General Relativity Homework 2

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(1) A metric $\bar{\mathbf{g}}$ is a rank-$(0,2)$ tensor which is symmetric and invertible. Which of the following is a metric tensor? State the reasons for your conclusion if it is not a metric tensor. (Reminder : tensor product $\otimes$ is not commutative.)
(i) $\bar{\mathbf{g}} = -\mathrm{d}t^2 + \mathrm{d}x^2 + xy\mathrm{d}x \otimes \mathrm{d}y + y^2\mathrm{d}y^2$
(ii) $\bar{\mathbf{g}} = -\mathrm{d}t^2 + e^{2t}(\mathrm{d}x^2 + \mathrm{d}x \otimes \mathrm{d}t + \mathrm{d}y \otimes \mathrm{d}x)$
(iii) $\bar{\mathbf{g}} = 2\mathrm{d}x^2 + 2\mathrm{d}y^2 + \mathrm{d}z^2 + \sqrt{2}(\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x) + (\mathrm{d}y \otimes \mathrm{d}z + \mathrm{d}z \otimes \mathrm{d}y)$
(iv) $\bar{\mathbf{g}} = -\mathrm{d}x^3 + \mathrm{d}y^3 + \mathrm{d}x \otimes (\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x)$

(2) Consider a geodesic parameterized by λ, $x^\mu(\lambda)$. The tangent vector at any point on this curve is given by $\mathbf{V} = V^\mu \partial_\mu$. Show that $V^\mu \nabla_\mu V^\nu = 0$ is the geodesic equation.

(3) (**Affine transformation.**) In class, we derived the geodesic equation for the Levi-Civita connection to be

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0. \tag{1}$$

(i) Show that the geodesic equation is invariant under the reparameterization

$$\tau' = a\tau + b \tag{2}$$

where $a \neq 0$ and b are real constants. Such a paramaterization is called an *affine transform* and τ is called an *affine parameter*.
(ii) Consider the non-affine transformation

$$\tau' = \tau^3. \tag{3}$$

Show that under this reparameterization, the geodesic equation is not invariant and has the form

$$\frac{d^2x^\mu}{d\tau'^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau'}\frac{dx^\sigma}{d\tau'} = f(\tau')\frac{dx^\nu}{d\tau'}. \tag{4}$$

Determine $f(\tau')$. Suppose that τ is the proper time which parameterizes the geodesic of a free-falling particle. Argue that under the reparameterization $\tau' = \tau^3$, if one consider τ' to be its proper time, then the geodesic equation describes the motion of a *non*-free-falling particle.

(4) Let x^μ and $x^{\mu'}$ be two overlapping coordinate systems on a two dimensional manifold $\mathcal{M}$, which are related by

$$x = e^{x'}\ ,\ y = x' + y' \tag{5}$$

with domains $x > 0$, $y > 0$, $x' > 0$ and $y' > 0$.
(i) Show that the coordinate transform is *smooth* and *analytic* in its domain.
(ii) State the rank of the following tensors and express them in terms of the primed basis. Write your answer in both abstract and component form.
- (a) $\mathbf{K} = \partial_x$
- (b) $\bar{\mathbf{L}} = \mathrm{d}x^2 + x\mathrm{d}x \otimes \mathrm{d}y + y\mathrm{d}y^2$
- (c) $\mathbf{M} = \mathrm{d}x \otimes \partial_x + \mathrm{d}x \otimes \partial_y + \mathrm{d}y \otimes \partial_y$
- (d) $\mathbf{N} = x^2\partial_x \otimes \partial_y + \partial_y \otimes \partial_y$

(iii) Using the tensors defined in (ii), calculate (as functions of x, y)

(a) $\mathbf{K} \otimes \mathbf{K}$
(b) $\bar{\mathbf{L}}(\mathbf{K}, \mathbf{K})$
(c) $\bar{\mathbf{M}}(\mathbf{K})$
(d) $\bar{\mathbf{L}}(\mathbf{N})$ (Contract the first index of $\mathbf{L}$ with the first index of $\mathbf{N}$.

(5) The metric of 3D Euclidean space in Cartesian coordinates (x, y, z) is given by

$$ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{6}$$

Consider the spherical coordinates (r, θ, ϕ), which is related to the Cartesian coordinates by

$$r = \sqrt{x^2 + y^2 + z^2}\ ,\ \ x = r \sin\theta \cos\phi\ ,\ \ y = r \sin\theta \sin\phi. \tag{7}$$

(i) The domain for x, y, z is $(-\infty, \infty)$. What is the (co)-domain for (r, θ, ϕ)?
(ii) Show that the metric in the spherical symmetric coordinate basis is given by

$$ds^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2). \tag{8}$$

(iii) The Christoffel symbols ($\Gamma^{\lambda}_{\mu\nu}$ in the Levi-Civita connection) is identically zero in the Cartesian basis. Calculate the Christoffel symbols in the spherical coordinate basis.
(iv) Consider the following curve in the Cartesian basis, parameterized by s

$$x(s) = a \sin s\ ,\ \ y(s) = b \cos s\ ,\ \ z(s) = c \tag{9}$$

where $a > 0$, $b > 0$, and c are real constants, and $0 \leq s < 2\pi$.

(a) Calculate the tangent vector, $V^\mu = dx^\mu/ds$, to this curve in the Cartesian basis.

(b) Find the parameterized curve in the spherical coordinate systems, i.e. find $r(s)$, $\theta(s)$ and $\phi(s)$. And hence calculate the tangent vector to this curve in spherical basis.

(c) Calculate the proper length (or distance) of the curve in the special case of $a = b$ and show that it is $2\pi a$.

(6) We argue in class that a vector $\mathbf{V}$ is a map from a function to another function, i.e.

$$\mathbf{V} : f \to h \tag{10}$$

where f and g are functions. This vector action obeys Leibnitz's rule

$$\mathbf{V}(fg) = f\mathbf{V}(g) + \mathbf{V}(f)g \tag{11}$$

where f and g are functions. Suppose $\mathbf{A}$ and $\mathbf{B}$ are two vectors. The composition action of these two vectors on some function f is given by $\mathbf{B}(\mathbf{A}(f))$.
(i) Prove that the composition action of $\mathbf{B}(\mathbf{A}(g))$ does not obey Leibnitz's Rule, and argue that the composition action of two vector is *not* a vector.
(ii) Show that the action

$$\mathbf{B}(\mathbf{A}(g)) - \mathbf{A}(\mathbf{B}(g)) = [\mathbf{B}, \mathbf{A}](g) \tag{12}$$

obeys Leibnitz's rule and hence defines a vector.

(7) The metric for a 3+1D spacetime is given by

$$ds^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{13}$$

where $a(t) > 0$ is a positive definite function of the coordinate t. This is the spatially flat Robertson-Walker (FR) metric.

(i) By finding the appropriate coordinate transforms, show that the metric signature for the RW metric is *Lorentzian.*
(ii) Consider a vector P^μ in this basis with components given by

$$P^\mu = (E(t), a^{-1}p^i) \tag{14}$$

where $i = 1, 2, 3$ denotes the components associated with the (x, y, z) basis, and $E(t) > 0$ is a function of t. Calculate the norm of this vector using the FR metric. What are the conditions for $E(t)$ such that P^μ is timelike, null or spacelike?
(iii) Consider another vector Q^μ given by

$$Q^\mu = (E(t), a^{-1}q^i). \tag{15}$$

What are the conditions for q^i and p^i such that P^μ and Q^μ is orthogonal?

(8) (**Relationship between differentiation and co-vectors**). Let (x, y) be coordinates on a 2-D manifold $\mathcal{M}$. Consider the following *differential equation*

$$\frac{dy}{dx} = 2x. \tag{16}$$

The solution to this equation is simply

$$y = x^2 + c \tag{17}$$

where c is an integration constant. Hence the solution to the differential equation is a *curve* through the manifold.
(i) Show that this curve can be parameterized by s via

$$x(s) = s\ ,\ y(s) = s^2 + c, \tag{18}$$

and hence show that the components of the tangent vector $\mathbf{V}$ to the curve in (i) is given by

$$V^\mu = (1, 2x). \tag{19}$$

(ii) In calculus, one solves the equation above by "doing algebra" on the differentials as

$$dy = 2xdx, \tag{20}$$

and then integrating both sides. One might be tempted to then write the following as a "tensorial" relation (note the non-italicized "d")

$$\mathrm{d}y - 2x\mathrm{d}x \stackrel{?}{=} 0. \tag{21}$$

Argue that this equation cannot be correct.
(iii) On other hand, if we define a new co-vector $\bar{\mathbf{W}}$ by

$$\mathrm{d}y - 2x\mathrm{d}x \equiv \bar{\mathbf{W}}, \tag{22}$$

show that

$$\bar{\mathbf{W}}(\mathbf{V}) = 0. \tag{23}$$

This is the meaning of the integral equation Eq. (20) – it is a differential equation *satisfied* by the solution $y = 2x + c$ which defines a curve on $\mathcal{M}$. In the language of differential geometry, the co-vector defined by this differential maps the tangent vector of this curve to zero.

(9) In class, we argue that the covariant derivative along a the basis vector $\mathbf{B} = \hat{\mathbf{e}}_{(\beta)}$ of a co-vector basis $\bar{\mathbf{A}} = \hat{\mathbf{e}}^{(\alpha)}$ is given by

$$\nabla_{\mathbf{B}}\bar{\mathbf{A}} = \nabla_{\hat{\mathbf{e}}_{(\beta)}}\hat{\mathbf{e}}^{(\alpha)} \equiv \sum_\mu \tilde{\Gamma}^\alpha_{\beta\mu}\hat{\mathbf{e}}^{(\mu)}. \tag{24}$$

(i) Derive the covariant derivative of any co-vector $\bar{\mathbf{W}}$ along the vector $\mathbf{V}$ in component form

$$\nabla_\nu W_\mu = \partial_\nu W_\mu + \tilde{\Gamma}^\sigma_{\nu\mu} W_\sigma. \tag{25}$$

(ii) As we discussed in class, the covariant derivative of a vector $\mathbf{T}$ along $\mathbf{V}$ is given in component form as

$$\nabla_\nu T^\mu = \partial_\nu T^\mu + \Gamma^\mu_{\nu\sigma} T^\sigma. \tag{26}$$

By writing the scalar f as the contraction of a vector and a co-vector, i.e. $f = T^\mu W_\mu$, prove that the requirement $\partial_\mu f = \nabla_\mu f$ for any scalar f requires that

$$\tilde{\Gamma}^\sigma_{\nu\mu} = -\Gamma^\sigma_{\nu\mu}. \tag{27}$$

(10) (**Torsion-free condition**). Let $(\nabla_\mu, \Gamma^\lambda_{\mu\nu})$ be a connection. The torsion is defined to be

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu}. \tag{28}$$

Suppose f is a scalar function. Show that imposing

$$\nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f = 0 \tag{29}$$

means that the torsion vanishes identically, and hence show by computing the torsion in two different coordinate bases that it is a rank-$(1, 2)$ tensor.