General Relativity Homework 3

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Week 5+6

(1) Consider the Levi-Civita connection. By expanding the covariant derivatives in terms of partial derivatives and Christoffel symbols, prove that

$$[\nabla_\mu, \nabla_\nu]V^\rho \equiv \nabla_\mu\nabla_\nu V^\rho - \nabla_\nu\nabla_\mu V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma \tag{1}$$

where V^μ is any vector. As a bonus (**), prove that

$$[\nabla_\mu, \nabla_\nu]F^\rho{}_\lambda = R^\rho{}_{\sigma\mu\nu}F^\sigma{}_\lambda + R_\lambda{}^\sigma{}_{\mu\nu}F^\rho{}_\sigma, \tag{2}$$

where $F^\rho{}_\lambda$ is a rank-(1,1) tensor and hence derive the following general formula for the action of the commutator for rank-(p,q) tensors

$$[\nabla_\mu, \nabla_\nu]F^{\rho_1\rho_2\cdots}{}_{\lambda_1\lambda_2\ldots} = R^{\rho_1}{}_{\sigma\mu\nu}F^{\sigma\rho_2\cdots}{}_{\lambda_1\lambda_2\ldots} + R^{\rho_2}{}_{\sigma\mu\nu}F^{\rho_1\sigma\cdots}{}_{\lambda_1\lambda_2\ldots} + R_{\lambda_1}{}^\sigma{}_{\mu\nu}F^{\rho_1\rho_2\cdots}{}_{\sigma\lambda_2\ldots} + R_{\lambda_2}{}^\sigma{}_{\mu\nu}F^{\rho_1\rho_2\cdots}{}_{\lambda_1\sigma\ldots} + \ldots \tag{3}$$

(2) (**Local Inertial Coordinates and Symmetries of Riemann Tensor**). Consider a 4*D Lorentzian* manifold $\mathcal{M}$ equipped with a metric $\bar{\mathbf{g}}$ which describes its curvature. In class, we argued that one can always choose a local coordinate system such that the *locally* the metric looks “flat”, i.e. the metric in the associate coordinate basis $\mathrm{d}x^\mu$

$$ds^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{4}$$

at some point $p \in \mathcal{M}$. Such a coordinate system is called a *local inertial coordinate system.*

(i) Suppose the spacetime is *not* Minkowski space, argue that in this coordinate system, the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ vanish but the partial derivatives

$$\partial_\alpha\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\alpha\partial_\mu g_{\sigma\nu} + \partial_\alpha\partial_\nu g_{\mu\sigma} - \partial_\alpha\partial_\sigma g_{\mu\nu}) \tag{5}$$

do not necessarily vanish.

(ii) Using your result in (i), show that the Riemann tensor in this basis is given by

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda}R^\lambda{}_{\beta\mu\nu} = \frac{1}{2}(\partial_\beta\partial_\mu g_{\alpha\nu} - \partial_\beta\partial_\nu g_{\alpha\mu} + \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\alpha\partial_\mu g_{\beta\nu}). \tag{6}$$

(iii) Using (ii), argue that the Riemann tensor has the following symmetries:

Anti-symmetric in the first 2 indices:

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}, \tag{7}$$

Anti-symmetric in the last 2 indices

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}. \tag{8}$$

Symmetric under the exchange of the first two with the last two indices

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}. \tag{9}$$

The sum of the permutations of the last 3 indices is zero

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0. \tag{10}$$

(3) (**Bianchi Identity**). Prove

$$\nabla_\tau R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\tau} + \nabla_\nu R_{\rho\sigma\tau\mu} = 0, \tag{11}$$

for the Levi-Civita connection. You may use all the symmetries of the Riemann Tensor without proof.

(4) (**3D Embedding of Ellipsoid**). We want to derive the metric for a 2D ellipsoid via its embedding in a 3D Euclidean space.
(i) Consider the 3D Euclidean space in the Cartesian basis

$$ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{12}$$

An embedding of a 2D ellipsoid into 3D Euclidean space is given by the restriction equations

$$x(\theta,\phi) = a\cos\theta\sin\phi\ ,\ y(\theta,\phi) = b\cos\theta\cos\phi\ ,\ z(\theta,\phi) = c\sin\theta, \tag{13}$$

where a, b, c are real and positive constants. Show that the equations obey the following equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{14}$$

which describes an ellipsoid.
(ii) Hence show that the metric in the (θ, ϕ) coordinate basis is given by

$$\begin{aligned} ds^2 &= \cos^2\theta(a^2\sin^2\phi + b^2\cos^2\phi)\mathrm{d}\phi^2 + \frac{b^2-a^2}{4}\sin 2\theta\sin 2\phi(\mathrm{d}\theta\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}\theta) \\ &+ [\sin^2(\theta)(a^2\cos^2\phi + b^2\sin^2\phi) + c^2\cos^2\theta]\mathrm{d}\theta^2. \end{aligned} \tag{15}$$

Verify that by setting $a = b = c$, you recover the metric for a 2D sphere.

(5) Suppose $\mathbf{A}$ and $\mathbf{B}$ are two rank-$(1,0)$ vectors, and $\bar{\mathbf{g}}$ be a metric tensor. If $\mathbf{A}$ and $\mathbf{B}$ are parallel transported along any curve $\mathcal{C}$, then show that their norm $\bar{\mathbf{g}}(\mathbf{A},\mathbf{B}) = g_{\mu\nu}A^\mu B^\nu$ is conserved (i.e. constant) along the curve $\mathcal{C}$. And hence argue that if a geodesic is spacelike/timelike/null, then it is spacelike/timelike/null everywhere.

(6) The metric for a 3D spacetime with coordinates (t, θ, ϕ) is given by

$$ds^2 = -\mathrm{d}t^2 + R(t)^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{16}$$

in the coordinate basis where $R(t) > 0$.
(i) Compute the Christoffel symbols for this metric.
(ii) How many independent components of the Riemann Tensor are there? Compute the Riemann Tensor for this metric.
(iii) Compute Ricci Tensor and the Ricci Scalar for this metric (hint: use the symmetries of the Riemann Tensor to simpify your work).

(7) Consider the following 2D metric

$$ds^2 = \bar{\mathbf{g}} = -r^2\mathrm{d}t^2 + \mathrm{d}r^2, \tag{17}$$

where (t, r) are coordinates and we have chosen the coordinate basis to represent the metric.
(i) Calculate the Christoffel symbols for this metric.

(ii) Calculate all the components of the Riemann Tensor for this metric. (Again, don't forget to use the symmetries of the Riemann Tensor to simplify your work.)
(iii) Find the coordinate transforms $T(r,t)$ and $X(r,t)$ such that Eq. (17) is reduced to the metric for 2D Minkowski space

$$ds^2 = -\mathrm{d}T^2 + \mathrm{d}X^2. \tag{18}$$

(8) Consider the 4D *weak field metric* in Cartesian coordinates and basis

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(1+\Phi(\mathbf{x}))\mathrm{d}t^2 + (1-\Phi(\mathbf{x}))(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{19}$$

where $|\Phi| \ll 1$, and μ, ν runs from 0 to 3.
(i) By writing the metric as

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(1+\Phi(\mathbf{x}))\mathrm{d}t^2 + (1-\Phi(\mathbf{x}))\gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j \tag{20}$$

where $\gamma_{ij} = diag(1,1,1)$ in its components and i,j runs from 1 to 3. Show that the inverse metric $g^{\mu\nu}$ has components

$$g^{\mu\nu} = \begin{pmatrix} -(1-\Phi(\mathbf{x})) & \\ & (1+\Phi(\mathbf{x}))\gamma^{ij} \end{pmatrix} \tag{21}$$

where $\gamma^{ij} \equiv (\gamma_{ij})^{-1}$ is the inverse of the such that $\gamma_{ik}\gamma^{kj} = \delta^i_j$.
(ii) Calculate all the components of the Christoffel symbols of Eq. (19) to first order in Φ. (You will find that it is much easier to work in terms of 0 and i when computing each individual components.)
(iii) Hence, calculate the Riemann Tensor to first order in Φ.