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QUANTUM MECHANICS

Example Sheet 1

Speed of light: $c = 3.00 \times 10^8\,\mathrm{m\,s^{-1}}$
Planck's constant: $\hbar = 1.055 \times 10^{-34}\,\mathrm{J\,s}$ $\quad (h = 2\pi\hbar = 6.63 \times 10^{-34}\,\mathrm{J\,s})$
Fine-structure constant: $\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$
Mass of electron: $m_e = 9.11 \times 10^{-31}$ kg
Mass of proton: $m_p = 1.67 \times 10^{-27}$ kg
Electron volt: 1 eV $= 1.60 \times 10^{-19}$ J
Bohr radius: $r_0 = 0.529 \times 10^{-10}$ m

1. When the surface of a sample of potassium is illuminated with light of wavelength 3×10^{-7} m it emits electrons with kinetic energy 2.1 eV. When the same sample is illuminated with light of wavelength 5×10^{-7} m it emits electrons with kinetic energy 0.5 eV. Use Einstein's explanation of this 'photoelectric' effect to obtain a value for Planck's constant $\hbar$, and find the minimum energy E_0 needed to free an electron from the surface of potassium.

2. Let $\psi_i(x)$, $i = 1, 2$, be two normalized stationary state wavefunctions. Assume that they are orthogonal, so that

$$\int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)\,dx = 0\,.$$

Show that the linear superposition $\alpha\psi_1 + \beta\psi_2$, for complex constants α and β is normalized if and only if $|\alpha|^2 + |\beta|^2 = 1$. Suppose now that ψ_1 and ψ_2 are normalized but not orthogonal. Show that there is a unique constant γ, with $|\gamma| \le 1$, such that $\psi = \psi_1 - \gamma\psi_2$ is orthogonal to ψ_2. Given that $|\gamma| < 1$ show that $\psi/\sqrt{1 - |\gamma|^2}$ is normalized.

3. Show that the operator

$$\hat{\mathcal{P}} \equiv \frac{\partial}{\partial x}$$

is not Hermitian. Show that it has purely imaginary eigenvalues. (Such an operator is called an *anti-Hermitian Operator*.)

4. Consider the generalized Boolean Operator for the qubit

$$\hat{N}_\theta = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that this operator is Hermitian and hence corresponds to an observable. Calculate the eigenvalues and normalized eigenvectors for $\hat{N}_\theta$.

- Consider the case where $\theta = \pi$. Write down its normalized eigenvectors χ_{up} and χ_{down}.
- A general qubit state can be written in this basis as

$$\psi = \alpha\chi_{\mathrm{up}} + \beta\chi_{\mathrm{down}}\ ,\ \alpha, \beta\ \in\ \mathbb{C}$$

with normalization $|\alpha|^2 + |\beta|^2 = 1$. We want to make a measurement associated with the this operator $\hat{N}_{\theta=\pi}$. What are the probabilities of measuring the states associated with χ_{up} and χ_{down}?

- Consider another operator $\hat{N}_{\theta=\pi/2}$. What are its eigenvalues and normalized eigenvectors? Suppose we begin with a state ψ and then a measurement of an observable associated with $\hat{N}_{\theta=\pi}$ was made and an eigenvalue of $+1$ was obtained. We now make another observation associated with $\hat{N}_{\theta=\pi/2}$ on this resulting state. Calculate the probability of obtaining an eigenvalue of $+1$.

- * The Uncertainty Principle of the qubit. Calculate the *commutator* of two Boolean Operators with real parameters $\theta = \alpha, \beta$

$$[\hat{N}_\alpha, \hat{N}_\beta] \equiv \hat{N}_\alpha \hat{N}_\beta - \hat{N}_\beta \hat{N}_\alpha .$$

  The commutator (as will be discussed in Chapter 7 of the lectures) of two observables quantifies our inability to measure with arbitrary precision simultaneously both observables, hence there exist an inherent uncertainty. Show that the condition for the commutator to vanish is $\alpha - \beta = n\pi$ where $n = 0, 1, 2, 3, \ldots$.

5. Consider a two-state system

$$\psi(t) = \alpha(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \ \alpha, \beta \in \mathbb{C}.$$

Its dynamics is described by the Schrodinger's Equation for a two-state system

$$i\hbar \frac{d\psi}{dt} = \hat{H}(t)\psi(t)$$

where the Hamiltonian matrix is

$$\hat{H} = \begin{pmatrix} E & -\epsilon \\ -\epsilon & E \end{pmatrix} \ , \ E \gg |\epsilon| \ , \ \epsilon, E \in \mathbb{R}.$$

Find the probability amplitudes $\alpha(t)$ and $\beta(t)$ as a function of time $t > 0$, given the initial condition $\alpha(0) = 0$ and $\beta(0) = 1$. Show that the total probability of this state is conserved as a function of time.

6. A particle with $m = \hbar$, moving freely in one dimension has wavefunction

$$\psi(x,t) = \frac{1}{\pi^{\frac{1}{4}} (1+it)^{\frac{1}{2}}} \exp\left(\frac{-x^2}{2(1+it)} \right) .$$

Verify that this wavefunction is normalized. Compute the probability density and probability current and verify that they are compatible with conservation of probability.

Consider the probability of finding the particle in an arbitrary finite interval $a \le x \le b$. Show that this probability vanishes in the limit $t \to \infty$ (with a and b held fixed).

7. Show that the stationary state wavefunctions of a particle in a potential $V(x)$ with $V(-x) = V(x)$ either have definite parity or can be chosen to have definite parity. Discuss the odd-parity bound states in the one-dimensional square well with potential $V = 0$ for $|x| > a$, $V = -U$ otherwise, where U is a positive constant. Use a graphical method to show that there is no odd-parity bound state if $2mU < (\hbar\pi/2a)^2$.

8. Sketch the potential

$$V = -\frac{\hbar^2}{m} \operatorname{sech}^2 x$$

and show that the time-independent Schrödinger equation for a particle in this potential can be written as

$$A^\dagger A \, \psi = (\varepsilon + 1)\psi$$

where $\varepsilon = 2mE/\hbar^2$ and

$$A = \frac{d}{dx} + \tanh x \, , \qquad A^\dagger = -\frac{d}{dx} + \tanh x \, .$$

Show, by integrating by parts, that for any normalized wavefunction ψ,

$$\int_{-\infty}^{\infty} \psi^* A^\dagger A \psi \, dx = \int_{-\infty}^{\infty} (A\psi)^* (A\psi) \, dx$$

and hence that the eigenvalues of $A^\dagger A$ are non-negative. Hence deduce that the ground state wavefunction must have $\varepsilon \geq -1$. Show that there is a wavefunction $\psi_0(x)$ with $\varepsilon = -1$, satisfying

$$\frac{d\psi_0}{dx} + \tanh x \, \psi_0 = 0 \, .$$

Find and sketch $\psi_0(x)$.

9. Write down the time-independent Schrödinger equation for the wavefunction ψ of a particle moving in a potential $V = -U\delta(x)$ for positive constant U (and $\delta(x)$ the Dirac delta function). Integrate the equation over the interval $-\epsilon < x < \epsilon$, for arbitrary positive constant ϵ, and hence show that there is a discontinuity at $x = 0$ in the derivative of $\psi(x)$:

$$\lim_{\epsilon \to 0} \left[\psi'(\epsilon) - \psi'(-\epsilon) \right] = -\frac{2mU}{\hbar^2} \psi(0) \, .$$

Show that there is a unique bound state ($E < 0$) solution $\psi_0(x)$. Find this ground state solution, and its energy.

10. Show that the Parity operator in one dimension

$$\hat{P}\psi(x) = \psi(-x)$$

is Hermitian and hence is an observable. Is a momentum eigenstate of eigenvalue p invariant under the action of the parity operator? If it is, calculate its eigenvalue associated with the parity operator. If it is not, explain in physical terms why not.