Solutions to Homework 4

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April 6, 2018

1) Using the symmetries of the Riemann tensor we find

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = g^{\rho\sigma} R_{\sigma\nu\rho\mu} = g^{\sigma\rho} R_{\sigma\nu\rho\mu} = R_{\nu\mu} \tag{1}$$

2) We find

$$\begin{aligned}
C_{\rho\sigma\mu\nu} &= R_{\rho\sigma\mu\nu} - \frac{1}{d-2}\left(g_{\rho\mu}R_{\nu\sigma} - g_{\rho\nu}R_{\mu\sigma} - g_{\sigma\mu}R_{\nu\rho} + g_{\sigma\nu}R_{\mu\rho}\right) \\
&\quad + \frac{1}{(d-1)(d-2)}\left(g_{\rho\mu}g_{\nu\sigma} - g_{\rho\nu}g_{\mu\sigma}\right)R \underbrace{=}_{\text{symmetries of Riemann, Ricci and metric}} \\
&= R_{\mu\nu\rho\sigma} - \frac{1}{d-2}\left(g_{\mu\rho}R_{\sigma\nu} - g_{\nu\rho}R_{\sigma\mu} - g_{\mu\sigma}R_{\rho\nu} + g_{\nu\sigma}R_{\rho\mu}\right) \\
&\quad + \frac{1}{(d-1)(d-2)}\left(g_{\mu\rho}g_{\sigma\nu} - g_{\rho\nu}g_{\mu\sigma}\right)R \underbrace{=}_{\text{exchange terms}} \\
&= R_{\mu\nu\rho\sigma} - \frac{1}{d-2}\left(g_{\mu\rho}R_{\sigma\nu} - g_{\mu\sigma}R_{\rho\nu} - g_{\nu\rho}R_{\sigma\mu} + g_{\nu\sigma}R_{\rho\mu}\right) \\
&\quad + \frac{1}{(d-1)(d-2)}\left(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu}\right)R \\
&= C_{\mu\nu\rho\sigma}.
\end{aligned} \tag{2}$$

We can see that the 5th and 6th line is equal to the 1st and 2nd by exchanging

$$\rho \to \nu \qquad \sigma \to \mu \quad , \tag{3}$$

and vice versa. We write

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = g^{\rho\sigma} R_{\sigma\nu\rho\mu} = g^{\sigma\rho} R_{\sigma\nu\rho\mu} = R_{\nu\mu}. \tag{4}$$

Additional

$$
\begin{aligned}
C_{\rho\sigma\mu\nu} &= R_{\rho\sigma\mu\nu} - \frac{1}{d-2}\left(g_{\rho\mu}R_{\nu\sigma} - g_{\rho\nu}R_{\mu\sigma} - g_{\sigma\mu}R_{\nu\rho} + g_{\sigma\nu}R_{\mu\rho}\right) \\
&+ \frac{1}{(d-1)(d-2)}\left(g_{\rho\mu}g_{\nu\sigma} - g_{\rho\nu}g_{\mu\sigma}\right)R \underbrace{=}_{\text{symmetries of Riemann, Ricci and metric}} \\
&= -R_{\sigma\rho\mu\nu} - \frac{1}{d-2}\left(g_{\rho\mu}R_{\nu\sigma} - g_{\rho\nu}R_{\mu\sigma} - g_{\sigma\mu}R_{\nu\rho} + g_{\sigma\nu}R_{\mu\rho}\right) \\
&+ \frac{1}{(d-1)(d-2)}\left(g_{\mu\rho}g_{\sigma\nu} - g_{\nu\rho}g_{\sigma\mu}\right)R \underbrace{=}_{\text{exchange terms}} \\
&= -R_{\sigma\rho\mu\nu} + \frac{1}{d-2}\left(g_{\sigma\mu}R_{\nu\rho} - g_{\sigma\nu}R_{\mu\rho} - g_{\rho\mu}R_{\nu\sigma} + g_{\rho\nu}R_{\mu\sigma}\right) \\
&- \frac{1}{(d-1)(d-2)}\left(g_{\sigma\mu}g_{\nu\rho} - g_{\sigma\nu}g_{\mu\rho}\right)R = \\
&= -C_{\sigma\rho\mu\nu}.
\end{aligned}
\tag{5}
$$

Using both symmetries from before, we show the last one

$$
C_{\rho\sigma\mu\nu} = C_{\mu\nu\rho\sigma} = -C_{\nu\mu\rho\sigma} = -C_{\rho\sigma\nu\mu}. \tag{6}
$$

In the second part of this exercise one must show that all contraction of the Weyl tensor vanish, some of these can be shown just using symmetries (1st and last in the exercise)

$$
\begin{aligned}
&g^{\mu\nu}C_{\rho\sigma\mu\nu} = \frac{1}{2}g^{\mu\nu}C_{\rho\sigma\mu\nu} + \frac{1}{2}g^{\mu\nu}C_{\rho\sigma\mu\nu} = \frac{1}{2}g^{\mu\nu}C_{\rho\sigma\mu\nu} - \frac{1}{2}g^{\nu\mu}C_{\rho\sigma\nu\mu} = \\
&\frac{1}{2}g^{\mu\nu}C_{\rho\sigma\mu\nu} - \underbrace{\frac{1}{2}g^{\mu\nu}C_{\rho\sigma\mu\nu}}_{\text{rename } \mu\to\nu} = 0.
\end{aligned}
\tag{7}
$$

In general, contractions between antisymmetric tensor symmetric ones are

equal zero. For other ones, we show explicitly

$$
\begin{aligned}
g^{\sigma\mu}C_{\rho\sigma\mu\nu} &= g^{\sigma\mu}R_{\rho\sigma\mu\nu} - \frac{1}{d-2}\left(g^{\sigma\mu}g_{\rho\mu}R_{\nu\sigma} - g^{\sigma\mu}g_{\rho\nu}R_{\mu\sigma} - g^{\sigma\mu}g_{\sigma\mu}R_{\nu\rho} + g^{\sigma\mu}g_{\sigma\nu}R_{\mu\rho}\right) \\
&+ \frac{1}{(d-1)(d-2)}\left(g^{\sigma\mu}g_{\rho\mu}g_{\nu\sigma} - g^{\sigma\mu}g_{\rho\nu}g_{\mu\sigma}\right)R \underbrace{=}_{g^{\mu\nu}g_{\mu\nu}=d} \\
&= -R_{\rho\nu} - \frac{1}{d-2}\left(\delta^{\sigma}{}_{\rho}R_{\nu\sigma} - g_{\rho\nu}R - dR_{\nu\rho} + \delta^{\mu}{}_{\nu}R_{\mu\rho}\right) \\
&+ \frac{1}{(d-1)(d-2)}\left(g_{\rho\mu}\delta^{\mu}{}_{\nu} - dg_{\rho\nu}\right)R = \\
&= -R_{\rho\nu} - \frac{1}{d-2}\left(R_{\nu\rho} - g_{\rho\nu}R - dR_{\nu\rho} + R_{\nu\rho}\right) \\
&+ \frac{1}{(d-1)(d-2)}\left(g_{\rho\nu} - dg_{\rho\nu}\right)R = \\
&= -R_{\rho\nu} + \frac{1}{d-2}g_{\rho\nu}R - \frac{1}{d-2}\left(2-d\right)R_{\nu\rho} \\
&- \frac{1}{(d-2)}Rg_{\rho\nu} = 0
\end{aligned}
\tag{8}
$$

all other can be obtained this result together with symmetries

$$
g^{\sigma\mu}C_{\rho\sigma\mu\nu} = -g^{\sigma\mu}C_{\sigma\rho\mu\nu} = -g^{\sigma\mu}C_{\rho\sigma\nu\mu} = g^{\sigma\mu}C_{\mu\nu\rho\sigma} = ... \tag{9}
$$

3) For exercise 7, we find that the Ricci tensor is zero. This is obvious, since all components of the Riemann tensor are already zero.

For the Ricci tensor exercise 8, we find the following non-zero components

$$
\begin{aligned}
R_{tt} &= \frac{1}{2}\epsilon\delta^{ab}\partial_a\partial_b\bar{\Phi} + \mathcal{O}(\epsilon^2) \\
R_{ti} &= \mathcal{O}(\epsilon^2) \\
R_{ij} &= \frac{1}{2}\epsilon\delta_{ij}\delta^{ab}\partial_a\partial_b\bar{\Phi} + \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{10}
$$

4i)We first get the area of the sphere by integrating over circumference of all possible circles (see fig. 1)with radii $\tilde{r}(\theta') = R\sin(\theta')$ of the cap

$$
\begin{aligned}
A(r) &= \int_{D\subset S^2} dA = \int_0^{\theta} 2\pi\tilde{r}(h)Rd\theta' = \int_0^{\theta} 2\pi R^2\sin(\theta')d\theta' \\
&= 2\pi R^2(1-\cos(\theta))
\end{aligned}
\tag{11}
$$

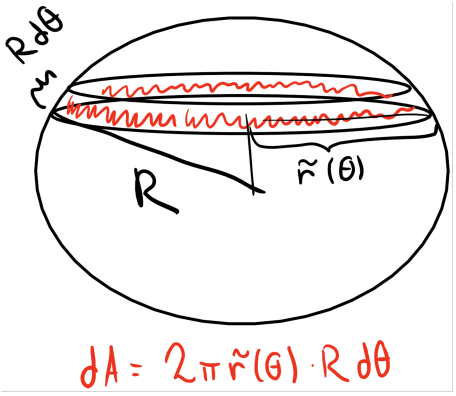

Figure 1: **Area of a section of** S^2 The areaelement dA is the circumfrence $2\pi\tilde{r}$ times the small length R dθ.

4ii)

$$\begin{aligned}\frac{\pi r^2 - A(r)}{24\pi r^4} &= \frac{\pi r^2 - 2\pi R^2(1-\cos(r/R))}{24\pi r^4} \\ &\approx \frac{\pi r^2 - 2\pi R^2\left(\frac{1}{2}\left(\frac{r}{R}\right)^2\right) - \frac{1}{4!}\left(\frac{r}{R}\right)^4 + \mathcal{O}(1/R^6)\Big)}{24\pi r^4} \\ &= \frac{1}{288}\frac{1}{R^2} + \mathcal{O}(1/R^4)\end{aligned} \tag{12}$$

4iii)Riemann tensor

$\{(1),(2),(3),(4)\}$	$R^{(1)}{}_{(2),(3),(4)}$
$\{\phi,\theta,\theta,\phi\}$	-1
$\{\phi,\theta,\phi,\theta\}$	1
$\{\theta,\phi,\phi,\theta\}$	$-\sin^2(\theta)$
$\{\theta,\phi,\theta,\phi\}$	$\sin^2(\theta)$

(13)

the Ricci scalar

$$R_{\mu\nu}g^{\mu\nu} = \frac{2}{R^2} \tag{14}$$

5)We first start to look at the Bianchi identity

$$\nabla_\tau R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\tau} + \nabla_\nu R_{\rho\sigma\tau\mu} = 0, \tag{15}$$

now contract the relation

$$g^{\rho\mu}g^{\sigma\nu}\left(\nabla_\tau R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\tau} + \nabla_\nu R_{\rho\sigma\tau\mu}\right) = 0, \tag{16}$$

which gives (using metric compatibility)

$$\nabla_\tau R - g^{\rho\mu}\nabla_\mu R_{\rho\tau} - g^{\sigma\nu}\nabla_\nu R_{\sigma\tau} = 0, \tag{17}$$

which finally gives

$$\nabla^\mu R_{\mu\tau} = \frac{1}{2}\nabla_\tau R. \tag{18}$$

We will use this relation to show that $\nabla^\mu G_{\mu\nu} = 0$,

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla_\nu R = 0. \tag{19}$$

6) We define

$$\begin{array}{lll} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} & , & |h_{\mu\nu}| << 1 \\ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} & , & h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma} \end{array} \tag{20}$$

we show [1]

$$\begin{aligned} g_{\delta\mu}g^{\mu\gamma} &= (\eta_{\delta\mu} + h_{\delta\mu})(\eta^{\mu\gamma} - h^{\mu\gamma}\ ) = \eta_{\delta\mu}\eta^{\mu\gamma} - \eta_{\delta\mu}h^{\mu\gamma} + \eta^{\mu\gamma}h_{\delta\mu} + \mathcal{O}(h^2) \\ &= \delta^\gamma{}_\delta - \underbrace{\eta_{\delta\mu}h^{\mu\gamma}}_{=\eta_{\delta\mu}\eta^{\mu\rho}\eta^{\gamma\sigma}h_{\rho\sigma}=\eta^{\gamma\sigma}h_{\delta\sigma}} + \eta^{\mu\gamma}h_{\delta\mu} + \mathcal{O}(h^2) = \delta^\gamma{}_\mu + \mathcal{O}(h^2) \end{aligned} \tag{21}$$

7i)We calculate the Ricci tensor

$$\begin{aligned} R_{\sigma\nu} = g^{\rho\mu}R_{\rho\sigma\mu\nu} &= \frac{K}{n(n-1)}g^{\rho\mu}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right) \\ &= \frac{K}{n(n-1)}\left(n-1\right)g_{\sigma\nu} = \frac{K}{n}g_{\sigma\nu} \end{aligned} \tag{22}$$

and the Ricci-scalar

$$R = g^{\sigma\nu}R_{\sigma\nu} = \frac{K}{n}g^{\sigma\nu}g_{\sigma\nu} = K \tag{23}$$

7ii) We found for the 2-sphere

$$R = \frac{2}{R^2} \tag{24}$$

[1] I ran out of indices and had to use δ as index, which might be confusing. Sorry.

we choose R = $\sqrt{2}$ and find

$$R = 1. \tag{25}$$

8i) The inverse

$$\gamma^{ij} = \begin{pmatrix} 1 - Kr^2 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin(\theta)^2} \end{pmatrix} \tag{26}$$

8ii) The Christoffel symbols

$\{(1),(2),(3)\}$	${\Gamma^1}_{(2)(3)}$
$\{\theta,r,\theta\}$ $\{\theta,\theta,r\}$ $\{\phi,r,\phi\}$ $\{\phi,\phi,r\}$	$\frac{1}{r}$
$\{r,r,r\}$	$\frac{Kr}{1-Kr^2}$
$\{r,\theta,\theta\}$	$r\left(Kr^2-1\right)$
$\{\phi,\theta,\phi\}$ $\{\phi,\phi,\theta\}$	$\cot(\theta)$
$\{\theta,\phi,\phi\}$	$\sin(\theta)(-\cos(\theta))$
$\{r,\phi,\phi\}$	$r\sin^2(\theta)\left(Kr^2-1\right)$
$\{r,t,r\}$ $\{r,r,t\}$ $\{\theta,t,\theta\}$ $\{\theta,\theta,t\}$ $\{\phi,t,\phi\}$ $\{\phi,\phi,t\}$	$\frac{a'(t)}{a(t)}$
$\{t,\theta,\theta\}$	$r^2a(t)a'(t)$
$\{t,r,r\}$	$\frac{a(t)a'(t)}{1-Kr^2}$
$\{t,\phi,\phi\}$	$r^2a(t)\sin^2(\theta)a'(t)$

(27)

8iii) The Riemann tensor

$\{(1),(2),(3),(4)\}$	$R^{(1)}{}_{(2),(3),(4)}$
$\{r,\theta,\theta,r\}$ $\{\phi,\theta,\theta,\phi\}$	$-r^2\left(a'(t)^2+K\right)$
$\{r,\theta,r,\theta\}$ $\{\phi,\theta,\phi,\theta\}$	$r^2\left(a'(t)^2+K\right)$
$\{\theta,r,\theta,r\}$ $\{\phi,r,\phi,r\}$	$\frac{a'(t)^2+K}{1-Kr^2}$
$\{\theta,r,r,\theta\}$ $\{\phi,r,r,\phi\}$	$\frac{a'(t)^2+K}{Kr^2-1}$
$\{r,\phi,\phi,r\}$ $\{\theta,\phi,\phi,\theta\}$	$-r^2\sin^2(\theta)\left(a'(t)^2+K\right)$
$\{r,\phi,r,\phi\}$ $\{\theta,\phi,\theta,\phi\}$	$r^2\sin^2(\theta)\left(a'(t)^2+K\right)$
$\{r,t,r,t\}$ $\{\theta,t,\theta,t\}$ $\{\phi,t,\phi,t\}$	$-\frac{a''(t)}{a(t)}$
$\{r,t,t,r\}$ $\{\theta,t,t,\theta\}$ $\{\phi,t,t,\phi\}$	$\frac{a''(t)}{a(t)}$
$\{t,\theta,\theta,t\}$	$-r^2a(t)a''(t)$
$\{t,\theta,t,\theta\}$	$r^2a(t)a''(t)$
$\{t,r,t,r\}$	$\frac{a(t)a''(t)}{1-Kr^2}$
$\{t,r,r,t\}$	$\frac{a(t)a''(t)}{Kr^2-1}$
$\{t,\phi,\phi,t\}$	$-r^2a(t)\sin^2(\theta)a''(t)$
$\{t,\phi,t,\phi\}$	$r^2a(t)\sin^2(\theta)a''(t)$

(28)

The Ricci tensor

$\{(1),(2)\}$	$R_{(1),(2)}$
$\{t,t\}$	$-\frac{3a''(t)}{a(t)}$
$\{\theta,\theta\}$	$r^2\left(a(t)a''(t)+2a'(t)^2+2K\right)$
$\{r,r\}$	$\frac{a(t)a''(t)+2a'(t)^2+2K}{1-Kr^2}$
$\{\phi,\phi\}$	$r^2\sin^2(\theta)\left(a(t)a''(t)+2a'(t)^2+2K\right)$

(29)

The Ricci scalar

$$R=\frac{\sin(\theta)\left(6r^2a(t)a''(t)+6r^2a'(t)^2+6Kr^2-2\right)-2\sin''(\theta)}{r^2a(t)^2\sin(\theta)} \tag{30}$$

The Einstein tensor

$\{(1),(2)\}$	$G_{(1),(2)}$
$\{t,t\}$	$\frac{3\left(a'(t)^2+K\right)}{a(t)^2}$
$\{\theta,\theta\}$	$-r^2\left(2a(t)a''(t)+a'(t)^2+K\right)$
$\{r,r\}$	$\frac{2a(t)a''(t)+a'(t)^2+K}{Kr^2-1}$
$\{\phi,\phi\}$	$-r^2\sin^2(\theta)\left(2a(t)a''(t)+a'(t)^2+K\right)$

(31)

8iv) For a co moving fluid

$$u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{32}$$

which gives the stress energy tensor

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & Pa(t)^2\gamma_{ij} & \\ 0 & & & \end{pmatrix} \tag{33}$$

we can now evalute the

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{34}$$

using the components derived in previous exercise. The tt component gives

$$\frac{a'(t)^2}{a(t)^2} = \frac{8}{3}\pi G\rho - \frac{K}{a(t)^2} \tag{35}$$

taking the $\theta\theta$ component we find

$$-\left(2a(t)a''(t)+a'(t)^2+K\right) = 8\pi G P a(t)^2 \tag{36}$$

we rewrite this as

$$\left(2\frac{a''(t)}{a(t)}\right) = -8\pi G P - \frac{a'(t)^2}{a(t)^2} - \frac{K}{a(t)^2} \tag{37}$$

and substituting in the tt component equation

$$\frac{a''(t)}{a(t)} = -\frac{4}{3}\pi G\left(3P+\rho\right) \tag{38}$$

8v) We have a particle with momenta

$$P^\mu = (E, a^{-1}p^i) \tag{39}$$

with

$$\partial_i P^\mu = 0 \tag{40}$$

using the godesic equation

$$\begin{aligned} 0 &= P^\mu \nabla_\mu P^\nu = P^\mu(\partial_\mu P^\nu + \Gamma^\nu{}_{\rho\mu} P^\rho) = P^t \partial_t P^\nu + \Gamma^\nu{}_{\rho\mu} P^\rho P^\mu \\ &= E\partial_t P^\nu + \Gamma^\nu_{\rho\mu} P^\rho P^\mu = (\star) \end{aligned} \tag{41}$$

We know for $\nu = t$

$$\Gamma^t{}_{ij} = a(t)a'(t)\gamma_{ij} \tag{42}$$

and

$$\Gamma^t{}_{ti} = 0 \tag{43}$$

and

$$\Gamma^t{}_{t\mu} = 0 \tag{44}$$

which finally gives

$$(\star) \Rightarrow 0 = -EE' + \frac{a'(t)}{a(t)}\gamma_{ij}p^i p^j. \tag{45}$$

We can rewrite this as

$$EE' = -\gamma^{ij}p_i p_j \frac{a'}{a},$$

Together with being massless

$$P^\mu P_\mu = 0 \quad \Rightarrow \quad E^2 = \gamma^{ij}p_i p_j,$$

which gives

$$\frac{E'}{E} = -\frac{a'}{a} \Rightarrow E \propto \frac{1}{a}$$

You see that a expanding universe decreases the energy of massless particles

8vi) We find for the tangent vector

$$T^\mu = \frac{d}{d\tau}\begin{pmatrix}\tau\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \tag{46}$$

we find

$$\theta = \nabla_\mu T^\mu = \partial_\mu T^\mu + \Gamma^\mu{}_{\rho\mu} T^\rho \tag{47}$$

with

$$\Gamma^i{}_{ti} = \frac{a'(t)}{a(t)}, \tag{48}$$

we find

$$\theta = 3\frac{a'(t)}{a(t)} = 3H. \tag{49}$$

9i) We know for Minkowski

$$R_{\rho\sigma\mu\nu} = 0 \tag{50}$$

and therefore

$$\begin{aligned} R_{\sigma\nu} &= g^{\rho\mu} R_{\rho\sigma\mu\nu} = 0 \\ R &= g^{\sigma\nu} R_{\sigma\nu} = 0 \\ G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} R = 0 \end{aligned} \tag{51}$$

since in Minkowski there is no matter $T_{\mu\nu} = 0$ the Einstein field equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{52}$$

is fulfilled.

10) T^μ is tangent field to time like congruence of geodesics, which is equivalent to

$$\begin{aligned} T_\mu T^\mu &= -1 \\ T^\mu \nabla_\mu T^\nu &= 0 \end{aligned} \tag{53}$$

They imply that

$$\begin{aligned} T^\mu B_{\mu\nu} &= T^\mu \nabla_\mu T_\nu = 0 \\ T^\nu B_{\mu\nu} &= T^\nu \nabla_\mu T_\nu = \nabla_\mu \underbrace{(T_\nu T^\nu)}_{-1} - T_\nu \nabla_\mu T^\nu \\ &\Rightarrow T^\nu \nabla_\mu T_\nu = -T_\nu \nabla_\mu T^\nu \\ &\Rightarrow T^\nu \nabla_\mu T_\nu = -T^\nu \nabla_\mu T_\nu \\ &\Rightarrow T^\nu \nabla_\mu T_\nu = 0, \end{aligned} \tag{54}$$

where the last line follows from zero being the only number that is equal to its negative.