e^{\beta E-\beta\mu}+1}dE \to \int \tilde{g}\frac{2\pi mA}{(2\pi\hbar)^2}EdE \tag{24}$$

which is clearly independent of T, so $C_V = 0$.

6)

(i) For $E_A = 5\epsilon$, we need to distribute the energy across 2 particles. The only possible combinations are ${}_A(0,5|x,x)_B$, ${}_A(1,4|x,x)_B$, ${}_A(5,0|x,x)_B$, ${}_A(4,1|x,x)_B$, ${}_A(2,3|x,x)_B$ and ${}_A(3,2|x,x)_B$, with statistical weight $\Omega(E_A) = 6$. [3 marks]. For $E_B = \epsilon$, the only two possible microstates are ${}_A(x,x|0,1)_B$ and ${}_A(x,x|1,0)_B$, with $\Omega(E_B) = 2$ [1 mark]. Hence the total number of microstates for the joint system is then $\Omega = \Omega(E_A)\Omega(E_B) = 12$. [1 mark]

(ii) In thermal equilibrium, the total energy of the system is the sum of $E = E_A + E_B = 6\epsilon$. Hence the total number of microstates is the number of ways we can distribute 6ϵ across 4 particles [2 marks]. The direct formula to calculate this is $\Omega = (6+4-1)!/(6!(4-1)!) = 84$ which is obtuse. But a semi-bruteforce method hinted at can be argued as follows (the first column denotes the energies of the possible microstate configuration)

particle distribution	permutations on the lattice	total
6, 0, 0, 0	${}^4C_1$	4
5, 1, 0, 0	${}^4C_1 \times {}^3C_1$	12
4, 2, 0, 0	${}^4C_1 \times {}^3C_1$	12
3, 3, 0, 0	${}^4C_2$	6
4, 1, 1, 0	${}^4C_1 \times {}^3C_2$	12
3, 2, 1, 0	${}^4C_1 \times {}^3C_1 \times {}^2C_1$	24
2, 2, 2, 0	${}^4C_3$	4
2, 2, 1, 1	${}^4C_2$	6
3, 1, 1, 1	${}^4C_3$	4

which adds up to 84. [5 marks]

(iii) Since the number of microstates for $E_A = 5\epsilon$ is 12 from (i), it is then easy to calculate that the probabilty is $P = 12/84 = 1/7$. [3 marks]. If the student made a mistake in (ii) and used the result to calculate (iii) "correctly", they should be given full marks.

7) The solution to this problem uses some of the algebraic tricks of Section 2.3.3 where we derived the Shannon entropy. Let's first derive some useful identities. From

$$\mathcal{Z}_{\mathbf{n}} = 1 + e^{-\beta(E_{\mathbf{n}}-\mu)} \tag{25}$$

so

$$N_{\mathbf{n}} = \frac{1}{e^{\beta(E_{\mathbf{n}}-\mu)}+1} = \frac{\mathcal{Z}_{\mathbf{n}}-1}{\mathcal{Z}_{\mathbf{n}}} \Rightarrow 1 - N_{\mathbf{n}} = \frac{1}{\mathcal{Z}_{\mathbf{n}}}\ , \tag{26}$$

and

$$N_{\mathbf{n}}\mathcal{Z}_{\mathbf{n}} = \mathcal{Z}_{\mathbf{n}} - 1 = e^{-\beta(E_{\mathbf{n}}-\mu)}\ . \tag{27}$$

Also, from $\beta = (k_bT)^{-1}$, we have

$$\frac{\partial}{\partial T} = -\frac{1}{k_bT^2}\frac{\partial}{\partial\beta}\ . \tag{28}$$

Now, the entropy is

$$S = \frac{\partial}{\partial T}(k_b T \ln \mathcal{Z}) = k_b \ln \mathcal{Z} + k_b T \frac{\partial}{\partial T} \ln \mathcal{Z} \ . \tag{29}$$

The first term is

$$k_b \ln \mathcal{Z} = k_b \ln \prod_{\mathbf{n}} \mathcal{Z}_{\mathbf{n}} = -\sum_{\mathbf{n}} k_b \ln(1 - N_{\mathbf{n}}) \ . \tag{30}$$

The second term is

$$\begin{aligned} k_b T \frac{\partial}{\partial T} \ln \mathcal{Z} &= -\beta k_b \frac{\partial}{\partial \beta} \ln \mathcal{Z} \\ &= -\beta k_b \frac{\partial}{\partial \beta} \left(\sum_{\mathbf{n}} \ln(1 + e^{-\beta(E_{\mathbf{n}} - \mu)}) \right) \\ &= k_b \left(\sum_{\mathbf{n}} \frac{\beta(E_{\mathbf{n}} - \mu) e^{-\beta(E_{\mathbf{n}} - \mu)}}{1 + e^{-\beta(E_{\mathbf{n}} - \mu)}} \right) \ . \end{aligned} \tag{31}$$

Now using the trick

$$\mathcal{Z}_{\mathbf{n}} - 1 = e^{-\beta(E_{\mathbf{n}} - \mu)} \rightarrow \ln(\mathcal{Z}_{\mathbf{n}} N_{\mathbf{n}}) = -\beta(E_{\mathbf{n}} - \mu) \tag{32}$$

where we have used the identity Eq. (27). So we get

$$\begin{aligned} k_b T \frac{\partial}{\partial T} \ln \mathcal{Z} &= -k_b \left(\sum_{\mathbf{n}} \frac{\ln(\mathcal{Z}_{\mathbf{n}} N_{\mathbf{n}}) \mathcal{Z}_{\mathbf{n}}}{\mathcal{Z}_{\mathbf{n}} - 1} \right) \\ &= -k_b \left(\sum_{\mathbf{n}} N_{\mathbf{n}} \ln(\mathcal{Z}_{\mathbf{n}} N_{\mathbf{n}}) \right) \\ &= -k_b \left(\sum_{\mathbf{n}} N_{\mathbf{n}} \ln \frac{N_{\mathbf{n}}}{1 - N_{\mathbf{n}}} \right) \ . \end{aligned} \tag{33}$$

In the 2nd line, we used identity Eq. (27) and the third line we used identity Eq. (26).

Putting both terms together, we get

$$\begin{aligned} S &= -k_b \sum_{\mathbf{n}} \left[\ln(1 - N_{\mathbf{n}}) + N_{\mathbf{n}} \ln \frac{N_{\mathbf{n}}}{1 - N_{\mathbf{n}}} \right] \\ &= -k_b \sum_{\mathbf{n}} \left[(1 - N_{\mathbf{n}}) \ln(1 - N_{\mathbf{n}}) + N_{\mathbf{n}} \ln N_{\mathbf{n}} \right] \end{aligned} \tag{34}$$

as required.

8) Using the results from Q1, the fermi energy is

$$E_F = \left(\frac{3N}{4\pi V \tilde{g}} \right)^{1/3} (2\pi \hbar c) \tag{35}$$

and now $N = M/m_p$ (since electron mass is very small we ignore it) and $V = 4\pi R^3/3$ we get

$$E_T = E_G + E_K = -\frac{3}{5} \frac{GM^2}{R} + (2\pi\hbar c) \left(\frac{9}{16\pi^2 \tilde{g}} \right)^{1/3} \left(\frac{M}{m_p} \right)^{4/3} \frac{1}{R} \propto \frac{1}{R} \ . \tag{36}$$

When $E_T < 0$, this means that the gravitational term $E_G > E_K$, and hence gravity will dominate over the kinetic energy, causing a collapse. This occurs at $E_T = 0$, which gives us

$$M_C = 5 \sqrt{\frac{5\pi}{6\tilde{g}}} \left(\frac{\hbar c}{G} \right)^{3/2} m_p^{-2} \ . \tag{37}$$

Plugging in the numbers we get $M_C \approx 10 M_{\odot}$.