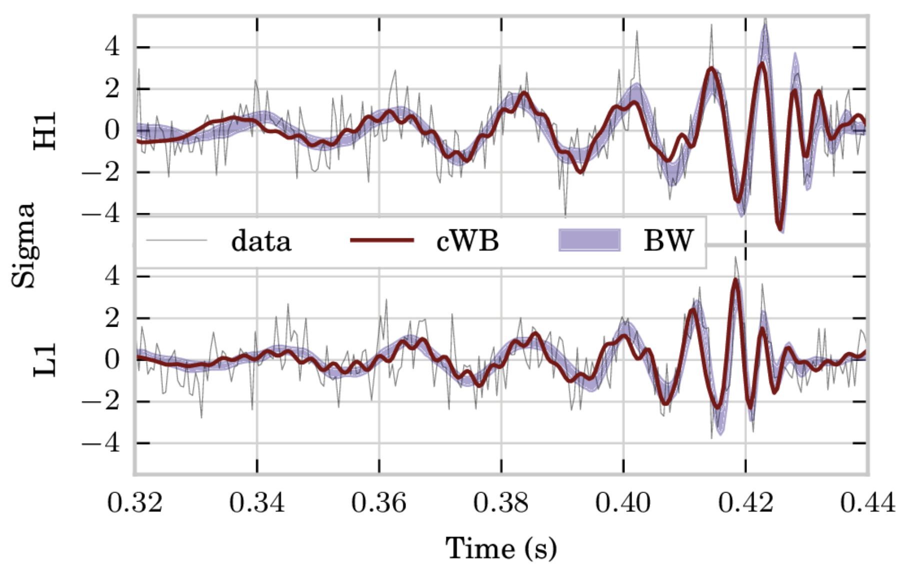
6CCP3630 General Relativity

Lecturer: Dr. Eugene A. Lim
2017-2018 Year 3 Semester 2 Theoretical Physics
Office : S7.09
King's College London
Department of Physics
August 10, 2024



Acknowledgments

First and foremost, I would like to thank my former PhD advisor Sean Carroll who, years ago, took a hard look at this completely unprepared ex-mechanical engineer and decided that I can be taught General Relativity while simultaneously trying to modify it. Sean's influence, through his teachings, his lecture notes and his GR book, permeates throughout the these notes. I am also very lucky to have known mentors, colleagues, students and friends with whom I have learned, and still learning, a lot of GR from through the years. I thought about listing your names down, but then that would sound like an awkward award acceptance speech that goes on for too long and who would want *that*? I would like to thank all the people who have written great books and lecture notes, whose material and ideas I have freely stolen for these lecture notes. Finally, I would like to thank Sophie Sampson for proof-reading these lecture notes, and also for putting up with me being obsessed about them for the past few months.

Figure on the previous page shows that strain signal from the gravitational wave event GW150914 from the two LIGO detectors at Hanford (H1) and Louisiana (L1). The source origin is the merger of a 36 solar mass black hole and a 29 solar mass black hole at the distance of 410 Megaparsec away. The merger resulted in a 62 solar mass black hole with 3 solar masses of energy radiated as gravitational waves. The signal was detected on 14 September 2015 by the LIGO observatory.

Contents

What did you just sign up for?

Why study GR in your undergraduate program? For those who are interested in pursuing a career in theoretical physics, GR is one of the cornerstones of our understanding of the universe, so that's clear. Even if you are not interested in a career in physics, GR is one of those beautiful theories that is entirely mathematically self-consistent, with nothing hidden under rugs. When I was a student, GR was often considered an "advanced" course, suitable to be taught only to post-graduate and PhD students. However, in recent years, it is increasingly taught as part of the undergraduate syllabus. Indeed, you can probably pick up a book and teach it to yourself – a friend of mine once said that "GR teaches itself" – and put people like me out of a job. With the increasing sophistication required of our undergraduates when they apply to postgraduate programs, this is a natural development.

The natural mathematical home for GR is differential geometry, which is probably something quite new to most 3rd year physics undergraduates. Many undergraduate treatments of General Relativity – worrying about the lack of mathematical nous of the students and in order to get to the sexy gravitational physics stuff quickly – usually skip over it, relying on instead a set of operational rules that teach you how to do "index algebra" (i.e. raising and lowering of tensor indices) and your intuition about curved spaces. While there is nothing wrong with this, advanced treatments of GR almost invariably start with a primer on differential geometry[1] – the full power and elegance of GR can only be appreciated when seen as a "geometric" theory. But this is not mere high-browing of a subject – differential geometry itself is a beautiful subject, and worthwhile of study on its own.

Thus in this module, I have decided to teach the course with a large dose of differential geometry. We will spend quite a bit of time learning it, albeit a slightly baby-sized version of it. Once you have learned it, you will never view the innocuous partial derivative as simply yet another boring old thing, but something deeply profound. While this means that we will get to General Relativity a bit slower than usual, when we eventually get there, you will find that the hard work you put into learning the true language of GR will be worth it. Also, it will prevent the formation of bad "mindless index manipulation" habits that permeate so many students looking to do PhDs in Physics – habits that take years to unlearn[2]. Since much of the "GR" part of the course will be covered again in any advanced GR course you might take in your post-graduate career, the hope is that by building the base better you will have less to unlearn and hence be better prepared. If this is the only GR course you are ever going to take, then I hope that it will prepare you better to read the advanced texts on GR.

So, is it possible to teach a more sophisticated GR course to undergraduates? My experience with teaching the 5CCP2332 Symmetry in Physics course to the 2nd year students at KCL has given me confidence that the students are more than capable of handling this course. On the other hand, if this whole thing blows up in my face, I hope you will forgive me!

How to use the notes

This is a set of lecture notes for the 3rd year undergraduate General Relativity module, a 10 week and 40 contact hours (including tutorials) physics course at King's College London. The lecture notes are written to accompany the actual lectures themselves, so these notes are not exhaustive and should not replace the suggested proper references (listed next page). The best way to use these notes is to read through them in the order presented, and work through the steps. While I try to be as self-contained as possible, it is impossible to go through the subject without taking some liberties, so if you find things

[1] Steven Weinberg's *General Relativity and Cosmology*, 1972 Wiley and Sons, is a rare exception where GR is taught from a non-Geometric point of view.

[2] Beyond GR, differential geometry is also the natural language for Thermodynamics, Classical Mechanics, Statistics, Electrodynamics and String Theory, which makes it a scandal that it is not taught in undergraduate courses.

that are not as well explained as you would have liked, ask me or read some of the references provided.

I will be teaching the course in the order the lectures are presented. But I suspect we will not have sufficient time to cover all the topics. So sometimes I will ask you to read sections that I don't have time to cover – so do try to keep up with your reading. There are additional material in these lecture notes which are non-examinable, and is marked with two asterisks **. Material marked with a single asterisk * is examinable, but is review material which will not be lectured on in class – so read ahead!

There are 5 homework sets to help you along, and the notes often refer you to these problems. The best way is to work through the problems when asked by the notes. GR, like most physics subjects, rewards those who practice, practice and practice. Fortunately, there are plenty of examples and problems that you can easily find on the internet. Unfortunately, that means that any problem I can come up with probably has a solution somewhere posted online. So, while you are encouraged to use the internet to help you study, try to resist checking the answers!

Finally, a word on notation. As you will soon see, there is going to be a lot of new notation introduced, and they can be quite daunting at first, but it is really quite intuitive once you get used to it. A word of warning when looking through other references (and the internet) though – everybody seems to have their own favorite notation, especially when it comes to representing abstract tensors and component tensors (if you don't know what these are, don't worry!), so it is easy to get confused. My advice is to pick a reference book or two, get used to their notation, and stick to them (and these lecture notes) and don't google too much on the internet.

Recommended Books

* is highly recommended for this course.
Online resources will be posted on the class webpage.

- B. Schutz, **A First Course in General Relativity, 2nd ed.*, Cambridge University Press 2009. This is an excellent book, one of the first books on GR aimed at undergraduates and still the best in my opinion!

- B. Schutz, **Geometrical Methods of Mathematical Physics*, Cambridge University Press 1985. Schutz's book on differential geometry is the best book on differential geometry aimed at a physicist. It has the right amount of rigour, and great explanations that is a hallmark of Schutz's books. Best of all – it spends very little time on GR, so it provides a nice counter balance to any GR book trying to teach differential geometry. Aimed at advanced undergraduates and beginning PhDs.

- B. Wald, *General Relativity*, The University of Chicago Press 1984. Aimed at PhDs and researchers, Wald's GR book has been around for 3+ decades but it still one of the best. Wald's mathematical approach can be quite dense sometimes, but it is accurate, precise and complete. The Wald Infallibility Principle : When in doubt, go Wald.

- Charles Misner, Kip Thorne and John Wheeler, *Gravitation*, Freeman 1973. The first modern book on GR, and what a tome. It tries to be super complete and goes a long way in achieving that goal. Personally, I don't use it as much though that's probably because I was not taught using it. It's a massive book, and good not just for GR, but also to kill pesky cockcroaches that infest dirty PhD offices.

- S. Carroll, **Spacetime and Geometry*, Addison-Wesley 2004. This is a relatively new addition to the many graduate level GR books, but has since gained a loyal following and is now one of the most popular books for instructors to teach GR from. It strikes a nice balance between the intuitive physics explanation of Schulz with the mathematical rigour of Wald. It is aimed at a beginning PhD student, but a motivated undergraduate will enjoy this.

- S. Weinberg, *Gravitation and Cosmology*, John Wiley 1972. This book by God, I mean, Weinberg, is the one book that bucks the trend of tying GR with differential geometry. Written by one of the most remarkable physicists of our generation who is *the* authority on field theory, this book teaches GR from a physically intuitive viewpoint with a generous dose of field theory. This may not be the one GR book you learn GR from, but it is full of deep insight into the theory that only God, I mean Weinberg, can provide.

- J. Hartle, **An Introduction to Einstein's General Relativity*, Pearson 2003. This book is aimed at the undergraduate level, approaching GR in a highly physically intuitive way. It deliberately avoids a lot of fancy mathematics. An excellent reference book for this course.

- A. Lightman, W. Press, R. Price, S. Teukolsky, **Problem Book in General Relativity*, Princeton University Press, 1975. This is a collection of GR problems with full solutions, causing much grief to instructors desperately trying to find new problems. Anyhow, it's a great resource, and you are encouraged to try some of the problems. In fact, it is now available online for free at http://apps.nrbook.com/relativity/index.html.

Chapter 1

Introduction
From Newton to Einstein

A high-brow is someone who looks at a sausage and thinks of Picasso.

A.P. Herbert

1.1 The Newtonian Worldview

I've got bad news. For many years in your life, you have been told several lies. The first lie is that a particle of mass m and position $\mathbf{x}$, under a force $\mathbf{F}$ moves under the Newton's Second Law of Motion

$$\mathbf{F} = m\frac{d^2\mathbf{x}}{dt^2} \tag{1.1}$$

where t is some absolute time that everyone in the entire Universe can agree upon. The second lie is that two particles of masses m and M will feel an attractive force towards each other whose amplitude is given by Newton's Law of Gravitation

$$F = \frac{GMm}{r^2} \tag{1.2}$$

where r is the distance between these two masses, and $G = 6.673 \times 10^{-11}\mathrm{Nm^2kg^{-2}s^{-2}}$ is Newton's constant. Equivalently, one can express the gravitational *potential* Φ of any mass m at point $\mathbf{x_0}$ as

$$\Phi(\mathbf{x}) = -\frac{Gm}{|\mathbf{x}-\mathbf{x_0}|}, \tag{1.3}$$

and then the gravitational acceleration $\mathbf{g}$ is simply the gradient of the potential

$$\mathbf{g} = -\nabla\Phi. \tag{1.4}$$

We usually call the potential "the gravitational field", whose gradient is the acceleration, although as we will see later it also makes sense to think of the acceleration as a "field". Equivalently, one can derive the potential Φ of any mass distribution specified by the density $\rho(\mathbf{x})$ via the **Poisson Equation**

$$\nabla^2\Phi = 4\pi G\rho(\mathbf{x}). \tag{1.5}$$

The Poisson Equation is exactly equivalent to Eq. (1.2).

You, and most people, bought these lies. The reason is that, like all successful lies, these lies are very good lies. These equations, combined with Newton's first and third laws of motion, allow us to compute the dynamics of the very small such as the curve of a football being struck by Roberto Carlos to the very large such as the motion of the planets around the Sun. We use these equations to accurately compute the trajectory our spacecraft must take to reach Pluto more than 4 billion km away and model the aerodynamics of an airplane. It unifies seemingly standalone concepts such as Galileo's Theory of Inertia and Kepler's Law of Planetary Motion, and asserts that the same laws apply to physics at *all scales* from the atomic scale to the galactic scale. By all measures, they are incredibly successful lies.

Newton's Laws are not just a bunch of equations where you plug in some initial conditions (usually a particle's initial position and velocity) to compute the result (either some future/past position and velocity), they actively promote the idea that space and time are separate entities, to be treated differently. In the Newtonian view of the Universe, we live in a 3-dimensional world. This 3D world then dynamically evolves, with the evolution govern by some quantity we call "time". Every being in this Universe, must agree on this time. Furthermore, all forces are instantaneous across infinite distances. The Laws, and their underlying ideas about how space and time are subdivided into their separate domains, are usually known collectively as **Newtonian Mechanics**.

Nevertheless, despite their successes, there are clues to why they are not quite the right picture of the Universe. These clues can be divided into two kinds : observational and theoretical. The (more obvious and commonly stated) observational clue is that the perihelion (the point of minimum distance from the Sun) of Mercury is precessing, i.e. the perihelion is changing by 43 arcseconds per century. While Newton's laws correctly predicted Mercury's orbital period and the fact that the orbit is an ellipse, it does not predict this precession. This precession was discovered by Urbain Le Verrier in 1859 (where he looked at data from 1697 to 1848), and recognized by him to be a problem for Newtonian mechanics. A totally Newtonian Mechanical solution to this problem is to postulate the existence of a yet unseen planet (named Vulcan) in order to provide the necessary gravitational tug to generate this effect – this is the Newtonian version of the Dark Matter problem. This ultimately failed hypothesis is not without merit: non-conformance of Uranus' orbit to Newton's Laws led to the prediction and subsequent discovery of Neptune in 1846.

Less well known and more subtly, the Newtonian picture of the Universe has theoretical issues as well. **Maxwell** had discovered his eponymous equations by 1861. These equations unify the theories of electric and magnetic fields, and predict that these fields propagate as *waves* at a finite speed of light. These equations are

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial_t \mathbf{E}, \tag{1.6}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{1.7}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0}, \tag{1.8}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{1.9}$$

Here $\epsilon_0 = 8.854 \times 10^{-12}\mathrm{kg}^{-1}\mathrm{m}^{-3}\mathrm{s}^4\mathrm{A}^2$ is the permittivity of free space and $\mu_0 = 4\pi \times 10^{-7}\mathrm{kgms}^{-2}\mathrm{A}^{-2}$ is the permeability of free space, and that we have used ρ_c for charge density. Note that we have used the notation $\partial_t \equiv \partial/\partial t$ – if you are not familiar with this you should probably start to get used to it as you will see it will save you a lot of writing in the future.

In **electrostatics** (i.e. the study of electromagnetism when all time derivatives are zero), Coulomb's law (discovered in 1781) states that the magnitude of the force between two particles of electric charges q_1 and q_2 separate by distance r, is given by

$$F = \frac{kq_1q_2}{r^2}, \tag{1.10}$$

where $k = 8.99 \times 10^{10}$ Nm^{2}C^{-2} is the Coulomb's constant. This force is attractive if the charges are of opposite sign, and repulsive otherwise. In analogy to the gravitational potential, we can also write the Coulomb's potential of a particle of charge q at $\mathbf{x}_0$

$$V(\mathbf{x}) = -\frac{kq}{|\mathbf{x} - \mathbf{x}_0|}. \tag{1.11}$$

The gradient of the potential in electrostatics is then simply the electric field $\mathbf{E}$ that you know and love

$$\mathbf{E} = -\nabla V(\mathbf{x}). \tag{1.12}$$

Comparing the above equation to Eq. (1.4), and Eq. (1.3) to Eq. (1.11) (the minus sign difference is due to the convention we chose for our potential V). Similarly, taking the gradient of the electric field[1] gives us one of Maxwell equations Eq. (1.8), which is the electromagnetic version of the Poisson Equation Eq. (1.5). The electromagnetic analog of Newton's 2nd law of motion is the Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.13}$$

where $\mathbf{v}$ is the velocity of the charge. This law was discovered experimentally, but have you ever wondered why this is so – in particular why is the force proportional to the electric field but proportional to the cross product of the particle *velocity* with the magnetic field $\mathbf{B}$? We will answer this question soon enough, but let's plow on.

Despite all these tantalizing analogies, the point of departure of Newton's gravity and electromagnetism is the presence of *waves*. The reason is that Maxwell equations go beyond electrostatics, but describe **electrodynamics**, i.e. where charges and fields evolve with time[2]. In the absence of any charges, Maxwell's equations Eq. (1.6) to Eq. (1.9) can be recast as a pair of *wave equations*

$$\partial_t^2 \mathbf{E} - \frac{1}{\epsilon_0 \mu_0} \nabla^2 \mathbf{E} = 0 \ , \ \partial_t^2 \mathbf{B} - \frac{1}{\epsilon_0 \mu_0} \nabla^2 \mathbf{B} = 0, \tag{1.14}$$

with solutions

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x} + kct)} \ , \ \mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x} + kct)} \tag{1.15}$$

where $\mathbf{k}$ is the wavevector and $k = |\mathbf{k}|$, which is simply waves propagating at velocity $c \equiv \sqrt{1/(\epsilon_0 \mu_0)} = 3 \times 10^{10}$cm s^{-1}, i.e. we (or more accurately, Maxwell) have discovered that electromagnetic waves travel at light speed. The question is then: why isn't there an equivalent "gravitational field" (or acceleration field) waves predicted in the Newton's Law of gravity too? The short answer is that Newton's Law of gravity Eq. (1.2) *has no time derivatives* in it, while the Coulomb Force Eq. (1.10) is supplemented by the other equations of Maxwell. This is our first purely theoretical hint that perhaps Newton's gravity is not the complete theory of gravity – maybe there exist similar "dynamical" equations for gravity too?

Now you might argue that gravity is fundamentally different from electromagnetism, so even though there are analogies between the two forces, Eq. (1.2) is really all there is to it. However, wave solutions also imply something important: since they propagate at some finite speed (namely c), their effects take time to propagate from one point to another. If the Sun suddenly goes out, it will take eight minutes or so for us to find out about it. Newton's law simply states that the effects of gravity propagate at infinite speed – along with its difficult philosophical consequences.

1.2 From Space and Time to Spacetime

In 1905, Einstein formulated his **Theory of Special Relativity**. You have studied special relativity in your previous modules, so this won't be new to you. However, let's review it again, but now in a

[1] One can also take the gradient of the magnetic field to give the equation $\nabla \cdot \mathbf{B} = 0$, where the magnetic field is not sourced – a consequence of the fact that we have no magnetic monopoles in nature.

[2] For example, Eq. (1.12) is really $\mathbf{E} = -\nabla \cdot V(\mathbf{x}) - \partial_t \mathbf{A}$ – we will discuss this in further detail in a moment.

deliberately high-brow language that you might not be familiar with. In fact, let's start with Newton's equations in high brow language. What follows is a completely unhistorical description – instead we'll focus on the modern view of things.

1.2.1 Newton's 2nd Law of Motion : Galilean Transformations, Space and Time

We begin by considering physics in 3 space and 1 time dimensions (usually shortened to "3+1D"). We want to define some **frame of reference** so that we can describe some physical situations, e.g. say "this particle is moving at velocity v *with respect to* X" where X can be some other particle, or more usually, a some frame of reference. The usual way to do this is to lay down some coordinates on this frame so that we can do some calculations on it. You can be quite creative with labeling your coordinates, but we want to impose additional restrictions: we want the coordinates to label the space *homogenously in space and time* and *isotropically*. What this means is that your coordinate system should look "the same" everywhere, and has no special directions. Such a coordinate frame is called an **inertial frame**. A simple choice of coordinates which satisfies these considerations are the usual **Cartesian coordinates** (t, x, y, z), where t labels time and (x, y, z) the 3 spatial coordinates. On the other hand, the spherical coordinate system (t, r, θ, ϕ) has a special point (the origin), so is *n*ot an inertial frame. Usually, non-inertial frames usually lead to "fictitious" forces such as the centripetal force or the Coriolis force.

Given such coordinates, we can "do physics", i.e. solve equations using it. For example, you can solve Newton's equations $d^2\mathbf{x}/dt^2 = \mathbf{F}/m$. For constant $\mathbf{F}$ and m, the solution, as you have done countless time, is

$$\mathbf{x} = \frac{\mathbf{F}}{2m}t^2 + \mathbf{v}t + \mathbf{x}_0 \tag{1.16}$$

where $\mathbf{v} = \dot{\mathbf{x}}(0)$ is the initial velocity at time $t = 0$ and $\mathbf{x}_0 = \mathbf{x}(0)$ is the position at time $t = 0$.

Suppose now I give you another inertial frame with a new coordinate system, say (t', x', y', z') which is related to (t, x, y, z) in some undefined way (for now). The question is: can you describe the same physics using this new coordinate system? Let's do a specific example, suppose the two coordinate systems are related by

$$t' = t\ ,\ \mathbf{x}' = \mathbf{x} - \mathbf{u}t. \tag{1.17}$$

In words, the primed frame is moving by some constant velocity $\mathbf{u}$ with respect to the unprimed frame. Such a coordinate transform is called a **boost**. It is easy to describe the physical quantities of velocity and acceleration in the new frame in terms of the quantities of the old frame, viz

$$\mathbf{v}' = \frac{d\mathbf{x}'}{dt'} = \frac{\partial t}{\partial t'}\frac{d(\mathbf{x} - \mathbf{u}t)}{dt} = \mathbf{v} - \mathbf{u} \tag{1.18}$$

and

$$\mathbf{a}' = \frac{d\mathbf{v}'}{dt'} = \frac{\partial t}{\partial t'}\frac{d(\mathbf{v} - \mathbf{u})}{dt} = \mathbf{a}. \tag{1.19}$$

Thus the velocity in the primed frame is shifted by the constant $\mathbf{u}$ while the acceleration remains the same – by the "same" we mean that the *value* remains the same in both frames. This is all very familiar.

In fact, from Eq. (1.19), we can see immediately that Newton's equation of motion Eq. (1.1) under the transformation Eq. (1.17), returns to its exact same functional form

$$\mathbf{F} = m\frac{d^2\mathbf{x}}{dt^2} \rightarrow m\frac{d^2\mathbf{x}'}{dt'^2}. \tag{1.20}$$

In high-brow language, we say that "Newton's 2nd Law of motion remains **invariant**[3] under the transformation Eq. (1.17)." This means that, it doesn't matter which of the inertial frames (primed or unprimed)

[3] For those who have taken the 5CCP2332 Symmetry in Physics module, this would be very familiar to you.

you are on, you can happily use Newton's 2nd Law to compute motion of particles and you will get the correct answer relative to your frame.

In addition to Eq. (1.17), what are the set of all possible **linear**[4] coordinate transformations which leave Newton's 2nd Law Eq. (1.1) invariant? We can shift the coordinates by constants, or a **translation**,

$$\mathbf{x}' = \mathbf{x} + \mathbf{x}_0 \ , \ t' = t + s, \tag{1.22}$$

it is easy to check that since

$$\frac{d}{dt} = \frac{\partial t'}{\partial t}\frac{d}{dt'} \ , \ d\mathbf{x}' = d\mathbf{x} \tag{1.23}$$

Newton's 2nd Law Eq. (1.1) remains invariant.

The other set of possible coordinate transformations are **Rotations**

$$\mathbf{x}' = \mathbf{R}\mathbf{x} \tag{1.24}$$

where $\mathbf{R}$ is some 3×3 square matrix called a **rotation matrix**. For example, a rotation of angle θ around the z axis would yield a rotation matrix

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.25}$$

You can imagine some more complicated rotations. Under rotation, Newton's 2nd law transforms as

$$\mathbf{F}' = m\frac{d^2(\mathbf{R}\mathbf{x})}{dt^2} \tag{1.26}$$

so as long as we rotate the force $\mathbf{F}' = \mathbf{R}\mathbf{F}$, we recover the Newton's 2nd Law.

It turns out these are *all* the possible continuous[5] coordinate tranformations that leave Newton's 2nd Law Eq. (1.1) invariant. These operations, the boosts Eq. (1.17), the translations Eq. (1.22), and the rotations Eq. (1.24) are known collectively as **Galilean Transformations**[6]. In this high-brow language, you can transform from one inertial frame to another by applying such Galilean transformations.

All this high-brow language (we did warn you) is saying something you are already familiar with when you do Newtonian Mechanics: *relative velocities* are calculated by adding or subtracting $\mathbf{u}$ (boosts), *relative positions* are calculated by adding or subtracting $\mathbf{x}_0$ (translations), and *relative angles* are calculated by rotating the system (rotations). More subtly, notice that we have somewhat relaxed the Newtonian notion of absolute space: inertial frames can be related to each other by boosts. But since the time coordinate is only translated $t' = t + s$, and never "boosted" (e.g. there is no $t' = t + \mathbf{u}\cdot\mathbf{x}$), hence it does not "mix" with the space coordinates. This means that while absolute space is no longer true, absolute time still holds supreme in the modern picture of Newtonian Mechanics. Can we relax this condition too?

[4] *Linear* means that the transformation can be written as a matrix equation, i.e.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{1.21}$$

where a_{ij} and b_i are in general complex coefficients, but in the case of coordinate transformations they are reals. So, a transformation such as $x' = 3x^2$ is *not* linear, while $x' = 2x + 10y + 3$ is linear.

[5] **We have sneaked in the qualifying term *continuous* here, because it turns out that there are 4 other transformations that will leave Newton's 2nd Law invariant. They are the 3 mirror reflections $\mathbf{x}' = -\mathbf{x}$ and the time reversal $\mathbf{t}' = -t$. We usually do not consider these *discrete* transforms for technical reasons (ok if you insist, they cannot be generated by infinitisimal generators). **

[6] **If you have taken 5CPP2332 Symmetry in Physics, you will not be surprised then to learn that these operations form a Group, called the **Galilean Group** $Gal(3)$. This is a 10 dimensional Lie Group.**

1.2.2 Special Relativity : Lorentz Transformation and Spacetime

The answer to the above question is of course "Yes". But how? There is an infinite number of possible transformations one can make up to get from the unprimed inertial frame to the primed inertial frame, so which one should we choose? In the previous section 1.2.1, we show that the Galilean Transformations leaves the Newton's 2nd Law of motion invariant, perhaps there are *other* sets of equations which we can use as a guide?

Indeed there is, and we have met the equations already: they are the Maxwell equations Eq. (1.6) to Eq. (1.9). But first, let's ask the question: *are the Maxwell equations invariant under the Galilean Transformation?* The answer is no. An easy way to see that you will run into trouble is to see that under boosts

$$\mathbf{x}' = \mathbf{x} - \mathbf{u}t \ , \ t' = t \tag{1.27}$$

and hence $\mathbf{v}' = \mathbf{v} - \mathbf{u}$ so the Lorentz force law Eq. (1.13) becomes

$$\mathbf{F}' = \mathbf{F} - \mathbf{u} \times \mathbf{B}. \tag{1.28}$$

This would have been OK if under boosts, the electric field also changes as $\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$ i.e. the electric and magnetic fields must "mix" in some way, but unfortunately Maxwell equations do not predict this. You can check this by bruteforce if you like, but here is a more slick argument as follows. From the free field solutions Eq. (1.14), we have shown that the electric and magnetic fields both propagate at light speed in vacuum (or some fixed velocity in some medium). Consider an inertial frame where some $\mathbf{E}$ field is propagating freely at c say in direction x, a boost of u in the x diretion would then imply that what we get is the same set of fields propagating at $c - u$. The magnetic fields behave exactly the same way, so under boosts $\mathbf{E}$ and $\mathbf{B}$ fields do not mix – besides the obvious point that $\mathbf{E}$ fields are experimentally observed to be always be propagating at the speed of light in a vacuum. So, it is clear that Maxwell equations is not invariant under Galilean Transformations.

We are now going to make an assertion: *Maxwell equations are invariant under (i) rotations (ii) space and time translations and (iii) Lorentz boosts.* Space and time translations are identical to their Galilean counterparts

$$\mathbf{x}' = \mathbf{x} + \mathbf{x}_0 \ , \ t' = t + s. \tag{1.29}$$

as are the rotations

$$\mathbf{x}' = \mathbf{R}\mathbf{x}. \tag{1.30}$$

On the hand, a **Lorentz boost**[7], in the x direction with the parameter $\beta < c$, is given by

$$\begin{aligned} t' &= \gamma\left(t - \frac{\beta x}{c^2}\right), \\ x' &= \gamma(x - \beta t), \\ y' &= y, \\ z' &= z \end{aligned} \tag{1.31}$$

where the **Lorentz Factor** is given by

$$\gamma = \left(1 - \frac{\beta^2}{c^2}\right)^{-1/2}. \tag{1.32}$$

Similar boosts for the y and z can constructed by replacing x' with y' and z' in the above equations. In terms of nomenclature, Lorentz boosts Eq. (1.31) and rotations Eq. (1.30) are collectively called **Lorentz**

[7] As opposed to the "Galilean boost" Eq. (1.17) we discussed earlier. In general, physicists call any transformation of the spatial coordinate with a velocity parameter a "boost".

Transformations[8]. Combining Lorentz Transformations with space and time translations gives us the set of transformations called **Poincaré Transformations**, although physicists (including yours truly) are usually careless and sometimes just call the whole thing "Lorentz Transformations". The important point here is that Lorentz boosts *mix* the spatial and time coordinates – and irrevocably unify space and time into **spacetime**.

It can be shown, with a lot of work and algebra, that Maxwell equations as written in Eq. (1.6) to Eq. (1.9) are invariant under Lorentz Transformations (those interested can read Chapter 26 of *The Feynman Lectures in Physics, Vol. II*). Fortunately, it turns out that there is a much more sophisticated way to show it, but it will requires you to learn some new fancy notation and ideas. In fact, we are going to be using this fancy notation throughout the rest of the module, so let's just dive right in.

1.3 The spacetime metric and Four Vectors

Let us now take a step back, and recall that Special Relativity tells us that there exists a quantity called the **spacetime interval** ds^2 given by[9]

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{1.33}$$

At this moment, you can simply (and correctly) think of the d as shorthand for "infinitisimal", i.e. $ds = \Delta s$ etc. – we will get more sophisticated in our understanding soon enough. We can equivalently write Eq. (1.33) in the following form

$$ds^2 = \sum_{\mu,\nu} \eta_{\mu\nu} dx^\mu dx^\nu \tag{1.34}$$

where the Greek indices μ , $\nu = 0, 1, 2, 3$ and the **spacetime metric**

$$\eta_{\mu\nu} = \begin{pmatrix} -c^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.35}$$

where the spaces are zeroes obviously. Note that, by convention, we have labeled the indices from $0, 1, 2, 3$ instead of $1, 2, 3, 4$ with 0 denoting the time component.

Also, you might wonder why we put some indices as superscript ("upper indices") and some indices as subscript ("lower indices"). This is because we want to sum over pairs of upper-lower indices, i.e. we sum over an "upper" index with its "lower". For now, you can use this as a mnemonic to remember which index should be summed over which – we will introduce a lot more of the mathematical structure behind this in the next Chapter. Since we will be doing a lot of these summations, we will use the **Einstein Summation convention** and drop the sum sign to write more succintly

$$ds^2 = \sum_{\mu,\nu} \eta_{\mu\nu} dx^\mu dx^\nu \Rightarrow \boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu.} \tag{1.36}$$

Now, $\eta_{\mu\nu}$ is a square matrix with a non-zero determinant, we can find its inverse

$$(\eta_{\mu\nu})^{-1} = \begin{pmatrix} -c^{-2} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{1.37}$$

[8] **The operators form a group $SO(3,1)$, as you have learned in your Symmetry in Physics module.**

[9] There is a great schism in theoretical physics in the choice of the signs of the c^2dt^2 and dx^2 terms. Half the field (namely particle physicists) like to use the convention $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ while the relativists like to use the convention of these lectures. There is a good reason behind the split, but that's a story for lecture time. In any case, be careful when you read other textbooks or lecture notes to check which convention they use.

Following the Einstein Summation convention, we can make the following definition

$$\eta^{\mu\nu} \equiv (\eta_{\mu\nu})^{-1}. \tag{1.38}$$

You can then check that $\eta^{\mu\nu}$ is indeed the inverse of $\eta_{\mu\nu}$ by multiplying the two matrices

$$\eta_{\mu\nu}(\eta_{\nu\rho})^{-1} = \eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu{}^\rho, \tag{1.39}$$

where $\delta_\mu{}^\rho$ is simply the identity matrix. Notice that, when performing matrix multiplication, you sum over the products of the row of the first matrix with the column of the second matrix – Einstein summation convention does that by telling you to sum over the ν index in this calculation. The upshot of this is that *index placement matters* – you should begin to track the placement religiously from now on. Given the definition Eq. (1.38), we can "convert" an upper index object into a lower index object

$$X_\mu \equiv \eta_{\mu\nu}X^\nu \tag{1.40}$$

and *vice versa*

$$X^\mu \equiv \eta^{\mu\nu}X_\nu. \tag{1.41}$$

You might have seen this before, and you might even have done some calculations like this. But it is important to emphasise that Eq. (1.40) and Eq. (1.41) are *definitions*, not a given. We will become a lot more intimate with these constructions (seemingly bizarre if you think deeply, seemingly obvious if do not) in the next Chapter (section 2.4), so let's for now push on with physics.

Special Relativity then states that

The spacetime interval is invariant under Lorentz transformations.

In other words, the spacetime interval ds^2 (Eq. (1.33), will remain the same value when the coordinates are transformed under Lorentz Transformations (the boosts Eq. (1.31) and rotations Eq. (1.30)).

Instead of checking the transformations one by one, let's now make use of our new slick notation to write the Lorentz transformations in following **linear** form[10],

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 & \Lambda^1{}_4 \\ \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 & \Lambda^2{}_4 \\ \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 & \Lambda^3{}_4 \\ \Lambda^4{}_1 & \Lambda^4{}_2 & \Lambda^4{}_3 & \Lambda^4{}_4 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{1.42}$$

where $\Lambda^{\mu'}{}_\nu$ is a 4×4 matrix with real components. Since the components are just constants, you can take their infinitisimals, and by using our fancy Einstein Summation index convention, we can write the above matrix multiplication as

$$dx^{\mu'} = \Lambda^{\mu'}{}_\nu dx^\nu \tag{1.43}$$

where we have put the primes in the index, i.e. we wrote $dx^{\mu'}$ instead of dx'^μ (it's a matter of taste). You should convince yourself by turning the crank on the algebra that, by summing over the μ index, you do get the right matrix multiplcation.

In this notation, the boost along the x axis Eq. (1.31) is then

$$\Lambda^{\mu'}{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta/c^2 & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.44}$$

[10]See section 2.1.3.

Under these transformations, the spacetime interval Eq. (1.33) becomes

$$\begin{aligned} ds'^2 &= \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'} \\ &= \eta_{\mu'\nu'} (\Lambda^{\mu'}{}_{\mu} dx^{\mu})(\Lambda^{\nu'}{}_{\nu} dx^{\nu}) \\ &= \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} \eta_{\mu'\nu'} dx^{\mu} dx^{\nu} \end{aligned} \tag{1.45}$$

and the requirement that the spacetime interval be invariant $ds'^2 = -c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2 = ds^2$ means that

$$\Lambda^{\mu'}{}_{\mu} \eta_{\mu'\nu'} \Lambda^{\nu'}{}_{\nu} = \eta_{\mu\nu} \tag{1.46}$$

and

$$\eta_{\mu'\nu'} = \begin{pmatrix} -c^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{1.47}$$

More explicitly, in matrix notation the condition Eq. (1.46) is

$$\Lambda^T \begin{pmatrix} -c^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Lambda = \begin{pmatrix} -c^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{1.48}$$

where Λ^T means the transpose matrix of Λ as usual. Let's unpack this equation. First look at the bottom right 3×3 components, i.e. the R_{ij} components as follows

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & R^1{}_1 & R^1{}_2 & R^1{}_3 \\ \dots & R^2{}_1 & R^2{}_2 & R^2{}_3 \\ \dots & R^3{}_1 & R^3{}_2 & R^3{}_3 \end{pmatrix}. \tag{1.49}$$

Recall that the 3 spatial rotation matrix Eq. (1.30). A general rotation in 3 spatial dimensions obeys the condition[11]

$$\mathbf{R}^T \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathbf{R} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}. \tag{1.50}$$

Indeed, since the spacetime metric is diagonal, the $R^i{}_j$ terms in the matrix $\Lambda^{\mu'}{}_{\nu}$ obey exactly the same condition. Which means that the $R^i{}_j$ terms are then simply rotations in 3D, e.g. rotations by θ around the z axis is given by

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} \tag{1.51}$$

and similarly a rotation of θ in x direction would be

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{pmatrix} \tag{1.52}$$

[11] You can easily check that rotations around the z axis Eq. (1.25) obey this condition.

and so on. You can check for yourself that these matrices obey the condition Eq. (1.48).

You can of course mix the rotations, i.e. rotate around x-axis, then rotate around y-axis etc, and you can check for yourself that the composition of these two rotations are given by a *product* of the two matrices, and this resultant matrix also obeys Eq. (1.48). (Homework).

The eagle-eyed amongst you will have noticed that the conditions Eq. (1.50) and Eq. (1.48) look extremely similar. The difference is that the $0-0$ component has a minus sign and a c^2, instead of simply being 1. Indeed if we replace $-c^2 \to 1$ in Eq. (1.48), then $\Lambda^{\mu'}{}_{\nu}$ are simply *rotations in 4 spatial dimensions!* In fact, we can do away with c and set it to one like all self-respecting theoretical physicists

$$\boxed{c=1}\ . \tag{1.53}$$

This is because c is just a historical accident in our choice of units to measure time (seconds) and space (centimeters), so by setting $c=1$ we are now saying that 1 second $= 3\times 10^{10}$ cm. So now, the condition Eq. (1.48) is

$$\Lambda^T \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Lambda = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{1.54}$$

On the other hand, we can't define away the minus sign – it tells us that time is slightly different from space in a deep and mysterious way. Leaving aside that mystery, we have seen that rotations in 3D mix the components of the different spatial directions together. For example, rotations around z axis mixes the x and y axes together e.g. $x' = x\cos\theta + y\sin\theta$ and $y' = -x\sin\theta + y\cos\theta$. What then is a "rotation" that mixes space and time together? It turns out that such a rotation can be executed by, instead of sines and cosines, by *hyperbolic* sines and cosines i.e.

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \cosh\theta & -\sinh\theta & & \\ -\sinh\theta & \cosh\theta & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.55}$$

denotes a rotation that mixes x and t together. (Homework) You can check that this operation obeys the condition Eq. (1.54). Now, if we define

$$\beta \equiv \frac{\sinh\theta}{\cosh\theta} = \tanh\theta \tag{1.56}$$

then some algebra shows that

$$t' = t\cosh\theta - x\sinh\theta = \gamma(t-\beta x) \tag{1.57}$$

$$x' = -t\sinh\theta + x\cosh\theta = \gamma(x-\beta t) \tag{1.58}$$

which is exactly the Lorentz boosts Eq. (1.31) that you have already seen in your studies on Special Relativity. Hence, the Lorentz transformations are really 3+1D generalizations of the 3D rotations that you know and love. When we execute a boost, we are rotating space into time and *vice versa*.

So the Lorentz transformation transforms physical properties from one frame to another, i.e. allowing us to transform from the infinitisimals dx^μ of the unprimed frame to the primed frame $dx^{\mu'}$

$$dx^{\mu'} = \Lambda^{\mu'}{}_{\mu} dx^\mu. \tag{1.59}$$

We can now ask : are there any physical quantities that we can do similar tranformations? The answer is yes!

Let's begin with the velocity. In usual 3D, the velocity in some inertial frame with coordinates (t, x, y, z) is given by

$$v^i = \frac{dx^i}{dt} \tag{1.60}$$

where $i = 1, 2, 3$ such that $x^1 = x, x^2 = y$ and $x^3 = z$. We want to now promote this 3D *spatial* vector into a 3+1D *spacetime* vector. A reasonable guess is

$$\boxed{U^\mu = \frac{dx^\mu}{d\tau}} \tag{1.61}$$

where $\mu = 0, 1, 2, 3$, where $x^0 = t$ as we have defined earlier, and the *proper time* $d\tau$ somewhat mystically defined as

$$d\tau^2 \equiv -ds^2. \tag{1.62}$$

Leaving aside the weird minus sign in Eq. (1.62) for now (we will come back to it soon enough), we have chosen to take the derivative with respect to the proper time $d\tau$ which is *invariant under Lorentz transforms just like* ds, instead of time – this makes sense since t is now a frame-dependent quantity but τ (and s) is independent of frame. The object Eq. (1.61) is called a **four velocity**, or 4-velocity for short.

The 0 component of U is

$$U^0 = \frac{dt}{d\tau}, \tag{1.63}$$

and now we want to relate $d\tau$ to dt. In the unprimed frame,

$$dx = v^x dt \ , \ dy = v^y dt \ , \ dz = v^z dt \tag{1.64}$$

and now using the Eq. (1.33) we get (recalling that $c = 1$ from now)

$$d\tau^2 = -ds^2 = -dt^2(-1 + v^{x2} + v^{y2} + v^{z2}), \tag{1.65}$$

so

$$d\tau = dt\sqrt{1 - v^2}. \tag{1.66}$$

Eq. (1.65) is the reason we have chosen in Eq. (1.62) $d\tau$ as our proper time instead of ds – since $v \leq 1$ (recall that $c = 1$), we will get $d\tau^2 \geq 0$ so $d\tau$ is real. Other than being annoying, this trick is really nothing mysterious – it is a consequence of our choice of convention for the invariant length Eq. (1.33) that has "more + than -" signs (if we have chosen $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$, like some people do, we won't have to resort to such trickery).

Having said all that, we can then compute

$$\frac{dt}{d\tau} = \gamma_v = \frac{1}{\sqrt{1 - v^2}} \ , \ v = \sqrt{v^{x2} + v^{y2} + v^{z2}} \tag{1.67}$$

where γ_v is the Lorentz factor of the velocity in this frame[12]. So then the zeroth component of U^μ is

$$U^0 = \frac{dt}{d\tau} = \gamma_v \tag{1.68}$$

For the spatial components, using the chain rule

$$U^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau}\frac{dx^i}{dt} = \gamma_v v^i. \tag{1.69}$$

[12]Don't mix up the Lorentz factor here with the Lorentz boost of Eq. (1.31). For any 3-velocity, we can always compute a Lorentz factor. The former Lorentz factor refers to the *particle*'s 3-velocity in its frame, while the latter refers to the *relative velocity* between the two inertial frames.

Hence the four-velocity in the unprimed frame is then

$$U^\mu = (\gamma_v, \gamma_v v^i) \tag{1.70}$$

where the v^i is just the usual 3-velocity in this frame Eq. (1.60). You can easily write the 4-velocity in the primed frame

$$U^{\mu'} = (\gamma_{v'}, \gamma_{v'} v'^i) \tag{1.71}$$

where $v'^i = dx'^i/dt'$ and with the resulting Lorentz factor $\gamma'_v = (1 - v'^2)^{-1/2}$. From Eq. (1.59), it is easy to relate the 4-velocities of the two different frames

$$\frac{dx^{\mu'}}{d\tau} = \Lambda^{\mu'}{}_\mu \frac{dx^\mu}{d\tau} \Rightarrow U^{\mu'} = \Lambda^{\mu'}{}_\mu U^\mu. \tag{1.72}$$

Given the 3-velocity of the unprimed frame, we can find the 3-velocity of the primed frame. Suppose the two frames are related to each other by a boost β (parameterized by γ) given by Eq. (1.44), using Eq. (1.72), you can compute

$$\gamma_{v'} = \gamma\gamma_v(1 - \beta v^x) \ , \ \gamma_{v'} v^{x'} = \gamma\gamma_v(-\beta + v^x), \tag{1.73}$$

which after some algebra you can derive the usual Lorentz transformation of the velocities that you have studied in your Nuclear physics course

$$v^{x'} = \frac{v^x - \beta}{1 - \beta v'}. \tag{1.74}$$

In general, you can transform *any* four-vector in one inertial frame to another by simply applying the Lorentz transformation

$$\boxed{V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu}. \tag{1.75}$$

For example, a particle of mass m moving with velocity $\dot{x}$ possess the momentum in the unprimed frame given by

$$\mathbf{p} = m\frac{d\mathbf{x}}{dt}, \tag{1.76}$$

which in our fancy index notation is

$$p^i = mv^i. \tag{1.77}$$

The **four momentum** is a straightforward generalization

$$P^\mu = mU^\mu. \tag{1.78}$$

4-momenta in one frame can be Lorentz transformed to another as usual

$$P^{\mu'} = \Lambda^{\mu'}{}_\mu P^\mu. \tag{1.79}$$

The computations follows as usual.

(Homework : show that timelike and spacelike properties of four-vectors is invariant under lorentz transformation).

1.4 Maxwell Equations in 4-vectors

Now that we have learned 4-vector notation, we are ready to demonstrate that the Maxwell equations Eq. (1.6) to Eq. (1.9) is invariant under Lorentz transformations. The trick is simple: we will rewrite Maxwell equations using four vectors, and then show that executing a Lorentz transformation on the resulting equation will get us back the same set of equations, but in the new primed frame.

To do this, we first recast the electric **E** and magnetic fields **B** in terms of their potentials. You might already have learned that, from Eq. (1.9), since $\nabla \cdot (\nabla \times \mathbf{X}) = 0$ for any vector **X**, we can rewrite the magnetic field **B** as the cross product of some **vector potential A**

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.80}$$

Similarly, since $\nabla \times (\nabla \cdot \phi) = 0$ for any scalar ϕ, we can rewrite Eq. (1.7) (also known as the **Faraday's Law**) as

$$\nabla \times (\mathbf{E} + \partial_{\mathbf{t}}\mathbf{A}) = \mathbf{0} \tag{1.81}$$

and hence

$$\mathbf{E} = -\nabla V - \partial_t \mathbf{A}. \tag{1.82}$$

We can (which at the moment seems unmotivated) combine this with the *scalar* electric potential V from Eq. (1.82) to get a four-vector called the **4-potential**

$$A^\mu = (V, \mathbf{A}). \tag{1.83}$$

We can use the metric $\eta_{\mu\nu}$ to lower the index (recall $c = 1$)

$$A_\mu = \eta_{\mu\nu} A^\nu = (-V, \mathbf{A}) \tag{1.84}$$

which result in the change of sign for the potential V. Given the 4-potential, we can construct an object called **electromagnetic tensor** $F_{\mu\nu}$

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{1.85}$$

where $\partial_\mu = (\partial_t, \partial_x, \partial_y, \partial_z)$. While we use the word "tensor" here (something which you will get very intimate with in this course), for now you can think of this object as a 4×4 matrix. It is a skew-symmetric or **anti-symmetric**, i.e. $F_{\mu\nu} = -F_{\nu\mu}$.

Despite being weird looking, the electromagnetic tensor $F_{\mu\nu}$ is simply a clever way of writing **E** and **B** fields *together in a single object.* For example, by comparing to Eq. (1.80) the spatial components encode the **B** fields

$$F_{xy} = \partial_x A_y - \partial_y A_x = B_z \ , \ F_{xz} = \partial_x A_z - \partial_z A_x = -B_y \ , \ F_{yz} = \partial_y A_z - \partial_z A_y = B_x \tag{1.86}$$

while the space-time components encode the **E** field (by comparing to Eq. (1.82))

$$F_{tx} = -E_x \ , \ F_{ty} = -E_y \ , \ F_{tz} = -E_z. \tag{1.87}$$

Again without motivation at the moment, we can also collect the charge density ρ and the current **J** into a **4-current** j^μ

$$j^\mu = (\rho, \mathbf{J}). \tag{1.88}$$

Given the electromagnetic tensor Eq. (1.85) the 4-current Eq. (1.88), we rewrite Eq. (1.6) and Eq. (1.8) of Maxwell equations in a single, concise equation,

$$\boxed{\partial_\mu F^{\nu\mu} = j^\nu}. \tag{1.89}$$

The electromagnetic tensor with upper indices are made out of raising the indices of $F_{\mu\nu}$ using the inverse matrix twice as per Eq. (1.41)

$$F^{\nu\mu} = \eta^{\nu\rho}\eta^{\mu\sigma}F_{\rho\sigma}. \tag{1.90}$$

Similarly, Eq. (1.7) and Eq. (1.9) can be rewritten as a single equation

$$\boxed{\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0.}. \tag{1.91}$$

You will show in a (Homework) problem that Eq. (1.89) and Eq. (1.91) indeed contain all the four equations of Maxwell's.

With this fancy notation, we can combine this with the Lorentz transformation $\Lambda^\mu{}_\nu$ to transform the quantities from one inertial frame to any inertial frame we like. For example, the 4-vector in an unprimed inertialframe can be transformed to primed frame simply by

$$A^{\mu'} = \Lambda^{\mu'}{}_\nu A^\nu \tag{1.92}$$

and similarly the 4-current

$$j^{\mu'} = \Lambda^{\mu'}{}_\nu j^\nu. \tag{1.93}$$

We can also transform the electromagnetic tensor by applying the Lorentz transformation twice (once on each index)

$$F^{\nu'\mu'} = \Lambda^{\nu'}{}_\rho \Lambda^{\mu'}{}_\sigma F^{\rho\sigma}. \tag{1.94}$$

In this notation, for example, under a Lorentz boost given by Eq. (1.44), the electric field $E^{x'}$ in the x' direction is given by the unprimed components as

$$\begin{aligned} F^{t'x'} &= \eta^{t't'}\eta^{x'x'}F_{t'x'} \\ &= E^{x'} \\ &= \Lambda^0{}_\alpha \Lambda^1{}_\rho F^{\alpha\rho} \\ &= \gamma^2(1-\beta^2)E^x \end{aligned} \tag{1.95}$$

or $E^{x'} = \gamma^2(1-\beta^2)$. More interestingly, the electric fields in the y' and z' directions are now *mixed* with the **B** field in the unprimed frame

$$E^{y'} = \gamma(E^y + \beta B^z)\ ,\ E^{z'} = \gamma(E^z - \beta B^y). \tag{1.96}$$

Recall that Galilean boosts do not mix the **E** and **B** fields.

Indeed, after this long detour, we are ready to unveil the proof that Maxwell equations are invariant under Lorentz Transformation, just as Newton's 2nd law is invariant under Galilean Transformations. As it turns out, we have actually already done all the hard work of this proof, by rewriting the Maxwell equations in 4-vector notation. The Lorentz transformation of the left hand side of Eq. (1.89) is

$$\partial_{\mu'} F^{\nu'\mu'} = \Lambda^{\nu'}{}_\nu \partial_\mu F^{\nu\mu}. \tag{1.97}$$

You might ask : why have we only transformed the ν variable? The reason is simple : in $\partial_\mu F^{\mu\nu}$ the μ index is already summed over! We will be super pedantic in a bit, but let's plow on. On the right hand side of Eq. (1.89), we have simply

$$j^{\nu'} = \Lambda^{\nu'}{}_\nu j^\nu \tag{1.98}$$

and hence the entire equation Eq. (1.89) transforms as

$$\partial_\mu F^{\nu\mu} = j^\nu \rightarrow \Lambda^{\nu'}{}_\nu \partial_\mu F^{\nu\mu} = \Lambda^{\nu'}{}_\nu j^\nu \rightarrow \partial_{\mu'} F^{\nu'\mu'} = j^{\nu'} \tag{1.99}$$

so the Maxwell equations appear in *exactly the same form in both primed and unprimed* inertial frames!

How did this sleight of hand happen? It happened because *it is possible to rewrite the Maxwell equations* in 4-vector form – in other words, the hard work is done when we derived Eq. (1.89) and Eq. (1.91). This is not a given: it is easy to convince yourself (and you should!) that Newton's 2nd Law *cannot* be rewritten in this form. Such a special 4-vector form is called a **covariant form**. In words, we say that Maxwell equations can be cast in a covariant form, while Newton's 2nd Law cannot be cast as such. You can go further, and write the covariant version of the Lorentz force law Eq. (1.13)

$$\frac{dp^\nu}{ds} = qF^{\mu\nu}V_\mu \tag{1.100}$$

where p^μ is the 4-momentum and V_μ is the 4-velocity of the charged particle. Hence the answer to the question we posed earlier in the paragraph below Eq. (1.13) is simply that this is the form that keeps the theory invariant under Lorentz Transformation. If we have transformed the frame where the particle is *not* in motion to one which the particle is in motion, then Lorentz transformation mixes up the **E** and **B** fields.

The fact that the Maxwell equations are Lorentz invariant, and that it predicts that light has a finite speed c led Einstein on his road to discovering the theory of Special Relativity. He was not the first person to recognize the Lorentz invariance of Maxwell's equations – in fact **Henrikh Lorentz** himself discovered the Lorentz transformation from studying Maxwell's equations! But he was the first person to make the connection that *all* physical theories should be Lorentz invariant.

Why are the Maxwell equations invariant under Lorentz transformation? Indeed, *all* known theories (including GR) are invariant under Lorentz transformation locally. The reason for this is unknown – for now it is just the way how nature works.

Before we finish with this section, let's tie up a loose end we have left hanging. We have shown that the inverse metric $(\eta_{\mu\nu})^{-1}$ can be written as an "upper index" counterpart $\eta^{\mu\nu}$ Eq. (1.38). What about the Lorentz Transformation matrix? We now assert that the inverse of the Lorentz transformation is

$$(\Lambda^{\mu'}{}_{\nu})^{-1} \equiv \Lambda_{\mu'}{}^{\nu} \tag{1.101}$$

i.e. we lower the raised index and *vice versa*. You can compute the components of this metric by using the metric to raise/lower the indices

$$\Lambda_{\nu'}{}^{\mu} = \eta_{\nu'\sigma'}\eta^{\mu\nu}\Lambda^{\sigma'}{}_{\nu} \tag{1.102}$$

which gets you

$$\Lambda_{\nu'}{}^{\mu} = \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.103}$$

which, not surprisingly, is a Lorentz boost in the x direction with the parameter $-\beta$. So a product of a β boost with a $-\beta$ boost should return you to the unboosted frame as expected.

You can explicitly check this is true as follows

$$\Lambda_{\mu'}{}^{\nu}\Lambda^{\mu'}{}_{\alpha} = \delta^{\nu}_{\alpha}\ ,\ \Lambda_{\mu'}{}^{\alpha}\Lambda^{\nu'}{}_{\alpha} = \delta^{\nu'}_{\mu'}. \tag{1.104}$$

Finally, we can show that for any summed over pair of indices

$$\begin{aligned} X_\mu Y^\mu &= (\Lambda_{\mu}{}^{\nu'}X_{\nu'})(\Lambda^{\mu}{}_{\rho'}X^{\rho'}) \\ &= \delta^{\nu'}_{\rho'}X_{\nu'}Y^{\rho'} \\ &= X_{\nu'}Y^{\nu'} \end{aligned} \tag{1.105}$$

is *invariant under Lorentz transformation.* This is why we can ignore the summed over index μ when we proved the invariance of the Maxwell equations under Lorentz transformation.

1.5 The Equivalence Principle and the Road to General Relativity

Let us now hark back to gravity, after the detour to electrodynamics (as an excuse to get you all acquainted with four-vectors and metrics). Recall Newton's 2nd law Eq. (1.1), which we now rewrite

$$\mathbf{F} = m\frac{d^2\mathbf{x}}{dt^2}. \tag{1.106}$$

Meanwhile, for any given gravitational field Φ, the acceleration is given by Eq. (1.4)

$$\mathbf{g} = -\nabla\Phi. \tag{1.107}$$

Now, as you have done countless times, if we want to calculate the motion, we replace the $\mathbf{a}$ in Eq. (1.106) with $\mathbf{g}$ and get

$$\mathbf{F} = m\mathbf{g} \tag{1.108}$$

and then set

$$\mathbf{g} = \frac{d^2\mathbf{x}}{dt^2} \tag{1.109}$$

and calculate away to solve for $\dot{\mathbf{x}}$ and $\mathbf{x}$ and so on. But wait! *Who says we are allowed to do that?* What if, the "mass" in Eq. (1.106) is different from the "mass" in Eq. (1.108). In other words, if we rewrite the two equations suggestively with $\mathbf{F} = m_{\rm inertial}\mathbf{a}$ and $\mathbf{F} = m_{\rm grav}\mathbf{g}$, then why is it true that

$$m_{\rm inertial}\mathbf{a} \stackrel{?}{=} m_{\rm grav}\mathbf{g}. \tag{1.110}$$

In words, there is no *a priori* reason that the *inertial mass* $m_{\rm inertial}$ (the mass associated with the inertia of any object) is the same as its *gravitational mass* $m_{\rm grav}$ (the mass associated with an object moving in a gravitational field). But of course, experimentally, they are the same mass $m_{\rm inertial} = m_{\rm grav}$, and this *underivable* fact is called the **Weak Equivalence Principle**, or WEP for short.

This means that, we can write pedantically

$$\ddot{\mathbf{x}} = \mathbf{g}(\mathbf{x}(t), t). \tag{1.111}$$

Notice that the gravitational field felt by the particle at time t depends on its location $\mathbf{x}(t)$, while the second t in the equation indicates that the gravitational field itself can be changing with time. Nowhere does m – nor indeed any information regarding *what* the particle is – appear in this equation, and you probably have done this calculation many times. But this is a huge deal: this means that, for any given gravitational field g, *any two particles with the same initial velocity and position will follow exactly the same trajectory, regardless of composition or mass* – it doesn't matter how massive or what charges these particles carry. A cat, an electron, planet Earth, all will follow the same trajectory as long as their initial velocities and position are identical.

In other words, the WEP tells us that every object in the universe is affected by gravity – unlike electromagnetic forces for example where uncharged particles are shielded from it. This implies something profound : there is *no unique global frame where you can measure the gravitational field like the way you measure the electromagnetic field.* To see why this is so, consider how you would measure the electromagnetic fields : you set up a group of electrically neutral detectors resting (i.e. unaccelerated) in their inertial frames (these are known as **inertial observers**), and then you release a charged test particle. The detectors will then measure motion of the particle, and from there we are can compute the electric (and magnetic) fields (say using the Lorenz force law). The detectors, since they are electrically neutral, will *remain inertial* and hence is not affected by the charged test particle. Once you make all the measurements, since the detectors are all in inertial frames, you know how to reconstruct the entire "global" electromagnetic field as we discussed in the previous section – pick some favorite inertial frame and then Lorentz transform every other frame to it. However, if you try to do the same thing with the gravitational field – release a test particle and measure its motion to compute the gravitational field, the WEP ensures that the *detectors themselves affect and will be affected by the gravitational field*, so they no longer stay inertial – once you release them they will accelerate towards each other and as we have studied before, they no longer stay inertial.

Einstein and many others initially thought that a dynamical formulation of gravity would be something like Electrodynamics – we simply write down a set of evolution equations that describe the gravitational potential, whose equations are Lorentz invariant, and then use the Lorenz equation to describe the forces exerted by these potentials. But, such a formulation will lead to the paradox described above. Instead of despairing, Einstein decided that our inability to construct a set of inertial observers in the sense of Special Relativity is not a bug, but a feature of gravity. He asserted that *it is not possible to construct inertial observers to measure the gravitational field*, and set himself the crazy task of finding a theory of gravity that does exactly that.

To solve all these problems all at once, he made the following radical proposal :

Gravity is not a force, but a property of the fact that spacetime is curved, and all free-falling (unaccelerated by any other means) objects simply follow the **geodesics** *– i.e. the shortest path between two points*[13]*. In spacetime, the distance is defined to be the spacetime interval we discussed earlier.*

He called this theory **General Relativity**, and we will be spending the entire term studying it.

Giving up global inertial observers is a big deal – what are we going to be measuring with respect to? Einstein "evaded" this problem by proposing an extension to the WEP

(Einstein) In a local *patch, the laws of physics reduce to that of special relativity.*

To illustrate this idea, let's consider the idea of **Tidal Forces** (see Figure 1.1). Suppose we lock you in a windowless lab, and drop the lab towards the Earth from some distance. Even though the lab is windowless, you can devise an experiment to check that you are not in a uniform gravitational field by checking that two bodies at each end of the lab are accelerating towards each other. These forces are known as tidal forces – which arises when there is a non-uniform gravitational field. Suppose now that, due to budget cuts of the government and without EU funding, you are now given a smaller and smaller lab, then this experiment becomes increasingly hard to do – sadly not a *gedenkenexperiment* in real world. Indeed, in the limit where the Lab is now *infinitisimally small*, then you will not be able to do this experiment at all. Such an infinitisimally small patch of spacetime is called a **local** patch. Inside such a tiny lab, you *would not be able to tell that you are moving through a gravitational field.* Indeed, as far as you are concerned, you will be "free-falling" and weightless. Of course, since your lab is windowless, you won't even know that you are "falling" towards the Earth. So you can happily define an inertial frame around yourself, confident that you are non-accelerating. Such a free falling inetial frame is called a **local inertial frame**.

The combination of the WEP, and Einstein's extension, is usually called (doh) the **Einstein Equivalence Principle** (EEP), although in practice they are almost equivalent (notice that in our Tidal forces example, the extension almost appears naturally without any additional conjectures) and people will not yell at you for mixing them up[14].

Local inertial frames allow us to define things like velocities, acceleration (say for example, you can measure the acceleration of an electron towards a proton inside your tiny lab) *locally* inside the lab.

[13] For example, on a sphere, the geodesic between two points is a section of a great circle.

[14] **Sometimes the EEP is called the "Strong Equivalence Principle", and sometimes people distinguish between the EEP and SEP by deciding some forces are "gravitational" and some forces are not. I find such distinctions confusing so I won't talk about the SEP.

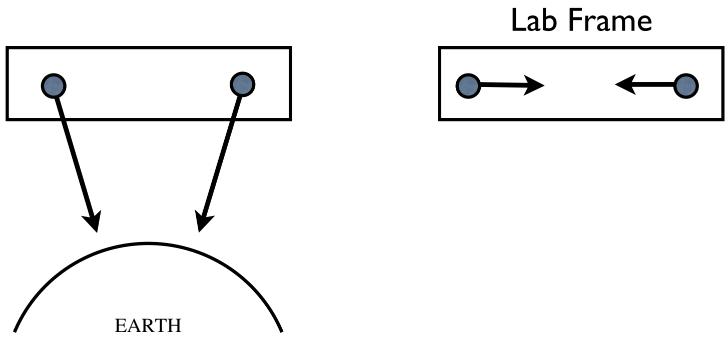


Figure 1.1: Tidal forces in a non-uniform gravitational field.

However, say, there is another physicist in an identical box, but falling towards the Earth on the opposite side (i.e. if you are falling towards the UK, she is falling towards New Zealand). Can we compare the relative velocities? In Special Relativity, this is easy : we simply use the Lorentz Transformation. But since we have given up global inertial observers, *there is no meaningful way that we can compare these velocities!* Put in another way, these physical variables are *only valid locally, and they have no global meaning.* Such ambiguity is the price of giving up global inertial frames.

All these bold statements are very new, very confusing, and possibly a bit scary. To describe all these new physics and new ways of looking at gravity, we need new mathematical tools. We have said that "spacetime is curved". How do we describe "curved space"? In our study of Special Relativity, we argued that the invariant length is given by the metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2, \tag{1.112}$$

which in spherical coordinates (t, r, θ, ϕ) is

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.113}$$

Without much elaboration, suppose we have a new metric $g_{\mu\nu}$ that looks like

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\left[1 + 2\Phi(t,r)\right] dt^2 + \left[1 - 2\Phi(t,r)\right] dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.114}$$

i.e. the metric is no longer simply $(-1, 1, 1, 1)$ but some functions of space and time, and $\Phi(t, \mathbf{x})$ is the usual Newtonian gravity potential. Furthemore, we make the assertion that ds^2 is still *still the quantity that is physical*, just like the invariant length of Special Relativity, even though ds^2 as expressed in Eq. (1.114) is no longer (globally) Lorentz invariant[15]. Since this is no longer an invariant, we give it a new name **Proper Time** defined to be the minus times ds^2

$$d\tau^2 = -ds^2. \tag{1.115}$$

How does this describe a gravity field? Consider the **Pound-Rebka Experiment**, Figure 1.2, where we place an emitter (Alice) on top of a tall tower, with the receiver (Bob) in the bottom. Since we are in Earth's gravitational field, $\Phi(t, r) \to \Phi(r)$ does not depend on time. Alice sends a photon down towards Bob at time t_1, and wait a short moment Δt_1, before sending another photon down towards Bob at time $t_1 + \Delta t_1$. Since Alice (and Bob) are stationary, $dr = 0$, the proper time between two emissions of photon is

$$d\tau_A^2 = -ds_A^2 = (1 + 2\Phi(r_A))dt^2 \Rightarrow ds_A \approx (1 + \Phi(r_A))dt \tag{1.116}$$

[15] It is still *locally* Lorentz invariant – by redefining the coordinates $dt' = \sqrt{1+2\Phi}dt$, $d\mathbf{x}' = \sqrt{1-2\Phi}d\mathbf{x}$.

where $\mathbf{x}_A$ is Alice's location. Similarly, Bob received the signals at t_B and $t_B + \Delta t$, so the proper time between the two photons at reception is

$$d\tau_B \approx (1 + \Phi(r_B))dt. \tag{1.117}$$

Eliminating dt, we get

$$d\tau_B = (1 + \Phi(r_B))(1 + \Phi(r_A))^{-1} d\tau_A \approx (1 + (\Phi(r_B) - \Phi(r_A))d\tau_A. \tag{1.118}$$

Since Bob is closer to Earth, he is in a deeper potential well, so $\Phi(r_A) > \Phi(r_B)$, and hence $d\tau_B < d\tau_A$.

The wavelength of the light is given by $\lambda = 1/d\tau$, so the wavelength of the light being emitted by Alice is *longer* than that which is received by Bob $\lambda_A > \lambda_B$. Which is to say, the light has been *blueshifted*. If instead, it was Bob who emitted the light towards Alice, then the light would be *redshifted*. This phenomenon is known as **Gravitational Redshift**. The Pound-Rebka Experiment was conducted in 1959, and gravitational redshifting was indeed observed.

Note that this is *not* a test of GR, since one can make exactly the same prediction using Newtonian gravity and the WEP alone – as you will show in a (Homework) problem. Nevertheless, it is *consistent* with the assertion that spacetime is curved – in the language of geometry, we say that light has taken the curved path through spacetime instead of light "feeling the effects" of the gravitational force in the Newtonian language.

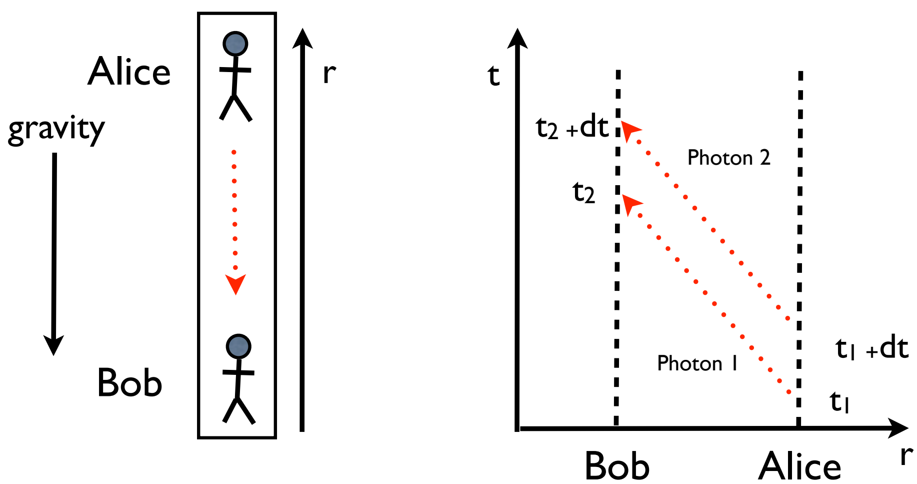


Figure 1.2: The Pound-Rebka Experiment.

In your skool days when you were young, you have studied geometry where you learned things like parallel lines and all the Euclid stuff, i.e. the behaviour of lines when the surface is flat. You might even have studied some *curved* surfaces like the surface of a sphere – there you were told that the angles of triangle do not add up to 180, and that "parallel" lines eventually meet somewhere. The study of surfaces is called **Differential Geometry**. This is the branch of mathematics that General Relativity (among other things) is built upon. Einstein realized quickly when he started formulating his theory that he needed to learn this mathematics, and he turned to his old friend **Marcel Grossman** for help. Grossman taught Einstein this mathematics, and together they laid out the mathematical structure of a theory of gravity that is purely geometrical in a groundbreaking paper in 1913[16]. Despite the popular media portrayal, Einstein was not a lone genius – he stood on shoulders of giants just as Newton did.

In the next Chapter, we will follow Einstein's footsteps, and embark on a study of Differential Geometry.

[16]Einstein, A.; Grossmann, M. (1913). "Outline of a Generalized Theory of Relativity and of a Theory of Gravitation"

Chapter 2

Baby Differential Geometry

How dare they teach thermodynamics to undergraduates without teaching them differential geometry first?

Sean M. Carroll

In this Chapter, we will study Differential Geometry, the mathematics that underlie General Relativity.

2.1 *Some Mathematics Preliminaries

In this section, we review some mathematics that you have probably learned elsewhere[1].

2.1.1 Maps

Given two sets A and B, we can define a link between them (Figure 2.1). Mathematically, we say that we want to find a **mapping** between two sets. The thing we use to do this is called a **map** or a **function**. If you have not thought about functions as maps, now is a time to take a private quiet moment to yourself and think about it. We defined maps by giving them *rules*. A rule is something that takes in an element of one set, and give you back an element of another set.

Example: Suppose $A = \mathbb{R}$ and $B = \mathbb{R}$, so

$$f : A \to B;\ f : x \mapsto x^2\ \forall\ x \in \mathbb{R} \tag{2.1}$$

where we have distinguished the arrows $\to$ to mean "maps to" while $\mapsto$ means "the rule is as follows" (or its **recipe**). The above described, in language you are familiar with, $f(x) = x^2$.

$$f : A \to B;\ f(x) = x^2\ \forall\ x \in \mathbb{R}. \tag{2.2}$$

(If you have not seen $\forall$ before, it is called "for all".)

The set of inputs A is called the **domain** of the map, while the set where the outputs live B is called the **codomain**. We call them dom(f) and cod(f) respectively. The subset of cod(f) that f is actually mapped to is called the **image** or im(f). If im(f) = cod(f) – i.e. every element in the codomain is "covered", then we say that f is **onto** or **surjective**. If, for every element in dom(f) is mapped to a unique element in cod(f), then the map is called **one-to-one** or **injective**.

[1]For those who had taken CP2332 Symmetry in Physics, this is truly a review. For those who had not, I encourage you to read Chapter 2 of the lecture notes http://damtp.cam.ac.uk/user/eal40/teach/symmetry/symroot.html

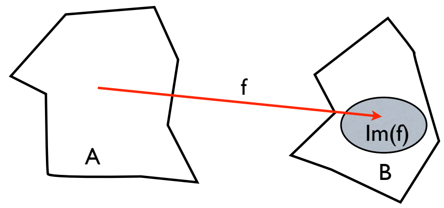


Figure 2.1: A map.

Maps which are both **onto** and **one-to-one** are called **bijective**, and such maps possess an **inverse maps**. The moral of the story is that *not all maps possess inverses* – in fact most maps don't possess inverses (e.g. $f(x) = x^2$ has no inverse, while $f(x) = 2x$ has an inverse map $f^{-1}(x) = x/2$.) This means that maps with inverses are precious, and we should love them accordingly.

You can **composed** maps together to make composite maps. Suppose we have two maps $f : A \to B$ and $g : B \to C$, then the composition map $g \circ f$ is described by

$$g \circ f : A \to C;\ \ g(f(x))\ \forall x \in A. \tag{2.3}$$

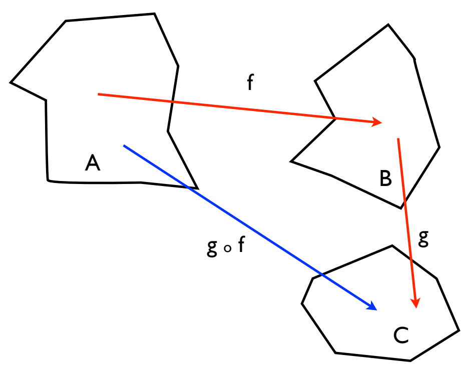


Figure 2.2: A composition of two maps f and g, which is written as $g \circ f$ – we write f to the right of g because the order of the operation goes from right to left.

2.1.2 Continuous, Smooth and Analytic Functions

Let $f(x)$ be a real function, with $x \in \mathbb{R}$, i.e. f maps the set of reals to another set of reals. A function is **continuous** if f is defined over its domain (there exists an element in its domain for every element x) and furthemore, in the limit of $x \to x_0$, $f(x) \to f(x_0)$ for all points x_0 in its domain. You are probably familiar with the former condition, but the latter condition ensures that "neighbouring points" are "connected" so to speak[2].

A function f is said to be **differentiable** if the function $f' = df/dx$ exists and is also continuous, where differentiation is defined as

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{2.4}$$

To illustrate this idea, consider the "kink" (see Fig. 2.3),

$$f(x) = \begin{cases} x + 1\ ,\ x < 0 \\ x - 1\ ,\ x \geq 0 \end{cases} \tag{2.5}$$

At $x = 0$, the derivative d^2f/dx^2 is "undefined". In other words, while $f(x)$ is differentiable, its

[2]If f only maps the set of integers to another set of integers, then it will not be continous for example.

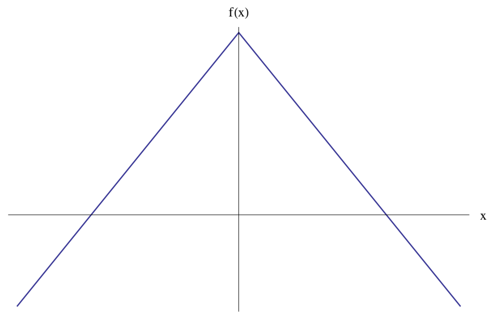


Figure 2.3: A kink is a continuous but non-smooth (at $x = 0$) function.

derivative $df(x)/dx$ is *not differentiable*. More generally, some functions are differentiable *everywhere* for up to n times, which we categorized them as C_n functions. A continuous C_∞ function is called a **Smooth function**.

A smooth function can be expanded around any point c as a **Taylor Series**

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x_0}(x - x_0) + \frac{1}{2!}\left.\frac{d^2f}{dx^2}\right|_{x_0}(x - x_0)^2 + \dots . \tag{2.6}$$

This series may or may not converge, depending on the exact form of f. Furthemore, if this series converges to its functional value *everywhere* in the (open) domain considered, then we say that the function f is **analytic** in M. Since we will be exclusively be daling with C_∞ functions, we will now drop the awkward term “infinitely” from now on.

These notions of smoothness and continuity can be generalized to functions whose domain are not the usual $\mathbb{R}$ or $\mathbb{C}$, but a type of mathematical structure called **Topological Spaces** which we will discuss later.

2.1.3 Linear Map and Transformation

A function f is said to be **Linear** if

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2) \ , \ \forall a, b \in \mathbb{C}. \tag{2.7}$$

This structure can be generalized. For example, in matrix multiplication, 3 dimensional column vectors acted upon by a 3×3 matrix M is given by

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = M \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{2.8}$$

Using the Einstein summation convention, we can write in index form

$$V'^j = M^j{}_i V^i. \tag{2.9}$$

In particular, we see that, for $a, b \in \mathbb{C}$, usual matrix multiplication rules give

$$M^j{}_i(aV^i + bW^i) = aV'^j + bW'^j \text{ where } V'^j = M^j{}_i V^i \ , \ W'^j = M^j{}_i W^i. \tag{2.10}$$

If you think of the square matrix $M^j{}_i$ as a map (playing the role of f in Eq. (2.1.3)), and the column matrices as its argument, then $M^j{}_i V^i$ takes V^i to another column matrix U^i, which is to say M maps column matrices to column matrices.

This means that the Lorentz transformation Eq. (1.42) is a linear map.

2.2 Deconstructing Vectors

In this section, we will discuss in loving detail, the idea of "vectors" which might feel like much ado about nothing for a moment, but the diligent will be rewarded!

You will doubtless be familiar with the ordinary 3-vectors (or 2-vectors in 2D) that you learned in skool. For example, you usually express the vectors in some choice of **basis vectors** – you can express any vector as $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$ where v_x and v_y are called the **components** of the vector. Notice that the $(\mathbf{i}, \mathbf{j})$ basis is *aligned* with the Cartesian coordinates (x, y) – such a basis is called a **Coordinate Basis**.

Let's introduce a new notation that will be useful in the future. Instead of $(\mathbf{i}, \mathbf{j})$, let's write this basis as a vector

$$\hat{\mathbf{e}}_{(i)} = (\hat{\mathbf{e}}_{(x)}, \hat{\mathbf{e}}_{(y)}) = (\mathbf{i}, \mathbf{j}). \tag{2.11}$$

You can of course easily generalize these to higher dimensions. The round brackets around the indices are a convention that is sometimes ignored[3], but let's be pedantic in a lecture.

We will now assert the following: the coordinate basis vectors are *really just partial derivatives*, i.e.

$$\boxed{\hat{\mathbf{e}}_{(i)} = \frac{\partial}{\partial x^i} = \partial_i}. \tag{2.12}$$

(Don't be disturbed by the fact that the LHS we have bracketed the index, and on the RHS we haven't.)

To show this, let's consider a continuous plane that is equipped with Cartesian coordinates (x, y) – the kind of plane that you have worked with since your high skool. In other words, each point on this plane can be labeled by a 2-tuple of real numbers (i.e. the coordinates) – we call such a space $\mathbb{R}^2$. As you have also learned, you can define any function g on this plane, the value of this function at any point on the plane is given by $g(x, y)$. Let this function be C_∞, and that it exists everywhere in the plane[4]. In other words, this function is a map from $\mathbb{R}^2$ to $\mathbb{R}$

$$g : \mathbb{R}^2 \to \mathbb{R}. \tag{2.13}$$

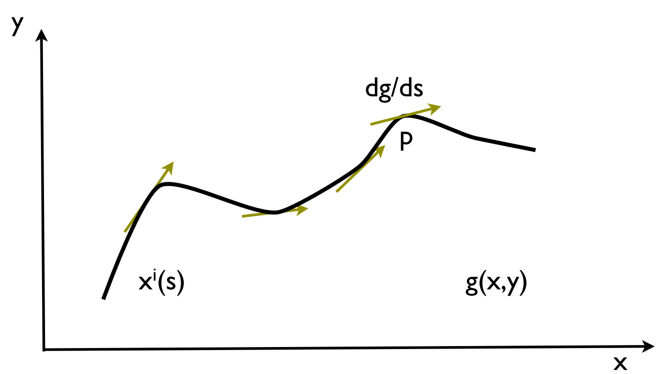


Figure 2.4: A smooth space is equipped with Cartesian coordinates (x, y). Let $x^i(s)$ be a smooth curve parameterized by s. This curve defines a vector d/ds at every point. Given any smooth function $g(x, y)$ on this space, we can calculate its gradient by dg/ds.

Meanwhile, we can also define a smooth **curve** $x^i(s)$, where s parameterized the location of the curve. For example, in the plane (x, y) the curve $y = x^2 + b$ can be equivalently described by

$$y = a^2s^2 + b \ , \ x = as \Rightarrow x^i(s) = (as, a^2s^2 + b) \ , \ \forall a, b \in \mathbb{R}. \tag{2.14}$$

[3]**The round brackets are to indicate that i label the *different* vectors $\hat{\mathbf{e}}_{(1)}$, $\hat{\mathbf{e}}_{(2)}$ etc, and not the *components* of the vectors V_1, V_2 etc.

[4]This requires the plane to be smooth – we will go into the deep details later, but for now we are relying on your experience with functions defined on plane.

The value of the function g at any point of this curve is then given by $g(x(s), y(s))$ or simply $g(s)$ (notice that this is composite map). We now ask : what is the **tangent vector** to the curve at any point $p \in s$? (See Figure 2.4). The rate of change of the function g along this curve is its derivative dg/ds, so using the chain rule (explicitly restoring the summation for now)

$$\left.\frac{dg}{ds}\right|_p = \sum_i \left.\frac{dx^i}{ds}\right|_p \frac{\partial g}{\partial x^i}. \tag{2.15}$$

But Eq. (2.15) is valid *independent of what the function g is!* So we can simply drop g and write the **tangent vector to any curve** $x^i(s)$ as

$$\left.\frac{d}{ds}\right|_p = \sum_i \left.\frac{dx^i}{ds}\right|_p \frac{\partial}{\partial x^i} = \sum_i \left.\frac{dx^i}{ds}\right|_p \partial_i. \tag{2.16}$$

Let's call

$$\frac{d}{ds} \equiv \mathbf{V}, \tag{2.17}$$

then at every point $p \in s$ is a vector with components

$$V_p^i \equiv \left.\frac{dx^i}{ds}\right|_p. \tag{2.18}$$

Double but: we haven't specified what the curve is either! Indeed, the curve itself is also mere scaffolding : I could have easily written a vector down, and then drawn a curve such that the vector is the curve's tangent at that point. So for *any vector*, we can express it as (using the Einstein Summation convention from now on)

$$\boxed{\mathbf{V} = V^i \partial_i}. \tag{2.19}$$

And everything works out exactly how you think it should. For any vector $\mathbf{V}$, we can write its *components*, explicitly labeling it with an index (an upper i here). The components of course depend on its basis vectors, which are now given by $\hat{\mathbf{e}}_{(i)} = \partial_i$ as we have asserted in Eq. (2.2). Harking back to your skool days, $\mathbf{i} = \partial_x$, $\mathbf{j} = \partial_y$ and $\mathbf{k} = \partial_z$, so a vector $\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ or $V^i = (a, b, c)$. *Don't confuse coordinates with the coordinate basis! The former is simply a n-tuple of numbers, while the latter is a vector basis that is natural to the coordinates.*

Some of you might be unfamiliar with equations which have derivatives hanging ∂_i in thin air, without anything to act on. This is an example of an *operator* expression – an operator is something that eats something and give you another, i.e. it is simply a map since it takes a function and give you another function as long as the former is differentiable. The derivative operator is a **linear operator**.

Given this elaborate construction, you are now ready to see vectors in a new light. We said in Eq. (2.2) that the tangent vector to any curve $x^i(s)$ is independent of what the function g is. If I give you any other smooth function, say h, acting on it with d/ds at point p

$$\left.\frac{dh}{ds}\right|_p = V_p^i \frac{\partial h}{\partial x^i}. \tag{2.20}$$

The result that you get when you do this is a number – here this number is the rate of change of the function h along the curve – so *vectors are* **maps** *that take a function and give you back a number or a function.* So you can equally write it in the "map" notation,

$$\mathbf{V}_p(h) = V_p^i \frac{\partial h}{\partial x^i} \in \mathbb{R} \text{ or } \mathbf{V}_p : h \to \mathbb{R}. \tag{2.21}$$

Furthermore, the map is *linear* (section 2.1.3), since given any two functions g and h,

$$\mathbf{V}_p(ag + bh) = a\mathbf{V}_p(g) + b\mathbf{V}_p(h) \ , \ \forall a, b \in \mathbb{C}, \tag{2.22}$$

which follows from the linearity of the derivative operator on smooth functions.

Next, we get to the notion of addition of vectors – hopefully you have seen enough up to now to wonder what other stuff you have learned in highskool that we are going to make fancy! We have said that the definition of a vector at point p does not depend on which curve we choose that goes through it, though implicitly different curves will give you different vectors. Now, you all have learned that you can add two vectors to get a third one in high skool, but for our construction to work out, this new vector must also be the tangent vector of some new curve.

Suppose we have two curves $x^i(s)$ and $y^i(s)$ intercepting at point p so $x^i(p) = y^i(p) = x^i_p$. Since the parameters are arbitrary we can choose them to be the same parameter s. At point p, let the tangent vectors of these two curves be X^i_p and Y^i_p. Meanwhile, we can *define* a new curve $z_i(s)$ which also intercepts point p using the two old curves via

$$z^i(s) = a(x^i(s) - x^i_p) + b(y^i(s) - x^i_p) + x^i_p \ , \ \forall a, b \in \mathbb{R}. \tag{2.23}$$

where the awkward insertions of $x^i_p \equiv x^i(p)$ are simply to ensure that the resulting curve $z^i(s)$ passes through p. The tangent vector to this curve is then, using Eq. (2.2),

$$\begin{aligned} \left.\frac{d}{ds}\right|_p &= \left.\frac{dz^i}{ds}\right|_p \frac{\partial}{\partial x^i} \\ &= \left(a \left.\frac{dx^i}{ds}\right|_p + b \left.\frac{dy^i}{ds}\right|_p\right) \frac{\partial}{\partial x^i} \\ &= (aX^i_p + bY^i_p)\partial_i \\ &\equiv Z^i_p \partial_i \end{aligned} \tag{2.24}$$

We have used the fact that x^i_p are constants in the 2nd equality, and the usual definitions for the tangent vector in the 3rd equality. The final equality shows that the gradient vector of the new curve $z_i(s)$ at p is $Z^i_p = aX^i_p + bY^i_p$ as we have asserted[5].

So, we have shown that, at every point p in the plane, we can define a *set* of vectors, whose addition (and subtraction) properties are properly defined. Adding two vectors at point p gives you another vector in p, just like you learned in high skool.

Now here is a very important point: we have been incredibly careful in saying "at point p" – this construction works when talking about vectors at the same point. *It makes no sense to discuss the addition of vectors at different points in the plane!* You might protest, "But wait, when I learned vectors at skool, my teacher told me that I can add a vector in point p to another vector in a different p'!". It turns out that this is only possible in very special situations[6]. In *curved* space, this is no longer true: remember that if you try to add vectors on different points on a sphere you will get nonsensical answers. Since we are going to move into curved spacetimes soon, this fact is extremely important to keep in mind. Indeed, the reason we are studying vectors in such excruciating detail is to prepare you for the general curved space case – you will find that when we study that (in the next section) you would have done much of the work here.

Taking stock, we have showed that

- Vectors are derivatives $d/d\lambda$ along a curve parameterized by λ.
- At each point p there exists a family of vectors which can be added to each other to make another vector at that point.

[5]Actually, we cheated a step here. Technically, we still need to make the last step of defining that the object $\mathbf{Z}_p = Z^i_p\partial_i$ is a linear map of any smooth function to $\mathbb{R}$, $\mathbf{Z}_p(f) \in \mathbb{R}$.

[6]Here the reason is that the plane is Euclidean, we will discuss this in section 2.3.

- Since the curves $x^i(s)$ are completely arbitrary, the components of the vectors dx^i/ds at any point p are also arbitrary, thus by choosing the values of the coefficients of the basis $\hat{\mathbf{e}}_{(i)} = (\partial_x, \partial_y)$ we can construct *any* 2-dimensional vectors. We can of course generalize this to any dimensions – in 3D Cartesian space, we will have a 3-D basis $\hat{\mathbf{e}}_{(i)} = (\partial_x, \partial_y, \partial_z)$ etc.

A set of vectors collected together, with the properties above, is called a **Vector Space**. A "space" in mathematical jargon usually means a set of things with some added mathematical structure. The things in this set are just the vectors, and the mathematical structure is the properties that these vectors obey such as addition, the existence of inverses etc.

Formally, a **Vector Space**, sometimes written as $\mathbb{V}$, is a space whose objects (the vectors) obey the following rules. For $U, V, W \in \mathbb{V}$,

- Associativity: $U + (V + W) = (U + V) + W$.
- Closure: $U + V \in \mathbb{V}$.
- Identity: $U + 0 = U$, where 0 here denotes a vector where all the entries are zeroes.
- Inverse: For every U, there exists an inverse element $-U$ such that $U + (-U) = 0$.

There is also the "scalar multiplication" operations. For any $a, b \in \mathbb{C}$,

- $a(V + U) = aV + aU$,
- $(a + b)V = aV + bV$,
- $a(bV) = (ab)V$.

Examples of vector spaces are ordinary 3-vectors at a point, the eigenfunctions of time-independent Schrödinger's equation and representations of Lie Groups.

In our plane, at each point p, there exists a vector space, which we will call[7] T_pM. You can do addition, subtraction etc of the vectors *only in the same space*. I.e. you cannot add vectors of T_pM to vectors of another point $T_{p'}M$ without extra rules ("adding additional mathematical structure"). Since, in our example here, these vectors are tangents to some curve passing through p, it is also called a **Tangent Space**.

On the other hand, since there is a vector space at every point, you can construct a **Vector Field** on the space. A Vector Field $\mathbf{V}$ or V^μ (with no point p subscript now) is simply a map from all points p to some vector which lives in the Vector Space of the point. See Figures 2.5 and 2.6. Other examples of vector fields you are familiar with are the electric (and magnetic fields), the acceleration field of Earth's gravity etc.

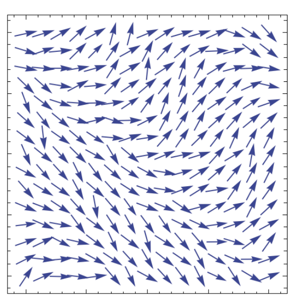


Figure 2.5: A vector field.

Finally, before we go towards the next section as we generalize all these to curved space, we will make contact with Chapter 1. There, we introduce four vectors (and also three vectors) V^μ, which are

[7]What's in the name : T means "tensor", "p" means the point p, and "M" means "manifold $\mathcal{M}$" which we will discuss in the next section. So T_pM means "tensor space at point p in manifold $\mathcal{M}$."

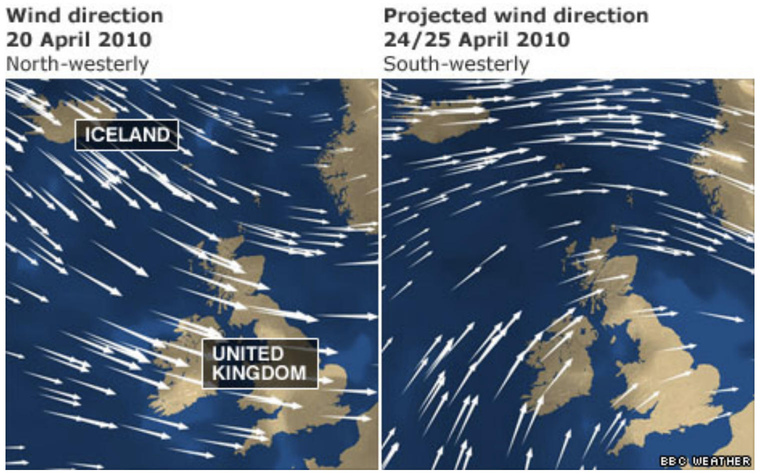


Figure 2.6: The weather map of wind speed and direction in a BBC weather forecast is a vector field. Also, in the process of writing this lecture, I just realized that "Northwesterly" means wind *from* the North west, not *to* the North west.

4 component objects $V^\mu = (V^0, V^i, V^2, V^3)$. Implicitly, these components are *basis dependent*, so the components of V^μ are vectors in the coordinate basis of the unprimed coordinates (t, x, y, z). In other words, the vector is really

$$\mathbf{V} = V^\mu \hat{\mathbf{e}}_{(\mu)} \ , \ \hat{\mathbf{e}}_{(\mu)} = (\partial_t, \partial_x, \partial_y, \partial_z). \tag{2.25}$$

What happens when we change coordinates, say from (t, x, y, z) to (t', x', y', z') where $x'(t, x, y, z)$ and $y'(t, x, y, z)$ etc? In the new primed coordinates, we can also define its corresponding new coordinate basis

$$\hat{\mathbf{e}}_{(\mu')} = (\hat{\mathbf{e}}_{(t')}, \hat{\mathbf{e}}_{(x')}, \hat{\mathbf{e}}_{(y')}, \hat{\mathbf{e}}_{(z')}) = (\partial_{t'}, \partial_{x'}, \partial_{y'}, \partial_{z'}). \tag{2.26}$$

We can go from the $\hat{\mathbf{e}}_{(\mu)}$ basis to the $\hat{\mathbf{e}}_{(\mu')}$ basis by executing a **Basis Transformation**[8]

$$\hat{\mathbf{e}}_{(\mu')} = \Lambda_{\mu'}{}^{\nu} \hat{\mathbf{e}}_{(\nu)}. \tag{2.27}$$

To find the components of $\Lambda_{\mu'}{}^{\nu}$, we use the chain rule. For example, for $\mu' = t'$, it is

$$\partial_{t'} = \frac{\partial_t}{\partial_{t'}} \partial_t + \frac{\partial_x}{\partial_{t'}} \partial_x + \frac{\partial_y}{\partial_{t'}} \partial_y + \frac{\partial_z}{\partial_{t'}} \partial_z \tag{2.28}$$

so $\Lambda_{\mu'}{}^{\nu}$ is a 4×4 square matrix with components

$$\boxed{\Lambda_{\mu'}{}^{\nu} \equiv \frac{\partial x^\nu}{\partial x^{\mu'}}.} \tag{2.29}$$

Indeed, you have seen such linear transformation before – in Chapter 1, the Lorentz Transformation is exactly such a coordinate transformation (we even used the same symbol Λ). Since the vector $\mathbf{V}$ is *independent of basis*, a conversion of the basis vector with Eq. (2.29) requires an inverse transform $V^{\mu'} = \Lambda^{\mu'}{}_{\alpha} V^\alpha$ where

$$\Lambda^{\mu'}{}_{\mu} = \frac{\partial x^{\mu'}}{\partial x^\mu} \tag{2.30}$$

[8]It is a Linear transformation because the transformation can be represented by a square matrix multiplication.

so

$$\begin{aligned}
\mathbf{V} &= V^{\mu'}\hat{\mathbf{e}}_{(\mu')} \\
&= V^{\mu'}\partial_{\mu'} \\
&= \Lambda^{\mu'}{}_{\alpha}V^{\alpha}\Lambda_{\mu'}{}^{\nu}\partial_{\nu} \\
&= V^{\alpha}\delta^{\nu}_{\alpha}\partial_{\nu} \\
&= V^{\nu}\partial_{\nu} \\
&= V^{\nu}\hat{\mathbf{e}}_{(\nu)}
\end{aligned} \tag{2.31}$$

where in the 3rd line we have used

$$\Lambda^{\mu'}{}_{\alpha}\Lambda_{\mu'}{}^{\nu} = \frac{\partial x^{\nu}}{\partial x^{\alpha}} = \delta^{\nu}_{\alpha} \tag{2.32}$$

which is a nice derivation and explanation of Eq. (1.4). Hence, everything checks out.

We end this section with several remarks on the notation.

- *Coordinate Bases*: In these lectures, we will work exclusively with coordinate bases $\hat{\mathbf{e}}_{(\mu)} = \partial/\partial x^{\mu}$ where x^{μ} are the coordinates. While the Cartesian coordinate basis $\hat{\mathbf{e}}_{(i)} = (\partial_x, \partial_y, \partial_z)$ on the flat plane which you are most probably familiar with is both orthogonal and orthonormal (i.e. the basis vectors are normalized to unity), many coordinate bases are *not orthogonal nor orthonormal* – as you will see when using spherical polar basis on a sphere in a (Homework) problem. There is nothing unusual about using bases which are not orthogonal (except maybe you are not familiar with it), but sometimes it is useful to explicitly use orthornomal non-coordinate bases, called **tetrads**. In this class however, we will always be using coordinate bases, and leave the study of tetrads to your advanced GR class..

- *Abstract and component vectors*: Both the "bold" form $\mathbf{V}$ and the "index" form V^{μ} or $V^{\mu'}$ are both called vectors: the former is an **abstract** object which is *basis independent*, while the latter is a *representation of the vector in some choice of basis whose component values change accordingly.* In these lectures (and when we discuss physics in general), it is usually evident what we meant, but to be clear we will emphasise the following here

$$\begin{aligned}
\mathbf{V} &: \quad \text{abstract vector} \\
V^{\mu} &: \quad \text{component vector in } \hat{\mathbf{e}}_{(\mu)} \text{ basis.}
\end{aligned} \tag{2.33}$$

  When you work with component form vectors, you must make sure that all the vectors are in the same basis. We will come back to these points in full force in the next section.

- *Indices Schwindices*: Not everything with an index is a vector! For example, the object $x^{\mu}(p)$ is simply the coordinate of point p, even though it has an upper μ to it. So you might be tempted to do a "coordinate transform" to the μ' coordinate with

$$x^{\mu'} \stackrel{?}{=} \Lambda^{\mu'}{}_{\mu}x^{\mu} \tag{2.34}$$

  but this is wrong, because general coordinate transformations $x^{\mu'}(x^{\nu})$ are not necessarily linear. (You might protest that you have been taught to do this in Special Relativity but the reason we can use this equation for Lorentz Transformations is because it is linear in Cartesian coordinates.)

2.3 Manifolds and Things that live on them

In the previous section 2.2, we mostly worked in a plane equipped with coordinates (x, y). You can extend this to any dimensions as you like, with coordinates $(x, y, z, \dots)$ etc. Points in such an $n-$dimension space then can be labeled by an n-tuple of real numbers (i.e. the coordinates), and is called $\mathbb{R}^n$.

Furthermore, in high skool, you have learned the following "truths" about the flat plane:

- The coordinates (x, y) "cover" (or overlay) the entire space, from $x, y = (-\infty, \infty)$.
- The distance between two points $p = (x_p, y_p)$ and $q = (x_q, y_q)$ is given by $L^2 = (x_p - x_q)^2 + (y_p - y_q)^2$.

These extra "truths" are actually additional structure which we impose on $\mathbb{R}^n$. Such an "enhanced" space is called an n-dimensional **Euclidean Space**, properly labeled $\mathbb{E}^n$. It is of course named after Εύ$\kappa\lambda\epsilon$ι$\delta\eta\sigma$, or **Euclides** (Euclid) – the father of geometry. Euclidean space has a lot of very nice special properties, one of the most intuitive and familiar to you is that "parallel lines stay parallel". In these lectures, it is *important* to distinguish between $\mathbb{E}^n$ and $\mathbb{R}^n$.

Here we want to make an important distinction between the (x, y) *coordinates* and the nature of the Euclidean space itself. I can give a young girl a flat (i.e. Euclidean) piece of paper, and she can draw parallel lines on it without any notion of coordinates. In fact, we can define some crazy coordinates on the paper and it'll still be fine. The (x, y) coordinates, however, is "natural" because it makes everything easy.

On the other hand, you have also worked with non-Euclidean spaces before – consider the 2D surface of a 3D ball (see Figure 2.7). In mathematical lingo, the 2D surface is called a 2-**sphere** $\mathbb{S}^2$ (while the entire 3D ball is called, not surprisingly, a 3-**ball** $\mathbb{B}^3$). You probably knew that parallel lines on spheres do not stay parallel – if you start two parallel lines from anyhere on the sphere they will eventually meet somewhere. Nevertheless, in small enough regions on the sphere, *locally things look like* $\mathbb{R}^2$ – your daily experience on living on Earth should convince you of this truth.

What about coordinates on $\mathbb{S}^2$? A "natural" choice of coordinates is to split the sphere into the azimuthal $0 \leq \phi < 2\pi$ and altitude $0 \leq \theta < \pi$. However, as the famous trick question goes "What timezone are the North or South poles in?", this choice of coordinates break down at the poles (i.e. what is the value of ϕ at $\theta = 0, \pi$?). Also the coordinates are discontinuous at $\phi = 0, 2\pi$ (what day is it at the international date line?). To fix these coordinate problems, we can use several *overlapping* coordinate

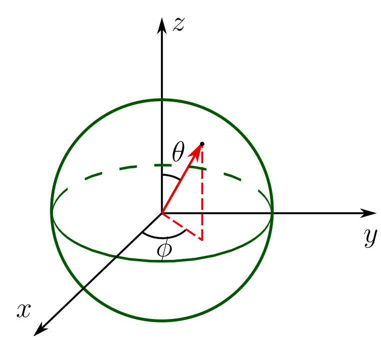


Figure 2.7: Spherical Polar Coordinates on $\mathbb{S}^2$.

systems, and devise rules to "stitch" them up together at the overlapping regions. For example, the British uses the Greenwich Meridian and the French uses Paris Meridian as the azimuthal coordinate, but it's perfectly find as long as they overlap (they do) and you can switch from one coordinate to the other wherever you like. In fact, in general more complicated curved spaces, *it is not possible to find a single set of coordinates that cover the entire space.*

The generalization of the sphere to more complicated geometries requires some extra mathematical structure called **differential manifolds**.

2.3.1 Manifolds

What is a Manifold?

A n-dimensional manifold is a **continuous** *space that* **looks locally like** $\mathbb{R}^n$.

More prosaicly, a n-dimensional manifold is a "continuous[9] space" where we can attach a set of real number $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ (called an n**-tuple**) on each element (or point) on it.

The Earth's surface is a 2D manifold. 3D Euclidean space is a manifold. A circle is a 1D manifold, so is an infinite line. But a "figure eight" is not a 1D manifold because at the crossing point it takes more than a number to describe it. In Special Relativity, the 3+1 spacetime is a manifold, called **Minkowski Space** – we will describe it properly later. In general relativity, general spacetimes are manifolds. In special and general relativity, each point on the manifold represents a spacetime **event**. At each point, we can have functions that live on them to describe actual physics, say e.g density, temperature, and, for this class, spacetime curvature. Colloquially we say that functions can "live" on the manifold in the sense that each point on the manifold can exist any kind of functions – just like functions can "live" on the 2D plane you are familiar with. You can loosely think of a manifold as a "mathematical container" where mathematical things like vectors, functions, numbers can "live in".

We started this discussion lamenting about coordinates, so let's put coordinates on the manifold.

Consider a manifold, $\mathcal{M}$. Separating the idea of coordinates from the space itself, coordinates are **bijective maps** that takes a point on the manifold and give you a set of numbers in $\mathbb{R}$. For example, say for some *open* region (or **neighbourhood**) $\mathcal{N} \subset \mathcal{M}$ of the manifold, mathematically this is

$$\phi : \mathcal{M} \to \mathbb{R}^n \quad \forall \mathcal{N} \subset \mathcal{M} \tag{2.35}$$

with n denoting the **dimensions** of the manifold. The coordinate system ϕ is called a **coordinate chart** or **chart** or **coordinates** for short. See Figure 2.8.

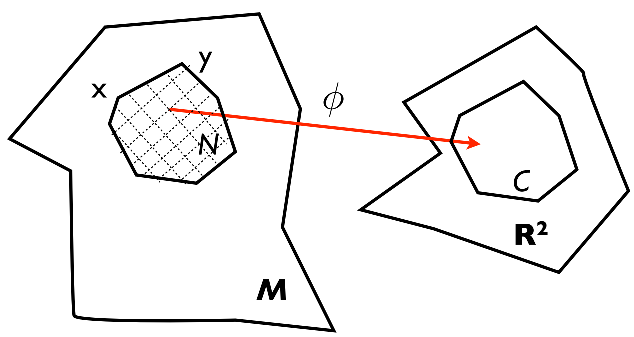


Figure 2.8: A chart from a neighbourhood $\mathcal{N}$ of $\mathcal{M}$ to $\mathbb{R}^2$. The chart is an bijective map from $\mathcal{N}$ to region C of $\mathbb{R}^2$. So for every point $p \in \mathcal{N}$, $\phi(p) = (x, y)$ gives you an ordered pair of coordinates. Since this is a bijective map, the inverse $\phi^{-1}(x, y) \in \mathcal{N}$ gives you a point on the manifold for that particular value of the coordinate.

In other words, we can associate each point in the neighbourhood $\mathcal{N}$ with an unique n-tuple number $(x^1, x^2, \ldots)$ – this is just a high-brow way of saying that we labeled each point in the region with a coordinate choice.

We can then cover the entire manifold with as many *overlapping* charts $\phi_1, \phi_2, \phi_3, \ldots$ as we need, until every point in the manifold is covered. The keyword *overlapping* is important, because we need to be able to move from one chart to another without any discontinuities. A collection of such a set of charts is called an **atlas** (obviously) – see Figure 2.9.

[9]**The notion of continuity is actually non-trivial, and is the subject of a whole new subfield of mathematics called **Point Set Topology**, but we will not go there but rely on your innate skool-train powers of intuition. Roughly speaking, it means that you can move from one point on the set to another point on the set by taking infinitisimally small steps. Note that it *doesn't* mean that the space has "no holes".

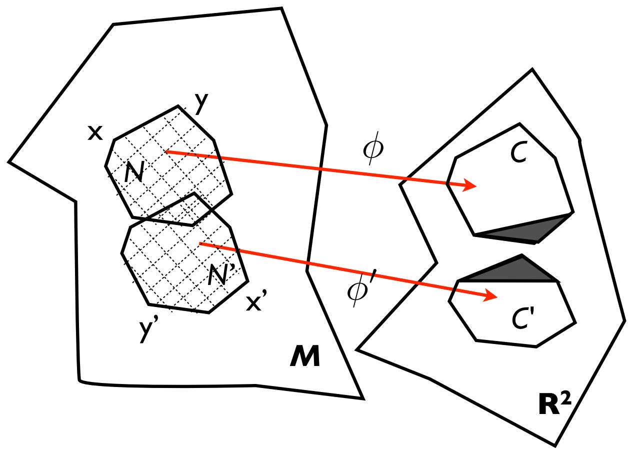


Figure 2.9: Two overlapping charts ϕ and ϕ' from two overlapping neighbourhoods $\mathcal{N}$ and $\mathcal{N}'$ of $\mathcal{M}$ to two *not necessarily overlapping* regions in $\mathbb{R}^2$. In the overlapping region, we can find a coordinate transform from C to C' in the usual, i.e. $x(x', y')$ and $y(x', y')$. The bijectiveness of the both charts mean that the inverse transform $x'(x, y)$ and $y'(x, y)$ exist.

Consider two charts ϕ and ϕ' with coordinates (x, y) and (x', y'). In the overlapping region, we can define a coordinate transform between the two coordinates systems (not necessarily overlapping in $\mathbb{R}^2$)

$$y' = y'(x, y)\ ,\ x' = x'(x, y), \tag{2.36}$$

and their inverses (which exist because ϕ and ϕ' are both bijective in the region).

You can think of Eq. (2.36) as functions of x and y. In general, these can be any functions, but we want to impose a further restriction: *these functions must be smooth*, i.e. C^∞ with respect to both x and y (see section 2.1.2 for definition of smoothness). While we have done this for 2 dimensional manifolds, it is easy to generalize this to as many dimensions as you want. In this course, we will often be dealing with 4-dimensional manifolds – for the 4 dimensions of space and time.

Given an atlas, and differentiable coordinate charts on a manifold $\mathcal{M}$ means that we can now do calculus on functions and other things that live on the manifold. Such a manifold is called a **Differentiable Manifold**. Since we will always be dealing with differentiable manifolds in this course, we will just call it the "manifold".

For example, a real function (that lives) on $\mathcal{M}$ is a map from any point to some real number

$$f : \mathcal{M} \to \mathbb{R}. \tag{2.37}$$

To write this function in the familiar form, at every region, we have a coordinate chart ϕ, which gives it a coordinate (x, y) so we can write simply $f(x, y)$. In a different region with a different chart ϕ', it would then be $f(x', y')$. In an overlapping region with charts ϕ and ϕ', then $f(x, y) = f(x', y')$ where $x'(x, y)$ and $y'(x, y)$, exactly as you would have intuitively guessed.

Let's pause for a moment to discuss notation. The *super* high brow way would be to explicitly carry this chart ϕ around in any calculation you do (which some books and lecture notes do): we write for any given point $p \in \mathcal{N}$ on the manifold, $\phi(p) = (x, y)$ gives you an ordered pair of coordinates. For any function f, its value at p would be to write $f(\phi(p)) = f(x, y)$. But this is super clumsy, so we usually just write $f(x, y)$ and skip the ϕ altogether. In order to keep track of different charts ϕ and ϕ', usually when we write the coordinatees (x, y) and (x', y') it's clear we are referring to different charts. Also, since vector components V^μ are also coordinate chart dependent if we use coordinate basis, it's clear when we write V^μ an $V^{\mu'}$ that the vector is using the coordinate basis of the respective charts.

2.3.2 Vectors and Co-vectors on Manifolds

In this section we will generalize our discussion on vectors in section 2.2. We have done so much of the work already that this generalization is almost trivial – everything simply follow through with the simple addition that the (flat) plane that we were working with is now replaced with some general manifold equipped with coordinates:

Consider a curve parameterized by s that lives on a 4 dimensional manifold $\mathcal{M}$, $x^\mu(s)$, see Figure 2.10. Furthemore, let g be a function on $\mathbb{M}$. This curve passes through a neighbourhood $\mathcal{N}$ which is covered by coordinates chart $x^\mu = (t, x, y, z)$. We want to ask: what is the rate of change of the value of g along this curve g at point $p \in \mathcal{M}$. Easy, take the derivative with respect to $s \in \mathcal{N}$

$$\left.\frac{dg}{ds}\right|_p = \left.\frac{dx^\mu}{ds}\right|_p \frac{\partial g}{\partial x^\mu} \tag{2.38}$$

exactly as Eq. (2.15). We can now see why we need the coordinate charts to be differentiable – it ensures dx^μ/ds exists. And it follows that since g is arbitrary, we can drop it to write

$$\mathbf{V}_p = \left.\frac{d}{ds}\right|_p = \left.\frac{dx^\mu}{ds}\right|_p \hat{\mathbf{e}}_{(\mu)} \ , \ \hat{\mathbf{e}}_{(\mu)} = \partial_\mu, \tag{2.39}$$

which is the same as Eq. (2.2). Indeed, everything else follows in exactly the same way:

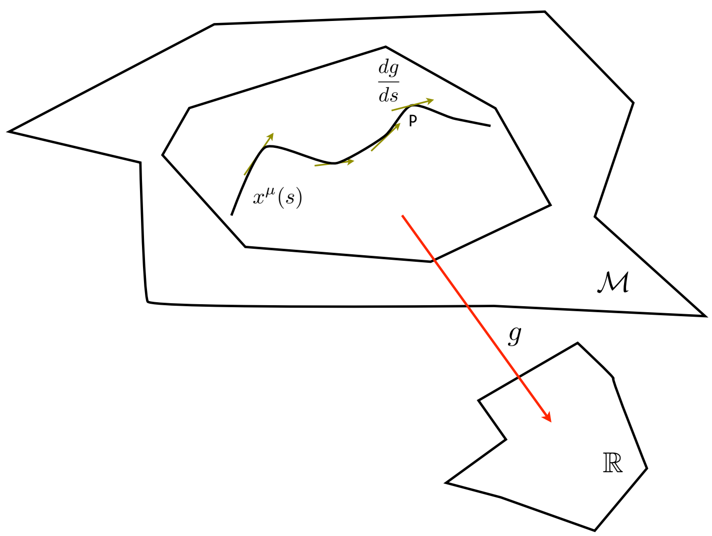


Figure 2.10: A curve on a manifold.

- The *component* form of the vector $\mathbf{V}_p$ in the μ coordinate chart is $V_p^\mu = dx^\mu/ds|_p$.
- At the overlapping regions between two coordinate charts, with their respective coordinate basis $\partial_\mu = (\partial_t, \partial_x, \partial_y, \partial_z)$ and $\partial_{\mu'} = (\partial_{t'}, \partial_{x'}, \partial_{y'}, \partial_{z'})$, we can represent the same vector with different components via $\mathbf{V} = V^\mu \partial_\mu = V^{\mu'} \partial_{\mu'}$, where $V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu$.
- At each point $p \in \mathcal{M}$, we can construct a 4-dimensional vector space T_pM. As usual, you *cannot* add or subtract vectors from different vector spaces T_pM and $T_{p'}M$ if $p \neq p'$. [10]
- Vector Field $\mathbf{V}$ is a map from all points on $\mathcal{M}$ to some vector which lives in the Vector Space of the point.

[10] **Sometimes, people combine all the vector spaces T_pM of all points in a ginormous manifold to make a new space, called a **fibre bundle** or in our special case where the vector spaces are derived from the original manifold itself, also known as **tangent bundle**. In the mad mind of a general relativist who had been locked up in a room for too long, the entire Universe is nothing but a gigantic tangent bundle and nothing else.

2.3.3 Co-vectors or dual vectors or one-forms

In Chapter 1, we mentioned that the index placement of the vectors are important in terms of the values of the components. For example in Eq. (1.83) and Eq. (1.84), we showed that A^μ and A_μ are vectors with different values, where the index had been "lowered" (or raised) the index with the metric, e.g. Eq. (1.40) and Eq. (1.41). We can call A_μ a "4-vector with lower indices", which would be correct. However, in this section, we will expand on this construction – setting up the scene where we will then explain why we can "raise" and "lower" the the indices with this mysterious object called the metric.

Consider two objects, one with raised index V^μ and one with lowered index U_μ. Let the components of these objects to be real numbers, so when we sum them up using Einstein's convention

$$U_\mu V^\mu \in \mathbb{R} \tag{2.40}$$

will get you a real number. Here is a less familiar way of looking at the above Eq. (2.40)

$$\bar{\mathbf{U}}(\mathbf{V}) \in \mathbb{R} \tag{2.41}$$

where $\mathbf{V} = V^\mu \partial_\mu$ is the (coordinate) basis independent form of V^μ and $\bar{\mathbf{U}}$ is the basis independent form of U_μ. Leaving aside the question of the basis for $\bar{\mathbf{U}}$ for the moment, Eq. (2.41) says that $\bar{\mathbf{U}}$ maps any four-vector $\mathbf{V}$ to a real number

$$\bar{\mathbf{U}} : \mathbf{V} \to \mathbb{R}, \tag{2.42}$$

or, in other words, $\bar{\mathbf{U}}$ is *really a function of (four)-vectors.* From Eq. (2.40), it's clear that this also means that vectors are also functions of these objects

$$\mathbf{V} : \bar{\mathbf{U}} \to \mathbb{R} \tag{2.43}$$

such that

$$\mathbf{V}(\bar{\mathbf{U}}) = \bar{\mathbf{U}}(\mathbf{V}). \tag{2.44}$$

The last equation is not surprising, since back in index form they are $V^\mu U_\mu = U_\mu V^\mu$.

Note that to distinguish them from vectors, we have added an overbar $\bar{\mathbf{X}}$. The old name for such a thing is called a **co-vector** or **dual vector** sometimes, but the fashionable swanky word for it is **one-form**, although the uncouth plebs who are not well-trained in the ancient arts of differential geometry call it "vector with lower index". You can (correctly) think of co-vectors as equal "partners" to the vectors (hence "co-") – the word we use is that they are **dual** to each other. Furthermore, co-vectors act linearly on vectors in the following way

$$\bar{\mathbf{U}}(a\mathbf{V} + b\mathbf{W}) = a\bar{\mathbf{U}}(\mathbf{V}) + b\bar{\mathbf{U}}(\mathbf{W}) \ , \ \forall \ a, b \in \mathbb{C}. \tag{2.45}$$

You can show that, with some algebra, the linearity of this action means that, similar to vectors where at every point $p \in \mathcal{M}$ there exists a vector space T_pM, at every point p there exists a **dual vector space** of co-vectors, called T_p^*M. This space is, as its name imply, also a vector space in the formal sense discussed in Page 33.

In fact, you have already spent a lot of your life working with co-vectors – you just call them by some other names. Some examples:

- *Matrix Multiplication.* Given two row matrices with real components

$$V = (v_1 \ v_2 \ v_3) \ , \ U = (u_1 \ u_2 \ u_3) \tag{2.46}$$

  the *dual* of U is its transpose

$$U^T = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{2.47}$$

so

$$U^T V = \sum_{i=1}^{3} u_i v_i = U^T(V) \in \mathbb{R}. \tag{2.48}$$

- *Quantum Mechanics.* The state of all possible quantum states of a system is a Complex Vector Space called the **Hilbert Space**, and each element of this space (a state) is represented by ket $|\psi\rangle$ (or if you prefer, some wave function $\psi(x)$). There exists a dual space, whose elements are represented by a bra $\langle\phi|$. As you have learned, taking the "bra-ket"

$$\langle\phi|\psi\rangle = \int dx\ \phi^\dagger\psi \in \mathbb{C} \tag{2.49}$$

gives you some complex number. In other words, in wave function language, the dual space of the ket, $\psi(x)$, is the **Hermitian conjugate** $\psi^\dagger(x)$.

Of course, if the components of the objects themselves are functions, then the map gives you some function

$$\mathbf{V} : \bar{\mathbf{U}} \to \text{some function}. \tag{2.50}$$

Like the vector, we can decompose them using some suitable basis to get their component form. As we have discussed in glorious detail in section 2.2, the vector $\mathbf{V} = V^\mu\partial_\mu$ is really a basis-independent object, but can be decomposed into its component form V^μ with some choice of coordinate basis given by the partials $\hat{\mathbf{e}}_{(\mu)} = \partial_\mu$. Similarly, we can decompose the co-vectors into its *coordinate basis* constructed out of the coordinates (t, x, y, z)

$$\boxed{\bar{\mathbf{U}} = U_\mu \mathrm{d}x^\mu \equiv U_\mu \hat{\mathbf{e}}^{(\mu)}} \tag{2.51}$$

where the basis co-vectors

$$\boxed{\hat{\mathbf{e}}^{(\mu)} = \mathrm{d}x^\mu}. \tag{2.52}$$

Wait! Look at Eq. (2.51) and Eq. (2.52) again! The "d" is not italicized d, actually is the non-italicized version. In fact, $\mathrm{d}f$ really means "the *gradient* of the function f", so $\mathrm{d}x^\mu$ means the gradient of the set of functions x^μ labeled by μ, and is *defined* by the formula

$$\boxed{\mathrm{d}f\frac{d}{d\lambda} \equiv \frac{df}{d\lambda}} \tag{2.53}$$

for any tangent vector $d/d\lambda$ at the same point.

Why is $\mathrm{d}f$ called the gradient? For any scalar function f, then in 3-dimensions

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i = \frac{\partial f}{\partial x^i}\hat{\mathbf{e}}^{(i)} \tag{2.54}$$

and since $\partial f/\partial x^i = \nabla_i f$, it is exactly the gradient for any function f that you know and love[11]. Given these, it's easy to show that, for the same coordinate basis $\mu = (t, x, y, z)$,

$$\begin{aligned} \bar{\mathbf{U}}(\mathbf{V}) &= U_\mu \mathrm{d}x^\mu V^\nu \frac{\partial}{\partial x^\nu} \\ &= U_\mu V^\nu \mathrm{d}x^\mu \frac{\partial}{\partial x^\nu} \\ &= U_\mu V^\nu \delta^\mu_\nu \\ &= U_\mu V^\mu \end{aligned} \tag{2.55}$$

where we have used

$$\mathrm{d}x^\mu \frac{\partial}{\partial x^\nu} \equiv \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu \tag{2.56}$$

[11] For those who have been confused by all those thermodynamic quantities with their myriad of partials and total derivatives, this Chapter's epitaph might make a bit more sense now.

in the 3rd line, which is derived using Eq. (2.53) – remember the x^μ is not a vector but simply a set of functions labeled by μ. Sometimes "d" is called the **exterior derivative**.

Basis transformations for co-vectors follow analogously to those of vectors Eq. (2.29)

$$\boxed{\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\mathrm{d}x^\mu} \tag{2.57}$$

so for any co-vector U_μ, its component transforms as

$$\boxed{U_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}U_\mu.} \tag{2.58}$$

Finally, you might be wondering about the action of the exterior derivative on a tensor, e.g. $\mathrm{d}\bar{\mathbf{U}}$ where $\bar{\mathbf{U}}$ is a co-vector. To define it we need to define a new way of taking products of two co-vectors called taking the **wedge product, and introduce a new subclass of co-vectors called **differential forms**. We won't have time to discuss this unfortunately in this lectures although they are quite important in GR. So far we have been talking about taking derivatives – what about doing *integration*? Differential forms are basically the objects where you integrate over. For a clear discussion, see B. Schutz *Geometrical Methods of Mathematical Physics*, Chapter 4.**

2.3.4 Tensors

We have discussed that, at any point $p \in \mathcal{M}$, there exists a vector space T_pM and its dual T_p^*M. We can construct product spaces by taking the **tensor product** of vector spaces

$$\mathcal{T} = T_pM \otimes T_pM. \tag{2.59}$$

The symbol $\otimes$ means that the spaces are "multiplied", and is called a **tensor product**.

A **tensor product** $\otimes$ is a binary operator, which takes two vector spaces $\mathbb{V}$ and $\mathbb{W}$ and give you a third vector space $\mathbb{Z}$ in the following way. Let $\hat{\mathbf{e}}_i^{(V)}$ and $\hat{\mathbf{e}}_i^{(W)}$ be the basis vectors of $\mathbb{V}$ and $\mathbb{W}$, then $\mathbb{Z}$ will be spanned by a basis which can be generated by $\hat{\mathbf{e}}_i^{(V)} \times \hat{\mathbf{e}}_i^{(W)}$, where strictly we have used $\times$ instead of $\otimes$ because it is a direct or Cartesian product[a]. Furthemore, this new basis is called a **tensor product basis** $\hat{\mathbf{e}}_i^{(V)} \otimes \hat{\mathbf{e}}_i^{(W)}$ when they are put together in a special way such that they obey the following conditions,

- Distributive with respective to addition
  $(v_1 + v_2) \otimes w_1 = v_1 \otimes w_1 + v_2 \otimes w_1$.
- Distributive with respective to scalar multiplication
  $\alpha(v_1 \otimes w_2) = (\alpha v_1) \otimes w_2 = v_1 \otimes (\alpha w_2)\ ,\ \forall \alpha \in \mathbb{C}$.
- Tensor products *are not necessarily commutative*, i.e.
  $\hat{\mathbf{e}}_i^{(V)} \otimes \hat{\mathbf{e}}_j^{(W)} \neq \hat{\mathbf{e}}_j^{(V)} \otimes \hat{\mathbf{e}}_j^{(W)}$

The resulting enlarged, or "multiplied", space $\mathbb{V} \otimes \mathbb{W}$ is also a vector space in the sense described in Page 33, except that they are labeled by two indices instead of one. An example is when we tensor product two column vectors together

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\ ,\ w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{2.60}$$

then

$$v \otimes w = \begin{pmatrix} v_1 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ v_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \end{pmatrix}. \tag{2.61}$$

[a]See Chapter 2 of CP2332 Symmetry in Physics lecture notes for a description.

Suppose $\mathbb{V}$ and $\mathbb{U}$ are two dimensional vector spaces, both which can be spanned by the same basis $(\hat{\mathbf{e}}_{(1)}, \hat{\mathbf{e}}_{(2)})$, then the resulting tensor product space vector space $\mathbb{V}\otimes\mathbb{U}$ will be spanned by a generated basis $(\hat{\mathbf{e}}_{(1)}\hat{\mathbf{e}}_{(1)}, \hat{\mathbf{e}}_{(1)}\hat{\mathbf{e}}_{(2)}, \hat{\mathbf{e}}_{(2)}\hat{\mathbf{e}}_{(1)}, \hat{\mathbf{e}}_{(2)}\hat{\mathbf{e}}_{(2)})$. You can also tensor product vector space $\mathbb{V}$ and co-vector space $\mathbb{U}^*$ together too, so the tensor product space would be spanned by the basis $(\hat{\mathbf{e}}_{(1)}\hat{\mathbf{e}}^{(1)}, \hat{\mathbf{e}}_{(1)}\hat{\mathbf{e}}^{(2)}, \hat{\mathbf{e}}_{(2)}\hat{\mathbf{e}}^{(1)}, \hat{\mathbf{e}}_{(2)}\hat{\mathbf{e}}^{(2)})$. You can tensor product as many vector (and co-vector) spaces together to make a really big space, by taking the product spaces of r times T_pM and q times T_p^*M

$$\mathcal{T} = T_pM \otimes T_pM \otimes T_pM \otimes \dots T_p^*M \otimes T_p^*M \otimes T_p^*M \dots \tag{2.62}$$

to make a very big space

This space is called a **rank-**(r, q) **Tensor Space** $\mathcal{T}$, and objects in it are called rank-(r, q) **tensors**. In component form, say a rank-$(2, 3)$ tensor would be

$$\mathbf{T} = T^{\mu\nu}{}_{\rho\sigma\gamma}\partial_\mu \otimes \partial_\nu \otimes \mathrm{d}x^\rho \otimes \mathrm{d}x^\sigma \otimes \mathrm{d}x^\gamma\ ,\ \mathbf{T} \in \mathcal{T} \tag{2.63}$$

So a tensor is generalization of the vector and dual vectors. For the plebs, you can call a tensor an object with some combination of upper and lower indices, and both vectors and co-vectors are tensors. *Be warned that not all objects with indices are tensors! This is such a common source of error that, in these lectures, we will be extremely anal about it.* Note that the fact that tensor products are *not commutative*, the *ordering* of how you tensor product spaces together is important – this is the same as saying that *index placement of the indices of the components are important.*

Tensors, like vectors, are also basis independent, i.e.

$$\mathbf{T} = T^{\mu'\nu'}{}_{\rho'\sigma'\gamma'}\partial_{\mu'} \otimes \partial_{\nu'} \otimes \mathrm{d}x^{\rho'} \otimes \mathrm{d}x^{\sigma'} \otimes \mathrm{d}x^{\gamma'}, \tag{2.64}$$

and while in their (coordinate basis) component form, the coordinate transformation law is, using Eq. (2.29) and Eq. (2.57)

$$\boxed{T^{\mu'\nu'}{}_{\rho'\sigma'\gamma'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\frac{\partial x^{\rho}}{\partial x^{\rho'}}\frac{\partial x^{\sigma}}{\partial x^{\sigma'}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}T^{\mu\nu}{}_{\rho\sigma\gamma}} \tag{2.65}$$

as you might have expected.

As we have discussed in the previous section 2.3.3 on co-vectors, co-vectors maps vectors to a function and *vice versa*, so *a rank-(p,q) tensor maps a rank-(q,p) tensor or a product of lower ranked tensors that add up to rank-(p,q) to a function.* For example, if $\mathbf{T}$ is a rank-$(1,1)$ tensor, then $\mathbf{T}(\mathbf{V},\bar{\mathbf{U}}) \in \mathbb{R}$ or $\mathbf{T}(\mathbf{W}) \in \mathbb{R}$ where $\mathbf{W}$ is a rank-$(1,1)$ tensor. In component form, this is the familiar

$$T_{\mu}{}^{\nu}V^{\mu}U_{\nu} = \text{some function} \tag{2.66}$$

You can also feed a rank-(p,q) tensor with a lower ranked tensor such as a rank-(r,s) tensor such that $r \leq q$ and $s \leq p$, and the result is a rank-$(p-s,q-r)$ tensor. For example, if $\mathbf{T}$ is a rank-$(1,2)$ tensor, and $\mathbf{V}$ is a vector, then

$$\mathbf{T}(\mathbf{V}) = \mathbf{W}\ ,\ \mathbf{W}\ \text{is rank}-(1,1) \tag{2.67}$$

or in the (possibly more familiar) component form

$$T^{\mu}{}_{\nu\sigma}V^{\nu} = W^{\mu}{}_{\sigma}. \tag{2.68}$$

In fact, when you "fully contract" the tensor, i.e. feed the tensor a combination of tensors such that it gives you just a function, then you can call this function a rank-$(0,0)$ tensor. Such a function is called a **Scalar**. Indeed, all this should not be a surprise to you – this is simply a formal description of the usual "index juggling" algebra that you have done before in Chapter 1.

Less obviously, if you feed the tensor with its basis vectors and co-vectors, the result are the components the tensor in that basis, i.e.

$$\mathbf{T}(\partial_{\mu},\partial_{\nu},\ldots,\mathrm{d}x^{\rho},\mathrm{d}x^{\sigma},\ldots) = T_{\mu\nu\ldots}{}^{\rho\sigma\ldots}. \tag{2.69}$$

You probably want to take a moment to think about Eq. (2.69)!

Since physics is independent of coordinates, many physical properties are profitably represented by tensors. Four momenta p^{μ} are simply rank-$(1,0)$ tensors. You'd probably have guessed that the **metric** $\eta_{\mu\nu}$ (Eq. (1.34)) and its gravitational counterpart $g_{\mu\nu}$ (Eq. (1.114)) are rank-$(0,2)$ tensors. But they are so special and central to General Relativity we will postpone their discussion to a special section (section 2.4).

One more note about notation: we will not always write the tensor product $\otimes$ explicitly – since we will be dealing with higher ranked tensors a lot we will often just drop it, i.e. $\mathrm{d}x \otimes \mathrm{d}y \to \mathrm{d}x\mathrm{d}y$. It should be clear in context when we are dealing with tensor product spaces. We will restore the notation when there is a chance of ambiguity.

2.3.5 The Energy-Momentum Tensor

An important object you might have already seen elsewhere is the **energy-momentum tensor** $T^{\mu\nu}$. We have already met the four-momentum p^{μ}, which is a vector (or a rank-$(1,0)$ tensor) which covariantly describe the energy and momentum of a particle. If we have a large number of such particles, while it is possible to describe this system of particles individually, it is often more convenient to treat them as a *continuum* and describe them as a **fluid** instead. For example, water is exactly such a fluid of particles of water molecules. Fluids have macroscopic properties such as density, pressure, viscosity etc.

Consider a small package of fluid in an infinitisimal volume $\Delta V = \Delta x \Delta y \Delta z$, where the coordinates (x, y, z) are chosen such that this package of fluid is at rest *momentarily*. Of course, this package of fluid is composed of many particles each with a different four-momenta, but the package can be at rest *on average*. Nevertheless, one can imagine that at the surfaces of constant x^ν of this package, momenta p^μ is flowing in and out. In fact, this allows us to define a rank-$(2,0)$ tensor $T^{\mu\nu}$ as momenta p^μ flowing in and out of surfaces of constant x^ν. $T^{\mu\nu}$ is also sometimes called the **stress-tensor**.

- T^{00} would be flux of p^0 (energy) in the x^0 (time) direction, i.e. the rest-frame energy density ρ.
- T^{0i} would be the flux of p^0 energy in the x^i direction which is the momentum density.
- T^{i0} would be momentum p^i in the time direction, which is also the momentum density.
- T^{ij} would be the p^i (momentum) through the constant x^j surfaces so if $i = j$ then it's the *pressure* P^i and if $i \neq j$ it is the *shear* stress σ^{ij}.

In the special case of special relativity (i.e. without any spacetime curvature), an important property of the energy-momentum tensor is that it obeys the **conservation law**

$$\boxed{\partial_\mu T^{\mu\nu} = 0 \text{ in special relativity}}. \tag{2.70}$$

Eq. (2.70) is satisfied as long as the equations of motion for the matter fields are satisfied – it is an expression of the dynamics of the matter fields being represented by $T^{\mu\nu}$. In general curved space, Eq. (2.70) is modified – to understand how we need to learn about curvature first so we postpone the discussion until section 4.2.

In the (coordinate) basis ∂_μ, the energy-momentum tensor looks like

$$\mathbf{T} = T^{\mu\nu}\partial_\mu\partial_\nu. \tag{2.71}$$

You can use Eq. (2.65) to transform this to any coordinate system

$$T^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}T^{\mu\nu}. \tag{2.72}$$

In a (Homework) problem, you will show that in a Lorentz Transformation, the energy-momentum tensor transform the physical properties of the fluid in exactly the way you expected. Note that the energy-momentum is **symmetric** under interchange of indices

$$T_{\mu\nu} = T_{\nu\mu}. \tag{2.73}$$

A very special kind of fluid is the **Perfect Fluid**. In such a fluid, the energy momentum tensor looks like

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix} \tag{2.74}$$

where ρ is the energy density and P is the pressure which is **isotropic** (i.e. equal in all spatial directions). We can also write Eq. (2.74)

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + p\eta^{\mu\nu}, \tag{2.75}$$

where u^μ is the **fluid four-velocity** and obey $\eta_{\mu\nu}u^\mu u^\nu = -1$ (**timelike**).

For a "fluid" of particles that are not moving very quickly, the pressure P is usually much smaller than the density, so $\rho \gg c^{-2}P$ (where we have restored the speed of light c to argue that pressure has to be very big to be comparable to energy density in magnitude). Such a fluid is called a **cold dust** and is given by $T_{\mu\nu} = (\rho, 0, 0, 0)$.

Nevertheless, even though we have motivated the energy-momentum tensor using the language of fluids, it is in fact a much more general object. For example, the energy-momentum tensor for a configuration of electro-magnetic fields is

$$T^{\mu\nu}_{\rm EM} = \eta_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta} - \frac{1}{4}\eta^{\mu\nu}|F^2| \ , \ |F^2| \equiv \eta_{\mu\alpha}\eta_{\nu\beta}F^{\mu\nu}F^{\alpha\beta} \tag{2.76}$$

where $F^{\mu\nu}$ is the electromagnetic tensor Eq. (1.90). You will show in a (Homework) problem that the conservation equation Eq. (2.70) is obeyed.

Speaking of which, you have already met **the electromagnetic tensor** Eq. (1.85) in Chapter 1,

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.77}$$

However, the appearance of the partial derivatives ∂_μ in the definition makes this tensor a tricky customer and we will have a lot more to say about this when we explore derivatives on curved space.

2.4 The Metric Tensor

We now discuss one of the most important object in Differential Geometry and particularly when applied to general relativity – the **metric tensor**. It is so important that we have made it into its own section. In the beginning of section 2.3, we took pains to emphasise that Euclidean space $\mathbb{E}^n$ is $\mathbb{R}^n$ *with additional structure*. In particular, in Euclidean space has the notion of distance between two points. We will now explain how this come about.

2.4.1 The Metric Tensor as a Duality Map

Formally, the metric is

$$\boxed{\bar{\mathbf{g}} = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu} \tag{2.78}$$

is a **non-degenerate symmetric** rank-$(0,2)$ tensor. This tensor "lives on the manifold" – it exists at every point on the manifold, i.e. it is a **tensor field**, just like a vector field (see figure 2.6). For now, despite its visible similarities to Eq. (1.36) and Eq. (1.114), hold your horses – notice that the "d" in Eq. (2.78) is not italicized unlike the latter equations. These non-italicized "d" denote the *basis* in co-vector space of the metric in the fancy highbrow way that we discussed in the last section. **Symmetricity** means that

$$\boxed{g_{\mu\nu} = g_{\nu\mu}}. \tag{2.79}$$

Non-degenerate means that $\bar{\mathbf{g}}(\mathbf{X}, \mathbf{Y}) = 0$ for all $\mathbf{X}$ if and only if $\mathbf{Y} = 0$ – a rather technical definition whose most important implication is that *none of the diagonal elements in any basis is zero.* This property means that the metric tensor *always* possesses an **inverse metric g** which is a rank-$(2,0)$ tensor which is found by inverting the $n \times n$ component matrix in whatever basis it is in. We can define the inverse of the metric as (just like Eq. (1.38))

$$(g_{\mu\nu})^{-1} \equiv g^{\mu\nu} \tag{2.80}$$

which means that the inverse metric maps $\bar{\mathbf{g}}$ to the identity and *vice versa*

$$g_{\mu\rho}g^{\rho\nu} = \delta^\nu_\mu. \tag{2.81}$$

As we have discussed in Page 43 , since the metric is a rank-$(0,2)$ tensor, it maps two rank-$(1,0)$ tensors (or a single rank-$(2,0)$ tensor) to a number, and give you a number. For example, suppose both vectors are the same vector V^μ, so

$$\begin{aligned}
\bar{\mathbf{g}}(\mathbf{V},\mathbf{V}) &= g_{\mu\nu}\mathrm{d}x^\mu V^\sigma \frac{\partial}{\partial x^\sigma}\mathrm{d}x^\nu V^\gamma \frac{\partial}{\partial x^\gamma} \\
&= g_{\mu\nu}V^\sigma \mathrm{d}x^\mu \frac{\partial}{\partial x^\sigma} V^\gamma \mathrm{d}x^\nu \frac{\partial}{\partial x^\gamma} \\
&= g_{\mu\nu}V^\sigma V^\gamma \delta^\mu_\sigma \delta^\nu_\gamma \\
&= g_{\mu\nu}V^\mu V^\nu
\end{aligned} \tag{2.82}$$

where we have used Eq. (2.56) twice in the 3rd line. Now, we *define*

$$\boxed{V_\mu \equiv g_{\mu\nu}V^\nu} \tag{2.83}$$

or in fancy tensor notation

$$\bar{\mathbf{V}} \equiv \bar{\mathbf{g}}(\mathbf{V}). \tag{2.84}$$

The inverse metric does the inverse (doh)

$$\boxed{V^\mu \equiv g^{\mu\nu}V_\nu}. \tag{2.85}$$

What we mean by "defining", is that we have chosen to call the co-vector of the vector mapped to it by the metric with the same symbol "V", instead of some other symbol. This emphasise the intimate relationship of the co-vector and the vector that is mapped to it by the metric tensor. Of course, any rank-$(0,2)$ tensor will map a vector to a co-vector, but we prescribe a special meaning to those mapped by the metric tensor. We call V^μ **covariant** and V_μ **contra-variant** respectively, and *assert that they describe exactly the same physical object*. Sometimes, we say that V^μ and V_μ are **dual** to each other. In other words,

The metric tensor provides a unique mapping between vectors and co-vectors in a 1-to-1 [12] *and onto manner.*

The reason the map is *bijective* (i.e. 1-to-1 and onto) is because the metric is by definition *invertible*. So, the metric provides a bijective map between T_pM and T^*_pM. Combined with the symmetricity $g_{\mu\nu}V^\nu = g_{\nu\mu}V^\nu$ property (and hence we don't have to worry about "which index we are contracting with"), this defines a unique mapping.

Some of you will have seen Eq. (2.83) being used, and indeed have probably done it without thinking too much about it. We can now come back to "raising" and "lowering" indices in Special Relativity that we discussed in Page 15 in Chapter 1

$$V_\mu = \eta_{\mu\nu}V^\nu \ , \ V^\mu = \eta^{\mu\nu}V_\nu, \tag{2.86}$$

where the metric we have used $\eta_{\mu\nu} = g_{\mu\nu}$. This ability to raise and lower the indices, or equivalently, map rank-(p,q) tensors to their rank-$(p,q-1)$, rank-$(p-1,q)$ etc using the metric and the inverse metric means that physics are equally described objects where their total rank is counted and not how the ranks are distributed between the "upper" or "lower" indices. For example, the tensors $X^{\mu\nu\sigma}$, $X^\mu{}_{\nu\sigma}$, $X_{\mu\nu\sigma}$, $X_\mu{}^\nu{}_\sigma$ etc all describe exactly the same physical object.

[12]**The 1 to 1 mapping behavior of the metric tensor is what allows us to say "they describe the same physical object". Since the metric tensor is present everywhere on the manifold, then vectors and co-vectors on this manifold can describe exactly the same physical objects because the metric tensor allows us to identify them.

2.4.2 Distances and Angles

One of the most important property of the metric is that *it allows us to define a notion of distances and lengths* on the manifold. Consider a curve parameterized by λ, $x^\mu(\lambda)$. We now ask, "What is the change in the length ds over a given change in the parameter $d\lambda$?". Using the gradient formula, Eq. (2.53), we can write

$$V^\mu = \frac{dx^\mu}{d\lambda}, \tag{2.87}$$

which is just the tangent vector at point p of the curve.

The length is a scalar – it shouldn't have any indices, so the one object we can construct from the vector Eq. (2.87) with the metric is the full contraction (again paying close attention to the italicized d and non-italicized "d" – we promise this madness will end soon)

$$\begin{aligned} g_{\mu\nu}V^\mu V^\nu &= g_{\mu\nu}\mathrm{d}x^\mu \frac{d}{d\lambda}\mathrm{d}x^\nu \frac{d}{d\lambda} \\ &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \left(\frac{d}{d\lambda}\right)^2 \end{aligned} \tag{2.88}$$

which should look very familiar to you. Indeed, if we *define* the square infinitisimal length "ds^2" as

$$ds^2 \equiv g_{\mu\nu}V^\mu V^\nu (d\lambda)^2 \tag{2.89}$$

we get the familiar looking formula

$$\boxed{ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}. \tag{2.90}$$

However, notice that it is the gradient "d" not the usual "d" that you first see in Eq. (1.36) (or indeed most books on general relativity). Notice that "ds^2" we stil use the italicized "d", because s is not the function of anything – ds is still an infinitisimal length and *independent of basis*.

Since $\mathrm{d}x^\mu$ is just the co-vector coordinate basis for the metric (see Eq. (2.64)), we could have just written

$$ds^2 \equiv \bar{\mathbf{g}}, \tag{2.91}$$

and all traces of the vector V^μ has vanished (yet another scaffolding). In other words, the *metric tensor is exactly the basis independent expression of the square length*! Of course, this is not a surprising fact – the word "metric" is derived from the French word *métre*, meaning "distance". In a way, we have "derived" the metric distance relation Eq. (2.90) in the completely opposite way – but that's OK in service of pedagogy.

Recalling that in skool, you were taught that the length of any vector $\mathbf{v}$ can be obtained by taking the **dot product** or **scalar product** $\mathbf{V}\cdot\mathbf{V} = |V|^2$. Notice the similarity to Eq. (2.88):

$$\mathbf{V}\cdot\mathbf{V} \leftrightarrow g_{\mu\nu}V^\mu V^\nu = \bar{\mathbf{g}}(\mathbf{V},\mathbf{V}) \tag{2.92}$$

where the last equality harks back to when we defined the metric in Eq. (2.82). Indeed, the result of $g_{\mu\nu}V^\mu V^\nu = \bar{\mathbf{g}}(\mathbf{V},\mathbf{V}) = |V|^2$ is known as the **norm** of V^μ. But recall that also in skool, you know how to find the *angle* of two vectors $\mathbf{V}$ and $\mathbf{U}$ by taking their dot product $\mathbf{V}\cdot\mathbf{U} = |V||U|\cos\theta$ where θ is the angle between the two angles, and $|V|$ and $|U|$ are norms of $\mathbf{V}$ and $\mathbf{U}$. We can do the same calculation here with the metric, by taking the "dot product" with the metric

$$\bar{\mathbf{g}}(\mathbf{V},\mathbf{U}) = g_{\mu\nu}V^\mu U^\nu \tag{2.93}$$

so the "cosine angle" would be

$$\frac{g_{\mu\nu}V^\mu U^\nu}{|V||U|} \equiv \text{cosine angle}. \tag{2.94}$$

Of course since we are in general curved space, this "angle" can be any units that you want to be – you can call it $\cos\theta$ if you like but there is no fixed convention.

This calculation is *exactly* the dot product angle formula if the metric is $g_{\mu\nu} = diag(1,1,1)$. In other metrics, the result will be different, but the point is that we can call the result of taking the norm of two vectors via the metric using Eq. (2.94) the "angle between **V** and **U**". In other words *the metric allows us to define a notion of angles between vectors.* In the flat plane (Euclidean space), vectors that are "right angles" to each other are called "orthogonal", i.e. $\mathbf{V}\cdot\mathbf{U} = 0$. Given the metric, we can equally define a notion of **orthogonality** via

$$g_{\mu\nu}V^\mu U^\nu = 0 \Rightarrow V^\mu \text{ and } U^\nu \text{ are orthogonal.} \tag{2.95}$$

2.4.3 Coordinate Transformations and Metric Signature

Now we face up to the madness of keeping track of "d" and "d". In Chapter 1, we introduced the "invariant length" $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, by appealing to your notion of infinitisimals – we (somewhat cavalierly) have simply call Δx to be dx and run with it. Now, given our new knowledge, we realize that, it is really $\mathrm{d}x$. The "d" is a gradient, and acts on the *any function* as in Eq. (2.53) and not just a coordinate[13]. This is not as hard as it sounds – it simply means that "d" is an operator, not just some notation to denote "small". In fact, you might have already secretly been using it as an operator when we change coordinates as we will see below. Notation wise, we will now use the proper "d" instead of "d", because we are cool cats – with the caveat that in most physics scientific literature (and books), this distinction is not really made.

In 3D Euclidean space $\mathbb{E}^3$, you have studied that, in Cartesian coordinates (x,y,z), the infinitisimal length is

$$ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{2.96}$$

so the **Euclidean Metric** is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.97}$$

which you have probably guessed. Note that $ds^2 \geq 0$ – the distance is **positive semi-definite**. With this piece of information, we can now retroactively define Euclidean space properly.

An n dimensional **Euclidean Space** $\mathbb{E}^n$ *is a differential manifold $\mathcal{M}$ which can be* **entirely** *mapped by a single coordinate chart such that the metric is $g_{\mu\nu} = diag(+1,+1,+1,\dots)$ everywhere. This chart is the Cartesian coordinates $(x,y,z,\dots)$.*

That's right folks, all your life, you have been working with differential geometry – the fixed plane you have been working on is secretly a differential manifold equipped with Cartesian coordinates and an Euclidean metric. We can of course use different coordinate charts on Euclidean Space. For example, in $\mathbb{E}^2$, we can choose spherical coordinates (r,θ) instead of the Cartesian coordinates (x,y)

$$x = r\sin\theta\ ,\ y = r\cos\theta\ ,\ r = \sqrt{x^2+y^2}. \tag{2.98}$$

We can easily calculate the components of the metric $g_{\mu'\nu'}$ in the (r,θ) coordinates using Eq. (2.65) which you will do in a (Homework) problem. But just to demonstrate that we can treat "d" just like a total derivative, let's calculate the coordinate transforms by using Eq. (2.57) instead

$$\mathrm{d}x = \sin\theta\mathrm{d}r + r\cos\theta\mathrm{d}\theta\ ,\ \mathrm{d}y = \cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta \tag{2.99}$$

[13] Remember that x itself is secretly a function – it is a coordinate chart that maps a point from the manifold to some $\mathbb{R}$.

and plugging these directly into $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ to get

$$\begin{aligned} ds^2 &= \mathrm{d}r\mathrm{d}r + r^2\theta\mathrm{d}\theta\mathrm{d}\theta + r\sin\theta\cos\theta(\mathrm{d}r\mathrm{d}\theta + \mathrm{d}\theta\mathrm{d}r) - r\sin\theta\cos\theta(\mathrm{d}r\mathrm{d}\theta + \mathrm{d}\theta\mathrm{d}r) \\ &= \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \end{aligned} \tag{2.100}$$

where we have in the first line pedantically showed the elements for $\mathrm{d}\theta\mathrm{d}r$ and $\mathrm{d}r\mathrm{d}\theta$ separately *because tensor products do not commute* – remember that $\mathrm{d}x^\mu\mathrm{d}x^\nu$ is secretly $\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ (see Page 43.) Hence in the (r, θ) coordinate basis, the metric is

$$g_{\mu'\nu'} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{2.101}$$

In the definition for Euclidean space above in Page 49, the qualifier "entirely mapped by a single coordinate chart" is important. Since we are free to choose coordinates on the manifold, it is possible that we can choose different coordinate charts at the different regions of the manifold such that they all look like $\eta_{\mu\nu} = diag(1, 1, 1, \dots)$ *locally*, but these are not Euclidean spaces as long as you use more than one chart. Indeed, this is not a surprising fact: recall (Page 37) that manifolds are defined such that it looks locally like $\mathcal{R}^n$, so locally we can always find a coordinate chart where it looks "Euclidean". Though we won't do it here, the last sentence is not very hard to prove rigorously (see Schultz *Geometrical Methods of Mathematical Physics*, section 2.29 if you are interested).

What about *finite* distances? For example, how do we calculate the length of a curve $x^\mu(\lambda)$ parameterized by λ? Recall that $\mathbf{V} = d/d\lambda$ is the rate of change along λ, then its norm $\bar{\mathbf{g}}(\mathbf{V}, \mathbf{V})$ must be the rate of change of distances *squared*. To find the distance along some length of the curve (say parameterized from $\lambda = \lambda_0$ to $\lambda = \lambda_1$, we integrate

$$\begin{aligned} \text{distance} &= \int_{\lambda_0}^{\lambda_1} \sqrt{\bar{\mathbf{g}}(\mathbf{V}, \mathbf{V})} d\lambda \\ &= \int_{\lambda_0}^{\lambda_1} \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} d\lambda. \end{aligned} \tag{2.102}$$

Since $g_{\mu\nu}$ are just *components*, and the curve $x^\mu(\lambda)$ is given so you can compute $dx^\mu/d\lambda$, the integral is well-defined. Notice that $d\lambda$ above is *not* a co-vector basis, i.e. the d is italicized, and hence we can integrate without worry. Sometimes, you can given the components of $\mathbf{V}$, $V^\mu(\lambda)$ instead, then Eq. (2.89) looks like a "square root" formula

$$\boxed{L = \int ds = \int \sqrt{g_{\mu\nu}V^\mu V^\nu} d\lambda} \tag{2.103}$$

where V^μ is the tangent vector of this curve at λ. For example, we want to calculate the length of a curve on a 2D Euclidean space given by $x = \lambda^2$ and $y = \lambda + 1$ from $\lambda = 0$ to $\lambda = 1$. The vector at λ is given by

$$V^\mu = \frac{dx^\mu}{d\lambda} = (2\lambda, 1) \tag{2.104}$$

so

$$\begin{aligned} L &= \int_0^1 \sqrt{4\lambda^2 + 1} d\lambda \\ &= \frac{1}{4}(2\lambda\sqrt{4\lambda^2 + 1} + \sinh^{-1}(2\lambda))|_0^1 \\ &= \frac{1}{4}(2\sqrt{5} + \sinh^{-1}(2)). \end{aligned} \tag{2.105}$$

Euclidean space has no natural "time" direction, but as we mentioned before the metric in special relativity $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ has a minus term which denotes time direction. Such a space time is called **Minkowski Space**

An n dimensional **Minkowski Space** *is a differential manifold $\mathcal{M}$ which can be* **entirely** *mapped by a single coordinate chart such that the metric is $g_{\mu\nu} = diag(-1,+1,+1,\dots)$ everywhere. This chart is the Cartesian coordinates $(t,x,y,z,\dots)$.*

The appearance of the minus sign means that the **invariant length**

$$ds^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{2.106}$$

is no longer positive definite – it can be positive, negative or zero. If we go back to Eq. (2.89),

$$ds^2 \equiv g_{\mu\nu}V^\mu V^\nu (d\lambda)^2 \tag{2.107}$$

we see that since $(d\lambda)^2$ is positive definite, we can classify vectors V^μ as

$$\begin{aligned} g_{\mu\nu}V^\mu V^\nu > 0 \quad &, \quad \text{spacelike} \\ g_{\mu\nu}V^\mu V^\nu = 0 \quad &, \quad \text{null} \\ g_{\mu\nu}V^\mu V^\nu < 0 \quad &, \quad \text{timelike.} \end{aligned} \tag{2.108}$$

These definitions allow us to physically describe the **causal** structure of spacetime – particles can only

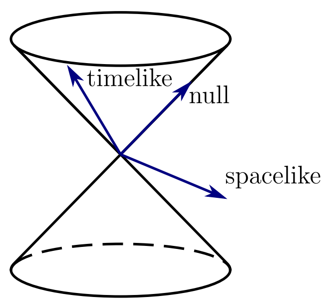


Figure 2.11: The light cone describes the causal structure of spacetime. Figure stolen from Harvel Reall's lecture notes.

move along timelike (or null if the particle is massless like the photon) directions. Figure 2.11 shows the **lightcone** – events at the vertex of the cone can only influence events that is within its *future lightcone*.

Some people choose the "wrong sign" metric $\eta_{\mu\nu} = (+1,-1,-1,-1)$ to be the Minkowski metric (just a convention), and in this case the classification is also swapped (i.e. timelike become spacelike etc). We can extend this to curves: a curve is called spacelike/null/timelike if the tangent vector of this curve is everywhere spacelike/null/timelike.

The indefiniteness of the metric gives us a slight trouble in defining lengths like we did for Euclidean space in Eq. (2.103), since square root of negative values is complex. To overcome this, for spacelike curves we use Eq. (2.103)

$$\text{Proper Distance} \ \ L = \int \sqrt{g_{\mu\nu}V^\mu V^\nu}d\lambda \tag{2.109}$$

and for timelike curves we use

$$\text{Proper Time} \ \ L = \int \sqrt{-g_{\mu\nu}V^\mu V^\nu}d\lambda. \tag{2.110}$$

For a curve that combines both timelike and spacelike sections, we use the appropriate formula for each section.

Notice that *null curves have zero lengths!* I can draw a null curve on a Minkowski manifold, and it *looks* like it has a finite non-zero length – indeed it can be parameterized by λ, say from $\lambda = 0$ to $\lambda = 1$

and clearly that's not zero. But of course, we know from special relativity that L is really the proper time – i.e. the time on the clock carried by the observer moving along this worldline. We also have learned that light travel in null curves, so light's proper time is zero. If you think about it, this means that light took no time to reach us from the furthest distance we know, which is the Big Bang. Funny isn't it, we are seeing the Big Bang – and everything else for that matter – exactly the way it had "happened", the photons haven't aged at all.

Similarly to the Euclidean space, it's only Minkowski space if the entire manifold is covered by a coordinate chart where the metric is $\eta_{\mu\nu} = (-1, 1, 1, 1)$. And we are also allowed to change to any coordinate charts in Minkowski space. The question is: can we find a set of coordinates on Minkowski space such that the metric is $\eta_{\mu\nu} = (1, 1, 1, 1)$, i.e. it looks "locally Euclidean"? The answer to this question is *no* (again see Schultz *Geometrical Methods of Mathematical Physics*, section 2.29 for the proof). It turns out that, even for a non-Euclidean metric, if you can find coordinates such that it is $(1, 1, 1, 1)$ then *you can find some other coordinates chart elsewhere that reduces it to* $(1, 1, 1, 1)$ *but never* $(-1, 1, 1, 1)$ and *vice versa*. If you like, the "$(1, 1, 1, 1)$-ness" and "$(-1, 1, 1, 1)$-ness" of the metric is fixed. This property is called the **signature** of the metric. A metric which is all $+1$ is called **Riemannian** and a metric which has a -1 is called **Lorentzian**. So a manifold equipped with a Euclidean (Minkowski) metric is called Riemannian (Lorentzian).

2.4.4 Why are Lorentzian and Riemannian Manifolds important to Physics?

In the previous section 2.4.3, we showed that in a manifold endowed with a metric tensor, we can always find a coordinate chart locally such that the metric looks like $(-1, 1, 1, 1)$ (Lorentzian) or $(1, 1, 1, 1)$ (Riemannian). This means that, as we take the limit of an infinitisimally small space, the manifold locally behaves more and more like Lorentzian/Riemannian spaces, e.g. orthogonal vectors are right angles to each other, parallel transport is like what Euclides taught etc.

For example, the Earth's surface is clearly a Riemannian manifold – "globally" we know it has curvature, but "locally" it looks very flat. We only find out about the curvature of Earth when we probe a large enough region of its surface, just as Magellan did.

What about spacetime? In special relativity, the metric of spacetime is everywhere Minkowski hence it is a Lorentzian manifold – we often say that special relativity is the "physics of flat space" – the extra "-1" notwithstanding, the 3D space of 3+1D Minkowski space is Euclidean flat. However, we have alluded to, in general relativity and general curved spacetimes, the metric is no longer Minkowski, but something more complicated though it still possess a Lorentzian signature. Recall from Page 24, the Einstein Equivalence Principle states that in a *local* patch, the laws of physics reduce to that of special relativity. Mathematically, this means that at every local patch of spacetime, we can find a coordinate chart which renders the metric $\eta_{\mu\nu} = (-1, 1, 1, 1)$. *But this is exactly what a Lorentzian manifold is!* In other words, the EEP automatically implies that spacetime is described by a Lorentzian manifold.

Note that *this does not mean that spacetime has no curvature* – the Earth's surface might look locally flat but it has curvature at every point! To make this notion clear, we need to study curvature in greater detail, which we will embark in the next section.

Chapter 3

Curvature

Spacetime tells matter how to move...
John Archibald Wheeler

In the last chapter, we studied the basics of differential geometry. Although we touched upon its uses in gravitation, we have not explored it in any detail. In this chapter, we will now focus on the one of its primary applications in physics (and indeed how most physicists got introduced to differential geometry) – the study of curved spaces, i.e. how it describes **curvature**.

While we have not yet seen the famous Einstein Equation, this chapter begins our study of General Relativity proper. Roughly speaking, we can split General Relativity into two main ideas.

1. *Spacetime curvature tells how matter moves.*

and

2. *Matter tells spacetime how to curve.*

In this chapter, we will focus on Idea 1. We will learn how to describe curvature in the language of differential geometry. We will learn how to calculate **geodesics** – the paths where free falling particles move – given any curved surface. In the next chapter 4, we will study how the spacetime evolves given some matter, i.e. the dynamics of spacetime itself.

3.1 Parallel Transport

In Euclidean space, as Euclides had told us, "parallel lines stay parallel".

What we have studied so far should now challenge our thinking here. What do we mean by "stay parallel"? Let's suppose we have a curve on $\mathbb{E}^2$ (see Figure 3.1), and we want to **transport** the green vector along the curve. In your Euclidean intuition, you can move the green vector along the curve, making sure that it is still "pointing in the same direction" – ensuring that the values of the components remain the same – on the plane, resulting in the brown colored vectors. But, I can also invent a new "rule", which is "I move the vector along the curve, and as I do, I rotate the vector in the clockwise direction by some small amount". The resulting vector (in blue) is now pointed in some other "direction". So there is an infinite number of ways to transport a vector – as many as you can make up – which one do we want to choose?

The moral of this story is that *we need to specify a* **rule** *when we transport vectors around.* Such a rule is called a **connection**. As we will soon see, the Euclidean space metric provide such rule which is very natural, *but it is not the only one.* We emphasise that the “intuitive” notion of **parallel transport** breaks down immediately when we move from our beloved Euclidean space.

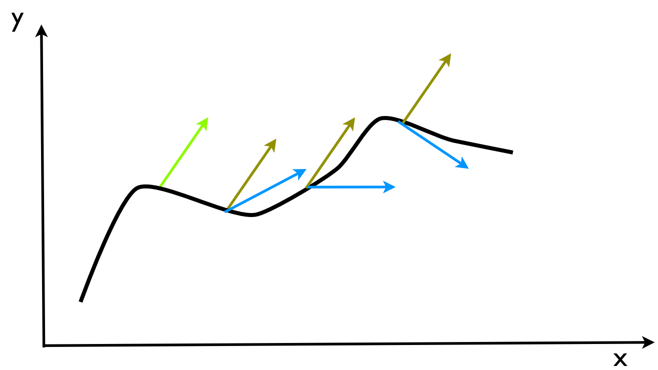


Figure 3.1: Transporting the origin green vector along a curve in $\mathbb{E}^2$. The brown vectors are using your naive Euclidean interpretation of making sure the “vectors stay parallel” while the blue vectors follow some screwy rule that I made up, just because I can.

Consider the 2D sphere $\mathbb{S}^2$ (see figure 3.2). We want to transport a vector from the N point to the S point along the great circle. One natural way (left diagram) of doing it is to say “we will transport the vector such that it is always tangent to the curve of the great circle”. Another, equally natural way (right diagram) is to “transport the curve such that it is perpendicular to the curve of the great circle”. But since these two great circles intersect each other at 90 degrees at the N and S points, the results of these two equally intuitive notions of transport give us a final vector that is anti-parallel to each other! So which one deserves to be called “the parallel transport” of a sphere?

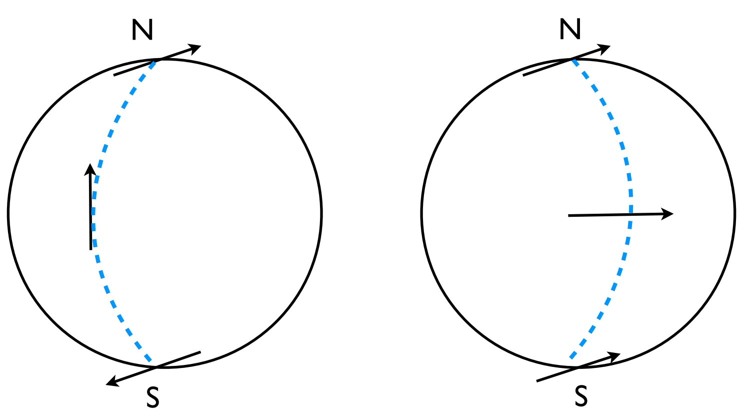


Figure 3.2: Transporting a vector along a sphere in two completely “natural intuitive” ways from the North pole to the South pole. In the left figure, the vector is tangent to the curve in which it is parallel transported in, and as we move along the curve, we insist that the vector is always kept tangent. In the right figure, the same vector is now moved along a *different* curve, and as we move along the curve we keep it parallel to the curve. Both are “parallel” transported, but the final vector at the South pole is opposites of each other.

Let’s begin with a basic question: What do we mean exactly by “transporting” a vector along the curve? Remember that at each point on the manifold, there exists a vector space (and the higher rank tensor space). So transporting vectors between two neighbouring points p and p' on the manifold means "a map from a vector in T_pM to another vector in $T_{p'}M$". Our goal is to define such a map.

Mathematically, *"curvature" is really given to us by the choice of connections* – of which there are infinitely many. In other words, we *define* curvature as how a vector is transported around curves on the manifold. This might be slightly counter-intuitive, or even sacrilegious to those who have taken some GR. Our human-born intuition tells us that curvature has something to do with distances between two points – for example, going from point A to point B on a map would be shorter if there is no mountains in between. So it seems natural that the metric tensor, which gives us the notion of distances and angles, should play central role in defining curvature. As it turns out, curvature is completely defined by a choice of connections, not by the metric tensor.

However, if the manifold has a metric tensor then *the metric allows us to uniquely define a particular choice of connection.* So, the statement that "the curvature is given by the metric" often stated in general relativity classes would be morally correct, but would technically be not quite true[1] (since you can have a metric *and* use another connection that has nothing to do with the metric). (Un?)-fortunately, since we are not doing a full blown differential geometry class, we will eventually follow the party line, and discuss this special connection given by the metric. And loosely say the words "the metric gives us the curvature".

Before we follow the party line, let's stay general for now.

3.1.1 The Covariant Derivative and the Connection

The first mathematical object that we need is the **covariant derivative** ∇. As the name implies, it is a *derivative* – so acting with ∇ on a tensor $\mathbf{T}$ will give you the rate of change of that tensor. But rate of change along what? Along a curve is one possibility – at each point on the curve we can then define a vector $\mathbf{V}$. Let this curve by parameterized by λ, then $\mathbf{V} = d/d\lambda$. Suppose now $\mathbf{T}$ is a vector field defined everywhere on this curve, we can evaluate $\mathbf{T}(\lambda)$ and $\mathbf{T}(\lambda+\epsilon)$, where ϵ is an infinitisimal. Since these two vectors are at different points $p(\lambda)$ and $p(\lambda+\epsilon)$ on the manifold, they live in different vector spaces $T_{p(\lambda)}M$ and $T_{p(\lambda+\epsilon)}M$ so we are not allowed to directly subtract them. Suppose I give you a prescription, allow you to construct a *new* vector in $T_{p(\lambda)}M$, called $\mathbf{T}_*(\lambda)$ whose value is related to $\mathbf{T}(\lambda+\epsilon)$ in some undefined way, then we can construct the derivative

$$\nabla_{\mathbf{V}}\mathbf{T} \equiv \lim_{\epsilon\rightarrow 0}\frac{\mathbf{T}_*(\lambda)-\mathbf{T}(\lambda)}{\epsilon}. \tag{3.1}$$

Eq. (3.1) is read as "derivative of $\mathbf{T}$ along the vector $\mathbf{V}$ at λ". Of course I have not told you how $\mathbf{T}_*(\lambda)$ is related to $\mathbf{T}(\lambda+\epsilon)$ – this is what we will be studying in this section. Eq. (3.1) is easily generalized to derivatives of higher ranked tensors.

But wait! We already know a derivative, the good old loyal partial derivative ∂_μ, why do we need to define a new derivative? The problem is that even through the partial derivative *is* a bona fide vector – recall that $\partial_\mu = \hat{\mathbf{e}}_{(\mu)}$, its *action on other tensors is basis dependent* hence the result is not "tensorial". To be explicitly, take a partial derivative on a vector, and do a basis transform on it, we see that

$$\begin{aligned}\partial_\mu V^\nu &= \frac{\partial x^{\mu'}}{\partial x^\mu}\partial_{\mu'}\left(\frac{\partial x^{\nu'}}{\partial x^\nu}V^{\nu'}\right)\\ &= \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}\partial_{\mu'}V^{\nu'}+\frac{\partial x^{\mu'}}{\partial x^\mu}\partial_{\mu'}\left(\frac{\partial x^{\nu'}}{\partial x^\nu}\right)V^{\nu'}\end{aligned} \tag{3.2}$$

[1]**Such a distinction might seem like hair-splitting, but if you are going to do research in physics, then understanding where things come from will save you a lot of embarrassment, when you for example, declare that you have discovered a new theory of gravity, when actually you are simply ignorant of this fact. True story.

and it's clear that the 2nd term on the RHS of Eq. (3.2) means that partial derivatives are not "tensorial", recalling that tensors are basis independent[2].

Let's now define the covariant derivative.

> The **covariant derivative** ∇ is defined such that it obeys the following rules, for vector fields $\mathbf{X}$, $\mathbf{W}$ and any two tensors $\mathbf{Y}$ and $\mathbf{Z}$
>
> - Linearity : $\nabla_{\mathbf{X}}(\mathbf{Y}+\mathbf{Z}) = \nabla_{\mathbf{X}}\mathbf{Y} + \nabla_{\mathbf{X}}\mathbf{Z}$.
> - Leibniz Rule : $\nabla_{\mathbf{X}}(\mathbf{Y}\otimes\mathbf{Z}) = \nabla_{\mathbf{X}}\mathbf{Y}\otimes\mathbf{Z} + \mathbf{Y}\otimes\nabla_{\mathbf{X}}\mathbf{Z}$.
> - Additiveness (of derivative directions) : $\nabla_{f\mathbf{X}+g\mathbf{W}}\mathbf{Z} = f\nabla_{\mathbf{X}}\mathbf{Z} + g\nabla_{\mathbf{W}}\mathbf{Z}$, where f and g are any functions.
> - Action on scalars : $\nabla_{\mathbf{X}}(f) = \mathbf{X}(f)$ for any scalar f.

Specializing to 4D for simplicity, let's now choose a (coordinate) basis[3] $\hat{\mathbf{e}}_{(\mu)} = (\partial_0, \partial_1, \partial_2, \partial_3)$. Since partials are still vectors, we want to ask "what is the rate of change of the vector of one direction, say α direction and call it $\mathbf{A}$, when we take the covariant derivative along the β (call it $\mathbf{B}$) direction". Or, following Eq. (3.1)

$$\boxed{\nabla_{\mathbf{B}}\mathbf{A} = \nabla_{\hat{\mathbf{e}}_{(\beta)}}\hat{\mathbf{e}}_{(\alpha)} \equiv \sum_{\mu}\Gamma^{\mu}_{\beta\alpha}\hat{\mathbf{e}}_{(\mu)}} \text{ and } \boxed{\nabla_{\hat{\mathbf{e}}_{(\beta)}} \equiv \nabla_{\beta}} \tag{3.3}$$

where α and β are *not* summed over (remember that they just pick out a direction), but μ is summed over. ∇ and its associated symbol $\Gamma^{\mu}_{\alpha\beta}$ is called the **connection**[4], and this is the connection that we were talking about when we discuss parallel transport in the beginning of this section. Eq. (3.3) says that the rate of change of $\mathbf{A}$ when transported along $\mathbf{B}$ is given to us by the connection.

Note that the 2nd equation of Eq. (3.3), $\nabla_{\hat{\mathbf{e}}_{(\beta)}} \equiv \nabla_{\beta}$ is a *notational definition* for the covariant derivative in component form, it is just notation for now and we have not told you what it should be when acting on other things – we will derive its action in the following.

The symbol $\Gamma^{\mu}_{\alpha\beta}$ is not a tensor! This is why we can be rather cavalier in the index placement. In other words, $\Gamma^{\mu}_{\alpha\beta}$ is an object you always see labeled by some basis indices – you can think of it as an array of functions labeled by three indices. In general, we are free to choose whatever connection we want to use here – this is exactly the freedom we have to move the vector around when we describe vector transport in $\mathbb{E}^2$ back in Page 53. While in general you can specify it to be whatever you want it to be, as we mentioned earlier, the metric allows us to define a special connection, which we will discuss in section 3.1.2. For now, let's keep it general.

Let's now derive a general formula in component form for Eq. (3.1). Let $\mathbf{V}$ and $\mathbf{T}$ be two rank-$(1,0)$ tensors, so expanding them in choice of basis $\mathbf{V} = V^{\mu}\hat{\mathbf{e}}_{(\mu)}$ an $\mathbf{T} = T^{\nu}\hat{\mathbf{e}}_{(\nu)}$, we have

$$\begin{aligned}
\nabla_{\mathbf{V}}\mathbf{T} &= \nabla_{\mathbf{V}}(T^{\mu}\hat{\mathbf{e}}_{(\mu)}) \\
&= (\nabla_{\mathbf{V}}T^{\mu})\hat{\mathbf{e}}_{(\mu)} + T^{\mu}\nabla_{\mathbf{V}}(\hat{\mathbf{e}}_{(\mu)}) \\
&= \mathbf{V}(T^{\mu})\hat{\mathbf{e}}_{(\mu)} + T^{\mu}\nabla_{V^{\nu}\hat{\mathbf{e}}_{(\nu)}}(\hat{\mathbf{e}}_{(\mu)}),
\end{aligned} \tag{3.4}$$

aaaand let's pause and unpack. Referring back to the rules on Page 56, the second line is using Leibnitz's Rule. The first term of the second line uses the Action on scalars: remember that T^{μ} is simply a column

[2] You can easily show that the partial derivative of a scalar is a tensor however.

[3] **We chose this for simplicity, but Eq. (3.3) is general for any basis.

[4] Nomenclature-wise, ∇ with its associated $\Gamma^{\mu}_{\alpha\beta}$ is called the "connection", but since ∇ itself is defined by whatever $\Gamma^{\mu}_{\alpha\beta}$ is, the names are interchangeable.

vector of functions (you should always remember this). Let's continue,

$$\begin{aligned}\nabla_{\mathbf{V}}\mathbf{T} &= V^{\nu}\hat{\mathbf{e}}_{(\nu)}(T^{\mu})\hat{\mathbf{e}}_{(\mu)} + V^{\nu}T^{\mu}\nabla_{\hat{\mathbf{e}}_{(\nu)}}(\hat{\mathbf{e}}_{(\mu)}) \\ &= V^{\nu}\hat{\mathbf{e}}_{(\nu)}(T^{\mu})\hat{\mathbf{e}}_{(\mu)} + V^{\nu}T^{\mu}\Gamma^{\alpha}_{\nu\mu}\hat{\mathbf{e}}_{(\alpha)} \\ &= V^{\nu}\hat{\mathbf{e}}_{(\nu)}(T^{\mu})\hat{\mathbf{e}}_{(\mu)} + V^{\nu}T^{\sigma}\Gamma^{\mu}_{\nu\sigma}\hat{\mathbf{e}}_{(\mu)} \\ &= V^{\nu}\left[\hat{\mathbf{e}}_{(\nu)}(T^{\mu}) + T^{\sigma}\Gamma^{\mu}_{\nu\sigma}\right]\hat{\mathbf{e}}_{(\mu)}. \end{aligned} \tag{3.5}$$

In the first line, at the second term we have again used the fact the V^{ν} is just a set of functions, and applied the Additiveness rule. The 2nd line we used Eq. (3.3) in the second term. In the 3rd line, we use the fact that we can relabel the indices. We collect the terms up suggestively in the final line.

Meanwhile on the LHS, we can also write it as

$$\begin{aligned}\nabla_{\mathbf{V}}\mathbf{T} &= \nabla_{V^{\mu}\hat{\mathbf{e}}_{(\mu)}}\mathbf{T} \\ &= V^{\nu}\nabla_{\hat{\mathbf{e}}_{(\nu)}}\mathbf{T} \\ &= V^{\nu}\nabla_{\nu}\mathbf{T}, \end{aligned} \tag{3.6}$$

now we can *define* the object $\nabla_{\nu}\mathbf{T} \equiv (\nabla_{\nu}T^{\mu})\hat{\mathbf{e}}_{(\mu)}$, to get

$$\nabla_{\mathbf{V}}\mathbf{T} = V^{\nu}(\nabla_{\nu}T^{\mu})\hat{\mathbf{e}}_{(\mu)} \tag{3.7}$$

so by comparing Eq. (3.7) with Eq. (3.5) we get

$$\nabla_{\nu}T^{\mu} = \hat{\mathbf{e}}_{(\nu)}(T^{\mu}) + \Gamma^{\mu}_{\nu\sigma}T^{\sigma} \tag{3.8}$$

and then finally using $\hat{\mathbf{e}}_{(\nu)} = \partial_{\nu}$ we get the **covariant derivative in component form**

$$\boxed{\nabla_{\nu}T^{\mu} = \partial_{\nu}T^{\mu} + \Gamma^{\mu}_{\nu\sigma}T^{\sigma}}. \tag{3.9}$$

Eq. (3.9) is so important that you should commit it to memory. This means that Eq. (3.7) can also be written as

$$\boxed{\nabla_{\mathbf{V}}\mathbf{T} = V^{\nu}(\partial_{\nu}T^{\mu} + \Gamma^{\mu}_{\nu\sigma}T^{\sigma})\hat{\mathbf{e}}_{(\mu)}}, \tag{3.10}$$

which is a convenient way to remember how to convert from the basis-independent abstract notation (the bolded stuff) to the more pedestrian component form[5]. The "$\hat{\mathbf{e}}_{(\mu)}$" at the end of Eq. (3.10) tells us that $\nabla_{\mathbf{V}}\mathbf{T}$ can be written as a "one upper index", i.e.

$$\nabla_{\mathbf{V}}\mathbf{T} = (\nabla_{\mathbf{V}}\mathbf{T})^{\mu}\hat{\mathbf{e}}_{(\mu)} \Rightarrow (\nabla_{\mathbf{V}}\mathbf{T})^{\mu} = V^{\nu}\nabla_{\nu}T^{\mu}. \tag{3.12}$$

Now that we have explicit expressions for $\nabla_{\mathbf{V}}\mathbf{T}$ in Eq. (3.9) and Eq. (3.10), we can define a notion of "parallel transport". Let T^{μ} be a vector at point p in the manifold. Suppose there is a curve labeled by λ, and we learned that at point p, this curve can generate a vector $\mathbf{V}(\lambda) = d/d\lambda|_p$. We can now transport T^{μ} along this curve using the covariant derivative Eq. (3.1). Intuitively, if we want the transported vector $\mathbf{T}$ at p' to be "parallel" then *the vector (and its the component values) must remain the same*, or in other words

$$\boxed{\nabla_{\mathbf{V}}\mathbf{T} = 0}\ \text{Parallel transport along the curve } x^{\mu} \text{ with tangent } \mathbf{V} = \frac{dx^{\mu}}{d\lambda}\partial_{\mu} \tag{3.13}$$

[5] In many treatments of GR, Eq. (3.9) is often motivated by insisting that $\nabla_{\mu}T^{\nu}$ "transform as a tensor", i.e. must obey the equation

$$\nabla_{\nu'}T^{\mu'} = \frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\nabla_{\nu}T^{\mu}. \tag{3.11}$$

Eq. (3.13) is exactly the definition for parallel transport for general manifolds and metrics. Using Eq. (3.10), we can then write down the the **equation for parallel transport**

$$\boxed{\nabla_{\mathbf{V}}\mathbf{T} = V^{\nu}(\partial_{\nu}T^{\mu} + \Gamma^{\mu}_{\nu\sigma}T^{\sigma})\hat{\mathbf{e}}_{(\mu)} = 0}, \tag{3.14}$$

or in component form,

$$\boxed{V^{\nu}\nabla_{\nu}T^{\mu} = 0}. \tag{3.15}$$

What about covariant derivative of co-vectors $\nabla_{\mathbf{B}}\bar{\mathbf{A}}$? Now, if we have a metric, then co-vectors are really just vectors since we can map them using the metric. But, it turns out we don't need the metric just yet – we can always define the connection analogously to Eq. (3.3), by choosing $\bar{\mathbf{A}}$ to be a basis co-vector, and then

$$\boxed{\nabla_{\mathbf{B}}\bar{\mathbf{A}} = \nabla_{\hat{\mathbf{e}}_{(\beta)}}\hat{\mathbf{e}}^{(\alpha)} \equiv \sum_{\mu}\tilde{\Gamma}^{\alpha}_{\beta\mu}\hat{\mathbf{e}}^{(\mu)}} \tag{3.16}$$

where $\tilde{\Gamma}^{\alpha}_{\beta\mu}$ is some other connection which for now bears no relation to $\Gamma^{\mu}_{\alpha\beta}$.

Redoing the entire calculation above (which you will do in a (Homework) problem), we find that in component form the covariant derivative of a co-vector $\nabla_{\mathbf{V}}\bar{\mathbf{W}}$ in component form is

$$\nabla_{\nu}W_{\mu} = \partial_{\nu}W_{\mu} + \tilde{\Gamma}^{\sigma}_{\nu\mu}W_{\sigma}. \tag{3.17}$$

But if we insist that *the covariant derivative reduces to the partial derivative* when acting on a scalar f, i.e.

$$\nabla_{\mu}f = \partial_{\mu}f \tag{3.18}$$

you can show that (Homework)

$$\tilde{\Gamma}^{\mu}_{\nu\sigma} = -\Gamma^{\mu}_{\nu\sigma} \tag{3.19}$$

$$\boxed{\nabla_{\nu}W_{\mu} = \partial_{\nu}W_{\mu} - \Gamma^{\sigma}_{\nu\mu}W_{\sigma}} \tag{3.20}$$

which again you should commit to memory. The generalization of Eq. (3.9) and Eq. (3.20) when acting on a general higher ranked tensor is straightforward using Leibnitz's rule,

$$\begin{aligned}\nabla_{\sigma}T^{\mu_1\mu_2\ldots}{}_{\nu_1\nu_2\ldots} &= \partial_{\sigma}T^{\mu_1\mu_2\ldots}{}_{\nu_1\nu_2\ldots} + \Gamma^{\mu_1}_{\sigma\rho}T^{\rho\mu_2\ldots}{}_{\nu_1\nu_2\ldots} + \Gamma^{\mu_2}_{\sigma\rho}T^{\mu_1\rho\ldots}{}_{\nu_1\nu_2\ldots} + \ldots \\ &\quad -\Gamma^{\rho}_{\sigma\nu_1}T^{\mu_1\mu_2\ldots}{}_{\rho\nu_2\ldots} - \Gamma^{\rho}_{\sigma\nu_2}T^{\mu_1\mu_2\ldots}{}_{\nu_1\rho\ldots} + \ldots\end{aligned} \tag{3.21}$$

In terms of index structure, the covariant derivative maps a rank-$(1,0)$ vector to a rank-$(1,1)$ tensor, or more generally

$$\nabla_{\mu} : \text{rank}-(p,q) \rightarrow \text{rank}-(p,q+1). \tag{3.22}$$

3.1.2 The Levi-Civita Connection

In the previous section 3.1.1, we have discussed covariant derivatives and connection, without even referring to the metric. Indeed, all the discussion is general and does not require a metric to make sense – and in a full blown differential geometry class, we will now proceed to define curvature using the connection (which is freely chosen). But in GR, the metric tensor plays a pivotal role – as we alluded to earlier, the existence of the metric allows us to define a unique choice of connection. This connection is so special it has a name: the **Levi-Civita connection**. In this section, we will now discuss this.

We begin with the famous **Fundamental Theorem of Riemannian Geometry** (although it really works for any signature).

Let $\mathcal{M}$ be a differential manifold, equipped with a metric tensor $\bar{\mathbf{g}}$. *Then there exists a unique* **torsion-free** *connection* ∇ *such that the* $\nabla\bar{\mathbf{g}} = 0$ *(covariantly conserved) always. This is the Levi-Civita connection.*

Let's unpack the theorem.

- The first new thing is **torsion**, which we can define in the following way. Suppose I have any tensor $\mathbf{T}$ and two vector fields $\mathbf{X}$ and $\mathbf{Y}$. We then ask, what is the difference in the result between $\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{T}$ and $\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{T}$? If we choose $\mathbf{T}$ to be a simple scalar f (i.e. rank-$(0,0)$ tensor), and choose $\mathbf{X}$ and $\mathbf{Y}$ to be the coordinate basis vectors $\hat{\mathbf{e}}_{(\mu)}$ and $\hat{\mathbf{e}}_{(\nu)}$, then it's easy to calculate

$$\begin{aligned}\nabla_\mu\nabla_\nu f &= \nabla_\mu\partial_\nu f \\ &= \partial_\mu\partial_\nu f - \Gamma^\lambda_{\mu\nu}\partial_\lambda f \end{aligned} \tag{3.23}$$

and similarly

$$\begin{aligned}\nabla_\nu\nabla_\mu f &= \nabla_\nu\partial_\mu f \\ &= \partial_\nu\partial_\mu f - \Gamma^\lambda_{\nu\mu}\partial_\lambda f. \end{aligned} \tag{3.24}$$

Taking the difference between Eq. (3.23) and Eq. (3.24), and using the fact that partial derivatives commute we get

$$\nabla_\mu\nabla_\nu f - \nabla_\nu\nabla_\mu f = \Gamma^\lambda_{\nu\mu}\partial_\lambda f - \Gamma^\lambda_{\mu\nu}\partial_\lambda f \equiv T^\lambda{}_{\mu\nu}\partial_\lambda f \tag{3.25}$$

where the new object $T^\lambda{}_{\mu\nu}$ is called the **torsion**. Notice that it has proper index placement – the torsion is a true tensor unlike the $\Gamma^\rho_{\mu\nu}$'s – as you will prove in a (Homework) problem.

Torsion-free means that

$$\boxed{T^\lambda{}_{\mu\nu} = 0 \Rightarrow \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}}. \tag{3.26}$$

In words, if the connection is torsion free, then it is *symmetric under exchange of the lower indices* $\mu \leftrightarrow \nu$. The Fundamental Theorem of Riemannian Geometry states that the Levi-Civita connection is torsion free.

- The second statement is that $\nabla\bar{\mathbf{g}} = 0$. In words, we say that the metric is "covariantly conserved", or more popularly, the "covariant derivative is **metric compatible**". In component form, this is

$$\boxed{\nabla_\sigma g_{\mu\nu} = 0.} \tag{3.27}$$

We will now prove that Eq. (3.27), combined with the torsion-free condition, uniquely defines the connection $\Gamma^\lambda_{\mu\nu}$. Expanding this equation, we get

$$\nabla_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma^\lambda_{\sigma\mu}g_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu}g_{\mu\lambda} = 0 \tag{3.28}$$

But we can also permute all the indices, to get

$$\nabla_\mu g_{\nu\sigma} = \partial_\mu g_{\nu\sigma} - \Gamma^\lambda_{\mu\nu}g_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma}g_{\nu\lambda} = 0 \tag{3.29}$$

and

$$\nabla_\nu g_{\sigma\mu} = \partial_\nu g_{\sigma\mu} - \Gamma^\lambda_{\nu\sigma}g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu}g_{\sigma\lambda} = 0. \tag{3.30}$$

Subtract the 2nd and the 3rd equations from the first, and use the torsion-free condition $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, we get

$$\partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma} - \partial_\nu g_{\sigma\mu} + 2\Gamma^\lambda_{\mu\nu}g_{\lambda\sigma} = 0. \tag{3.31}$$

Multiplying by the inverse metric $g^{\sigma\rho}$, we get an equation for $\Gamma^\rho_{\mu\nu}$ in terms of partial derivatives of the metric

$$\boxed{\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)} \tag{3.32}$$

which is an equation you should commit to memory. Eq. (3.32) tells us that, for a given metric $g_{\mu\nu}$, the Levi-Civita connection ∇ can be uniquely calculated – the $\Gamma^\rho_{\mu\nu}$ of this very special connection are often called **Christoffel symbols** especially in GR literature. Note that the Christoffel symbols, like any regular Torsion-free connection is symmetric under the interchange of its lower indices $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$.

Let's end this section by showing how the presence of the Euclidean metric in a manifold, "parallel lines stay parallel". In component form, we use Eq. (3.14) and the Levi-Civita connection to obtain

$$\nabla_\nu T^\mu = \partial_\nu T^\mu + \Gamma^\mu_{\sigma\nu} T^\sigma = 0. \tag{3.33}$$

Now, since the Euclidean metric (in 2D) in Cartesian coordinates basis, is $\eta_{\mu\nu} = (1, 1)$, using Eq. (3.32), it is easy to show that the Christoffel symbols all vanish, giving us

$$\partial_\nu T^\mu = 0, \tag{3.34}$$

which says that T^μ is constant along the curve – and hence remains parallel as Euclides had claimed millienia ago.

3.2 Geodesics in Curved Spaces

Way back in Page 24, we told you that Einstein asserted that *all free-falling objects follow* **geodesics**. A geodesic is the shortest path between two points on any surfaces. In a flat Euclidean space, the geodesic is simply a straight line, while on a sphere it is a great circle. We will now study how to compute geodesics from any general curved spaces.

3.2.1 The Geodesic Equation

In both the cases described above, notice that, as you move along the geodesic, *the tangent to the geodesic is always parallel transported along the geodesic.* This gives us a fancy way of defining geodesics without using the metric (we'll have a more pedestrian version in section 3.2.3).

A curve $\mathcal{C}$ is a geodesic if its tangent vector is parallel transported along it.

Using this, we will now derive a differential equation whose solution is the geodesic. Suppose $x^\mu(\lambda)$ is a curve $\mathcal{C}$ parameterized by λ which is **monotonic**, and $\mathbf{V} = V^\mu \hat{\mathbf{e}}_{(\mu)}$ is the tangent vector $V^\mu = dx^\mu/d\lambda$ to $\mathcal{C}$. From Eq. (3.13), parallel transporting the tangent vector $\mathbf{V}$ along the curve is

$$\nabla_{\mathbf{V}} \mathbf{V} = 0. \tag{3.35}$$

In component form, this is

$$\begin{aligned} \nabla_{\mathbf{V}} \mathbf{V} &= V^\nu (\partial_\nu V^\mu + V^\sigma \Gamma^\mu_{\nu\sigma}) \hat{\mathbf{e}}_{(\mu)} \\ &= (V^\nu \partial_\nu V^\mu + V^\nu V^\sigma \Gamma^\mu_{\nu\sigma}) \hat{\mathbf{e}}_{(\mu)} \\ &= \left[\frac{d}{d\lambda} \left(\frac{dx^\mu}{d\lambda} \right) + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} \right] \hat{\mathbf{e}}_{(\mu)} \end{aligned} \tag{3.36}$$

and setting this to zero, we get the **Geodesic Equation**

$$\boxed{\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0}. \tag{3.37}$$

Since Eq. (3.37) is a second order ODE, it is guaranteed to have a solution given initial conditions $(x^\mu(\lambda_0), dx^\mu(\lambda_0)/d\lambda)$. Given any connection, we can compute its geodesics via Eq. (3.37). Of course, if we have a metric, we can compute the Levi-Civita connection $\Gamma^\mu_{\nu\sigma}$, and then solve this equation to find the geodesics of the manifold.

The solution to Eq. (3.37) $x^\mu(\lambda)$ is the geodesic, parameterized by λ. In principle, λ can be any **monotonic parameter** (i.e. increasing in one direction along the curve), but since it is monotonic, for *timelike particles*, it makes sense to think of the parameter as some kind of "clock" as we move along it so it is very convenient to use the *proper time* τ as the parameter. This way Eq. (3.37) is even more physical – its solution $x^\mu(\tau)$ is the location of the timelike observer at *her clock time* τ. Hence, since the geodesic is the observer's trajectory through the spacetime, it is called her **worldline**.

On the other hand, for *null-like geodesics*, e.g. the geodesics of light rays, as we discussed in section 2.4.3, the proper time is always zero, so the parameter λ has no physical meaning beyond being a label.

In most GR literature, the parameter λ is often called the **affine parameter**. A parameter is called "affine" when, under the **affine transformation**

$$\lambda \rightarrow a\lambda + b, \tag{3.38}$$

for some real constants a and b, the geodesic equation is invariant. This is (perhaps not so obviously) true in our case here – if we try a general reparameterization $\lambda \rightarrow f(\lambda)$ you not guaranteed to recover Eq. (3.37) (try $\lambda \rightarrow \lambda^3$! (Homework)), so adding "affine" to the name does not really add a lot of information other than to remind you that "you can change parameters but only if it obeys Eq. (3.38)". Unless they are geodesics, you can parameterize curves on the manifold in any kind of parameter you like, so we don't really use the terminology here (although there is nothing wrong with using it.) Of course, if you call the parameter of a curve that is not a geodesic "affine", then you just look silly[6].

(Homework) Show that $V^\mu \nabla_\mu V^\nu = 0$ is the geodesic equation for a curve generated by the vector V^μ.

3.2.2 Geodesics on 2-sphere

Let's explicitly derive the great circle of the unit 2-sphere $\mathbb{S}^2$. The metric that describes $\mathbb{S}^2$, in the coordinate basis with coordinates (θ, ϕ), is

$$ds^2 = \mathrm{d}\theta^2 + (\sin\theta)^2 \mathrm{d}\phi. \tag{3.40}$$

The coordinates have domain $0 \leq \theta < \pi$ and $0 \leq \phi < 2\pi$, which as we discussed earlier in section 2.3, does not cover the entire 2-sphere, but as long as we stay away from the poles we will be fine. In "matrix" form, this is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & (\sin\theta)^2 \end{pmatrix} \tag{3.41}$$

[6]**What is the physical meaning of the affine-ness of a parameter? As you will show in a homework set, suppose one makes a non-affine transformation $\lambda' = f(\lambda)$, then the geodesic equation becomes

$$\frac{d^2x^\mu}{d\lambda'^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda'}\frac{dx^\sigma}{d\lambda'} = \frac{f''}{f'}\frac{dx^\mu}{d\lambda'}. \tag{3.39}$$

In flat space, $\Gamma^\mu_{\nu\sigma} = 0$, then we get $\ddot{x}^\mu = f''/f'\dot{x}^\mu$ where dot is $d/d\lambda'$, i.e. in this parameterization, the acceleration is proportional to the velocity. We can always eliminate this non-zero "acceleration" by choosing a clever reparameterization of the time coordinate, i.e. the original parameter λ. In words, when we choose a non-affine parameterization, we have chosen a weird choice of time coordinate which give us the illusion that acceleration is non-zero. Aficionados of Symmetry in Physics might wonder if the invariance of the geodesic equation under the affine transformation has any physical meaning – the answer is yes, and we will discuss this on Page 98.

and the inverse metric is

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & (\sin\theta)^{-2} \end{pmatrix}. \tag{3.42}$$

The Christoffel symbols for the Levi-Civita connections can be calculated using Eq. (3.32) problem. Let's work through this in detail to see how a lot of these calculations don't look as hard as they seems. From Eq. (3.32), $\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$, we can split the calculation up into $\rho = \theta$ and $\rho = \phi$ terms:

- $\rho = \theta$. We get

$$\Gamma^\theta_{\mu\nu} = \frac{1}{2} g^{\theta\theta}(\partial_\mu g_{\theta\nu} + \partial_\nu g_{\mu\theta} - \partial_\theta g_{\mu\nu}) + \frac{1}{2}\cancelto{0}{g^{\theta\phi}(\text{stuff})} \tag{3.43}$$

where we don't even have to think about "stuff" because $g^{\theta\phi} = 0$. Now, we can systematically compute for μ and ν through θ and ϕ. For $\mu = \nu = \theta$,

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2}(\partial_\theta g_{\theta\theta} + \partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta}) = 0 \tag{3.44}$$

since $g_{\theta\theta} = 1$, so any derivative w.r.t. must be zero. For $\mu = \nu = \phi$, we have

$$\begin{aligned} \Gamma^\theta_{\phi\phi} &= \frac{1}{2}(\cancelto{0}{\partial_\phi g_{\phi\theta}} + \cancelto{0}{\partial_\phi g_{\theta\phi}} - \partial_\theta g_{\phi\phi}) \\ &= \frac{1}{2}(-2\sin\theta\cos\theta) \\ &= -\sin\theta\cos\theta. \end{aligned} \tag{3.45}$$

Finally, for $\mu = \theta$ and $\nu = \phi$ (and *vice versa* since the Christoffel symbol is symmetric under interchange of the two lower indices), using symmetry of the two lower indices of the connection, we have

$$\Gamma^\theta_{\phi\theta} = \Gamma^\theta_{\theta\phi} = \frac{1}{2} g^{\theta\theta}(\partial_\phi g_{\theta\theta} + \partial_\theta g_{\phi\theta} - \partial_\theta g_{\theta\phi}) = 0 \tag{3.46}$$

since $g_{\theta\theta} = 1$ so any derivative is zero, and $g_{\theta\phi} = 0$.

- $\rho = \phi$. We get

$$\Gamma^\phi_{\mu\nu} = \frac{1}{2} g^{\phi\phi}(\partial_\mu g_{\phi\nu} + \partial_\nu g_{\mu\phi} - \partial_\phi g_{\mu\nu}) + \frac{1}{2}\cancelto{0}{g^{\phi\theta}(\text{stuff})} \tag{3.47}$$

where again we used $g^{\phi\theta} = 0$. Now, as before, we systematically compute the other terms

$$\Gamma^\phi_{\theta\theta} = \frac{1}{2} g^{\phi\phi}(\partial_\theta g_{\phi\theta} + \partial_\theta g_{\theta\phi} - \partial_\phi g_{\theta\theta}) = 0 \tag{3.48}$$

which again as usual we have used the fact that $g_{\theta\theta} = 1$ and vanishes under any derivative. Similarly,

$$\Gamma^\phi_{\phi\phi} = \frac{1}{2} g^{\phi\phi}(\partial_\phi g_{\phi\phi} + \partial_\phi g_{\phi\phi} - \partial_\phi g_{\phi\phi}) = 0. \tag{3.49}$$

The final term is

$$\begin{aligned} \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} &= \frac{1}{2} g^{\phi\phi}(\cancelto{0}{\partial_\phi g_{\phi\theta}} + \partial_\theta g_{\phi\phi} - \cancelto{0}{\partial_\phi g_{\phi\theta})} \\ &= \frac{1}{2}\frac{1}{\sin^2\theta}(2\sin\theta\cos\theta) \\ &= \frac{\cos\theta}{\sin\theta}. \end{aligned} \tag{3.50}$$

Hence, we find that the only non-zero terms are the $\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi}$ and $\Gamma^\theta_{\phi\phi}$ terms. The geodesic equation is then a pair of coupled second order ODEs

$$\frac{d^2\theta}{d\lambda^2} + \frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda}\Gamma^\theta_{\phi\phi} = \frac{d^2\theta}{d\lambda^2} - \left(\frac{d\phi}{d\lambda}\right)^2 \sin\theta\cos\theta = 0 \tag{3.51}$$

and

$$\frac{d^2\phi}{d\lambda^2} + \frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda}\Gamma^{\phi}_{\phi\theta} + \frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda}\Gamma^{\phi}_{\theta\phi} = \frac{d^2\phi}{d\lambda^2} + 2\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda}\frac{\cos\theta}{\sin\theta} = 0, \tag{3.52}$$

where we have been careful to note that even though $\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi}$, they are two separate terms and have to be summed over accordingly.

Eq. (3.51) and Eq. (3.52) are not easy to solve in general for any generic initial conditions $(\theta(0), \theta'(0))$ and $(\phi(0), \phi'(0))$ where primes mean derivative with respect to λ. However, we can take advantage of the symmetry of the sphere itself, and our freedom to choose coordinates such that the initial conditions become

$$\theta(0) = \frac{\pi}{2}\ ,\ \theta'(0) = 0\ ,\ \phi(0) = 0\ ,\ \phi'(0) = \omega \tag{3.53}$$

where ω is the initial velocity of the curve. It is then easy to show that the following *ansatz*

$$\theta(\lambda) = \frac{\pi}{2}\ ,\ \phi(\lambda) = \omega\lambda \tag{3.54}$$

is a solution to the geodesic equations and initial conditions Eq. (3.53) (plug it in and see for yourself). With a lot more work, you can show that the following **great circle equation**

$$\cot\phi = a\cos(\theta - \theta_0) \tag{3.55}$$

where a and θ_0 depend on the initial conditions, is a solution to geodesic equations of the $\mathbb{S}^2$, though we leave this to the reader to work it out for her/himself.

3.2.3 Geodesics as Curves of Minimum distance

In section 3.2.1, we derived the geodesic equation by defining it to be the geodesic where its tangent vector is parallel transported. This definition is general in the sense that it does not rely on the notion of distances, which as we have discussed in section 2.4.2 requires the metric tensor. But then how is it then the *shortest path* between two points if we have not used the metric tensor to define it? We will now rederive the geodesic equation, and show that it is indeed the curve of shortest distance given a metric.

Recall that the definition of the invariant distance for Euclidean metric Eq. (2.103),

$$L = \int ds = \int \sqrt{g_{\mu\nu}V^{\mu}V^{\nu}}d\lambda. \tag{3.56}$$

Defining dots to be $\dot{} = d/d\lambda$, so $V^{\mu} = dx^{\mu}/d\lambda = \dot{x}^{\mu}$, this become

$$\begin{aligned}\int ds &= \int \sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}d\lambda \\ &= \int G(x^{\mu}, \dot{x}^{\mu})d\lambda\end{aligned} \tag{3.57}$$

where $G(x^{\mu}, \dot{x}^{\mu}) \equiv \sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}$ is a **functional**[7] of $x^{\mu}(\lambda)$ and $\dot{x}^{\mu}(\lambda)$, which themselves are functions of λ. The usual prescription is now to find a curve $x^{\mu}(\lambda)$ such that L is *minimum* between the two points on the manifold, labeled by $\lambda = 0$ and $\lambda = 1$. Such a solution obeys the so-called **Euler-Lagrange equation**[8]

$$\frac{d}{d\lambda}\frac{\partial G}{\partial \dot{x}^{\mu}} - \frac{\partial G}{\partial x^{\mu}} = 0. \tag{3.58}$$

Now, it can be shown that by solving Eq. (3.58), we will obtain the geodesic equation with the Levi-Civita connetion. However, the square root in Eq. (3.57) is rather annoying to deal with, so we will instead consider the following *invariant squared distance*

$$|L|^2 = \int g_{\mu\nu}V^{\mu}V^{\nu}d\lambda, \tag{3.59}$$

[7] A functional is a function of functions.

[8] This minimization/maximization procedure is known as the **Action Principle** – L is called the *action* and we say "that the solution to the Euler-Lagrange equations minimize/maximize the action".

and apply the Euler-Lagrange equations Eq. (3.58) to it. The reason that this will recover the geodesic equation as much as using Eq. (3.56) is because if a curve which minimizes Eq. (3.56) exists, it *must* also minimnizes its square Eq. (3.59). For pedants, we provide the derivation using Eq. (3.56) in Appendix A.

Proceeding with the squared distance Eq. (3.59), we define dots to be $\dot{} = d/d\lambda$, such that $V^\mu = dx^\mu/d\lambda = \dot{x}^\mu$, the integrand functional then becomes

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{3.60}$$

We plug this into the Euler-Lagrange equation

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0, \tag{3.61}$$

whose solution should give us a line $x^\mu(\lambda)$ that minimizes $\int \mathcal{L}\, d\lambda$.

Working out the easier second term of the Euler-Lagrange equation first, this is

$$\frac{\partial\mathcal{L}}{\partial x^\mu} = \partial_\mu g_{\sigma\nu}\dot{x}^\sigma\dot{x}^\nu, \tag{3.62}$$

since $g_{\mu\nu}$ is clearly a function of the position x^α.

For the first term, we first calculate[9]

$$\begin{aligned}\frac{\partial}{\partial\dot{x}^\mu}\mathcal{L} = \frac{\partial}{\partial\dot{x}^\mu}g_{\sigma\nu}\dot{x}^\sigma\dot{x}^\nu &= g_{\sigma\nu}\dot{x}^\nu\delta^\sigma_\mu + g_{\sigma\nu}\dot{x}^\sigma\delta^\nu_\mu \\ &= 2g_{\mu\nu}\dot{x}^\nu \end{aligned}\tag{3.63}$$

to get

$$\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = 2g_{\mu\nu}\dot{x}^\nu. \tag{3.64}$$

Taking total derivative of Eq. (3.65), we have

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) = 2\frac{d}{d\lambda}\left(g_{\mu\nu}\dot{x}^\nu\right) = 2(\partial_\sigma g_{\mu\nu}\dot{x}^\sigma\dot{x}^\nu + g_{\mu\nu}\ddot{x}^\nu), \tag{3.65}$$

where we have used the fact that $g_{\mu\nu}$ is a function of $x^\sigma(\lambda)$, and apply the chain rule accordingly.

Putting Eq. (3.62) and Eq. (3.65) together into the Euler-Lagrange equation Eq. (3.61), we get

$$\begin{aligned}\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} - \frac{\partial\mathcal{L}}{\partial x^\mu} &= 2\left(g_{\mu\nu}\ddot{x}^\nu + \partial_\sigma g_{\mu\nu}\dot{x}^\sigma\dot{x}^\nu - \frac{1}{2}\partial_\mu g_{\sigma\nu}\dot{x}^\sigma\dot{x}^\nu\right) \\ &= 2\left(g_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}\partial_\sigma g_{\mu\nu}\dot{x}^\sigma\dot{x}^\nu + \frac{1}{2}\partial_\nu g_{\sigma\mu}\dot{x}^\sigma\dot{x}^\nu - \frac{1}{2}\partial_\mu g_{\sigma\nu}\dot{x}^\sigma\dot{x}^\nu\right) = 0. \end{aligned}\tag{3.66}$$

where in the second line we have used the following **symmetrization** procedure (you should convince yourself that this is actually correct)

$$V^\mu V^\sigma X_{\mu\sigma\ldots} = V^\mu V^\sigma X_{\sigma\mu\ldots}. \tag{3.67}$$

Finally, multiplying by inverse metric $g^{\mu\rho}$, we have

$$\ddot{x}^\rho + \frac{1}{2}g^{\rho\mu}\left[\partial_\sigma g_{\mu\nu}\dot{x}^\sigma\dot{x}^\nu + \partial_\nu g_{\sigma\mu}\dot{x}^\sigma\dot{x}^\nu - \partial_\mu g_{\sigma\nu}\dot{x}^\sigma\dot{x}^\nu\right] = 0. \tag{3.68}$$

or

$$\ddot{x}^\rho + \Gamma^\rho_{\sigma\nu}\dot{x}^\sigma\dot{x}^\nu = 0. \tag{3.69}$$

[9]Pay attention to the indices – resist the temptation to write $\partial/\partial\dot{x}^\mu g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ and go horribly wrong.

which is the geodesic equation Eq. (3.37) as promised (restoring the $d/d\lambda$ for the dots)

$$\frac{d^2x^\rho}{d\lambda^2} + \Gamma^\rho_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0, \tag{3.70}$$

where the Christoffel symbols $\Gamma^\rho_{\nu\sigma}$ are given by the Levi-Civita connection Eq. (3.32).

Hence, we have shown that the geodesic equation is indeed the equation whose solution is the path of minimal distance. Notice that since we have chosen to minimized the distance as defined by the metric, we automatically pick up the Levi-Civita connection.

3.2.4 Calculating Christoffel Symbols via the Action Principle

The prescription described in section 3.2.3 is more than mere interesting side-show, it provides to us an extremely efficient way of computing the Christoffel symbols quickly. Compare the Euler-Lagrange equation Eq. (3.61)

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\rho} - \frac{\partial\mathcal{L}}{\partial x^\rho} = 0, \tag{3.71}$$

with Eq. (3.69)

$$\ddot{x}^\rho + \Gamma^\rho_{\sigma\nu}\dot{x}^\sigma\dot{x}^\nu = 0, \tag{3.72}$$

we see that, if we have used the Euler-Lagrange equation on individual coordinates x^ρ we can systematically and quickly compute the Christoffel symbols. The best way to illustrate this is to show an example. Let's consider the 2-sphere again that we have encountered in section 3.2.2, and compute its Christoffel symbols using the Euler-Lagrange method.

The metric for the 2-sphere is as Eq. (3.40)

$$ds^2 = \mathrm{d}\theta^2 + (\sin\theta)^2\mathrm{d}\phi. \tag{3.73}$$

There are two coordinates $x^\mu = (\theta,\phi)$, hence the gradient along the geodesic is $V^\mu = dx^\mu/d\lambda = (\dot\theta,\dot\phi)$, so the invariant squared length Eq. (3.59) is

$$|L|^2 = \int g_{\mu\nu}V^\mu V^\nu d\lambda = \int \dot\theta^2 + \sin^2\theta\dot\phi^2\ d\lambda. \tag{3.74}$$

We now use the Euler-Lagrange Eq. (3.71).

- Coordinate $\rho = \theta$. We have

$$\frac{\partial\mathcal{L}}{\partial\theta} = 2\sin\theta\cos\theta\dot\phi^2\ ,\ \frac{\partial\mathcal{L}}{\partial\dot\theta} = 2\dot\theta, \tag{3.75}$$

so

$$\begin{aligned} \frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) - \frac{\partial\mathcal{L}}{\partial\theta} &= 0 \\ \frac{d}{d\lambda}\left(2\dot\theta\right) - 2\sin\theta\cos\theta\dot\phi^2 &= 0 \\ 2\ddot\theta - 2\sin\theta\cos\theta\dot\phi^2 &= 0 \end{aligned} \tag{3.76}$$

or

$$\ddot\theta - \sin\theta\cos\theta\dot\phi\dot\phi = 0. \tag{3.77}$$

Comparing this Eq. (3.77) to Eq. (3.72), we can immediately read off the Christoffel symbols

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\ ,\ \Gamma^\theta_{\phi\theta} = \Gamma^\theta_{\theta\phi} = 0\ ,\ \Gamma^\theta_{\theta\theta} = 0. \tag{3.78}$$

- Coordinate $\rho = \phi$. We have

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \ , \ \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 2\sin^2\theta\dot{\phi}, \tag{3.79}$$

or

$$\begin{aligned} \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} &= 0 \\ \frac{d}{d\lambda}\left(2\sin^2\theta\dot{\phi}\right) - 0 &= 0 \\ 2(2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + \sin^2\theta\ddot{\phi}) &= 0 \end{aligned} \tag{3.80}$$

where in the 3rd line we have made sure to remember that all the coordinates are functions of λ, so a total derivative acts through the chain rule $df(\theta)/d\lambda = (\partial f/d\theta)\dot{\theta}$. Rearranging the last line we get

$$\begin{aligned} \ddot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\phi}\dot{\theta} &= 0 \\ \ddot{\phi} + \frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} + \frac{\cos\theta}{\sin\theta}\dot{\phi}\dot{\theta} &= 0 \end{aligned} \tag{3.81}$$

where in the 2nd line we recall that the Christoffel symbols are symmetric under the interchange of its two lower indices. Reading off the Christoffel symbols by comparing it this to Eq. (3.77) again, we get

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{\cos\theta}{\sin\theta} \ , \ \Gamma^{\phi}_{\theta\theta} = \Gamma^{\phi}_{\phi\phi} = 0. \tag{3.82}$$

You can check that these results are identical to those of section 3.2.2 computed using the usual Christoffel symbol formula Eq. (3.32).

Which method you use to compute the Christoffel symbols is really up to you. In general, the action principle method outlined in this section is most efficient when you want to compute all the Christoffel symbols, and when the metric possesses a lot of symmetries. On the other hand, the formulaic method using Eq. (3.32) is faster when you only need one or two symbols (e.g. when you just want the symbols needed in a geodesic equation).

3.3 The Riemann Curvature Tensor

So far, we have extensively discussed the connection with respect to parallel transport of tensors on the manifold and how its choice (whether using the Levi-Civita connection or otherwise) "defines curvature". In this section, we will now make concrete this notion.

The main idea is as follows. If we draw a finite sized closed loop around some manifold with some connection, then in general transporting any vector around this loop will *not* return the same vector. This difference gives us a way to quantify **curvature** in a general manner – in other words we want to measure the *path dependence* of parallel transport - see figure 3.3. Clearly, we want our quantity to be coordinate independent, so it must be a tensor. What is the rank of this tensor? Let's guess: the final answer we want the difference in the vector δV^{μ} which is another vector, and a loop can be constructed out of a minimum of two vectors X^{μ} and Y^{μ}, and including the original vector V^{μ} we guess that this object must have the form

$$\delta V^{\rho} \stackrel{?}{=} R^{\rho}{}_{\mu\nu\sigma}Y^{\mu}X^{\nu}V^{\sigma} \tag{3.83}$$

i.e. this object must be a rank-$(1,3)$ tensor. This is indeed correct – and $R^{\rho}{}_{\mu\nu\sigma}$ is called the **Riemann Curvature Tensor** and is one of the most important objects in general relativity so we will study it in some detail.

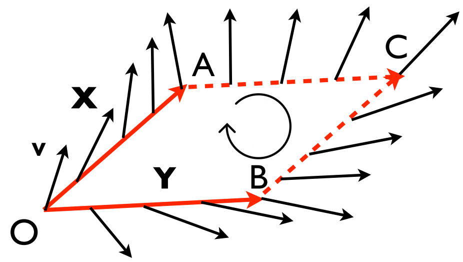


Figure 3.3: Transporting a vector $\mathbf{V}$ around a closed loop from the origin O does not return the same vector in general. The Riemann curvature tensor measures the difference of the vector $\mathbf{V}$ when transported along the path OAC and the path OBC.

3.3.1 Commutator of Vector Fields

Let's take a step back. Instead of transporting a vector around a loop, what about scalars? Now, since scalars are not vectors, it doesn't make sense to "parallel" transport them. But remember (see section 2.2) the vector $\mathbf{X}$ *maps a function to another function*, i.e. $\mathbf{X}(f) \mapsto g$, so if we now have two vectors $\mathbf{X}$ and $\mathbf{Y}$, is $\mathbf{X}(\mathbf{Y}(f))$ also a vector since it also maps a function to another function?

The answer to this question is *no* in general. However, we can define the *difference* as a **commutator**

$$\boxed{\mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f)) \equiv [\mathbf{X}, \mathbf{Y}](f)}. \tag{3.84}$$

Now this commmutator takes a scalar f and give you another scalar g such that the map can be represented by another vector field $\mathbf{Z}$, i.e.

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{Z}. \tag{3.85}$$

In other words, the commutator of two vector fields is another vector field. Note that the commutator is *not* a tensor product of $\mathbf{X}$ and $\mathbf{Y}$ i.e. $[\mathbf{X}, \mathbf{Y}] \neq \mathbf{X} \otimes \mathbf{Y} - \mathbf{Y} \otimes \mathbf{X}$ which would make it a rank-$(2,0)$ tensor instead of a vector. We will see later that this commutator notation is abused (unfortunately) where it *is* a tensor product, so be careful!

A good question to ask is : why do we need to define a commutator? Since the composition map $\mathbf{X}(\mathbf{Y}(f))$ maps a scalar to another scalar, isn't it another vector? The answer is *no* – since this map does not obey Leibnitz's rule as you will show in a (Homework) problem. The commutator is however, a perfectly good vector.

You will show in a (Homework) problem that, if the connection ∇ is torsion-free, then the commutator has the following nice identity

$$[\mathbf{U}, \mathbf{V}] = \nabla_{\mathbf{U}}\mathbf{V} - \nabla_{\mathbf{V}}\mathbf{U}\ ,\ \text{if } \nabla \text{ is torsion free.} \tag{3.86}$$

A special case arises when $\mathbf{X} = \hat{\mathbf{e}}_{(\mu)}$ and $\mathbf{Y} = \hat{\mathbf{e}}_{(\nu)}$ are coordinate basis vectors, then their commutator

$$[\mathbf{X}, \mathbf{Y}] = [\hat{\mathbf{e}}_{(\mu)}, \hat{\mathbf{e}}_{(\nu)}] = \partial_\mu \partial_\nu - \partial_\nu \partial_\mu = 0 \tag{3.87}$$

vanishes since partials commute[10].

3.3.2 Riemann Curvature Tensor Defined

Harking back to figure 3.3, let's make the loop smaller and smaller, until it is an infinitisimally small loop. Suppose at the origin O, there are two vectors $\mathbf{X}$ and $\mathbf{Y}$ which we can use to construct a "parallegram

[10]**Eq. (3.87) is both the necessary and sufficient condition for any set of vector fields to define a coordinate basis.

loop". We know that parallel transporting a vector $\mathbf{V}$ around this loop does not return the same vector. We now want to quantify this.

It turns out that instead of parallel transporting around the loop, it is more convenient to parallel transport $\mathbf{V}$ using $\mathbf{X}$ then $\mathbf{Y}$ (path OAC) and $\mathbf{Y}$ then $\mathbf{X}$ (path OBC), and then compute the difference $\delta\mathbf{V}$ at point C. So one might expect the difference between these two different routes of parallel transport can be written as

$$\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{V} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{V} \stackrel{?}{=} \delta\mathbf{V}. \tag{3.88}$$

Eq. (3.88) would be a perfectly fine definition except for the fact that the result is something in general not quite wieldy since it is a *derivative* operator.

Without going into too much detail on why, it turns out the following combination

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{V} \equiv \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{V} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{V} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{V} = \delta\mathbf{V} \tag{3.89}$$

is easier to use though it looks kinda scary for the moment – so let's unpack it. The object

$$\mathbf{R}(\mathbf{X},\mathbf{Y}) \equiv \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \nabla_{[\mathbf{X},\mathbf{Y}]} \tag{3.90}$$

should be read as "$\mathbf{R}$ is a function of two vectors, but if it acts on a vector $\mathbf{V}$ then according to Eq. (3.89), it results in a vector $\delta\mathbf{V}$". This means that $\mathbf{R}$ is really a rank-$(1,1)$ tensor – despite the fact that you feed with two rank-$(1,0)$ vectors.

To connect this to the previous definition of the Riemann Tensor (a rank-$(1,3)$ tensor) in Eq. (3.83), let's calculate Eq. (3.89) in a coordinate basis. We choose $\mathbf{X} = \hat{\mathbf{e}}_{(\mu)}$, $\mathbf{Y} = \hat{\mathbf{e}}_{(\nu)}$ and $\mathbf{Z} = \hat{\mathbf{e}}_{(\rho)}$, then using Eq. (2.69), we get (remember that $\nabla_\mu = \nabla_{\hat{\mathbf{e}}_{(\mu)}}$)

$$\begin{aligned}
\mathbf{R}(\hat{\mathbf{e}}_{(\mu)},\hat{\mathbf{e}}_{(\nu)})\hat{\mathbf{e}}_{(\rho)} &= \nabla_{\hat{\mathbf{e}}_{(\mu)}}\nabla_{\hat{\mathbf{e}}_{(\nu)}}\hat{\mathbf{e}}_{(\rho)} - \nabla_{\hat{\mathbf{e}}_{(\nu)}}\nabla_{\hat{\mathbf{e}}_{(\mu)}}\hat{\mathbf{e}}_{(\rho)} - \nabla_{[\hat{\mathbf{e}}_{(\mu)},\hat{\mathbf{e}}_{(\nu)}]}\hat{\mathbf{e}}_{(\rho)} \\
&= \nabla_\mu(\Gamma^\tau_{\nu\rho}\hat{\mathbf{e}}_{(\tau)}) - \nabla_\nu(\Gamma^\tau_{\mu\rho}\hat{\mathbf{e}}_{(\tau)}) \\
&= \nabla_\mu\Gamma^\tau_{\nu\rho}\hat{\mathbf{e}}_{(\tau)} + \Gamma^\tau_{\nu\rho}\nabla_\mu\hat{\mathbf{e}}_{(\tau)} - \nabla_\nu\Gamma^\tau_{\mu\rho}\hat{\mathbf{e}}_{(\tau)} - \Gamma^\tau_{\mu\rho}\nabla_\nu\hat{\mathbf{e}}_{(\tau)} \\
&= \nabla_\mu\Gamma^\tau_{\nu\rho}\hat{\mathbf{e}}_{(\tau)} + \Gamma^\tau_{\nu\rho}\Gamma^\sigma_{\mu\tau}\hat{\mathbf{e}}_{(\sigma)} - \nabla_\nu\Gamma^\tau_{\mu\rho}\hat{\mathbf{e}}_{(\tau)} - \Gamma^\tau_{\mu\rho}\Gamma^\sigma_{\nu\tau}\hat{\mathbf{e}}_{(\sigma)} \\
&= \nabla_\mu\Gamma^\sigma_{\nu\rho}\hat{\mathbf{e}}_{(\sigma)} + \Gamma^\tau_{\nu\rho}\Gamma^\sigma_{\mu\tau}\hat{\mathbf{e}}_{(\sigma)} - \nabla_\nu\Gamma^\sigma_{\mu\rho}\hat{\mathbf{e}}_{(\sigma)} - \Gamma^\tau_{\mu\rho}\Gamma^\sigma_{\nu\tau}\hat{\mathbf{e}}_{(\sigma)}
\end{aligned} \tag{3.91}$$

In the 2nd line we have used Eq. (3.3) and the fact that $[\hat{\mathbf{e}}_{(\mu)},\hat{\mathbf{e}}_{(\nu)}] = 0$ since partial derivatives commute to get rid of the last term. In the 3rd line we have again used Eq. (3.3) and Leibnitz's law. In fourth line, we have used Eq. (3.3) again. In the last line we relabeled some indices (since they are summed over we can do that). Now recall that the Christoffel symbols $\Gamma^\tau_{\nu\rho}$ are not tensors but just an array of functions, hence the covariant derivative of Christoffel symbols is a partial derivative, i.e.

$$\nabla_\mu\Gamma^\tau_{\nu\rho} = \partial_\mu\Gamma^\tau_{\nu\rho} \tag{3.92}$$

we finally get,

$$\boxed{R^\tau{}_{\rho\mu\nu} = \partial_\mu\Gamma^\tau_{\nu\rho} + \Gamma^\tau_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\tau_{\mu\rho} - \Gamma^\tau_{\nu\sigma}\Gamma^\sigma_{\mu\rho}}. \tag{3.93}$$

Eq. (3.93) is the component form of the Riemann tensor, and it is a rank-$(1,3)$ tensor. Whether you call $\mathbf{R}(,)$ or $R^\tau{}_{\rho\mu\nu}$ the Riemann tensor is really a matter of taste – the latter is simply the former in when we feed $\mathbf{R}$ with the basis vectors. Since in component form, Riemann tensor has 4 indices, sometimes it is defined, for general X^μ and Y^ν, in relation to Eq. (3.89), as

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{V} \equiv R^\rho{}_{\sigma\mu\nu}V^\sigma X^\mu Y^\nu\hat{\mathbf{e}}_{(\rho)} = \delta V^\rho\hat{\mathbf{e}}_{(\rho)}. \tag{3.94}$$

In GR literature, the Riemann tensor is also sometimes defined as

$$[\nabla_\mu,\nabla_\nu]V^\rho \equiv \nabla_\mu\nabla_\nu V^\rho - \nabla_\nu\nabla_\mu V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda{}_{\mu\nu}\nabla_\lambda V^\rho \tag{3.95}$$

for any vector V^ρ where $T^\lambda{}_{\mu\nu}$ is the torsion as defined in Eq. (3.25), while

$$[\nabla_\mu, \nabla_\nu]V_\rho \equiv \nabla_\mu\nabla_\nu V_\rho - \nabla_\nu\nabla_\mu V_\rho = R_\rho{}^\sigma{}_{\mu\nu}V_\sigma - T^\lambda{}_{\mu\nu}\nabla_\lambda V_\rho \tag{3.96}$$

for any co-vector V_ρ. You will derive this in a (Homework) problem.

The reason that the Riemann tensor Eq. (3.93) is more useful than Eq. (3.88) is because, despite being defined as commutators of covariant derivatives, it is really a *multiplicative* operator – we don't need to know what $R^\rho{}_{\sigma\mu\nu}$ is acting on to calculate its values, it depends on just the connections. In GR, given a metric we can calculate the Levi-Civita connections and hence compute the Riemann tensor. If the metric exists everywhere on the manifold, so will the Riemann tensor[11].

Finally, since the Riemann tensor is constructed out of Christoffel symbols and its derivatives, *it is identically zero for Euclidean and Minkowski spaces since both these spaces possess a single coordinate chart such that all the Christoffel symbols vanish everywhere.* Notice that this is *not* the same as saying that the Riemann tensor vanishes because the Christoffel symbols vanish – one can always find a coordinate chart such that *locally* the Christoffel symbols vanish but since the Riemann tensor contains *derivatives* of the Christoffel symbols, non-Minkowski/Euclidean spaces do not necessarily vanish. In addition, one can always find a coordinate charts for Euclidean/Minkowski spaces where the Christoffel symbols do not vanish everywhere – as you will show in a (Homework set) when you are asked to calculate the Christoffel symbols for Euclidean space in spherical polar coordinates.

3.3.3 Symmetries of the Riemann Tensor in Levi-Civita Connection

From looking at the component form of the Riemann tensor Eq. (3.93), it's clear that it is anti-symmetric under the change of the last two indices, i.e.

$$\boxed{R^\tau{}_{\rho\mu\nu} = -R^\tau{}_{\rho\nu\mu}}. \tag{3.97}$$

Without any more assumptions about the connection, this is the only symmetry the Riemann Tensor possesses.

In GR, we specialise to the Levi-Civita connection, and due to the symmetric nature of the metric $g_{\mu\nu} = g_{\nu\mu}$, the Riemann tensor gains a few additional identities (which you should memorise). Using the "lowered indices" tensor (which you will prove in (Homework))

$$R_{\rho\sigma\mu\nu} = g_{\rho\gamma}R^\gamma{}_{\sigma\mu\nu}, \tag{3.98}$$

we can show the following is true:

- It is anti-symmetric in the first two indices

$$\boxed{R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}}, \tag{3.99}$$

  in addition to being anti-symmetric in the last two indices as we already shown in Eq. (3.97)

$$\boxed{R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}}. \tag{3.100}$$

- It is symmetric under the exchange of the first two with the last two indices

$$\boxed{R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}}. \tag{3.101}$$

- The sum of the permutations of the last 3 indices is zero

$$\boxed{R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0}. \tag{3.102}$$

[11] Most elementary treatments of GR hence usually say that "the metric defines the Riemann curvature tensor", but you are now wise to realize that this is not quite true.

- It obeys the **Bianchi Identity** (Homework)

$$\boxed{\nabla_\tau R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\tau} + \nabla_\nu R_{\rho\sigma\tau\mu} = 0.} \tag{3.103}$$

Given all these identities[12] , it is useful to ask how many *independent* quantities are there in the Riemann tensor. Let d be the dimension of the manifold. The first property means that there are $M = (1/2)d(d-1)$ ways of choosing the $\rho\sigma$ pair. Similarly for the second property, there are also $M = (1/2)d(d-1)$ ways of choosing the $\mu\nu$ pair. The 3rd property means that there the first two choices are symmetric, and hence there are $(1/2)M(M+1)$ ways in total. The last property tells us that not all these choices are independent. It turns out (which we will state without proof, but you can show to yourself) that Eq. (3.102) implies that the sum of the **anti-symmetrized** indices is zero

$$R_{\rho\sigma\mu\nu} - R_{\rho\sigma\nu\mu} + R_{\rho\nu\sigma\mu} - R_{\nu\rho\sigma\mu} + R_{\nu\rho\mu\sigma} - R_{\nu\mu\rho\sigma} + R_{\mu\nu\rho\sigma} - \cdots = 0 \tag{3.104}$$

This is sometimes written as

$$R_{[\rho\sigma\mu\nu]} = 0 \tag{3.105}$$

where the square brackets in the indices indicate **anti-symmetrization** – we won't use this notation here. This means that out of there are ${}^dC_4 = d!/(4!(d-4)!)$ terms which are not independent but subject to this constraint, making a total of

$$\frac{1}{2}M(M+1) - {}^dC_4 = \frac{1}{12}d^2(d^2-1) \tag{3.106}$$

independent components of the Riemann tensor.

In a 4D manifold, there are a total of 20 independent components – this will be what we usually be working with since spacetime is 4D. In 2D, the total number of independent components is 1. For example, the 2-sphere, $\mathbb{S}^2$ can be completely described by a single function $R(p)$ where $p \in \mathcal{M}$, which is the radius of curvature for the sphere – i.e. its radius. You will show in a (Homework) problem that if we "deform" the sphere into a dumb-bell (think of a deflated football for example), then this single parameter describes the local radius of curvature. In 1D, there are zero components – 1D manifolds do not possess curvature. You might wonder why a circle is "curved" – it "looks curvy" when *embedded* in a 2D surface where we drew it; 1D person trapped on a circle can only go forward or backwards, and hence would have *no way of measuring curvature* – read section 3.4 if you'd like to learn more!

In 3D, the total number of independent components is 6. It will turn out that this implies that gravity has no dynamics in 3D, but first we will have to discuss gravitational dynamics so we will postpone this discussion to Chapter 4

3.3.4 Geodesic Deviation

With the Riemann Tensor in hand, we can finally quantify what we meant by "parallel lines don't stay parallel in general" in curved spaces. In section 3.1, we discuss briefly how in Euclidean space parallel lines do stay parallel. In a general curved space (such as the sphere), there is no notion of "what is parallel", so the way we quantify this is to consider how *nearby geodesics move towards or away from each other* – we call this the **geodesic deviation**.

Consider a family of geodesics labeled by the parameter $s \in \mathbb{R}$, and each of these geodesic is parameterized by $t \in \mathbb{R}$ – see figure 3.4. Since we are always free to choose how we assign the labels and parameters to the curves, we assign them such that in a coordinate chart x^μ, the geodesics are specified by $x^\mu(s,t)$ *smoothly* (so that we can take derivatives of them w.r.t. s and t). This means that this family of geodesics trace out a 2D surface Σ in the manifold $\mathcal{M}$. You can think of this surface as "traced out" by curves of constant s and constant t. Note that the curves are not necessarily orthogonal to each other.

[12] To be completely precise, the last two identities require just that the connection be torsion-free, which the Levi-Civita certainly obeys.

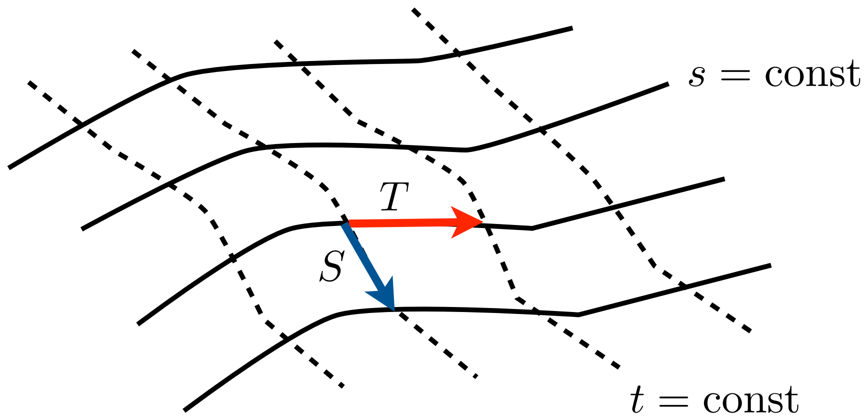


Figure 3.4: A family of geodesics labeled by s and parameterized by t. This set of curves trace out a 2D surface Σ in the manifold $\mathcal{M}$. At any point on this set of, we can construct the tangent vectors to these set of parameters $\mathbf{T} = d/dt$ and $\mathbf{S} = d/ds$. The relative rate of change of $\mathbf{S}$ along the geodesic is then $\nabla_{\mathbf{T}}\mathbf{S}$ and the relative acceleration is $\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\mathbf{S}$.

At any point p on this surface Σ, we can construct the vectors $\mathbf{T} = d/dt$ and $\mathbf{S} = d/ds$, which are tangent vectors to the t and s curves respectively. In the coordinate x^μ, $\mathbf{S} = dx^\mu/ds\partial_\mu$, and hence Taylor expanding the curve around s with a small perturbation δs, i.e.

$$x^\mu(s+\delta s, t) = x^\mu(s,t) + \delta s S^\mu + \mathcal{O}(\delta s^2) \tag{3.107}$$

we can see that S^μ points towards the neighbouring geodesic. From our definition of distances using the metric, Eq. (2.89) we see that $S^\mu \delta s$ measure the (infinitisimal) distance

$$dl^2 = g_{\mu\nu} S^\mu S^\nu (ds)^2. \tag{3.108}$$

We can now ask: how does $\mathbf{S}$ change as we move along the geodesics? This is the *rate of change* of $\mathbf{S}$ along $\mathbf{T}$, i.e.

$$\nabla_{\mathbf{T}}\mathbf{S} = (\nabla_{\mathbf{T}}\mathbf{S})^\mu \hat{\mathbf{e}}_{(\mu)} \equiv V^\mu \hat{\mathbf{e}}_{(\mu)}, \tag{3.109}$$

where $(\nabla_{\mathbf{T}}\mathbf{S})^\mu = V^\mu$ is the *relative velocity* between neighbouring geodesics.

We can also define the *relative acceleration* between neighbouring geodesics (in analogy to $\ddot{x} = a$ in Newtonian Mechanics), using Eq. (3.12),

$$\begin{aligned}
\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\mathbf{S} &= T^\alpha(\nabla_\alpha(\nabla_{\mathbf{T}}\mathbf{S})^\mu)\hat{\mathbf{e}}_{(\mu)} \\
&= T^\alpha\nabla_\alpha(T^\nu\nabla_\nu S^\mu)\hat{\mathbf{e}}_{(\mu)} \\
&\equiv A^\mu \hat{\mathbf{e}}_{(\mu)},
\end{aligned} \tag{3.110}$$

where A^μ is known as the **geodesic deviation vector** and Eq. (3.110) is known as the **geodesic deviation equation**.

If the connection ∇ is torsion-free, as it would be if we use the Levi-Civita connection in GR, then from Eq. (3.86), we have

$$[\mathbf{T},\mathbf{S}] = 0 \Rightarrow \nabla_{\mathbf{T}}\mathbf{S} = \nabla_{\mathbf{S}}\mathbf{T} \tag{3.111}$$

hence we can write the geodesic deviation vector the following equivalent ways

$$A^\mu = T^\alpha\nabla_\alpha(T^\nu\nabla_\nu S^\mu) = T^\alpha\nabla_\alpha(S^\nu\nabla_\nu T^\mu). \tag{3.112}$$

We can express A^μ in terms of the Riemann tensor as follows. Since $[\mathbf{T},\mathbf{S}] = 0$, using the definition of the Riemann tensor Eq. (3.89), we get

$$\mathbf{R}(\mathbf{T},\mathbf{S})\mathbf{T} = \nabla_{\mathbf{T}}\nabla_{\mathbf{S}}\mathbf{T} - \nabla_{\mathbf{S}}\nabla_{\mathbf{T}}\mathbf{T} - \cancelto{0}{\nabla_{[\mathbf{T},\mathbf{S}]}\mathbf{T}} \tag{3.113}$$

or

$$\nabla_{\mathbf{T}}\nabla_{\mathbf{S}}\mathbf{T} = \mathbf{R}(\mathbf{T},\mathbf{S})\mathbf{T} + \nabla_{\mathbf{S}}\nabla_{\mathbf{T}}\mathbf{T}. \tag{3.114}$$

Using again Eq. (3.111) to replace the LHS term, we get

$$\begin{aligned} A^\rho \hat{\mathbf{e}}_\rho &= \nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\mathbf{S} \\ &= \mathbf{R}(\mathbf{T},\mathbf{S})\mathbf{T} + \nabla_{\mathbf{S}}\nabla_{\mathbf{T}}\mathbf{T} \\ &= \mathbf{R}(\mathbf{T},\mathbf{S})\mathbf{T} \end{aligned} \tag{3.115}$$

where we have used the fact that $\nabla_{\mathbf{T}}\mathbf{T} = 0$ since $\mathbf{T}$ is the tangent to the geodesic. Using Eq. (3.94), in component form this is

$$\boxed{A^\rho = R^\rho{}_{\sigma\mu\nu} T^\sigma T^\mu S^\nu}. \tag{3.116}$$

Since free-falling objects follow geodesics, A^μ allow us to measure how neighbouring objects "accelerate" relatively towards each other. Eq. (3.116) then directly relate the motion of free-falling particles to the Riemann curvature tensor – so if we want to measure curvature of spacetime all we need to do is to let a bunch of particles free-fall, measuring how they move towards or away from each other would allow us to directly obtain the components of the Riemann tensor. Put it another way, it measures how spacetime "stretches" or "compresses". This *relative* motion of free-falling particles is the general relativistic explanation of the Tidal Forces we talked about way back in Chapter 1. We will elaborate further on this in section 4.5.

As you will see later when we talk about gravitation radiation in section 5.3, this is exactly how we characterise and measure gravitational waves.

3.4 Submanifolds and Embeddings

The problem with human beings is that we are terrible at imagining things that are not in 3D. When we talk about the 2-sphere $\mathbb{S}^2$, we draw the 2-sphere as a sphere inside a "ambient" 3D box (i.e. 3D Euclidean space $\mathbb{E}^3$), just like in figure 2.7. Our eyes (or our imagination) sees that the sphere's surface "curves" with respect to the 3D "ambient space". But this can be very misleading – curvature with respect to "ambient space" is *not* the same as **intrinsic** curvature of the sphere itself. The Riemann tensor describes the **intrinsic curvature** of the sphere. Having said that, there is a notion of "curving with respect to a higher-dimensional ambient space", and we call that **extrinsic curvature**.

A more illustrative example is that of 1-sphere $\mathbb{S}$, which is just the regular old circle. We can "draw" the circle on a flat piece of paper, i.e. the "ambient space" is 2D Euclidean space $\mathbb{E}^2$. In 2D, the circle obviously looks "curved", and we can calculate its extrinsic curvature – it is simply the radius of the circle. However, if you are stuck on the circle itself (if you take too much pot, this is the kind of dream you have as a physicist), then *there is no way you can make this measurement* since you can't go "out" of the circle. All you can do is move forward and backward on the circle – even sideways motion is not allowed – and if you move in one direction long enough, you will find that you are back at the same spot. But since there is no notion of "left" or "right", there is no notion of "turning". In fact, there is *no intrinsic curvature* – from Eq. (3.106) the Riemann tensor has *zero* components since $d = 1$ so it is identically zero $R_{1111} = 0$. Indeed, this is a general result – 1D manifolds have no intrinsic curvature.

Let's put this in slightly more precise terms and with proper jargon. When we put a circle $\mathbb{S}$ (a 1D manifold) in a 2D box $\mathbb{E}^2$ (a 2D manifold), we say that we **embed** $\mathbb{S}$ into $\mathbb{E}^2$. Then $\mathbb{S}$ is a **sub-manifold** of the **embedding space** $\mathbb{E}^2$. The precise definition of an embedding is beyond what we have time for. But roughly speaking it is a injective (1-to-1) map from the lower (d_M) dimensional manifold $\mathcal{M}$ to the higher (d_N) dimensional manifold[13] $\mathcal{N}$

$$f : \mathcal{M} \to \mathcal{N} \tag{3.117}$$

such that the vector spaces of $\mathcal{M}$, T_pM, are also mapped in an injective manner to a d_M **subspace** of $T_{f(p)}N$. In even less precise language, an embedding maps not just the manifold, but also its vector spaces (and hence its tensor spaces) to the higher dimensional manifold. If you like, it "preserves the structure" of the submanifold inside the embedding space.

Embeddings allow us to construct metrics of lower dimensional manifolds if we know how it "looks" like in the higher dimensional manifold. Let's see this by considering the how the embeddings of $\mathbb{S}^2$ in $\mathbb{R}^3$ allow us to find the metric for $\mathbb{S}^2$, since we already have the figure 2.7 conveniently plotted out.

In Cartesian coordinates, 3D Euclidean space has the metric $g_{\mu\nu} = (1, 1, 1)$ in Cartesian coordinates (x, y, z), but in spherical coordinates (r, θ, ϕ) the metric is

$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2. \tag{3.118}$$

The sphere $\mathbb{S}^2$ is then the submanifold of $\mathcal{E}^3$ with some fixed radius $r = R = \mathrm{const}$, and we get

$$ds^2 \equiv d\Omega^2 = R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\phi^2 \tag{3.119}$$

which we can write as

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \tag{3.120}$$

which is then the metric of an R-unit $\mathbb{S}^2$ – you can set $R = 1$ to get the unit sphere we considered earlier in section 3.2.2. Notice that we have used the fact that the metric (which is a rank-$(0, 2)$ tensor) in $\mathbb{S}^2$ has a 1-to-1 map to the subspace of the metric tensor in $\mathbb{E}^3$ – this is the reason why we have to be pedantic about not just mapping the manifold, but also the tensor spaces that come with it[14].

In more advanced treatments of GR, we can write down the formulas for the extrinsic curvature of any submanifold in its embedding space, but unfortunately we won't have the time to do that in these lecture notes. Those interested can refer to the GR books suggested in the beginning of the notes.

[13] To be super precise, it is actually OK as long as $d_M \leq d_N$, not just $d_M < d_N$.

[14] g_{ij} is in the subspace of $g_{\mu\nu}$ – it spans the subset of the basis vectors without the component ∂_r.

Chapter 4

Gravitation and the Einstein Equation

...matter tells spacetime how to curve.
John Archibald Wheeler

We have spent a lot of time discussing the mathematics of curved spaces, and have learned a bunch of tools to deal with them – how things move on them, how to characterize how "curved" they are etc.

Recall that we can split General Relativity into two main ideas.

1. Spacetime curvature tells how matter moves.

and

2. Matter tells spacetime how to curve.

In the previous Chapter 3, we studied Idea 1. We are now ready to study Idea 2, which is the *dynamics* of curvature – how curvature changes in response to matter fields.

Again we will be working with the Levi-Civita connection unless explicitly specified.

4.1 Decomposing the Riemann Tensor

It should be clear to you that the Riemann tensor plays a central role in the physics of curved spaces. Being a 4-index object, the Riemann tensor contains a lot of information about the spacetime curvature. We will now break it down into several pieces.

4.1.1 The Ricci Tensor and the Ricci Scalar

Without too much elaboration at the moment, we can *define* the **Ricci Tensor** by contracting the Riemann tensor with the (inverse) metric

$$\boxed{R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}}. \tag{4.1}$$

You will show in a (Homework) problem that the Ricci tensor is symmetric under interchange of its two indices

$$R_{\mu\nu} = R_{\nu\mu}. \tag{4.2}$$

We can further contract the Ricci tensor to a scalar by contracting its two indices to obtain the **Ricci Scalar** or **Ricci curvature**

$$\boxed{R \equiv g^{\mu\nu} R_{\mu\nu}}. \tag{4.3}$$

The Ricci scalar has a very nice physical interpretation: it exactly quantifies the difference in the *d-volume* from completely flat (Euclidean/Minkowski) space. To see this, let's consider a 2D Riemannian manifold, i.e. $d = 2$. Since it is a Riemannian manifold, we know there is a metric. The metric allows us to define a notion of distances and angles as we discussed in section 2.4.2. Leaving aside the angles for now, with distances we can define a notion of volume ("square of the distances"). At any point x in this manifold, we can draw a circle with radius r (again we can define lengths). In Euclidean space, this area is πr^2. See figure 4.1. But in a curved space, due to the curvature of the space, this area would not be πr^2. Clearly, this difference is bigger when r is bigger – we see more deviation from flat Euclidean space as we probe more and more of the curved spacetime. What happens when we take $r \to 0$? It turns out that the difference is exactly the Ricci scalar at point x

$$R(x) = \lim_{r\to 0} \frac{\pi r^2 - A(x,r)}{24\pi r^4}. \tag{4.4}$$

If $R < 0$, then the area is said to be **negatively curved**, like a saddle (or a Pringle). While if $R > 0$, the area is **positively curved** – like the surface of a sphere as in figure 4.1.

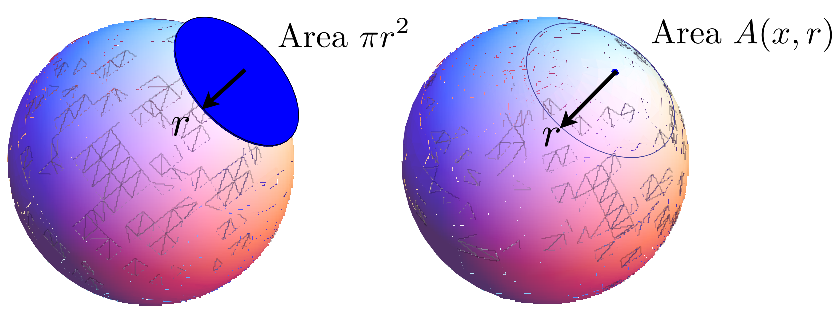


Figure 4.1: The Ricci scalar is defined as the difference in the area (volume in higher dimensions) of the curved space *projected onto a plane* (right), when compared to the flat space (left blue disk). Since the right space is curved, the *projected* area A will be *smaller* compared to the unprojected flat (left blue disk).

In 2D, the Ricci tensor also has a very nice physical interpretation – instead of the area within a disk, consider the area of an angular section (i.e. a pizza slice) given by $A(x, r, \theta, \mathbf{v})$ where θ is the angle subtended by the section and $\mathbf{v}$ is the vector pointing at the direction of the section (i.e. which slice of pizza). In Euclidean space, the area of the section is given by the "section area formula" $(1/2)\theta r^2$. The Ricci tensor is then given by the limit of the difference between the curved section and the Euclidean section when we take r and θ to zero, i.e.

$$R_{\mu\nu} v^\mu v^\mu = \lim_{r\to 0} \lim_{\theta\to 0} \frac{((1/2)\theta r^2 - A(x, r, \theta, \mathbf{v}))}{\theta r^4/24}. \tag{4.5}$$

It turns out that in 2D, the Ricci tensor and the Ricci scalar contain exactly the same information. In more than 2D however, the Ricci tensor can contain more information. You can imagine replacing 2D sections with the appropriate 3 or higher dimensions objects, such as cones and hypercones, and the extra information allows spacetime to be positively curved in one direction and negatively curved in another and so forth.

Roughly speaking, the Ricci tensor tells you how the *volume changes* as you move along different directions[1]. In Lorentzian manifolds of spacetime, the Ricci tensor encodes how "spacetime volume" changes as we move around the manifold. In particular, if we move along the time direction, then it encodes how the *spatial volume changes in time* – we will soon see (section 4.2) that this fact allows us to intuit how matter tells how spacetime to curve.

(Homework) Prove the Ricci Identity

4.1.2 The Weyl Tensor

Both the Ricci tensor and Ricci scalar are contractions of the Riemann tensor – so one would correctly expect that they do *not* fully capture all the information of the Riemann tensor. For example, in d dimensions, the Ricci tensor has $1/2d(d+1)$ independent components. So in 4D, it has 10 independent components. Compare this to the 20 independent components of the 4D Riemann tensor, we have lost half the information. This "lost" information is captured by the **Weyl tensor** $C_{\rho\sigma\mu\nu}$, which is a rank-$(0,4)$ tensor defined by (no need to memorize this)

$$\begin{aligned} C_{\rho\sigma\mu\nu} &= R_{\rho\sigma\mu\nu} - \frac{1}{d-2}(g_{\rho\mu}R_{\nu\sigma} - g_{\rho\nu}R_{\mu\sigma} - g_{\sigma\mu}R_{\nu\rho} + g_{\sigma\nu}R_{\mu\rho}) \\ &\quad + \frac{1}{(d-1)(d-2)}(g_{\rho\mu}g_{\nu\sigma} - g_{\rho\nu}g_{\mu\sigma})R. \end{aligned} \tag{4.6}$$

You will show in a (Homework) problem that all the contractions of the Weyl tensor vanish.

Ricci tensor is the "trace" part of the Riemann tensor, while the Weyl tensor is the "trace-free" part of the Riemann tensor. This is analogous to the way we decompose any 3-vector, $v_i = \tilde{v}_i + \partial_i v$, into its trace-free $\partial_i \tilde{v}_i = 0$ component and trace component v.

While the Ricci tensor (and scalar) encodes how volume changes in the manifold, the Weyl tensor encodes *how curvature changes without any volume changes*. Without too much elaboration for now, it stores information on physical effects such as tidal forces and gravitational waves.

4.2 The Einstein Equation

In 1915, Einstein wrote down his eponymous equation of motion for gravitation

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}} \tag{4.7}$$

where G is the Newton's constant and $T_{\mu\nu}$ is the energy-momentum tensor that we discussed in section 2.3.5 (we have lowered the indices with the metric $T_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}$). Since the Ricci tensor and Ricci scalar describe how spacetime curvature changes in its volume, and the energy-momentum tensor $T_{\mu\nu}$ encodes the density, pressure and stresses of the matter, Eq. (4.7) roughly states that

$$\text{How spacetime volume changes} = \text{State of the matter}. \tag{4.8}$$

In other words, the **Einstein Equation** tells us how spacetime curvature changes dynamically in response to the presence of matter fields.

4.2.1 Conservation of Energy-Momentum and Principle of Minimal Coupling

The LHS of Eq. (4.7) is called the **Einstein tensor** $G_{\mu\nu}$

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R}. \tag{4.9}$$

[1] **We will actually derive the necessarily equations to prove this fact in section 4.5.1 although we won't show it here – you are encouraged to convince yourself of this fact by using Eq. (4.82).

You will show in a (Homework) problem that the contracted form of the Bianchi Identity Eq. (3.103) implies

$$\boxed{\nabla^\mu G_{\mu\nu} = 0}, \tag{4.10}$$

which is sometimes called the **Contracted Bianchi Identity**. This means that the energy-momentum tensor must also obey the **conservation law** $\nabla^\mu T_{\mu\nu} = 0$, or after raising/lowering with $g_{\mu\nu}$ and $g^{\mu\nu}$,

$$\boxed{\nabla_\mu T^{\mu\nu} = 0}. \tag{4.11}$$

Furthermore, the energy-momentum tensor of perfect fluids Eq. (2.75) also becomes

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}, \tag{4.12}$$

where we have also replaced $\eta^{\mu\nu} \to g^{\mu\nu}$ in the last term.

Compared to special relativity version of the conservation law Eq. (2.70), we see that Eq. (4.11) is different by the replacement

$$\partial_\mu \to \nabla_\mu \ , \ \eta_{\mu\nu} \to g_{\mu\nu} \tag{4.13}$$

to obtain the curved space generalization. This prescription, sometimes called **the principle of minimal coupling** is actually quite general. Often, *but not always*, we can get the "curved space generalization" of "special relativity covariant" equations by doing the following simply prescribing Eq. (4.13) – unfortunately we also call the curved space generalization equations **covariant form**. For example, the curved space equivalent of the Maxwell equation $\partial_\mu F^{\mu\nu} = J^\mu$ is $\nabla_\mu F^{\mu\nu} = J^\mu$ etc. The reason we can do this is because of EEP – physical laws must reduce to their flat space equivalent around local inertial frame. You can use this as a rule of thumb, but be warned that this prescription does not alway mean you get the correct equations[2].

4.3 Postulates of General Relativity

We have developed much of Einstein's formulation of gravity throughout the lectures, but they are somewhat all over the place because we have to develop the mathematics to understand it along the way – like learning to fly while building the plane at the same time. Let's now put them all together in one place here, now with high-brow language.

- *Spacetime is a Lorentzian manifold equipped with the Levi-Civita connection. In other words, it is equipped with a metric with a Lorentzian signature which also defines distances and angles on the manifold.*
- *Minkowski space describes a spacetime with no gravitational fields. Gravitational fields are described by deviations of the metric from Minkowski.*
- *Free-falling matter travels on null or time-like geodesics.*
- *Matter are represented by the energy-momentum tensor $T^{\mu\nu}$ which lives on the manifold, which obeys the conservation equation $\nabla_\mu T^{\mu\nu} = 0$.*
- *The curvature of spacetime dynamically evolves according to the Einstein Equation*

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{4.14}$$

  where $G_{\mu\nu}$ is the Einstein tensor which obeys the contracted Bianchi Identity $\nabla^\mu G_{\mu\nu} = 0$.

[2] **A oft-made error is the electromagnetic tensor Eq. (2.77) $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. It seems tempting to write it as $F_{\mu\nu} \stackrel{?}{=} \nabla_\mu A_\nu - \nabla_\nu A_\mu$ but it would be wrong – it would gain a torsion term that may or may not be zero depending on connections. The electomagnetic tensor is secretly something that can exist without the need of a connection. In proper jargon, $\bar{\mathbf{F}}$ is an exact 2-form which we can integrate over the manifold without needing any extraneous structure like the metric or connections etc.

4.3.1 The Newtonian Limit

We know from experience that Newtonian Mechanics is very accurate, so the Einstein equation Eq. (4.14) must reduce to the Newton's 2nd law Eq. (1.1) and Eq. (1.2) (or equivalently the Poisson Eq. (1.5)) in some limit.

Specializing to the Cartesian coordinates (and using the coordinate basis), these limits are

- *Non-relativistic motion*: Particles are traveling at velocities much smaller than the speed of light $v \ll 1$ (remember we set $c = 1$). This reduces to Newton's 2nd law of motion in the presence of a gravitational field. To see this, let's consider the geodesic equation Eq. (3.37) for a particle

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0, \tag{4.15}$$

where we have chosen the *proper time* τ to parameterize the geodesic, i.e. this is the time the particle's watch is showing, while the (t, x, y, z) is the coordinates we chose. A frame-independent notion of "moving slowly" is given by the condition

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}. \tag{4.16}$$

Using Eq. (1.67) and Eq. (1.69) (with τ replacing s), we can see that Eq. (4.16) implies that

$$\gamma_v v^i \ll \gamma_v \Rightarrow v^i \ll 1 \tag{4.17}$$

which is to say the velocity of the particle $v^i = dx^i/dt$ with respect to the inertial frame is small (i.e. $v \ll 1$). In this limit, the geodesic equation then becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0. \tag{4.18}$$

- *Weak gravitational fields*: In the absence of gravity, Einstein's Postulates state that the metric is simply the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$. With gravity, spacetime will curve, and in a Lorentzian manifold, this means that the metric is no longer Minkowski. If the gravitational field is *weak*, then it means that the metric is just a small deviation from the Minkowski metric, i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\ ,\ |h_{\mu\nu}| \ll 1. \tag{4.19}$$

But wait! You might say, well what happened if we change basis such that the Minkowski metric is no longer $\eta_{\mu\nu} = (-1, 1, 1, 1)$, then what happens to the "smallness condition" $|h_{\mu\nu}| \ll 1$? Indeed, Eq. (4.19) is a component equation, so the decomposition *is basis dependent.* In arbitrary basis we have to be a lot more careful, which we will be when we discuss **gauge conditions** in section 5.3.1. In Cartesian coordinates and basis, Eq. (4.19) is a good enough condition for now, so let's plow on. We will also need the inverse metric $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$, which from Eq. (4.19) is (as you will show in a (Homework) problem)

$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\nu} - \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma} + \mathcal{O}(h^{\mu\nu})^2 \\ &= \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^{\mu\nu})^2\ ,\ h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma} \end{aligned} \tag{4.20}$$

to first order. Notice the minus sign change! In this limit, the Christoffel symbol is

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2}\eta^{\mu\lambda}(\partial_\nu h_{\lambda\sigma} + \partial_\sigma h_{\nu\lambda} - \partial_\lambda h_{\nu\sigma}) + \mathcal{O}(h^2) \tag{4.21}$$

so the Γ^μ_{00} term is

$$\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\lambda}(\partial_0 h_{\lambda 0} + \partial_0 h_{0\lambda} - \partial_\lambda h_{00}) + \mathcal{O}(h^2), \tag{4.22}$$

where we have used the fact that $\partial_\rho\eta_{\mu\nu} = 0$, and ignore higher order terms in h.

- *Static gravitational field*: This means that the time derivative of the metric is zero, i.e. $\partial_0 g_{\mu\nu} = \partial_0 h_{\mu\nu} = 0$, so Eq. (4.22) finally becomes

$$\begin{aligned} \Gamma^{\mu}_{00} &= -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} \\ &= -\frac{1}{2}\eta^{\mu i}\partial_i h_{00}, \end{aligned} \tag{4.23}$$

where we have used $\partial_0 h_{00} = 0$ and $i = 1, 2, 3$ run over spatial components as usual.

Plugging Eq. (4.23) back into Eq. (4.18), we get the geodesic equation in the Newtonian limit

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \eta^{\mu i}\partial_i h_{00} \tag{4.24}$$

or by multiplying both sides by $(d\tau/dt)^2$,

$$\frac{d^2 x^\mu}{dt^2} = \frac{1}{2}\eta^{\mu i}\partial_i h_{00}. \tag{4.25}$$

The $\mu = 0$ component of Eq. (4.25) is trivial

$$\frac{d^2 t}{dt^2} = 0 \tag{4.26}$$

since $\eta^{0i} = 0$. The $\mu = i$ component of Eq. (4.25) is

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}. \tag{4.27}$$

If we now identify

$$h_{00} \equiv -2\Phi \tag{4.28}$$

Eq. (4.27) becomes the familiar Newtonian equation for the acceleration of a particle under the gravitational field Φ

$$\mathbf{a} = -\nabla\Phi. \tag{4.29}$$

Notice that we have *not* used the Einstein equation (just the geodesic equation), but we still have yet to show that Φ obeys the Poisson equation $\nabla^2\Phi = 4\pi G\rho$. Notice that the Poisson equation relates the gravitational field Φ to matter density ρ, so it's clear that we have to recover this from the Einstein equation. We first take the trace of the Einstein equation Eq. (4.14)

$$g^{\mu\nu}G_{\mu\nu} = (8\pi G)g^{\mu\nu}T_{\mu\nu} \tag{4.30}$$

to get

$$R = -8\pi G T \tag{4.31}$$

where the traces are given by $R = g^{\mu\nu}R_{\mu\nu}$ and $T = g^{\mu\nu}T_{\mu\nu}$ (note that $g^{\mu\nu}g_{\mu\nu} = \delta^\nu_\nu = 4$ in 4D). We can then replace R in the Einstein equation with T using Eq. (4.31) to get a *completely equivalent* formula for the Einstein equation

$$\boxed{R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right)}. \tag{4.32}$$

Let's consider a perfect fluid for the energy-momentum tensor Eq. (2.75)

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}, \tag{4.33}$$

where $u^\mu = (u^0, u^1, u^2, u^3)$ is the **fluid four-velocity**, and we have use the Principle of Minimal Coupling (section 4.2.1) to convert $\eta_{\mu\nu} \to g_{\mu\nu}$. In the Newtonian limit, things are moving slowly, so as we discussed the pressure is tiny since it is suppressed by c (i.e. $\rho \gg c^{-2}P$), we can ignore the pressure, so we have

$$T_{\mu\nu} = \rho u_\mu u_\nu. \tag{4.34}$$

Furthermore, we will choose a coordinate frame where the matter is at rest, so the fluid velocity $u^i = 0$ and $u^0 = 1 + \mathcal{O}(h)$ – the extra h comes from the fact that u^μ needs to satisfy the timelike condition $g_{\mu\nu}u^\mu u^\nu = -1$ as we discussed before, but it will turn out to be not too important so we won't keep track of it too carefully. Lowering the index with the metric, we get that $u_0 = -1 + \mathcal{O}(h)$ so

$$T_{\mu\nu} = \rho u_\mu u_\nu \Rightarrow T_{00} = \rho(1 + \mathcal{O}(h)) \tag{4.35}$$

The trace of the energy momentum tensor $T = g^{\mu\nu}T_{\mu\nu} = -T_{00}(1 + \mathcal{O}(h)) = -\rho(1 + \mathcal{O}(h))$. Plugging T_{00} and T into Eq. (4.32), we get (and keeping only the time-time component)

$$R_{00} = 4\pi\rho(1 + \mathcal{O}(h)). \tag{4.36}$$

Next, we work out what R_{00} is in the Newtonian limit. From the component form of the Riemann tensor Eq. (3.93), we have

$$R^\mu{}_{0\mu 0} = \partial_\mu \Gamma^\mu_{00} + \Gamma^\sigma_{00}\Gamma^\mu_{\mu\sigma} - \partial_0\Gamma^\mu_{\mu 0} - \Gamma^\sigma_{\mu 0}\Gamma^\mu_{0\sigma} \tag{4.37}$$

but since $R^0{}_{000} = 0$, Eq. (4.37) becomes

$$R^i{}_{0i0} = \partial_i \Gamma^i_{00} + \Gamma^\sigma_{00}\Gamma^i_{i\sigma} - \partial_0\Gamma^i_{i0} - \Gamma^\sigma_{i0}\Gamma^i_{0\sigma}. \tag{4.38}$$

Recall that the Christoffel symbols for Minkowski space vanish, so the Christoffel symbols of a weak field metric must behave like $\Gamma \sim \mathcal{O}(h)$, so $(\Gamma)^2$ terms is higher order in h, while the static gravitational field assumption means that $\partial_0 \Gamma^\mu_{\mu 0} = 0$, leaving us with

$$\begin{aligned} R_{00} &= R^i{}_{0i0} \\ &= \partial_i \Gamma^i_{00} \\ &= -\partial_i \left(\frac{1}{2}\eta^{i\lambda}(\partial_\lambda h_{00})\right) \\ &= -\frac{1}{2}\eta^{ij}\partial_i\partial_j h_{00} \end{aligned} \tag{4.39}$$

where we have used Eq. (4.23) in the 3rd line. Combining Eq. (4.39) with Eq. (4.36), we get

$$\nabla^2 h_{00} = -8\pi G\rho(1 + \mathcal{O}(h)). \tag{4.40}$$

But since $h \ll 1$ in the weak gravitational field limit we can drop the last term since it is being compared to 1, i.e. $1 + \mathcal{O}(h) \approx 1$. Note that you can't "drop" the h_{00} on the LHS because we are comparing to 0. Using $h_{00} = -2\Phi$ from our definition Eq. (4.28) we finally get

$$\nabla^2\Phi = 4\pi G\rho \tag{4.41}$$

which is the Poisson equation as advertised.

4.4 Cosmology

The Einstein equation is a 2nd order non-linear partial differential equation. For a general matter distribution $T_{\mu\nu}$, the solution for the metric $g_{\mu\nu}$ is in general very hard to find. Indeed, there are very few known analytic solutions to the Einstein equation (and if you can find one, you get your name attached to it).

In these lectures, we will study two of the most famous analytic solutions – the Friedmann-Lemaitré-Robertson-Walker (FLRW) model of cosmology and the Schwarzschild spacetime (black holes). We leave the latter to Chapter 5, but now let's finally use the full power of the Einstein equation to investigate the evolution of the cosmos.

4.4.1 The Robertson-Walker metric

At the largest scales – around 10 Megaparsecs (Mpc, or 10^6 parsecs) and above, the Universe is to a good approximation **spatially homogenous**. Loosely speaking, homogeneity means that, an observer transported across space to another point that is about 10 Mpc or further away, she would find that her surroundings looks about "the same" – the same distribution of galaxies around her, the same kind of stars, and possibly even the same kind of cats. The Universe is also **spatially isotropic** – it "looks" more or less the same in every direction for *at least one* observer in her local inertial frame[3]. Put another way, isotropy means that there is no special direction at that point. A Universe which is isotropic at all points is automatically also homogenous, but the converse is not true – think of a homogenous pattern of stripes.

Our task is now to construct a metric which would describe such a Universe. Obviously, the Universe is neither homogenous nor isotropic in time – the Universe looks very different in the past and in the future. It makes sense then to "chop" the spacetime manifold into non-overlapping **hyperslicings** Σ_t of space, with each hyperslice labeled by some "time" coordinate t (see Figure 4.2). Such a "chopping" is called a **foliation**. We also *foliate* the spacetime such that the vectors that are tangent to the hypersurface is spacelike while timelike vectors are oriented "forward" in time – this is represented by the light cone (see Figure 2.11).

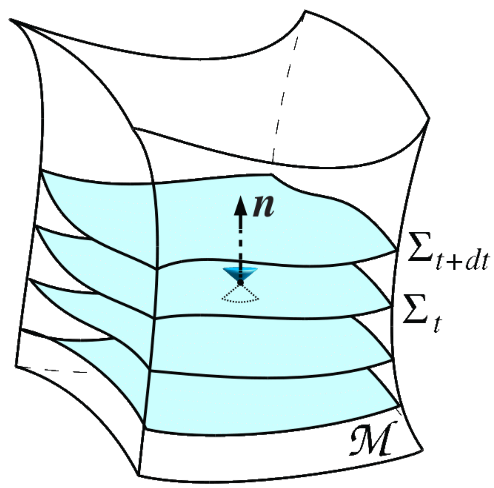


Figure 4.2: A foliation of the manifold $\mathcal{M}$ into spatial hyperslicings Σ_t labeled by a time coordinate t. Figure stolen from Gourgoulhon's *3+1 Formalism in General Relativity.*

In the language of section 3.4, each hypersurface Σ_t is a 3D *submanifold* of the spacetime. In general, we can foliate the spacetime any way we like, but since we are trying to describe cosmology, we want to foliate it in a clever way. As we said, at any time t the Universe looks homogenous and isotropic, but the Universe is also expanding as time passes – galaxies move away from each other equally, or "space stretches". So we can foliate the spacetime such that the hyperslicings that are homogenous and isotropic but its overall "scale" changes with time. If γ_{ij} is the metric (sometimes called the **3-metric**) for a homogenous and isotropic space then we can write the spacetime metric as

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{4.42}$$

where $a(t)$ is called the **scale factor** and is a measure of how much space has expanded (or even contracted). Then $a^2(t)\gamma_{ij}$ is the metric of each Σ_t at any time t – note that γ_{ij} is not a function

[3]Obviously, we can have two observers at the same point, and one is moving at some non-zero velocity with respect to the other. If the Universe is isotropic with respect to the observer at rest, then it would *not* be isotropic with respect to the moving observer.

of t since we have foliated spacetime in such a way that the overall expansion of the Universe is the same everywhere on Σ_t. Coordinates on the submanifold are then "co-moving" with the evolution of the hyperslices Σ_t, and are creatively called **co-moving coordinates** so physical distances on the hyperslice is given by $dl^2 = a(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$. At each coordinate point on Σ_t, we can put an observer on it – such observers are called **co-moving observers**. Note that we don't to need to add a factor $b^2(t)$ in front of $\mathrm{d}t^2$ because we can always absorb it in a reparameterization of the time coordinate $\mathrm{d}t' = b(t)\mathrm{d}t$.

Let's now consider the symmetries of an isotropic and homogenous metric γ_{ij} at some fixed time t (so we can drop a from the discussion). Spatial isotropy and homogeneity then means that γ_{ij} possesses some symmetries. What do we mean by "possessing symmetries"? Consider some hyperslice Σ_t equipped with some coordinate chart x^i, then the 3-metric on this manifold in the component form is γ_{ij}. At any point y^i on the manifold, the metric's components in the coordinate basis then has the values γ_{ij}. Homogeneity means that the metric looks the same when $y^k \to y^k + \Delta y^k$, i.e. a translation the spatial coordinates. While isotropy means that the metric looks the same when $y^k \to R^k{}_j y^j$ where $R^k{}_j$ is a rotation matrix[4].

Now, we will make an assertion which we shall not prove : *homogeneity and isotropy means that spatial metric γ_{ij} has the maximum number of symmetries.* Such a spacetime is called **maximally symmetric**.

An n-dimensional **maximally symmetric** *geometry has $\frac{1}{2}n(n+1)$ number of symmetries.*

So, for $n = 3$, we have 6 symmetries which is equal to the 3 translations and 3 rotations we discussed above. Furthemore, a maximally symmetric possess a Riemann tensor of the following form

$$\boxed{R_{\rho\sigma\mu\nu} = \frac{K}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})} \tag{4.43}$$

where K is a constant throughout the manifold and is called the **curvature constant**. Maximally symmetric spaces are also called **spaces of constant curvature**. Specializing to our 3-metric, this means that our hyperslicings Σ_t are spaces of constant curvature and must obey

$$R^{(3)}_{ijkl} = \frac{K}{6}(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}). \tag{4.44}$$

Again, we will assert but not prove, the most general maximally symmetric 3-metric in spherical coordinates (r, θ, ϕ) is given by

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2, \tag{4.45}$$

which you will verify in a (Homework) problem satisfies Eq. (4.44).

When $K = 0$, then Eq. (4.45) reduces to

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \tag{4.46}$$

which is just 3D Euclidean space $\mathbb{E}^3$ in spherical coordinates. In other words, $K = 0$ means that the spatial hyperslicing Σ_t is **flat**. When $K > 0$, Eq. (4.45) becomes

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2. \tag{4.47}$$

[4]**The attentive student might ask "how do we compare tensors at different points on the manifold, since they live in different tensor spaces!" This is completely fair. More technically, a symmetry of the spacetime is a statement that "the metric is unchanged when we move along this symmetry" on the manifold, which means we need to define a notion of "moving along", i.e. a derivative. Recall that we have defined the covariant derivative to teach us how to compare (or map) tensors from some point to other neighbouring points in the manifold, but if we used the Levi-Civita connection then this derivative depends on the metric itself, hence it is a bit perverse and circular to define a symmetry of the metric using itself. To do so, we need a metric-independent definition of derivative, called the **Lie derivative**. See the Appendix B.1 for more information.

You will show in a (Homework) problem that, this is the metric for a 3-sphere $\mathbb{S}^3$. Just like its lower dimensional counterpart the 2-sphere, the space is finite since if you go far enough in one direction you'll end in the same spot hence a positively curved hyperslicing Σ_t is called **closed**.

Finally, for $K < 0$, Eq. (4.45) becomes

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{\mathrm{d}r^2}{1+|K|r^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2. \tag{4.48}$$

In the same homework problem above, this is a hyperboloid, and can extend to infinite distances, hence a negatively curved Σ_t is called **open**.

Harking back to our foliation of spacetime metric Eq. (4.42), the cosmological spacetime metric in the basis for coordinates (t, r, θ, ϕ) is then

$$\boxed{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right)}. \tag{4.49}$$

Eq. (4.49) is called the **Robertson-Walker** metric, and it describes an expanding (or contracting) Universe which is homogenously and isotropic in space. Sometimes we write $\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 \equiv d\Omega^2$ which is simply the metric of the 2-sphere (notice the italicized d in front of Ω to indicate that Ω is not a coordinate or a function) which we discussed in section 3.4.

We can calculate the Christoffel symbols for this metric which you'll do in a (Homework) problem

$$\begin{gathered}
\Gamma^0_{11} = \tfrac{a\dot{a}}{1-Kr^2}\ ,\ \Gamma^1_{11} = \tfrac{Kr}{1-Kr^2}\ ,\ \Gamma^0_{22} = a\dot{a}r^2 \\
\Gamma^0_{33} = a\dot{a}r^2\sin^2\theta\ ,\ \Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \tfrac{\dot{a}}{a} \\
\Gamma^1_{22} = -r(1-Kr^2)\ ,\ \Gamma^1_{33} = -r(1-Kr^2)\sin^2\theta\ ,\ \Gamma^2_{12} = \Gamma^3_{13} = \tfrac{1}{r} \\
\Gamma^2_{33} = -\sin\theta\cos\theta\ ,\ \Gamma^3_{23} = \cot\theta,
\end{gathered} \tag{4.50}$$

with the corresponding Ricci tensor

$$\begin{gathered}
R_{00} = -3\tfrac{\ddot{a}}{a}\ , \qquad R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2K}{1-Kr^2} \\
R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2K)\ ,\quad R_{33} = r^2(a\ddot{a} + 2\dot{a}^2 + 2K)\sin^2\theta,
\end{gathered} \tag{4.51}$$

and the Ricci scalar

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right]. \tag{4.52}$$

4.4.2 Cosmological Evolution: the Friedmann Equation

As Einstein said, matter tells spacetime how to curve. So, the cosmological evolution of spacetime depends on the matter content of the entire Universe, as given by the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$.

So what's the matter content of the Universe $T_{\mu\nu}$? From our optical telescopes, we can see that stuff in the Universe is filled with all sorts of galaxies, stars, intergalactic gas, planets, cats etc. This might seem impossible to model, but since we are dealing with cosmological scales (i.e. Mpc scales), all these stuff can be seen as some form of fluid. In particular, we can model them as perfect fluids Eq. (4.12)

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \tag{4.53}$$

and u^μ is fluid four-velocity (i.e. the general "flow" of the galaxies and other stuff in the Universe).

What is the u^μ? In general, this can be quite complicated but since our goal is find to fluids that will result in an isotropic Universe, we choose the fluid to be at rest with the isotropic frame of a hyperslice Σ_t

– i.e. the frame where all co-moving observers on the hyperslice see an isotropic fluid. In the coordinate basis that we have used to defined the RW metric Eq. (4.49), this means that

$$u^\mu = (1,0,0,0), \tag{4.54}$$

which means that Eq. (4.53) becomes

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & a^2\gamma_{ij}P & \\ 0 & & & \end{pmatrix}. \tag{4.55}$$

Such a fluid is sometimes called a **co-moving fluid**.

The matter must be conserved $\nabla_\mu T^\mu{}_\nu = 0$ as we discussed in section 4.2.1. The $\nu = 0$ component

$$\nabla_\mu T^\mu{}_0 = \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\sigma} T^\sigma{}_0 - \Gamma^\sigma_{\mu 0} T^\mu{}_\sigma \tag{4.56}$$

which gives us the **continuity equation**

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0}. \tag{4.57}$$

The continuity equation encodes the dynamics of matter fields. If we have several components of perfect fluid matter, i.e.

$$T_{\mu\nu} = \sum_i T^{(i)}_{\mu\nu} \tag{4.58}$$

with their equivalent densities and pressures $\rho^{(i)}$ and $p^{(i)}$ then only the *total* density and pressure $\rho = \sum_i \rho^{(i)}$ and $p = \sum_i p^{(i)}$ satisfy the continuity equation and *not individually.* Having said that, if the components do not interact, then *usually* the individual equations of motion for the matter components will impose the individual satisfaction[5] of their respective continuity equations – but this is additional information and not a result of the Bianchi identity.

With $T_{\mu\nu}$ and $R_{\mu\nu}$ (and R) from section 4.4.1, we are ready to derive the cosmological evolution equations. The nice thing about Eq. (4.49) is that it has only a single free function $a(t)$ which is the scale factor – the Universe's evolution can be described by how this scale factor evolves, so what we are aiming for is to derive an evolution equation of a as a function of the matter content encoded in ρ and P. Using the Einstein equation, the G_{00} component of the Einstein tensor is

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2}g_{00}R \\ &= 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3K}{a^2} \end{aligned} \tag{4.59}$$

which when combined with $T_{00} = \rho$ gives us

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}}. \tag{4.60}$$

The factor

$$\boxed{\frac{\dot{a}}{a} \equiv H} \tag{4.61}$$

is called the **Hubble Parameter** and it quantifies the (normalized) **expansion rate** of the Universe, and is a *directly measurable quantity.* Much money and angst has been thrown in a bid to measure H

[5] I know. But this is grammatically correct.

accurately, and our current best estimate on its value *today* depends on which experiment you believe in : $H = 67$ (km/s)/Mpc using the Cosmic Microwave Background satellite PLANCK data, or $H = 73$ (km/s)/Mpc using the Hubble Space Telescope.

Meanwhile the G_{11} component is

$$G_{11} = \frac{a^2}{1-Kr^2}\left[-2\ddot{a}a - \left(\frac{\dot{a}}{a}\right)^2 - \frac{K}{a^2}\right] \tag{4.62}$$

which when combined with $T_{11} = a^2/(1-Kr^2)P$ and Eq. (4.60) to get rid of the $\dot{a}/a$ term, we get

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}. \tag{4.63}$$

The G_{22} and G_{33} components of the Einstein equation reproduce Eq. (4.63) because space is isotropic, so they contain no further information (as you can check yourself).

The pair of equations Eq. (4.60) and Eq. (4.63) are known as the **Friedmann Equations**, and they tell us how the scale factor a evolves as a function of the matter content encoded by the perfect fluids with ρ and P. Why a pair and not just one? Most good things in life come in pairs, but this is not the situation here. The "true" equation of motion for the scale factor is Eq. (4.63), since it is 2nd order in time derivatives (of a) while Eq. (4.60) has only one time derivative[6]. To fully specify the initial conditions at time $t = t_0$ for cosmological evolution, we have to choose $a(t_0)$ and $\dot{a}(t_0)$, which is freely specified.

The first equation Eq. (4.60) is actually a **constraint equation** – solutions $a(t)$ of the 2nd equation Eq. (4.63) must satisfy it, but it adds no additional information to the dynamical system, with a caveat as we will state below.

Of course since ρ and P themselves are time-dependent quantities, to solve Eq. (4.63) we need to know how ρ and P evolve – which will depend on the actual matter being described so in general will require independent equations for them. However, in the special case of **Baryotropic** fluids, where the density ρ is just a function of the pressure P, $\rho(P)$, the continuity equation Eq. (4.57) is sufficient. Cosmological fluids are generally baryotropic fluids of the following "equation of state"

$$\boxed{P = w\rho}, \tag{4.64}$$

where w is a constant and is called **the equation of state parameter** which is a terrible abuse of jargon but we don't get to choose the names here. Since w is a constant, we can immediately plug Eq. (4.64) back into the continuity equation Eq. (4.57) to get

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \tag{4.65}$$

which we can immediately integrate to get ρ as a function of a,

$$\rho \propto a^{-3(1+w)}. \tag{4.66}$$

This immediately tells us what P is given Eq. (4.64).

The reason the parameterization Eq. (4.64) is useful is because it allows us to describe some very common matter. For example, **matter** is stuff that moves non-relativistically (with respect to the isotropic frame) as we have discussed in section 4.3.1 on Newtonian limits (see Eq. (4.35)) have negligible pressure $P = 0$, so then

$$w_m = 0 \tag{4.67}$$

[6]Recall that in Newton's 2nd law $\ddot{x} = F/m$, the equation is 2nd order in time.

and matter then evolves as

$$\rho_m \propto a^{-3}. \tag{4.68}$$

This is exactly what we expect: if we put a bunch of n balls with mass m in a box of size L, its density is then $\rho = nm/L^3$. As the Universe expands, $L \to La(t)$, and the density then dilutes as $\rho(t) = nm/(La(t)^3) \propto a^{-3}$.

The other common cosmological fluid is **radiation**, which in cosmologist-speak is simply stuff that moves relativistically $v \approx c$ but you can think of it as a fluid of photons. Such a fluid has radiation pressure (I refer you to your Thermal Physics module) and an equation of state $p_r = \rho_r/3$, or

$$w_r = \frac{1}{3}. \tag{4.69}$$

Radiation density then evolves as

$$\rho_r \propto a^{-4}. \tag{4.70}$$

Which again is consistent with our intuition – the extra factor of a comes from the redshifting of the wavelength of the photons in addition to simple dilution of number density.

With ρ and P as functions of a, we can immediately plug them into the Friedmann equations to solve for the evolution of $a(t)$ as a function of time t. However, it turns out that if we know ρ as a function of a, all we need to do is to use the first Friedmann equation Eq. (4.60)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 a^{-3(1+w)} - \frac{K}{a^2} \tag{4.71}$$

where ρ_0 is the density at time t_0, which we can integrate immediately. The details of these derivations we will leave to some (Homework) problems. But we want to warn you that this is one of the few cases where we can ignore the 2nd Friedmann equation. The reason we can do that is because we can express $\rho(a)$ by using the continuity equation which is not always possible – even though a is described by a second order equation, we sneakily did one of the integration when we integrated the continuity equation in Eq. (4.65) which leaves just one more integration to do in Eq. (4.71).

4.5 Physical Meaning of the Einstein Equation

Notice that the LHS of Eq. (4.7) contains only the Ricci tensor, and not the Riemann tensor. In 4D, the Ricci tensor only contains "half" the information of the Riemann tensor – why isn't the Riemann tensor coupled to $T_{\mu\nu}$ instead? There are several ways to motivate for this[7], but the reality is that Eq. (4.7) is a *postulate* and we simply run with it in the previous sections. Indeed it is as much as postulate of the dynamics of gravitation as the Newton's equation $F = m\ddot{x}$ or the Maxwell's equation.

However, it will be nice to get an intuitive sense of why Eq. (4.7) is the way it is – in particular why matter "sources" spacetime changes that is volume-related. We will follow a really nice argument which I first learned from John Baez and Ted Bunn[8].

Consider a cloud of very light test particles – so light that we can ignore their masses. Let $V(t)$ be the *volume* traced out by the particles. We then put this cloud in a spacetime which contain some perfect fluid $T_{\mu\nu} = (\rho, P, P, P)$ (Eq. (2.74)). Spacetime will curve in response to the presence of this perfect fluid, and the particles will then travel on the geodesics of this curving spacetime – the cloud stretches and compresses and contorts in response. Now here is the surprising assertion (maybe even for those who

[7] For a very formal symmetry-based argument see S. Weinberg's *Gravitation and Cosmology : Principles and Applications of the General Theory of Relativity*

[8] https://arxiv.org/abs/gr-qc/0103044

knows some GR), but it can be shown that Einstein Equation actually states that volume $V(t)$ of this cloud will change according to

$$\frac{\ddot{V}}{V} = -4\pi G(\rho + 3P). \tag{4.72}$$

Eq. (4.72) is known as the **Raychaudhuri Equation**, expressed in the inertial frame of the cloud of particles. The derivation of this equation is technical, so it won't be in the exams, but we will do in the next section 4.5.1 anyhow because it's so beautiful. You can skip right to section 4.5.2 if you are a savage and simply want to get to the dessert (and miss the amazing main course).

4.5.1 **The Raychaudhuri Equation

We begin by reminding you of geodesic deviation from section 3.3.4. In that section, we consider a family of geodesics that trace our a 2D surface, and study the "velocity" deviation vector Eq. (3.109) and the "acceleration" deviation vector Eq. (3.110) between the two neighbouring geodesics. Now instead of considering simply a 2D surface of geodesics, we want to consider *all* the geodesics that are neighbours to each other, which looks like a densely packed "bunch" of *non-intersecting* geodesics (see the top diagram from Figure 4.3). Such a collection of non-intersecting geodesics is called a **congruence** – they are exactly the geodesics of the cloud of test particles in free fall. Studying the evolution of this congruence of geodesics will then tell us how the cloud of test particles will evolve as time passes.

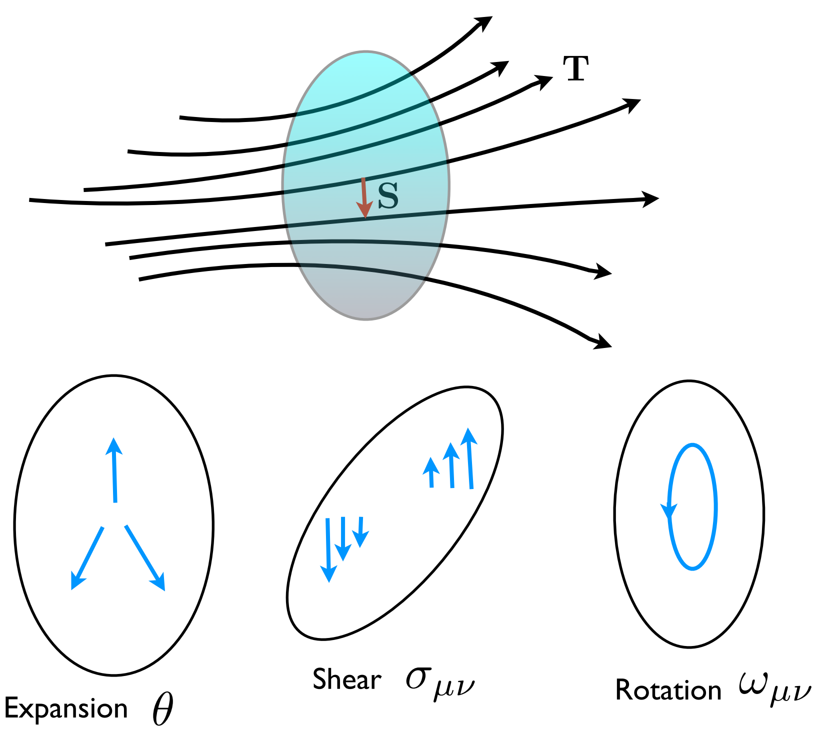


Figure 4.3: A congruence of geodesics, which defines a vector field **T**. We have suppressed a space dimension, so at any time τ we can construct a disk which **T** is normal. The "deviation tensor" $B_{\mu\nu} = \nabla_\nu T_\mu$ is tangent to this disk, and describes the how the geodesics propagate with respect to each other. It can be decomposed into an expansion, a shear and a rotation component

The congruence of geodesics can be described by a vector *field* T^μ (now no longer just a single curve). As usual, the particles are timelike, so $T^\mu T_\mu = -1$ and they obey the geodesic equation $T^\nu \nabla_\nu T^\mu = 0$. Each geodesic can be parameterized by some time parameter τ.

Consider some time τ_0. We can consider some 3D subspace (the green shaded area in Figure 4.3, where we have suppressed a dimension so it looks like a disk) which is defined by insisting that all T^μ is *normal* (or orthogonal) to it. For any given vector, we can project it onto this subspace by using a **Projection tensor**

$$P^\mu{}_\nu \equiv \delta^\mu{}_\nu + T^\mu T_\nu. \tag{4.73}$$

You can check that "projecting" T^μ returns a zero as expected $P^\mu{}_\nu T^\nu = 0$ by construction.

The vector *field* $\mathbf{S}$ which points from these geodesics to their neighbours then must lie along this 3D subspace (or on the 2D disk in Figure 4.3). On this 3D subspace, we want to calculate the "velocity" and "acceleration" deviation for the vector $\mathbf{S}$, from Eq. (3.109) $\nabla_\mathbf{T}\mathbf{S}$. But since we are using the Levi-Civita connection, which is torsion free, Eq. (3.111) allows us to write it as

$$\begin{aligned}\nabla_\mathbf{T}\mathbf{S} &= \nabla_\mathbf{S}\mathbf{T} \\ &= S^\nu\nabla_\nu T^\mu \hat{\mathbf{e}}_{(\mu)}.\end{aligned} \tag{4.74}$$

We now define the "deviation tensor"

$$B_{\mu\nu} \equiv \nabla_\nu T_\mu, \tag{4.75}$$

so Eq. (4.74) becomes

$$S^\nu\nabla_\nu T^\mu = S^\nu B_\mu{}^\nu. \tag{4.76}$$

As expected, the deviation tensor $B_{\mu\nu}$ is orthogonal to T^μ, which you will show in a (Homework) problem

$$T^\mu B_{\mu\nu} = 0\ ,\ T^\nu B_{\mu\nu} = 0. \tag{4.77}$$

$B_{\mu\nu}$ encodes how the geodesics move with respect to one another. We can decompose it into a trace, trace-free symmetric and trace-free antisymmetric part as follows

$$B_{\mu\nu} \equiv \frac{1}{3}\theta P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{4.78}$$

where we have defined the **expansion** using

$$\boxed{\theta \equiv \nabla_\mu T^\mu}. \tag{4.79}$$

The expansion θ describes how geodesics move "away" from each other – indeed you will show in a (Homework) set that in a Friedmann-Lemaitré-Robertson-Walker spacetime, θ is exactly 3 time of the Hubble parameter $\theta = 3H$. If you like, it describes "local uniform expansion" – see the bottom left figure of Figure 4.3. So it tells us how the volume of our cloud of test particles changes.

The trace-free *symmetric* term $\sigma_{\mu\nu}$ is called the **shear**. As the name implies, it describes the "shear deformation" of our cloud of particles (e.g. from a cube cloud into a parallelepiped) – see the middle figure of Figure 4.3.

Finally the anti-symmetric term $\omega_{\mu\nu} = -\omega_{\nu\mu}$ describes **rotation** (see bottom right figure of Figure 4.3. The particles in the cloud of particles will "spiral" as it propagates in the presence of non-zero rotation.

How does $B_{\mu\nu}$ evolve then? We simply calculate its rate of change along the congruence, i.e.

$$\begin{aligned}\nabla_\mathbf{T} B_{\mu\nu} &= T^\sigma\nabla_\sigma B_{\mu\nu} \\ &= T^\sigma\nabla_\sigma\nabla_\nu T_\mu \\ &= T^\sigma\nabla_\nu\nabla_\sigma T_\mu + R_\mu{}^\rho{}_{\sigma\nu}T_\rho T^\sigma \\ &= T^\sigma\nabla_\nu\nabla_\sigma T_\mu + R_{\mu\rho\sigma\nu}T^\rho T^\sigma \\ &= T^\sigma\nabla_\nu\nabla_\sigma T_\mu - R_{\rho\mu\sigma\nu}T^\rho T^\sigma\end{aligned} \tag{4.80}$$

where we have used Eq. (3.96) with the torsion free condition in the 3rd line, and the antisymmetry condition Eq. (3.97) in the final line. Meanwhile, the first term of the final line can be written as $T^\sigma \nabla_\nu \nabla_\sigma T_\mu = \nabla_\nu (T^\sigma \nabla_\sigma T_\mu) - (\nabla_\nu T^\sigma)(\nabla_\sigma T_\mu)$, and the first term is zero via the geodesic equation so we finally get the general **Raychaudhuri's Equation**

$$\boxed{\nabla_{\mathbf{T}} B_{\mu\nu} = -B^\sigma{}_\nu B_{\mu\sigma} - R_{\rho\mu\sigma\nu} T^\sigma T^\rho}. \tag{4.81}$$

The Raychaudhuri equation describes the evolution of a congruence of geodesics, or in more prosaic terms, how a cloud of test particles will change its shape and velocities in a free fall, given a manifold with curvature described by some Riemann Tensor $R^\sigma{}_{\mu\nu\rho}$.

Let's show that Eq. (4.81) is indeed Eq. (4.72) in the limit we are considering. Tracing over the μ and ν components with $g^{\mu\nu}$ we obtain an equation for the evolution of the expansion

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}T^\mu T^\nu. \tag{4.82}$$

So far it is all geometry – the Einstein equation plays no part in the derivation so far. But now we connect this equation to the Einstein equation by replacing $R_{\mu\nu}$ in Eq. (4.82) with the "Ricci tensor" only version of the Einstein equation Eq. (4.32) $R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$ to get

$$\boxed{\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - 8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)T^\mu T^\nu}, \tag{4.83}$$

with the reminder not to confuse the geodesic T^μ with the traced over energy momentum tensor $T = g_{\mu\nu}T^{\mu\nu}$. Eq. (4.83) now connects the evolution of geometry to the Einstein equation, and *it contains all the information of the dynamics of Einstein equation.*

The cloud of particles are initially at rest at $t = t_0$ with respect to each other so $\sigma_{\mu\nu} = \omega_{\mu\nu} = \theta = 0$ *initially* in all coordinates, leaving us with

$$\left.\frac{d\theta}{d\tau}\right|_{t=t_0} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)T^\mu T^\nu. \tag{4.84}$$

Furthermore, we can also choose coordinates (and hence basis) such that the cloud of particles are initially rest – note that this is *not* the same as saying the particles are at rest *with each other* (the fact we used above), so $T^\mu = (1, 0, 0, 0)$. Since this is also the local inertial frame, $g_{\mu\nu} = \eta_{\mu\nu}$ as Einstein insisted and hence we get

$$\left.\frac{d\theta}{d\tau}\right|_{t=t_0} = -8\pi G\left(T_{00} - \frac{1}{2}Tg_{00}\right) \tag{4.85}$$

and since $T = -\rho + 3P$ (for perfect fluid) and $T_{00} = \rho$, we finally get

$$\left.\frac{d\theta}{d\tau}\right|_{t=t_0} = -4\pi G(\rho + 3P). \tag{4.86}$$

But $\theta = \nabla_\mu T^\mu \equiv dV/d\tau$, we have derived Eq. (4.72).

4.5.2 Implications of the Raychaudhuri Equation

Let's now consider the implications of Eq. (4.72)

- **Gravity in empty space : Tidal Forces**. Consider **empty space**, so $T_{\mu\nu} = 0$ hence $\rho = P = 0$, then Eq. (4.72) becomes quite simply

$$\frac{\ddot{V}}{V} = 0. \tag{4.87}$$

Integrating, we get

$$\dot{V} = C_1(x) \ , \ V = C_1(x)t + C_2(x) \tag{4.88}$$

where C_1 and C_2 are functions of space x which depend on initial conditions and are generally not zero. This means something quite profound: *spacetime can still dynamically curve in the absence of matter!* Actually this should not be too surprising – in Chapter 1 tidal forces can occur even in empty space as long as there is some matter *nearby*. For example, if you go outside the Earth's atmosphere into the vacuum of space and then set up the cloud of particles as we described earlier, then this cloud of particles will become ellipsoidal due to two effects. First, particles closer to Earth will be attracted more so will fall slightly faster than particles further away (vertical tidal forces), while due to the spherical shape of Earth, particles will converge horizontally. Both these effects will change the volume V of the particles, but in such a way that $\ddot{V} = 0$. In fact, you can show this in Newtonian Mechanics!

- **Attractiveness of Gravity and the Accelerating Universe**. In a "normal" fluid, both the density $\rho > 0$ and the pressure $p \geq 0$ are positive (semi)-definite, so $\rho + 3P > 0$. Eq. (4.72) then says that

$$\frac{\ddot{V}}{V} < 0 \tag{4.89}$$

or, in words, the cloud of particles will accelerate towards each other and the volume will eventually *shrink – gravity is attractive*. We say "normal" fluid above, but how sure are we? Can we find "weird stuff" where $\rho + 3P < 0$ such that $\ddot{V} > 0$ hence gravity becomes *repulsive*? Actually, as you no doubt know by now, we have recently found that the "Universe's expansion is accelerating". If you think of all the galaxies in the Universe as the "cloud of particles" we talked above, then we can think of V as the volume of some ginormous region of the Universe. The "accelerating expansion" then means that $\ddot{V} > 0$ implying that $\rho + 3P < 0$ – so we have indeed "discovered" some weird stuff[9] in the Universe! Since $\rho > 0$ by construction, then this weird stuff has "negative pressure" $P < 0$. This stuff is sometimes called **Dark Energy**, and we have no idea what it is. One possible candidate for this stuff is the **Cosmological Constant**.

- **Newton's Law of Gravity**. Consider a spherical planet with mass M and initial volume V_P. We can surround this planet with a (larger) imaginary sphere $V_S > V_P$. We want to investigate what Eq. (4.72) has to say about how the volume V_S changes in the presence of this planet. We fill up the entire large sphere V_S with the cloud of (massless) particles – so the planet is also filled up since it is inside V_S. The planet is made out of rocks that are not moving, so $\rho > 0$ but $P = 0$ and hence *inside* the planet, Eq. (4.72) says

$$\text{Inside}: \ \frac{\ddot{V}}{V} = -4\pi G\rho \tag{4.90}$$

while outside it is

$$\text{Outside}: \ \frac{\ddot{V}}{V} = 0 \tag{4.91}$$

since it is *in vacuo*.

After some short time δt, as we shown above the according to

$$\text{Inside}: \ \frac{\delta V}{V} = -\frac{1}{2}4\pi G\rho(\delta t)^2 \tag{4.92}$$

where the 1/2 comes from Taylor expanding $V(t+\delta t) = V(t) + \dot{V}\delta t + (1/2)\ddot{V}(\delta t)^2 + \ldots$ to 2nd order and assuming that $\dot{V} = 0$ initially (recall that we say that cloud of particles is initially at

[9]**Physicists like to classify matter roughly according to the positivity or negativity of the thermodynamics quantities ρ and P as **Energy Conditions**. The condition $\rho + 3P > 0$ is known as the **Strong Energy Condition**.

rest). The *total* change of the V_S then must be equal to δV_P, i.e.

$$\begin{aligned}
\delta V_S &= \delta V_P \\
&= \frac{\delta V}{V} V_P \\
&= -2\pi G(\rho V_P)(\delta t)^2 \\
&= -2\pi G M(\delta t)^2
\end{aligned} \tag{4.93}$$

where we have used $M \equiv \rho V_P$ in the last line. If r is the radius of V_S, then the change in r due to the shrinking is $\delta V_S = 4\pi r^2 \delta r$, which we can plug into Eq. (4.93) to get

$$\delta r = -\frac{GM}{2r^2}(\delta t)^2. \tag{4.94}$$

But now, given a constant acceleration g, and initial velocity of zero, we know from Newton's law of motion that $\delta r = (1/2)g(\delta t)^2$, which we means that

$$g = -\frac{GM}{r^2} \tag{4.95}$$

which is exactly Newton's Law of Gravity.

In the first and third examples above, we have assumed the that we are in the Newtonian Limit (sect 4.3.1). First, the gravitational field is "weak", so that it makes sense to talk about a cloud of particles "initially at rest with each other". Why is this important? Recall that, in GR, it makes no sense to compare velocities of particles in different points on the manifold! So "initially at rest with each other" does not make sense in GR, but this notion is OK as long as gravitational field is weak or the particles are packed sufficiently close to each other. Secondly, we have assumed that the particles are moving very slowly compared to the speed of light $v \ll c$ – the so-called **non-relativistic limit**, so that we can use Newton's Law of Motion.

Finally, you might ask "what about the Weyl tensor?" Why is it not present in the Einstein Equation? After all, full curvature is described by the Riemann tensor and not just the Ricci tensor. Does this mean that the Weyl tensor components are redundant and do not describe physics? No. As we have already seen above, *empty space* can still curve else there would be no tidal forces, and the moon still feels the Earth's gravitational pull through the vacuum of space, gravitational waves will still travelled through empty space from the black hole merger GW150914 to the LIGO detectors on Sept 14 2015 etc. In other words, the Weyl tensor contains important *observable* information about the spacetime curvature. How does this come about from Einstein Equation?

There are two equivalent ways to answer this question. The first more mathematical way is to notice that we are not allowed to specify the components of the Riemann tensor *freely in the entire manifold* – they are constrained by the Bianchi Identity Eq. (3.103). Since the Bianchi Identity is a *derivative* identity, it means that the component values of *neighbouring* Riemann tensors must be related to each other. This is not a very satisfying "physics" argument, since it relies on some mathematical facts of the Riemann tensor (which at first glance has nothing to do with physics).

More physically speaking, the Einstein Equation *does not fully specify what the curvature should look like for any given energy-momentum tensor* $T_{\mu\nu}$. This is not a very profound statement – but this fact is so often forgotten even by professional physicists that it bears repeating: *for a configuration of matter field* $T_{\mu\nu}$ *the spacetime curvature is* **not** *uniquely specified.* Let's consider a simple but extremely pertinent example – that of empty space $T_{\mu\nu} = 0$, so the Einstein equation becomes $G_{\mu\nu} = 0$. A solution to this equation (as you will show in a (Homework) set is Minkowski space, i.e. the metric is $g_{\mu\nu} = (-1, 1, 1, 1)$ and $R_{\mu\nu\sigma\rho} = 0$ everywhere. A *different* solution to this equation is one filled with gravitational waves (which we will study in detail in Chapter 5). Yet another different solution is that of black holes! Which solution is the right solution depends on the initial and boundary conditions.

Why is this not profound? It is because you have seen this before – when you solved the sourceless Maxwell equations $\partial_\mu F^{\mu\nu} = 0$ (i.e. the 4-current $J^\mu = 0$ – analogous to $T_{\mu\nu} = 0$), there exist solutions where $F^{\mu\nu} = 0$ everywhere, or where $F^{\mu\nu}$ describes a set of freely propagating electricmagnetic radiation – the actual solution depends on initial and boundary conditions.

To be super explicit, although the Weyl tensor is not coupled to the energy-momentum tensor algebraic-ly via the Einstein Equation, you can show in a (Homework) problem that by using the Bianchi Identity Eq. (3.103) and Einstein equation, it is coupled via derivatives (no need to memorize this one)

$$g^{\rho\beta}\nabla_\beta C_{\rho\sigma\mu\nu} = 4\pi G\left(\nabla_\mu T_{\nu\sigma} - \nabla_\nu T_{\mu\sigma} + \frac{1}{3}g_{\sigma\mu}\nabla_\nu T - \frac{1}{3}g_{\sigma\nu}\nabla_\mu T\right) \tag{4.96}$$

where $T = g_{\mu\nu}T^{\mu\nu}$ is the trace of the energy-momentum tensor.

With this, we have completed our 2nd part of General Relativity: matter tells spacetime how to curve. Indeed, with the discussion of the Raychaudhuri equation, we have come full circle – matter tells matter how to move, through its influence on the curvature.

Chapter 5

Black Holes and Gravitational Radiation

Black holes are the harmonic oscillators of the 21st Century.

A. Strominger

In this final chapter, we study two of the most important solutions of GR.

The first solution is the **Schwarzschild spacetime**, which is the (spatially) spherically symmetric and *empty* spacetime in the presence of a matter source. Such a spacetime approximately describes the empty space around any spherically symmetric matter distribution, such as the Solar System, the local space around the Earth, or even the intergalactic space around the Galaxy. Furthermore, it also describes static black holes which are some of the most interesting objects in nature.

The second solution is that of **gravitational waves**, which are propagating wave solutions of GR. The very recent direct detection of such waves by the LIGO experiment in 2015 has further underlined its importance.

5.1 The Schwarzschild Spacetime

The Schwarzschild spacetime is the solution to Einstein equation for a **vacuum** or empty spacetime, i.e. it is a solution to the equation

$$G_{\mu\nu} = 0 \tag{5.1}$$

where $T_{\mu\nu} = 0$.

As we have discussed earlier, empty doesn't necessarily mean the spacetime is Minkowski space. Eq. (5.1) is a *point-wise* equation, which is to say, it is solved at every point in spacetime. Since the Einstein equation is a 2nd order differential equation, then the solution will depend on both boundary and initial conditions.

This is analogous to solving the "empty" $\rho = 0$ Poisson equation $\nabla^2\Phi = 0$ – which has the solution $\Phi(r) = -C_1/r + C_2$ in spherical polar coordinates where C_1 and C_2 are constants that depend on boundary conditions. For example, for $C_1 = GM \neq 0$ and $C_2 = 0$, the solution is that of empty space outside a planet of mass M – obviously there is still a non-zero gravitational field.

5.1.1 Birkhoff's Theorem and the Schwarzschild Metric

We begin with the famous **Birkhoff's Theorem**:

The most general **spherically symmetric empty spacetime** *is the* **Schwarzschild metric** *which has the form*

$$\boxed{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = ds^2 = -\left(1-\frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)}. \tag{5.2}$$

Here M is a real positive semi-definite constant and usually means the **mass** *of the system. We have chosen the coordinate choice (t, r, θ, ϕ) where θ and ϕ are coordinates on the 2-sphere (see section 3.4)*

$$d\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2. \tag{5.3}$$

We will not prove this theorem (for a human-being understandable proof, see S. Carroll *Spacetime and Geometry*). Here are some of its general properties.

- The **static** condition is enforced by the fact that the Eq. (5.2) is independent of the time coordinate t. This means that the metric remains the same as time passes, i.e. the metric is *symmetric in time*, or $\partial_t g_{\mu\nu} = 0$.
- **Spherically symmetric** means that the metric is invariant under 3D rotation – it is clear from Eq. (5.2) that for fixed θ, the metric is independent of ϕ, so $\partial_\phi g_{\mu\nu} = 0$. It is slightly more complicated to show (Homework) that this symmetry extend to non-fixed θ.
- If $M = 0$, we recover Minkowski space in spherical symmetric coordinates. For $M \neq 0$, the metric describe the curvature of empty space *outside* some central mass distribution with total mass M. So if the border of this mass distribution is R, the metric is valid for $r > R$.
- The coordinate r is *not* physical radial distance from the center (you will get to show why it is not in a (Homework) problem). The way it is defined is as follows. Notice that the metric Eq. (5.2) has the term $r^2 d\Omega^2$; in words r^2 multiplies the unit sphere metric $d\Omega^2$, so $r^2 d\Omega^2$ tells you the **surface area** $4\pi r^2$ of the 2-spheres of fixed r and t. This is how r is physically *defined* – to know what "r" is, you have to measure the surface area A and then calculate r from it. Such a coordinate system where spacetime is foliated into 2-spheres labeled by r, with r defined to be $r \equiv \sqrt{A/4\pi}$ is called the **Schwarzschild Coordinates** (not to be confused with the metric itself), and the r coordinate is known as the **Areal coordinate**.
- In the limit of large $r \to \infty$, $GM/r \to 0$, and the metric again asymptotically approaches Minkowski space. We say that the metric is **asymptotically Minkowski** or **asymptotically flat** – spacetime curves less the further we are to the central mass distribution.
- The fact the metric is static and spherically symmetric does *not* imply that the central mass distribution is static nor spherically symmetric. For example, the Sun is not static nor spherically symmetric since it is rotating about an axis (which breaks both symmetries).
- **In the high-brow language of Appendix B.1, the Schwarzschild metric possesses isometries of the group $\mathbb{R} \times SO(3)$. The subgroup $\mathbb{R}$ generates the time-like Killing vector field $\mathbf{K} = \partial_t$ (in the coordinate basis). The rotation group $SO(3)$ generates the spherical symmetry, and acts on only the 2-sphere metric $d\Omega^2$ – the group rotates the 2-sphere into itself and does not act on r and t, so the Killing vector field $\mathbf{R}$ is spacelike and 3-dimensional.

Historically, this is the first non-trivial solution of the Einstein equation, found by **Karl Schwarzschild** in 1916 a few months after Einstein published his famous paper.

You will show in a (Homework) problem that the 2-sphere metric Eq. (5.3) is invariant under rotations[1].

[1] **In the language of symmetries, we say that the isometry group of Eq. (5.3) is $SO(3)$.

5.1.2 The Weak Gravitational Field Limit and Gravitational Redshift

As we have discussed in section 4.3.1 on Newtonian limits, the weak gravitational field limit means that we are close to Minkowski spacetime. From Eq. (5.2), we see that if

$$\frac{2GM}{r} \ll 1 \tag{5.4}$$

then the g_{rr} term of the metric Eq. (5.2) can be Taylor expanded to become

$$\left(1-\frac{2GM}{r}\right)^{-1} \approx \left(1+\frac{2GM}{r}\right)+\mathcal{O}\left(\frac{GM}{r}\right)^2+\ldots \tag{5.5}$$

and hence the metric is approximately

$$ds^2 \approx -\left(1-\frac{2GM}{r}\right)\mathrm{d}t^2+\left(1+\frac{2GM}{r}\right)\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2) \tag{5.6}$$

and if we replace $2GM/r$ with its "Newtonian potential"

$$\Phi(r) \equiv -\frac{2GM}{r} \tag{5.7}$$

we get

$$ds^2 \approx -(1+2\Phi(r))\,\mathrm{d}t^2+(1-2\Phi(r))\,\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2) \tag{5.8}$$

which is the metric we first saw in section 1.5 when we discussed gravitational redshift and the Pound-Rebka experiment – and hence justifying the discussion there (which is still completely valid and correct).

5.1.3 The Geodesic equation and Conserved quantities of Schwarzschild spacetime

Our next task is to explore the geodesics of the Schwarzschild metric, in particular we want to study the **orbits** of test particles of mass $m \ll M$. Analogous to Newtonian gravity, where you have studied conic section orbits (circles, parabolas, hyperbola etc) of a central mass source, we are also interested in the orbits of their GR counterparts.

The equation of motion for test particles is the geodesic equation Eq. (3.37), so a "strategy" is to blindly plug in the metric and try to find solution for the orbits $x^\mu(\tau)$. But, as you have learned from solving a ton of Newtonian mechanics problems, we can often simplify our task by using **conservation laws** such as the conservation of energy, the conservation of momentum or angular momentum (or conservation of charges in electrodynamics) etc. Things are no different in GR, but first we have to identify these consevation laws.

We begin with the geodesic equation Eq. (3.37),

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}=0, \tag{5.9}$$

where τ parameterizes the geodesic, which we can use the proper time without any loss of generalization.

We would need the Christoffel symbols for the Schwarzschild metric, which you will calculate in a (Homework) problem to be

$$\begin{aligned}
\Gamma^r_{tt}&=\frac{GM}{r^3}(r-2GM)\ , \qquad \Gamma^\theta_{r\theta}=\Gamma^\theta_{\theta r}=\tfrac{1}{r}\ , \qquad \Gamma^r_{\phi\phi}=-(r-2GM)\sin^2\theta\\
\Gamma^r_{rr}&=\frac{-GM}{r(r-2GM)}\ , \quad \Gamma^r_{\theta\theta}=-(r-2GM)\ , \quad \Gamma^\theta_{\phi\phi}=-\sin\theta\cos\theta\\
\Gamma^t_{tr}=\Gamma^t_{rt}&=\frac{GM}{r(r-2GM)}\ , \qquad \Gamma^\phi_{r\phi}=\Gamma^\phi_{\phi r}=\tfrac{1}{r}\ , \qquad \Gamma^\phi_{\phi\theta}=\Gamma^\phi_{\theta\phi}=\cot\theta.
\end{aligned} \tag{5.10}$$

The $x^0 = t$ component of the geodesic equation Eq. (5.9) is then

$$\frac{d^2t}{d\tau^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0, \tag{5.11}$$

which we can rearrange to get (as you will show in a (Homework) problem)

$$\frac{d}{d\tau}\left[\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}\right] = 0. \tag{5.12}$$

Eq. (5.12) is a total derivative, which can be immediately integrated to get

$$\boxed{\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} = E}, \tag{5.13}$$

where E is an integration constant which depends on the initial conditions. In fact, E is the **Energy per unit mass** of the test particle, and Eq. (5.12) expresses a **conservation law**. It is a very general principle of physics, that the presence of conservation laws and **conserved quantities** indicates the existence of some symmetries in the equations. Indeed, these are the very symmetries of the Schwarzschild metric. The conservation of energy is usually associated with some form of symmetry in time, and in this case it is the *the invariance of the metric under* $\partial_t g_{\mu\nu} = 0$. In other words, the isometries of the metric leads to conservation laws of the dynamics of test particles. To see this, consider the four momentum of a test particle of mass m

$$p^\mu = m\frac{dx^\mu}{d\tau}, \tag{5.14}$$

and using the geodesic "vector" equation (see below Eq. (3.37) or your Homework problem) in terms of Eq. (5.14) we get

$$p^\nu \nabla_\nu p^\mu = 0 \tag{5.15}$$

which we can use $g_{\rho\mu}\nabla_\nu p^\mu = \nabla_\nu p_\rho$ since $\nabla_\nu g_{\rho\mu} = 0$ as the Levi-Civita connection is metric compatible, we get

$$p^\nu \partial_\nu p_\mu - p^\nu \Gamma^\rho_{\nu\mu} p_\rho = 0. \tag{5.16}$$

The first term is

$$\begin{aligned} p^\nu \partial_\nu p_\mu &= m\frac{dx^\nu}{d\tau}\partial_\nu p_\mu \\ &= m\frac{dp_\mu}{d\tau}, \end{aligned} \tag{5.17}$$

where in the second line we have used Eq. (2.20). Meanwhile the 2nd term is

$$\begin{aligned} \Gamma^\rho_{\nu\mu} p^\nu p_\rho &= \frac{1}{2} g^{\rho\lambda}(\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\nu\mu}) p^\nu p_\rho \\ &= \frac{1}{2}(\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\nu\mu}) p^\nu p^\lambda \\ &= \frac{1}{2}\partial_\mu g_{\lambda\nu} p^\nu p^\lambda + \frac{1}{2}\cancelto{0}{(\partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\nu\mu}) p^\nu p^\lambda} \end{aligned} \tag{5.18}$$

where the last term cancels because $p^\nu p^\lambda$ is symmetric while the partials of the metric are anti-symmetric under interchange of $\nu \leftrightarrow \lambda$. Putting everything together we get

$$m\frac{dp_\mu}{d\tau} = \frac{1}{2}(\partial_\mu g_{\lambda\nu}) p^\nu p^\lambda. \tag{5.19}$$

or

$$m\frac{d(g_{\mu\sigma}p^\sigma)}{d\tau} = \frac{1}{2}(\partial_\mu g_{\lambda\nu}) p^\nu p^\lambda. \tag{5.20}$$

To make contact with Eq. (5.13), recall that p^μ/m is the four-velocity, so $p^0/m = dt/d\tau$ harking all the way back to Eq. (1.68). We can rewrite Eq. (5.19) to be

The $\mu = 0$ component of Eq. (5.19) is then

$$m\frac{d(g_{00}p^0)}{d\tau} = \frac{1}{2}(\partial_t g_{\lambda\nu})p^\nu p^\lambda \tag{5.21}$$

and since the isometry of the metric imposes $\partial_t g_{\lambda\nu} = 0$ we can immediately integrate to get

$$\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} = \text{constant} \tag{5.22}$$

and if we call the constant E, we then recover Eq. (5.13). Hence we have shown that *symmetries of the metric, or isometries, lead to conserved quantities.*

What about the rotation symmetry of the Schwarzschild metric? You can probably guess that the conserved quantity here is the **Angular momentum per unit mass**, and you would indeed be correct. To see this explicitly, we compute the $\mu = \phi$ component of the geodesic equation Eq. (5.9) which is

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau} + 2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} = 0 \tag{5.23}$$

which we can again rearrange as a total derivative

$$\frac{1}{r^2\sin^2\theta}\frac{d}{d\tau}\left[r^2\sin^2\theta\frac{d\phi}{d\tau}\right] = 0. \tag{5.24}$$

Integrating this equation yields

$$\boxed{r^2\sin^2\theta\frac{d\phi}{d\tau} = L} \tag{5.25}$$

where L is the *conserved* angular momentum of the particle as promised. It is a little bit more tricky to see from Eq. (5.20) since we are in the spherical polar basis $\partial_\mu = (\partial_t, \partial_r, \partial_\theta, \partial_\phi)$ so you have to do a wee bit more work that L is indeed the angular momentum of the system. But you can see that if $\theta = \pi/2$, i.e. at the equator, then $d\phi/d\tau = \omega$ is the angular velocity. Setting $\omega = v/r$ where v is the speed of the particle, we get $r^2\omega = rv$ which is then the angular momentum per unit mass. That there exists conserved quantities like energy and angular momentum in spherically symmetric gravitational system is not a surprise of course – they already exist in spherically symmetric Newtonian gravity systems as you doubtless know. What we have found are their GR equivalents[2].

Before we proceed to finally solve the geodesic equation in full, we take a step back to point out that although we called E the "energy", we have *secretly* used the special relativistic notion of the four-momentum and call $p^0 = E$ etc. This is true when the metric is Minkowski as it is in special relativity, but it is not true in general non-flat metrics. However, since Schwarzschild is asymptoptically flat at $r \to \infty$ so far away from the center it *is* Minkowski space, we can *define* E to be "the energy per unit rest mass if observed at infinity". Similar arguments apply for the angular momentum. If the particle is massless, as it will be for photons, then while E and L are conserved, they don't have any particular physical meaning since the nullness of their geodesics allow us to rescale $\tau \to a\tau$ where a is some constant. However, the ratio L/E *is* physical, as we will see when we discussed null like geodesics in section 5.1.4.

Carrying on with the geodesic equation, the $x^\mu = \theta$ term is

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0. \tag{5.26}$$

[2] The fact that isometries lead to conserved quantities is a general one and not just for Schwarzschild. Unfortunately we won't have the time to study this in detail, but it is such a pretty topic that I cannot resist putting a discussion of it in the Appendix B – have a look, it is really not that hard.

We can use the conserved angular momentum Eq. (5.25) to eliminate the $(d\phi/d\tau)^2$ term, and rewrite the first two terms to get

$$r^2\frac{d}{d\tau}\left[r^2\frac{d\theta}{d\tau}\right]-\frac{\cos\theta}{\sin^3\theta}L^2=0. \tag{5.27}$$

Recall that we always have a freedom to choose spherical polar coordinates for the sphere $\mathbb{S}^2$. In particular, we can choose coordinates such that $\theta(\tau_0)=\pi/2$ and $d\theta/d\tau=0(\tau_0)=0$ at some initial time $\tau_0=0$. Using these initial conditions, Eq. (5.27) is trivial to solve and yields the solution $\theta(\tau)=\pi/2$ – in other words the orbit is on some fixed plane defined by $\theta=\pi/2$. This means that we only need to solve for $r(\tau)$. The obvious thing to do now is to use the final geodesic equation, that of $x^\mu=r$,

$$\frac{d^2r}{d\tau^2}+\frac{GM(r-2GM)}{r^3}\left(\frac{dt}{d\tau}\right)^2-\frac{GM}{r(r-2GM)}\left(\frac{dr}{d\tau}\right)^2-(r-2GM)\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]=0 \tag{5.28}$$

and use the conserved energy Eq. (5.13) and angular momentum Eq. (5.25) to eliminate the $dt/d\tau$ and $d\phi/d\tau$ terms and fix $\theta=\pi/2$ to get

$$\frac{d^2r}{d\tau^2}+\frac{GM}{r(r-2GM)}E^2-\frac{GM}{r(r-2GM)}\left(\frac{dr}{d\tau}\right)^2-(r-2GM)\frac{L^2}{r^4}=0, \tag{5.29}$$

which is a 2nd order ODE for $r(\tau)$. This is still quite daunting though it can be solved analytically. Fortunately, we have one *final* trick up our sleeve – there is actually another *conservation law* hidden in the geodesic equation. It will turn out that the quantity

$$\boxed{\sigma\equiv-g_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}} \tag{5.30}$$

is conserved along the geodesic $x^\mu(\tau)$, i.e. if $\mathbf{T}=dx^\mu/d\tau\partial_\mu$ is the tangent vector to the geodesic, then

$$\frac{d\sigma}{d\tau}=\nabla_{\mathbf{T}}\sigma(\tau)=0 \tag{5.31}$$

where in the second equality we have used the fact that σ is a scalar hence $\partial_\mu\sigma=\nabla_\mu\sigma$. Then,

$$\begin{aligned}\nabla_{\mathbf{T}}\sigma(\tau) &= -T^\rho\nabla_\rho(g_{\mu\nu}T^\mu T^\nu)\\ &= -T_\mu T^\rho\nabla_\rho T^\mu-T^\mu T^\alpha\nabla_\alpha T_\mu\\ &= 0\end{aligned} \tag{5.32}$$

using the fact that $T^\mu\nabla_\mu T^\nu=0$ since it is the tangent to the geodesic[3].

Using the Schwarzschild metric, Eq. (5.30) is then

$$-\sigma=-\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2+\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dt}{dr}\right)^2+r^2\left(\frac{d\theta}{d\tau}\right)^2+r^2\sin\theta^2\left(\frac{d\phi}{d\tau}\right)^2 \tag{5.33}$$

which we can again use our energy and momentum conservation Eq. (5.13) and Eq. (5.25), and impose $\theta=\pi/2$ to get the very nice looking equation

$$\boxed{\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+V(r)=\frac{1}{2}E^2} \tag{5.34}$$

where the **effective potential** $V(r)$ is

$$\boxed{V(r)\equiv\frac{1}{2}\sigma-\sigma\frac{GM}{r}+\frac{L^2}{2r^2}-\frac{GML^2}{r^3}} \tag{5.35}$$

[3]Conservation of the norm of the tangent vector is a consequence of the fact that we have chosen an affine parameter to parameterize our geodesics, as we discussed in 61 earlier.

The equation of motion Eq. (5.34) is written in a particularly comforting way – it looks like the equation of motion of a unit mass particle moving in a potential $V(r)$ with energy $E^2/2$. All the GR secret sauce is now hidden in the potential term $V(r)$ where we have gained an extra term GML^2/r^3 when compared to the Newtonian version. This term scales as r^{-3}, and hence decays faster at large r when compared to the other terms, while as r approaches the origin it starts to dominate. This means that *the largest deviations from Newtonian Mechanics occur at the small r limit* – or when gravity is strongest, as we might have expected.

5.1.4 Orbits of the Schwarzschild spacetime

We can now explore the geodesics – or orbits – of particles around the Schwarzschild spacetime. In your Mechanics course, you studied only non-relativistic orbits $v \ll c$, but now since we are doing GR, orbits can be relativistic. In particular, massless particles travel at $v = c$, and hence are null-like $\sigma = 0$ while massive particles are timelike ($\sigma > 0$). For massive particles, without any loss of generality, we can always find an affine transformation $\tau \to a\tau'$ such that $\sigma = 1$. As it turns out, there are some important qualitative differences between null-like and time-like solutions, so we will explore them separately.

Massive particles : timelike orbits

We plot the potentials for $L = 1, 3, 5, 7, 9$ in units of $GM = 1$ and $\sigma = 1$ in Figure 5.1. From the equation of motion Eq. (5.34), we can see that, given $V(r)$, what $dr/d\tau$ will do depends on the energy of the particle $E^2/2$. Think of a "ball" rolling on the potential from the right towards the left. It will climb the potential until it reaches $V(r) = E^2/2$ whereby $dr/d\tau = 0$, and "turns around' and rolls back towards the right. Of course, $dr/d\tau$ is just the *radial* velocity – the angular velocity depends on L.

For circular orbits, $dr/d\tau = 0$, occuring at $r = r_c$, and when $dV/dr = 0$ (else it will want to roll away from r_c). Differenting the potential Eq. (5.35), we get

$$\mathrm{GR}\ :\ \sigma GMr_c^2 - L^2 r_c + 3GML^2 = 0 \tag{5.36}$$

and

$$\mathrm{Newton}\ :\ \sigma GMr_c - L^2 = 0. \tag{5.37}$$

For Newtonian gravity, there *always* exists a stable circular orbit at $r_c = L^2/GM\sigma$, as you have no doubt learned. For large angular velocity (i.e. L is large), the circular orbit is at greater radius r_c.

For GR, things are similar to Newtonian gravity at large $r \gg GM$. But at small r, things are very different. First, the potential $V(r)$ *always always reaches zero at $r = 2GM$ irregardless of L or σ*, and continue to $\lim_{r\to 0} V = -\infty$. At $r = 2GM$, as we will study in the next section on black holes, is the **black hole event horizon**. Of course the Schwarzschild metric is valid for non-black holes such as the Earth – in this case, the surface of the object must be $R_{\text{surface}} > 2GM$, so the metric will no longer be valid (particles just hit the surface and stop moving or explode or whatever).

The circular orbit radii is found by solving the quadratic equation Eq. (5.36), to get

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12(GML)^2}}{2GM}. \tag{5.38}$$

If $L^2 < 12(GM)^2$, then there is no turning point and hence no stable orbit – the particle will spiral into $r = 2GM$. For $L^2 > 12(GM)^2$, there are two solutions, an inner orbit r_{inner} and outer orbit r_{outer}. It's easy to check that r_{inner} is *unstable* since it is a maxima, while r_{outer} is *stable* since it is a minima. Both orbits coincide when $L^2 = 12(GM)^2$ which occurs at $r_c = 6GM$. In the limit of $L \to \infty$, the finite solution of Eq. (5.38) gives $r_c \to 3GM$. This means that the orbits are bounded by

$$3GM < r_{\text{inner}} < 6GM < r_{\text{outer}}. \tag{5.39}$$

Notice that since the orbits coincide at $6GM$, this is the **innermost stable circular orbit**, or ISCO. Any orbit $r < 6GM$ will be unstable under small perturbation and fall into $r = 2GM$ (and beyond).

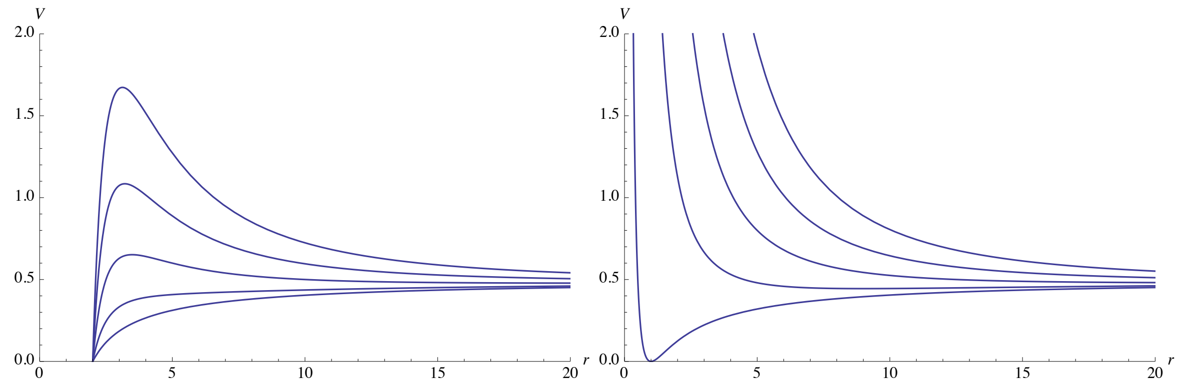


Figure 5.1: The effective potential $V(r)$ for GR (left) and Newton's laws (right) for **massive** particles, with $L = 9, 7, 5, 3, 1$ from top to bottom, with units set such that $GM = \sigma = 1$.

Massless particles : null-like orbits

We plot the potentials for $L = 1, 3, 5, 7, 9$ in units of $GM = 1$ and $\sigma = 0$ in Figure 5.2.

From Newtonian gravity, there is no stable circular orbit – you can check by setting $\sigma = 0$ in Eq. (5.37). This is not surprising either – light will travel in straight lines through a gravitational potential in Newtonian gravity.

For GR, the potential again reaches zero at the black hole horizon $r = 2GM$. Also interestingly, the circular orbit is *independent of* L, at $r_c = 3GM$. Clearly, this orbit is unstable since it is a maxima – under perturbation it will either fall into $r = 2GM$ or fly away into $r = \infty$.

At this point, the value of $V(r = 3GM) = L^2/54GM$. So if $E^2/2 > L^2/54GM$, the massless particle will "roll over the top" and fall into the horizon at $r = 2GM$. As we discussed earlier in section 5.1.3, the quantities E and L have no physical meaning for massless particles, but the ratio L/E is physical. In particular, the critical value

$$\frac{L}{E} = \sqrt{27GM}, \tag{5.40}$$

tells us whether the particle will fall into $2GM$ or escape to infinity. L/E is called the **impact parameter**. If $L/E > \sqrt{27GM}$ then incoming light rays will not fall into $2GM$, but instead will be "bent" and then sent back to infinity – i.e. it is gravitationally lensed[4]. If $L/E < \sqrt{27GM}$, light rays will spiral into $2GM$. For a geometric interpretation, see R. Wald *General Relativity*.

5.2 Schwarzschild Black Holes

In the previous sections, we have alluded to the fact that a "black hole horizon" occurs at $r = 2GM$ for the Schwarzschild metric. We will now study this in greater detail. I mean, **Black Holes**, man!

5.2.1 Physics outside the Black hole $r > 2GM$

The factors $(1 - 2GM/r)$ in the Schwarzschild metric Eq. (5.2) seem to suggest that the metric blows up at $r = 2GM$, in addition to the fact that the potential V invariably goes to zero at $r = 2GM$. This "blow-up" in the metric is actually not a problem – it is the result of the fact that the Schwarzschild coordinates

[4]To see this explicitly, we have to solve the geodesic equation for ϕ.

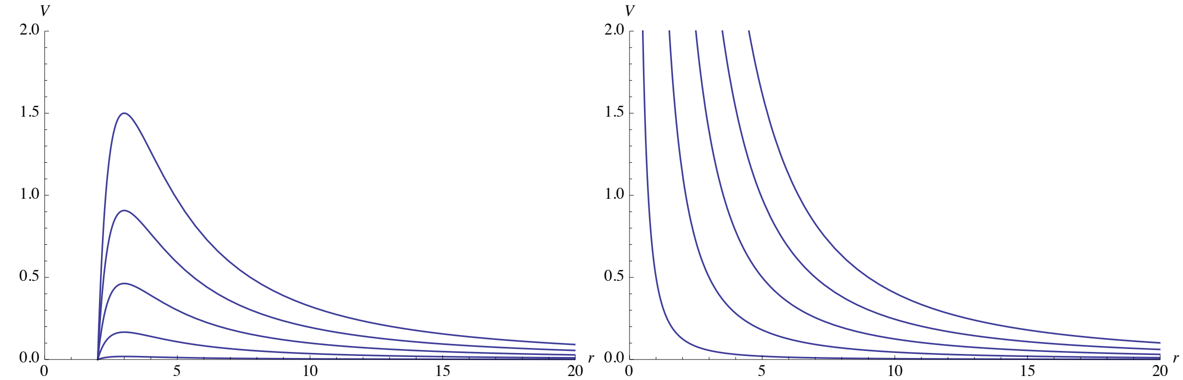


Figure 5.2: The effective potential $V(r)$ for GR (left) and Newton's laws (right) for **massless** particles, with $L = 9, 7, 5, 3, 1$ from top to bottom, with units set such that $GM = 1$ and $\sigma = 0$.

that we have chosen in Eq. (5.2) are bad at this point. Recall from section 2.3 about coordinate systems, general curved spacetimes often require several different coordinate systems to describe them fully. This blow-up at $r = 2GM$ is a failure of the coordinate system – known as a **coordinate singularity**, so it is not physical. The solution is of course to find another coordinate system, which we will do shortly.

Nevertheless, the point $r = 2GM$ is incredibly interesting physically – it is the location of the **black hole event horizon**. The surface $r = 2GM$ *is* the hole that is black (and not the singularity that actually resides at $r = 0$). Fortunately, most of the physics outside the black hole can be explored using the Schwarzschild coordinates.

As we mentioned before, at $r \to \infty$, the Schwarzschild metric reduces to that of Minkowski spacetime. So for a static observer at $r \to \infty$, (held up by some rocket engines from falling into the black hole), called (creatively) **an observer at infinity** – let's call her Alice – her proper time τ_A is synchronized with the time coordinate t, i.e.

$$\text{At } r \to \infty : ds^2 = -dt^2 \to d\tau_A = dt \tag{5.41}$$

since Alice is static so the $dx^i = 0$. So we can use her watch as a timer. Now suppose Alice drops a beeping beacon into the black hole, falling *radially*, and this beacon is set such that it sends off a light signal at uniform period of $\Delta\tau_c$ where τ_c is its proper time (i.e. the beacon's own clock). The geodesic of the beacon then follows that of the (once integrated) geodesic equation Eq. (5.13)

$$\frac{dt}{d\tau_c} = \left(1 - \frac{2GM}{r}\right)^{-1} \tag{5.42}$$

where we have rescaled τ such that $E = 1$. The radial geodesic equation is given by Eq. (5.34), with $L = 0$ (since the beacon is falling radially) and $\sigma = 1$ (with $E = 1$) is

$$\frac{dr}{d\tau_c} = \sqrt{\frac{2GM}{r}} \tag{5.43}$$

which can be easily integrated to obtain

$$\tau_c = \frac{1}{3}\sqrt{\frac{2}{GM}}(r_0^{3/2} - r_*^{3/2}) \tag{5.44}$$

where $r_0^{3/2}$ is the initial altitude of the beacon which is very large but finite distance, and τ_c is the total time for the beacon to fall to radius $r = r_*$. Notice that τ_c is *finite* for both $r_* = 2GM$ (the horizon) and $r_* = 0$. In other words, the beacon will take a finite proper time (according to its clock) to fall to the horizon and the center. The $r = 2GM$ horizon is not some barrier for things fallings in.

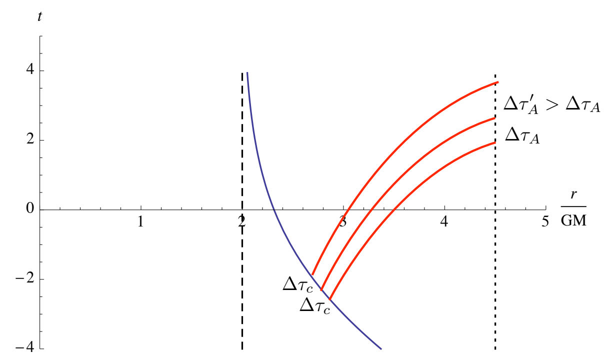


Figure 5.3: Gravitational redshifting of a beacon falling into a black hole. The blue thick curve is the trajectory of the beacon and the red lines are the null geodesics of the periodic beeps of the beacon. The beacon has proper time τ_c (which parameterizes the blue curve), and hence its beeps are periodic in $\Delta\tau_c$. We can't draw Alice at $r \to \infty$, but a static observer further away at $r = 4.5GM$ will receive the beeps of the beacon at increasingly longer intervals.

What does Alice see? While the beacon is near her, she will see roughly beeps at constant intervals $\Delta t \approx \Delta\tau_c$. But, as Eq. (5.42) show, as $r \to 2GM$, $\Delta t = \Delta\tau_A$ becomes *larger* for the same amount of $\Delta\tau_c$, since the integrand $\lim_{r\to 2GM}(1-2GM/r)^{-1} = \infty$. So, as the beacon falls closer and closer to $r = 2GM$, the interval between beeps become longer and longer, until it takes an infinite amount of time according to Alice to see the next beep. See Figure 5.3. You will compute the trajectory of the beacon as a function of t and r in a (Homework) problem.

Now, instead of beeps, the beacon sends a light ray up at frequency λ. The beeps are then replaced by the peaks (and troughs) of the light electromagnetic wave, and hence the frequency of light that Alice receives as the beacon falls become longer and longer until it becomes infinitely long. In other words, light undergoes **Gravitational Redshift** (according to Alice) as it falls into the black hole. Alice will see her beacon becomes redder and redder, until it is so red that it is no longer visible.

5.2.2 Physics inside the Black hole $r < 2GM$

As we mentioned earlier, to study the physics of the black hole near and inside the horizon $r \leq 2GM$, we need to find a better coordinate system. To see what is the problem with the Schwarzschild coordinates, consider a radially infalling null geodesic, i.e.

$$ds^2 = 0 = -\left(1-\frac{2GM}{r}\right)\mathrm{d}r^2 + \left(1-\frac{2GM}{r}\right)^{-1}\mathrm{d}t^2 \tag{5.45}$$

or

$$\frac{dt}{dr} = \pm\left(1-\frac{2GM}{r}\right)^{-1}. \tag{5.46}$$

From Eq. (5.46), we can see that as $r \to 2GM$, dt/dr blows up. That means that it takes the t coordinate is being "stretched" to infinity as $r \to 2GM$, or that t is changing too rapidly for small r changes. If we integrate Eq. (5.46), we get

$$t = \pm\left[r - 2GM\ln(2GM - r)\right] + t_0 \tag{5.47}$$

which shows that t is log divergent as $r \to 2GM$. To evade that, we define a new radial coordinate, the so-called **Regge-Wheeler** or **tortoise** coordinate,

$$r_* \equiv r + 2GM\ln\left|\frac{r}{2GM} - 1\right|, \tag{5.48}$$

which the log term now happily eliminates the original log divergence. In these coordinates, the Schwarzschild metric becomes (Homework)

$$ds^2 = \left(1 - \frac{2GM}{r(r_*)}\right)(-\mathrm{d}t^2 + \mathrm{d}r_*^2) + r^2 d\Omega^2. \tag{5.49}$$

In these coordinates, $r_* \approx r$ for $r \gg 2GM$ but $r_* \to -\infty$ as $r \to 2GM$, so we are not quite there yet (since we want our metric to go across $2GM$). However, the nice thing about the metric Eq. (5.49) is that radial null-like geodesics is now

$$ds^2 = \left(1 - \frac{2GM}{r(r_*)}\right)(-\mathrm{d}t^2 + \mathrm{d}r_*^2) = 0 \tag{5.50}$$

or

$$\frac{dr_*}{dt} = \pm 1 \tag{5.51}$$

so

$$t \pm r_* = \text{constant}. \tag{5.52}$$

In other words, null rays travel on curves where $t \pm r_* =$ constant, with the $\pm$ signs indicating the ingoing (+) and outgoing rays (-). We can then define the **ingoing Eddington-Finkelstein** coordinates by

$$v \equiv t + r_*, \tag{5.53}$$

whereby the Schwarzschild metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r(r_*)}\right)\mathrm{d}v^2 + (\mathrm{d}v\mathrm{d}r + \mathrm{d}r\mathrm{d}v) + r^2 d\Omega^2. \tag{5.54}$$

The metric[5] Eq. (5.54) now is *smooth* and regular all the way from $r = \infty$ to $r \to 0$ since the determinant $g = -r^4 \sin^2\theta$ it has an inverse for $r \neq 0$ and everything is good. In these coordinates, *ingoing null* geodesics are simply lines of constant v.

If you like, the ingoing Eddington-Finkelstein coordinates is a "better" coordinate system than the Schwarzschild coordinates if all we care is how much of the spacetime it covers – the coordinate singularity at $r = 2GM$ in the Schwarzschild coordinates is not present here. But, it is easier to interprete the physics in the Schwarzschild coordinates, so it remains the more popular.

What about $r = 0$? Here, the determinant $g = 0$ so the metric is not invertible at $r = 0$, and hence is not a metric anymore. In fact, though $R_{\mu\nu} = R = 0$ for the Schwarzschild metric, the Riemann tensor $R^\sigma{}_{\mu\nu\rho}$ is not vanishing. Of course, the components of $R^\sigma{}_{\mu\nu\rho}$ are basis dependent so that might not mean anything. However, we can compute a basis independent – and hence *physical* – quantity by contracting the Riemann tensor with itself, i.e (which you will do in a (Homework) problem)

$$R_{\sigma\mu\nu\rho}R^{\sigma\mu\nu\rho} = \frac{12G^2M^2}{r^6} \tag{5.55}$$

which clearly diverges at $r = 0$. Basis independent measures of curvature like Eq. (5.55) are known as **curvature invariants**. Other examples are $R_{\mu\nu}R^{\mu\nu}$, $C_{\sigma\mu\nu\rho}C^{\sigma\mu\nu\rho}$ etc. The divergence at $r = 0$ is physical, and is known as the **Singularity**. Here the general relativistic description of physics breaks down, and we need a new theory to describe reality.

What about the *outgoing* geodesics? We know from Eq. (5.52) that outgoing light rays obey $t - r_* =$ constant, which we can write using Eq. (5.53) as $v = 2r_* +$ constant. In figure 5.4, we plot the geodesics of the incoming and outgoing rays as a function of r and a new coordinate t' defined to be

$$t' \equiv v - r. \tag{5.56}$$

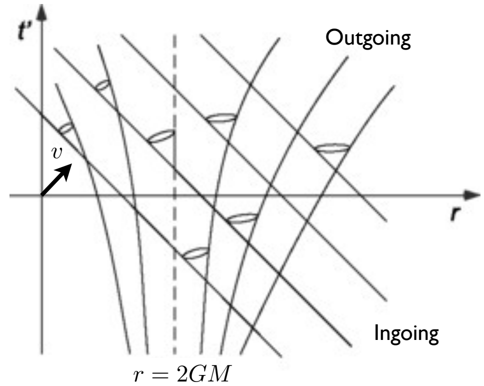


Figure 5.4: The **Finkelstein diagram** shows the light cone structure of a Schwarzschild black hole near the horizon. Incoming radial light rays are 45° to the r axis, and fall through the horizon with no issues. Outgoing light rays that begin at $r > 2GM$ will gradually fall away from the horizon into infinity. Outgoing light rays that begin at $r < 2GM$ actually keep falling towards $r = 0$ – which belies their name "outgoing". The "cones" are light cones. (Figure modified from N. Straussman.)

From figure 5.4, since we know both the geodesics of ingoing and outgoing radial null geodesics, we can draw "light cones" on it. You can see that, on $r = 2GM$, the light cone is always facing towards $r = 0$ n the coordinates r and t', ingoing radial null geodesics are straight diagonal lines that are 45° to r – they crosses the horizon $r = 2GM$ without any issues. However, how outgoing radial null geodesics curve depends on whether they begin outside or inside $r = 2GM$. If they begin *outside*, they curve *outwards away* towards $r = \infty$. If they begin *inside*, their r actually *always decreases* (despite being "outgoing"), and will fall into $r = 0$ in finite proper time. From figure 5.4, since we know both the geodesics of ingoing and outgoing radial null geodesics, we can draw "light cones" on it. You can see that, on $r \leq 2GM$, the light cone is always facing towards $r = 0$. This is what we meant when we say "light cannot escape the horizon". Indeed, it can be shown (Homework) that *the r for timelike geodesics that began inside the black hole $r < 2GM$ also always decreases whether they are ingoing or outgoing* – everything inside $r < 2GM$ will fall into the singularity in finite proper time. This is the reason why black holes are black.

Finally, if they begin at $r = 2GM$, they will simply travel on $r = 2GM$ forever – *the Black hole horizon is a null surface.*

5.3 Gravitational Radiation

For our final section, we will now discuss one of the most exciting things in physics today: **gravitational waves**. On Sept 14 2015, the LIGO gravitational wave detectors at both Hanford and Louisiana picked up the gravitational waves of a black holes merger event that occurred 410Mpc away, an event now known as GW150914. The two black holes which were initially 29 and 36 times the mass of the sun, spiralled into each other and merged. After the merger they formed a single black hole of 62 times the mass of the sun. The total gravitational wave energy released during this event was 3 times the mass of the sun, in less than 0.01 seconds. Using $E = mc^2$, the total released energy in this short time is 1.8×10^{47} Joules of energy, so the power is about 10^{49} Watts. For comparison, the luminosity of the Milky Way is 10^{36} Watts. The total luminosity of all the galaxies in the observable universe is about 10^{48} Watts – this single event was brighter than the entire universe for one brilliant moment.

[5]We were pedantic in writing $\mathrm{d}v\mathrm{d}r + \mathrm{d}r\mathrm{d}v$ instead of the more common $2\mathrm{d}v\mathrm{d}r$ because the basis co-vectors $\mathrm{d}v\mathrm{d}r = \mathrm{d}v \otimes \mathrm{d}r$ in general do not commmute.

Roughly speaking, gravitational waves are generated when matter is accelerated – just like electromagnetic waves are generated when charged particles such as the electrons are accelerated. We will first study the GW themselves in sction 5.3.1, and then study their generation in section 5.3.4.

5.3.1 Weak Gravity Limit Again : Linearized Gravity

In general, gravitational waves are propagating modes of curvature that can exist in any kind of spacetime. Indeed, one often has to solve the full non-linear Einstein equation to calculate the gravitational waves generated by motion of matter fields. In the GW150914 event, the equations are so complicated that the exact waveforms are computed using a supercomputer – after a heroic 30+ years of research and development in understanding how to solve such complicated equations on a computer.

Nevertheless, all is not lost. We can always study gravitational waves that are propagating in *almost Minkowski space*. Recall from our discussion of Einstein equation in section 4.5 at gravitational waves can exist on its own without a source – again much like electromagnetic waves can propagate in vacuum. In particular, for simplicity, we want to study gravitational waves in the *weak gravitational field limit*. When we discussed the Newtonian Limit in section 4.3.1, we already talked about this particular limit, which is that the metric $\bar{\mathbf{g}}$ can be decomposed into

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5.57}$$

where $\eta_{\mu\nu}$ is the flat space metric (called the **background metric**) and the **metric perturbation** $h_{\mu\nu} = h_{\nu\mu}$ is symmetric. We *choose* a coordinate system such that in its coordinate basis, $\eta_{\mu\nu} = (-1, 1, 1, 1)$, in which $|h_{\mu\nu}| \ll 1$. In this section, we will discuss the Einstein equation in this limit (and this limit only – we don't impose non-relativistic nor static gravitational field limit). Let's expand a little bit more about what we are really doing here.

In the weak field limit, the *physical metric* is still $g_{\mu\nu}$ (including the perturbation) – particles travel on geodesics defined by $g_{\mu\nu}$. So morally speaking, tensor indices are still lowered by $g_{\mu\nu}$ and raised by the inverse metric $g^{\mu\nu}$. However, as we have shown in Eq. (4.20), the inverse metric can be written as

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2). \tag{5.58}$$

Note that $g_{\mu\nu}$ and $\eta_{\mu\nu}$ are *both* metric tensors on the manifold, so we can use either of them to lower (and raise) indices on tensors. Which one should we use? Since $g_{\mu\nu}$ is the physical metric, morally speaking we should use it to raise/lower indices. However, the difference between using $\eta_{\mu\nu}$ and $g_{\mu\nu}$ to raise/lower indices is $\mathcal{O}(h)$. If we now think of the metric perturbation $h_{\mu\nu}$ (which is *not* a metric) a rank-$(0,2)$ tensor field on the manifold, then the difference between raising and lowering it with $g_{\mu\nu}$ and $\eta_{\mu\nu}$ is

$$\begin{aligned} g_{\mu\nu}h^{\mu\sigma} - \eta_{\mu\nu}h^{\mu\sigma} &= h_\nu{}^\sigma + h_{\mu\nu}h^{\mu\sigma} - h_\nu{}^\sigma \\ &= h_{\mu\nu}h^{\mu\sigma} \propto \mathcal{O}(h^2) \end{aligned} \tag{5.59}$$

i.e. it is 2nd order in h. So as long as we are simply raising and lowering $h_{\mu\nu}$ and keeping only up to $\mathcal{O}(h)$ in our expansion, we can use $\eta_{\mu\nu}$ to raise and lower the indices. Another way of thinking about this system is that the perturbation $h_{\mu\nu}$ is a Lorentzian tensor – it is raised and lowered by $\eta_{\mu\nu}$. Nevertheless, we emphasised that test particles still travel on geodesics defined by $g_{\mu\nu}$ not $\eta_{\mu\nu}$, i.e. particles still feel the effects of curvature as Einstein proposed.

What about the Einstein equation? Since we are in vacuum, $T_{\mu\nu} = 0$, so we want to now expand $G_{\mu\nu}$ to first order in h (clearly at the zeroth order $G_{\mu\nu} = 0$ since the metric at zeroth order is $\eta_{\mu\nu}$.

We begin with the Christoffel symbol Eq. (3.32), which is to first order in $\mathcal{O}(h)$

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2}\eta^{\mu\lambda}(\partial_\nu h_{\lambda\sigma} + \partial_\sigma h_{\nu\lambda} - \partial_\lambda h_{\nu\sigma}) + \mathcal{O}(h^2), \tag{5.60}$$

so at leading order $\Gamma^{\mu}_{\nu\sigma} = \mathcal{O}(h)$. We can then expand the Riemann tensor in component form Eq. (3.93)

$$\begin{aligned} R_{\sigma\rho\mu\nu} &= g_{\sigma\tau} R^{\tau}{}_{\rho\mu\nu} \\ &= g_{\sigma\tau}\left[\partial_\mu \Gamma^{\tau}_{\nu\rho} + \Gamma^{\sigma}_{\nu\rho}\Gamma^{\tau}_{\mu\sigma} - \partial_\nu \Gamma^{\tau}_{\mu\rho} - \Gamma^{\sigma}_{\mu\rho}\Gamma^{\tau}_{\nu\sigma}\right] \\ &= \eta_{\sigma\tau}\left[\partial_\mu \Gamma^{\tau}_{\nu\rho} - \partial_\nu \Gamma^{\tau}_{\mu\rho}\right] + \mathcal{O}(h^2) \end{aligned} \tag{5.61}$$

where we have used the fact that Γ^2 terms are $\mathcal{O}(h^2)$. Inserting Eq. (5.60) back into the equation, we get the Riemann tensor to first order in h

$$R_{\sigma\rho\mu\nu} = \frac{1}{2}\left(`\partial_\mu\partial_\nu h_{\sigma\rho} + \partial_\sigma\partial_\rho h_{\mu\nu} - \partial_\rho\partial_\nu h_{\sigma\mu} - \partial_\sigma\partial_\mu h_{\rho\nu}\right)`. \tag{5.62}$$

Notice that Eq. (5.62) still obey the symmetries of the Riemann tensor as we studied in section 3.3.3, as it should.

The Ricci tensor, contracted using $\eta_{\mu\nu}$, is (as you will show in a (Homework) problem)

$$R_{\mu\nu} = \eta^{\sigma\mu} R_{\sigma\rho\mu\nu} = \frac{1}{2}(\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h) \tag{5.63}$$

where $\Box \equiv g^{\mu\nu}\partial_\mu\partial_\nu$ is the **de Alembertian** and $h = \eta_{\mu\nu}h^{\mu\nu}$ is the trace of the perturbation.

Finally, the Ricci scalar is (Homework)

$$R = \partial^\rho\partial^\mu h_{\mu\rho} - \Box h. \tag{5.64}$$

The Einstein tensor to first order is then obtained by putting together Eq. (5.63) and Eq. (5.64), to get

$$\boxed{G_{\mu\nu} = \frac{1}{2}(\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h) - \frac{1}{2}\eta_{\mu\nu}(\partial^\rho\partial^\mu h_{\mu\rho} - \Box h)}. \tag{5.65}$$

Eq. (5.65) is called the **Linearized Einstein Tensor**, and since we are only keeping track of the metric up to first order in h, we say we are working in the regime of **Linear Gravity**.

To make further progress, we have to hark back to our discussion below Eq. (4.19) – we alluded to the fact that the decomposition $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ depends on coordinates. In other words, while the perturbation $\mathbf{h}$ is a rank-$(0,2)$ tensor and hence is basis independent, the *values* of the components depend on basis obviously. This is normally not an issue, but here we are insisting that $h_{\mu\nu} \ll g_{\mu\nu}$ – i.e. we are using the fact that "the difference in the component values" are small to truncate our equations so now it matters in linearized gravity.

The good news is that, since we are doing perturbations around Minkowski space, we can *always* choose the basis such that the background metric is $\eta_{\mu\nu} = (-1, 1, 1, 1)$.

The bad news is that, since $h_{\mu\nu}$ is a "small" perturbation, there is still a large freedom in the choice of basis. For example, suppose in one basis, $g_{\mu\nu} = \mathrm{diag}(-0.99, 0.99, 0.99, 1)$, but in another basis $g_{\mu'\nu'} = \mathrm{diag}(-1, 1, 1.01, 1.01)$. Both bases then can be decomposed into their respective $\eta_{\mu\nu}$ and $\eta_{\mu'\nu'}$ and $h_{\mu\nu} = \mathrm{diag}(-0.01, 0.01, 0.01, 0)$ and $h_{\mu'\nu'} = \mathrm{diag}(0, 0, -0.01, -0.01)$ (of course $h_{\mu\nu}$ is not necessarily diagonal, but this is a simple example). So what is the right value for the perturbation tensor $\bar{\mathbf{h}}$?

The origin of this ambiguity is obviously the freedom we have in choosing coordinates (and hence its associated coordinate basis) on the manifold. Recall that a general coordinate transform from x^μ chart to $x^{\mu'}$ chart is $x^{\mu'}(x^\mu)$ (e.g. $x'(x, y)$ and $y'(x, y)$ for 2D). But we have already picked the background metric basis to give us $\eta_{\mu\nu}$, so any coordinate change must still keep this fact true. In other words, we need is *infinitisimal* coordinate transforms, i.e. it is a derivative and hence it is a vector (remember from section 2.2 that vectors are derivatives). Let's call this vector ξ_μ, in the basis where $\eta_{\mu\nu} = (-1, 1, 1, 1)$. The 4 components of the ξ_μ encodes the 4 infinitisimal coordinate transforms. The choice of basis is called a **gauge**, and these transformations are called **gauge transformations** or **gauge conditions**.

In other words, there is a *redundancy* in our description of the same physics. This is closely analogous to the gauge freedom for the vector potential in electrodynamics $A_\mu \to A_\mu + \partial_\mu \alpha$.

We now want to construct a formula for $h_{\mu\nu}$ which allows us to go from one basis to another. There is a proper way to do this, but unfortunately we haven't covered the necessary mathematics to do it[6]. Nevertheless, we can motivate it. What we want is to find a transformation for $h_{\mu\nu}$

$$h_{\mu\nu} \to h_{\mu\nu} + f_{\mu\nu}, \tag{5.66}$$

where $f_{\mu\nu}$ is constructed out of ξ_μ and ∂_μ. Since $h_{\mu\nu}$ is symmetric, then the symmetric combinations are

$$f_{\mu\nu} = c_1(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) + c_2 \partial_\rho \xi^\rho \eta_{\mu\nu} + c_3 \xi_\mu \xi_\nu + \mathcal{O}(\xi^3) + \ldots \tag{5.67}$$

where c_1, c_2 and c_3 are constants. Since we want *small* deviations from $h_{\mu\nu}$, we drop the c_3 term as it is second order in ξ_μ, and we can absorb c_1 into ξ_ν. The only term we cannot justify using our symmetry argument is that $c_2 = 0$. Given these, we get the **gauge transformation equation**

$$\boxed{h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu}. \tag{5.68}$$

What Eq. (5.68) means is simple: there is a freedom in choosing the basis, encoded in 4 functions ξ_μ. *Any* two sets of $h_{\mu\nu}$ that can be related to each other by this equation *describes exactly the same physics*. This "gauge freedom" is both annoying and awesome. Annoying because we have to make sure that when we compare quantites, we are comparing them "in the same gauge" (i.e. in the same coordinate basis). However, it is also awesome, because it allows us to simplify our equations as we can choose clever coordinates (and hence coordinate bases) to eliminate terms, as we will soon see below.

Going back to our linearized Einstein tensor Eq. (5.65), we first simplify it slightly by defining a new perturbation variable

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}, \tag{5.69}$$

where $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h$ so it is sometimes known as the "trace-reversed" metric perturbation. Obviously it contain exactly the same amount of information as $h_{\mu\nu}$. In this new variable, Eq. (5.65) is slightly simplified

$$G_{\mu\nu} = -\frac{1}{2}\left[\Box \bar{h}_{\mu\nu} - \partial^\rho \partial_\mu \bar{h}_{\nu\rho} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma}\right]. \tag{5.70}$$

The "trace-reversed" metric perturbation under gauge transformation (Homework) is

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial_\rho \xi^\rho \eta_{\mu\nu}. \tag{5.71}$$

We now use freedom to choose ξ^μ to simplify our equations. We choose ξ^μ such that it is a solution to the following equation

$$\Box \xi_\mu = -\partial_\rho \bar{h}^\rho{}_\mu \tag{5.72}$$

then plugging this into Eq. (5.71), we get the **Lorenz Gauge condition**[7]

$$\boxed{\partial_\mu \bar{h}^{\mu\nu} = 0}. \tag{5.73}$$

This is completely analogous to the Lorenz gauge $\partial_\mu A^\mu = 0$ of electrodynamics. This condition does *not* eliminate all the gauge freedom we have – but we will come back very soon to this in the next section 5.3.2. Using the fact that partials commute, the Lorenz condition renders the last three terms in Einstein tensor Eq. (5.70) trivial, and give us the super nice equation

$$\boxed{\Box \bar{h}_{\mu\nu} = 0}, \tag{5.74}$$

[6]**The proper mathematics is discussed in the Appendix B.1. The coordinate transformation is a diffeomorphism along the vector ξ_μ, so the perturbation tensor gauge transforms as $h_{\mu\nu} \to h_{\mu\nu} + \mathcal{L}_\xi \eta_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$.

[7]After Ludvig Lorenz (no t), and different from Henrikh Lorentz (with t), but often confused.

which is just the **sourceless wave equation**. This is analogous to the vacuum electromagnetic wave equations $\Box \mathbf{E} = 0$ and $\Box \mathbf{B} = 0$. If we restore the matter energy-momentum tensor $T_{\mu\nu}$, we get the **Linearized Einstein equation**

$$\boxed{\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}}, \tag{5.75}$$

where the extra factor of -2 on the RHS comes from keeping track of all the redefinitions. You might worry that the energy-momentum tensor $T_{\mu\nu}$ might also depend on gauge conditions – fortunately by *assumption* for objects which vanish in empty space (such as $T_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$), the additional terms we gain from gauge transforming them are always higher order so *the energy momentum tensor is gauge invariant to first order.*

5.3.2 Gravitational Waves

Since Eq. (5.74) is a wave equation, we can try for plane wave solutions

$$\bar{h}_{\mu\nu} = H_{\mu\nu} e^{ik_\rho x^\rho} \tag{5.76}$$

where $H_{\mu\nu}$ is symmetric rank-$(0, 2)$ tensor with complex constant components, describing the **polarization and amplitude** of the wave solution, and k^μ describes the **wave vector** as usual. Like all wave solutions, we only take the real component of Eq. (5.76) although we won't explicitly write it down. Plugging this into the wave equation Eq. (5.74) gets us

$$k^\mu k_\mu = 0 \tag{5.77}$$

which tells us that gravitational wave vectors are null-like, hence just like light, *gravitational waves propagate at the speed of light in a Minkowski spacetime*[8].

The Lorenz gauge condition Eq. (5.73) becomes

$$k_\mu H^{\mu\nu} = 0, \tag{5.78}$$

or, using $\eta_{\sigma\nu}$ to lower the ν index, this is

$$k_\mu H^\mu{}_\sigma = 0 \tag{5.79}$$

which means that the wave is **traverse** to the propagating direction k^μ. Eq. (5.79) is sometimes called the **traverse condition**.

We can plug Eq. (5.79) into Eq. (5.72) ξ_μ to get

$$\Box \xi_\mu = 0 \tag{5.80}$$

which is *also* a wave equation! So it has wave solutions

$$\xi_\mu = X_\mu e^{ik^\mu x_\mu}, \tag{5.81}$$

where X_μ is some constant co-vector with complex components and the k^μ is the *same* as the one in Eq. (5.76), so Eq. (5.77) ensures that it satisfies Eq. (5.80). We *define* k^μ to be a vector that is raised and lowered by the Minkowski metric $\eta_{\mu\nu}$.

We pause a bit here to discuss what it means to have a wave solution for the gauge variable ξ_μ. Remember that gauge choices are secretly coordinate choices, and in differential geometry we are free to choose coordinates at any point on the manifold. You can of course go all hipster-y and custom design your atlas, but you can also be clever and *choose coordinates in response to what the physics*

[8]And like light, it can propagate at a speed slower than c through other media – it is an interesting and largely unexplored area of physics studying the wave properties of gravitational waves in non-vacuum media.

is doing. In this case, Eq. (5.72) basically "choose the coordinates in response to the solution for the metric perturbation $h_{\mu\nu}$". But this does not fully "specify the gauge" – i.e. the manifold is still not fully charted. To completely "fix the gauge", we have to add extra conditions – we are still free to choose X_μ. In fact, we *must* choose X_μ else there will be ambiguity in our final solutions. Such ambiguities are such a dastardly scourge for general relativists that it has a name: **unphysical gauge modes**.

With such foreboding words, let's see how we can totally fix X_μ. It is easier to see how this works if we consider the case of a wave traveling in the z direction, i.e.

$$k^\mu = \omega(1,0,0,1) \tag{5.82}$$

here ω is the frequency of the gravitational wave as we can see from the plane wave mode

$$\bar{h}_{\mu\nu} = H_{\mu\nu}e^{i\omega(-t+z)}. \tag{5.83}$$

The traverse property of $H_{\mu\nu}$ Eq. (5.79) constraints the values of $H_{\mu\nu}$ and k^μ, so in this case

$$H_{\mu 0} = H_{\mu z}. \tag{5.84}$$

Meanwhile, under gauge transformation, $H_{\mu\nu}$ transforms as Eq. (5.71) (and *not* Eq. (5.68)!), so this becomes

$$H_{\mu\nu} \to H_{\mu\nu} + i(k_\mu X_\nu + k_\nu X_\mu - k^\rho X_\rho \eta_{\mu\nu}). \tag{5.85}$$

From Eq. (5.85), we have sufficient freedom in X_μ to impose the **longitudinal condition**

$$H_{0\nu} = 0. \tag{5.86}$$

In other words, we can choose X_μ such that Eq. (5.86) is always obeyed. To see this, expand the $\nu = 0$ component of Eq. (5.86). We start with the $\nu = x, y$ components to get

$$H_{0x} \to H_{0x} + i(k_0 X_x) \ , \ H_{0y} \to H_{0y} + i(k_0 X_y) \ . \tag{5.87}$$

so it's clear that we can choose X_x and X_y such that $H_{0x} = H_{0y} = 0$ – in other words, whatever value of H_{0x} (or H_{0y}), we can choose X_x and X_y to cancel it such that we get $H_{0x} = H_{0y} = 0$. This leaves X_0 and X_z. The $\nu = 0$ component transform as

$$H_{00} \to H_{00} + i(2k_0 X_0 + k^\rho X_\rho) = H_{00} + i(k_0 X_0 + k_z X_z) \tag{5.88}$$

recalling that we raise and lower indices with the Minkowski metric, while the $\nu = z$ component of Eq. (5.86) transforms as

$$H_{0z} \to H_{0z} + i(k_0 X_z + k_z X_0) \ . \tag{5.89}$$

Now the traverse condition Eq. (5.84) tells us that $H_{00} = H_{0z}$ (with $\mu = 0$), and since we know $k_0 = k_z = \omega$ (from Eq. (5.82)), the pair of equations Eq. (5.88) and Eq. (5.89) unfortunately contain the same information, so we can only solve for one of the two unknowns X_0 or X_z. In other words, imposing the longitudinal condition does not fully specify the gauge.

To solve for the final condition, we need to impose the final **traceless condition**

$$H = \eta^{\mu\nu}H_{\mu\nu} = H^\mu{}_\mu = 0. \tag{5.90}$$

which in our case give us

$$H \to H + 2i(k_0 X_0 - k_z X_z). \tag{5.91}$$

So with Eq. (5.91) and Eq. (5.88) (or Eq. (5.89)), we have two different equations for the two unknowns X_0 and X_z and we have completely fixed the gauge and vanquished the evil unphysical gauge modes.

The two conditions, the traverse $k_\mu H^{\mu\nu} = 0$ and the traceless $H = 0$ conditions are known collectively as the **Traceless-Traverse condition** or **Traceless-Traverse gauge**. Of course, these two conditions alone are not sufficient unless we also have already imposed the Lorenz gauge.

With all these sorted out, the traceless-traverse conditions mean that, for a gravitational wave propagating in the z direction, the components of $H_{\mu\nu}$ has the following form

$$H^{\mathrm{TT}}_{\mu\nu} = \begin{pmatrix} 0 & & & \\ & H_+ & H_\times & \\ & H_\times & -H_+ & \\ & & & 0 \end{pmatrix} \tag{5.92}$$

where we $H_\times$ and H_+ are the two polarization modes. Note that $H_{22} = -H_{33}$ by the trace-free condition Eq. (5.90) while $H_{23} = H_{32}$ by the symmetricity of $H_{\mu\nu}$. We have attached the subscript "TT" to indicate that we have chosen the traceless-traverse gauge.

You will show in a (Homework) problem that, in this gauge, the metric perturbation $h_{\mu\nu}$ is actually equal to the trace-reversed version $\bar{h}_{\mu\nu}$, i.e.

$$h^{\mathrm{TT}}_{\mu\nu} = \bar{h}^{\mathrm{TT}}_{\mu\nu}. \tag{5.93}$$

So, as long as we keep to this gauge, we can use the nice weak field equation we have derived for the Riemann tensor in terms of $h_{\mu\nu}$ (Eq. (5.62)) by simply plugging in $H_{\mu\nu}$.

5.3.3 Effects of Gravitational Waves on Test Particles

How do we detect gravitational waves? Since it is gravitational fields after all, our discussion about the physical meaning of the Einstein equation in section 4.5 informs us that the best way to do this is to set up a cloud of test particles and observe how they change shape as gravitational waves pass through them. In fact, since the gravitational waves are *plane wave* solutions, we just need to set up a plane of test particles instead of a cloud.

We now want to figure out how such a plane of particles move *with respect to each other* when interacting with a gravitational wave. We have studied how to do this before – we want to calculate the evolution of the *geodesic deviation vector* for this set of test particle as we did in section 3.3.4.

Let's set up the particles on a plane, and let $\mathbf{S}$ be the spacelike separation vector between the two particles as defined in Eq. (3.107) – see figure 5.5. Since the background metric is Minkowski, it is totally kosher to think of its components S^μ as the actual physical distance between the particles – to see this replace $g_{\mu\nu}$ with $\eta_{\mu\nu}$ in Eq. (3.108). We want to compute the evolution of S^μ, using the geodesic deviation equation Eq. (3.116)

$$T^\alpha \nabla_\alpha (T^\nu \nabla_\nu S^\rho) = R^\rho{}_{\sigma\mu\nu} T^\sigma T^\mu S^\nu \tag{5.94}$$

where T^μ is the timelike geodesic of the individual test particles.

If the particles are initially at rest, then in the Minkowski background, all of them have the velocity vector

$$T^\mu = (1, 0, 0, 0). \tag{5.95}$$

Now as a gravitational wave passes through our test particles, we expect them to get pushed around. But since the metric perturbations are $\mathcal{O}(h)$, then the velocity vector would be roughly $T^\mu = (1 + \mathcal{O}(h), \mathcal{O}(h), \mathcal{O}(h), 0)$ (where $T_z = 0$ still since the wave is traverse to the z direction). The linearized Riemann tensor Eq. (5.62) is also first order in h, so if we want to keep the LHS of Eq. (5.94) in h, we have to make a choice on "which h to ignore". A proper way to do this would be to carefully show that there always exists a basis such that $T^\mu = (1, 0, 0, 0)$, but that would require learning about

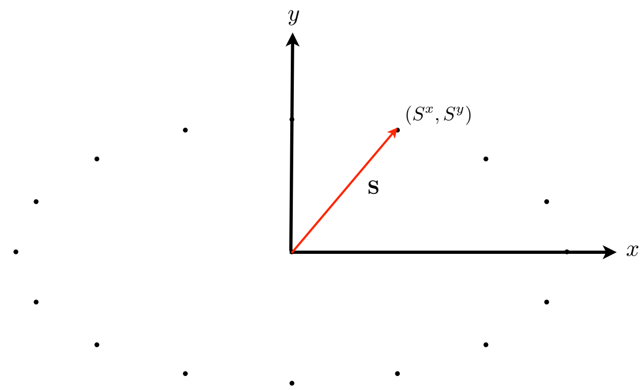


Figure 5.5: The displacement vector **S** indicates the distance between some particle from the origin $(0,0)$ on the $x-y$ plane for a gravitational wave that is propagating in the z direction. Our goal is to compute how **S** changes with times for all the particles.

non-coordinate bases so we will have to make do with the slight cheat by arguing that the norm of $|T| = \sqrt{T^\mu T_\mu} = 1 > \mathcal{O}(h)$ while for the Riemann tensor the leading term is $\mathcal{O}(h)$ so we can ignore the h term and assume $T^\mu = (1,0,0,0)$ throughout.

Given this, then Eq. (5.94) becomes (using the fact that our background is Minkowski so $\nabla_\mu \to \partial_\mu$)

$$\frac{\partial^2 S^\rho}{\partial t^2} = R^\rho{}_{00\nu} S^\nu. \tag{5.96}$$

Now in the traceless-traverse gauge, Eq. (5.93) holds, so we can drop the bars from h and use Eq. (5.62). In this gauge, $h^{\mathrm{TT}}_{\mu 0} = 0$, so the linearized Riemann tensor becomes

$$R^\rho{}_{00\nu} = \frac{1}{2}\frac{\partial^2}{\partial t^2}(h^\rho{}_\nu)^{\mathrm{TT}} \tag{5.97}$$

whereby the geodesic deviation equation becomes the nice form

$$\frac{\partial^2 S^\rho}{\partial t^2} = \frac{1}{2}\frac{\partial^2}{\partial t^2}(h^\rho{}_\nu)^{\mathrm{TT}} S^\nu. \tag{5.98}$$

Using Eq. (5.92), we can write down the evolution equations for for S^x and S^y ($S^t = S^z = 0$ obviously).

Let's consider the "+" mode, where $H_+ \neq 0$ and $H_\times = 0$. The equations are

$$\frac{\partial^2 S^x}{\partial t^2} = \frac{1}{2} S^x \frac{\partial^2}{\partial t^2}(H_+ e^{i\omega(-t+z)}) \tag{5.99}$$

and

$$\frac{\partial^2 S^y}{\partial t^2} = -\frac{1}{2} S^y \frac{\partial^2}{\partial t^2}(H_+ e^{i\omega(-t+z)}). \tag{5.100}$$

The solutions for Eq. (5.99) and Eq. (5.100) are easily found to be

$$S^x_+ = \left(1 + \frac{1}{2}H_+ e^{i\omega(-t+z)}\right) S^x(0)\ ,\ S^y_+ = \left(1 - \frac{1}{2}H_+ e^{i\omega(-t+z)}\right) S^y(0). \tag{5.101}$$

The time evolution of the + mode is shown in figure 5.6.

You will show in a (Homework) problem that for the "$\times$" mode, where $H_\times \neq 0$ and $H_+ = 0$, the solutions are

$$S^x_\times = S^x(0) + \frac{1}{2}H_\times e^{i\omega(-t+z)} S^y(0)\ ,\ S^y_\times = S^y(0) + \frac{1}{2}H_\times e^{i\omega(-t+z)} S^x(0). \tag{5.102}$$

The time evolution of the $\times$ mode is shown in figure 5.7.

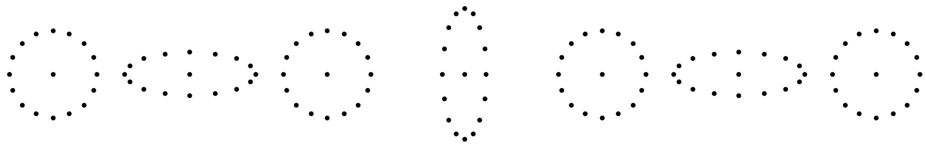


Figure 5.6: The evolution of the + mode gravitational wave. Time moves from left to right in the figure, and the circle's deformation oscillates like a "+" sign.

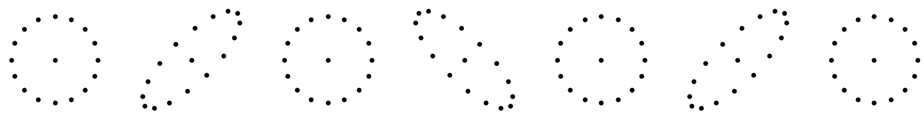


Figure 5.7: The evolution of the × mode gravitational wave. Time moves from left to right in the figure, and the circle's deformation oscillates like a "×" sign.

5.3.4 Generation of Gravitational Waves

In this final section of our lectures, we will discuss how GW are generated. Just like electrodynamics, where electromagnetic waves are generated when charge particles (like electrons) are accelerated, in GR, GW are generated when the matter distribution is accelerated. So we expect the our final solution to look something like

$$h_{\mu\nu} \overset{?}{\propto} \frac{\partial^2}{\partial t^2} \int T_{\mu\nu}(t,x) dV \tag{5.103}$$

where $\int dV$ is the volume integral over some compact mass distibution. As we will see, this is not quite far from the actual answer.

Staying in the linear regime, we begin with the linearized Einstein equation with non-zero energy-momentum tensor Eq. (5.75),

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{5.104}$$

Even though it is a linear equation, Eq. (5.104) is very difficult to solve in general. We can simplify our task while keeping the essential physics by making the following approximations. See figure 5.8.

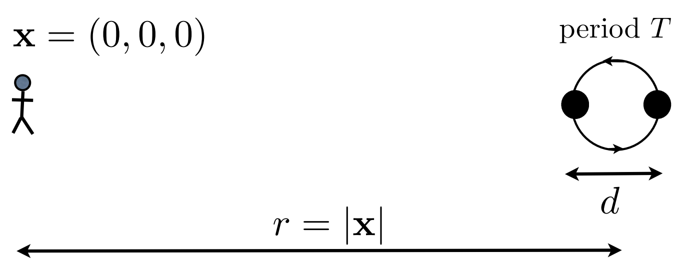


Figure 5.8: GW generation from a far away compact source (here depicted as an orbiting binary with orbital period T and distance d, such that $d/T \ll 1$ (non-relativistic). The origin of the spatial coordinate $(0,0,0)$ is located at the centre of the source. The observer is located at spatial coordinate $\mathbf{x}$, and the source is at $r = |\mathbf{x}| \gg d$ from her.

- Matter distribution occupies only a small region of physical size $d \ll r$, where r is the distance of the GW observer.

- The matter is moving non-relativistically. If T is roughly the time-scale of the matter motion – for example in a binary system T would be the orbital period – then non-relativistic motion means that "light will cross the size of the system much faster than T". Hence if d is the size of the system (e.g. the orbital diameter), so this limit means that $d/T \ll 1$ (in the units where $c = 1$). This means that

$$\frac{\partial T_{\mu\nu}}{\partial t} \sim \mathcal{O}(1/T) T_{\mu\nu}. \tag{5.105}$$

Given these assumptions, we can quickly write down a solution for Eq. (5.104) using the retarded Green's function that you learned in electrodynamics

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4G \int_V \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x})}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \tag{5.106}$$

Let's parse the equation (see figure 5.8). The integral on the RHS is over the volume V where the source has support, using the coordinate $\mathbf{x}'$. The origin $(0, 0, 0)$ of the coordinate system is at the observer far away from the source so $|\mathbf{x} - \mathbf{x}'| \approx |\mathbf{x}| = r$, while the integral only picks up points around the source $|\mathbf{x}'| \sim d$. These approximations allow us to Taylor expand $T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x})$ to get

$$T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}) = T_{\mu\nu}(t - r, \mathbf{x}') + \mathcal{O}(d/T) T_{\mu\nu}(t - r, \mathbf{x}') + \dots \tag{5.107}$$

and since $d/T \ll 1$ we can drop it and obtain the solution

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4G \int_V \frac{T_{\mu\nu}(t - r, \mathbf{x}')}{r} d^3x'. \tag{5.108}$$

From our gauge conditions we fixed in the previous section 5.3.2, we are interested in the spatial components of $\mathbf{h}_{ij}$, i.e.

$$\bar{h}_{ij}(t, \mathbf{x}) = 4G \int_V \frac{T_{ij}(t - r, \mathbf{x}')}{r} d^3x'. \tag{5.109}$$

The integral on the RHS is straightforward to do with a few useful tricks (dropping the primes for simplicity)

$$\begin{aligned} \int_V T_{ij} d^3x = \int_V T^{ij} d^3x &= \int_V d^3x \; \partial_k (T^{ik} x^j) - (\partial_k T^{ik}) x^j \;, \text{ integrate by parts} \\ &= \int_V d^3x \; \cancel{\partial_k (T^{ik} x^j)} - (\partial_k T^{ik}) x^j \;, \text{ drop surface term} \end{aligned} \tag{5.110}$$

Where we have used the fact that we can raise and lower the spatial indices with delta function δ_{ij} since the background metric is Minkowski. Now we use energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ (recalling that our background metric is now Minkowski so we can set $\nabla_\mu \to \partial_\mu$) to get the relation

$$\partial_k T^{ik} + \partial_0 T^{i0} = 0 \tag{5.111}$$

which we can plug into the last line to get

$$\int_V T_{ij} d^3x = \partial_0 \int_V d^3x \; (T^{i0}) x^j. \tag{5.112}$$

We have managed to "convert" one of the spatial index i in T^{ij} into a time derivative. We can convert the other index in a very similar way, which you will do in a (Homework) problem, to get our final answer

$$\int_V T_{ij} d^3x = \frac{1}{2} \partial_0 \partial_0 \int_V d^3x \; T^{00} x^i x^j. \tag{5.113}$$

The quantity

$$\boxed{\int_V d^3x\; T^{00}x^ix^j \equiv I^{ij}} \tag{5.114}$$

is called the **second moment of the energy density**[9]. The linearized Einstein equation Eq. (5.108) (restoring primes) is then

$$\boxed{\bar{h}_{ij}(t,\mathbf{x}) = \frac{2G}{r}\ddot{I}_{ij}(t-r).} \tag{5.115}$$

Eq. (5.115) is known as the **quadrupole formula**. It says that, for a non-relativistic compact source of size d, and observer that is far away $r \gg d$ will see gravitational waves $\bar{h}_{ij}$ at her location according to this formula.

Let's now apply Eq. (5.115) to that of two planets of mass M orbiting each other at distance d from each other. Let the orbit be on the $x-y$ plane, and the "local" time coordinate to be $t' = t - r$, so the energy-momentum tensor is then

$$T^{00}(t',\mathbf{x}') = M\delta(z)\left[\delta\left(x - \frac{d}{2}\cos\Omega t'\right)\delta\left(y - \frac{d}{2}\sin\Omega t'\right) + \delta\left(x + \frac{d}{2}\cos\Omega t'\right)\delta\left(y + \frac{d}{2}\sin\Omega t'\right)\right] \tag{5.116}$$

where the two sets of products of delta functions are the locations of the two planets with respect to the origin $(0,0)$ defined to be the center of mass of the orbit. The delta functions make the integral Eq. (5.114) easy to do (so you get to do them in a (Homework) problem), giving us the final answer

$$\bar{h}_{ij}(t,\mathbf{x}) = \frac{2GM}{r}\Omega^2 d^2 s_{ij} \tag{5.117}$$

where

$$s_{ij} = \begin{pmatrix} -\cos 2\Omega t' & -\sin 2\Omega t' & 0 \\ -\sin 2\Omega t' & \cos 2\Omega t' & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.118}$$

Comparing this to Eq. (5.92), the gravitational wave generated has a wave vector that is *perpendicular* to the plane of orbit (so edge on, we won't see any GW). The other components of the metric $\bar{h}_{\mu\nu}$ can be computed using the gauge conditions, which you can do in the comfort and privacy of your own home.

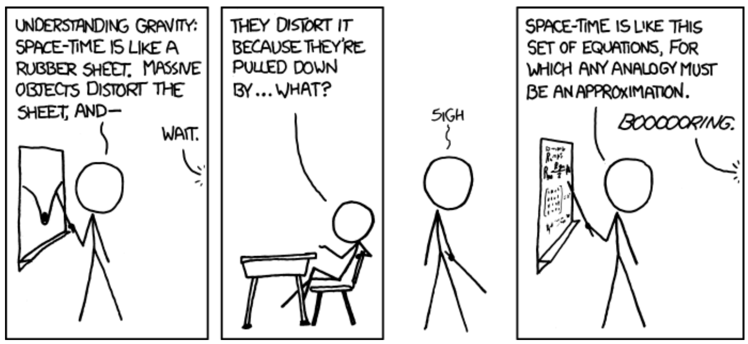
One can further compute the energy loss due to gravitational waves emission, but unfortunately, we have run out of time.

[9] The zeroth moment is $\int T^{00}d^3x$, the first moment is $\int T^{00}x^id^3x$ etc.

Epilogue

So that's the end of our lectures. Obviously, there are plenty of other things to learn in GR. If you are seeking to continue your studies to do a PhD, then I hope that the lectures have given you a good base to make the jump from undergraduate GR into the more hardcore post-graduate GR. If not, I hope that not only have you enjoyed the journey, but you have also learned some interesting facts about how the universe works beyond simply a bunch of equations.

Finally, relevant XKCD (credit Randall Munroe http://www.xkcd.com/).



Appendix A

**Geodesic Equation via Principle of Minimal Action

We will show that we can derive the geodesic equation with the Levi-Civita connection by varying the invariant distance (called the *action*)

$$L = \int ds = \int \sqrt{g_{\mu\nu} V^\mu V^\nu} d\lambda = \int G(x^\mu, \dot{x}^\mu)\ d\lambda \tag{A.1}$$

with respect to the coordinate x^μ, where $G(x^\mu, \dot{x}^\mu) \equiv \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$ is a **functional** of $x^\mu(\lambda)$ and $\dot{x}^\mu(\lambda)$, which (importantly) themselves are functions of λ.

We now want to find a curve $x^\mu(\lambda)$ such that L is minimum between the two points on the manifold, labeled by $\lambda = 0$ and $\lambda = 1$. Such a solution obeys the so-called **Euler-Lagrange equation**

$$\frac{d}{d\lambda}\frac{\partial G}{\partial \dot{x}^\mu} - \frac{\partial G}{\partial x^\mu} = 0. \tag{A.2}$$

Working out the easier second term of the Euler-Lagrange equation first, this is

$$\frac{\partial G}{\partial x^\mu} = \frac{1}{2G}\partial_\mu g_{\sigma\nu}\dot{x}^\sigma \dot{x}^\nu. \tag{A.3}$$

For the first term, we calculate

$$\begin{aligned} \frac{\partial}{\partial \dot{x}^\mu} g_{\sigma\nu}\dot{x}^\sigma \dot{x}^\nu &= g_{\sigma\nu}\dot{x}^\nu \delta^\sigma_\mu + g_{\sigma\nu}\dot{x}^\sigma \delta^\nu_\mu \\ &= 2g_{\mu\nu}\dot{x}^\nu \end{aligned} \tag{A.4}$$

to get

$$\frac{\partial G}{\partial \dot{x}^\mu} = G^{-1} g_{\mu\nu}\dot{x}^\nu. \tag{A.5}$$

To calculate $d/d\lambda \left(G^{-1} g_{\mu\nu}\dot{x}^\nu\right)$, we use a trick by first noting from Eq. (A.1) that $ds = G d\lambda$, or

$$\frac{d}{d\lambda} = G\frac{d}{ds} \tag{A.6}$$

so

$$G^{-1}\dot{x}^\mu = G^{-1}\frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{ds}. \tag{A.7}$$

Then the first term of the Euler-Lagrange equation is

$$\begin{aligned}
\frac{d}{d\lambda}\left(Gg_{\mu\nu}\dot{x}^\nu\right) &= G\frac{d}{ds}\left(g_{\mu\nu}\frac{dx^\nu}{ds}\right) \\
&= G\left(g_{\mu\nu}\frac{d^2x^\nu}{ds^2}+\frac{dx^\nu}{ds}\frac{d}{ds}g_{\mu\nu}\right) \\
&= G\left(g_{\mu\nu}\frac{d^2x^\nu}{ds^2}+\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds}\partial_\sigma g_{\mu\nu}\right) \\
&= G\left(g_{\mu\nu}\frac{d^2x^\nu}{ds^2}+\frac{1}{2}\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds}\partial_\sigma g_{\mu\nu}+\frac{1}{2}\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds}\partial_\nu g_{\mu\sigma}\right),
\end{aligned} \tag{A.8}$$

where we have used $d/ds = (dx^\sigma/ds)\partial_\sigma$ in the 3rd line, and symmetrized in the 4th line.

Meanwhile, the 2nd term of the Euler-Lagrange equation Eq. (A.9) is

$$\frac{\partial G}{\partial x^\mu} = \frac{G}{2}\partial_\mu g_{\sigma\nu}\frac{dx^\sigma}{ds}\frac{dx^\nu}{ds}. \tag{A.9}$$

Putting everything together, the Euler-Lagrange equation is then after cancelling G's.

$$g_{\mu\nu}\frac{d^2x^\nu}{ds^2}+\frac{1}{2}\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds}\partial_\sigma g_{\mu\nu}+\frac{1}{2}\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds}\partial_\nu g_{\mu\sigma}-\frac{1}{2}\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds}\partial_\mu g_{\nu\sigma}=0 \tag{A.10}$$

and multiplying by inverse metric $g^{\mu\rho}$, we recover the geodesic equation Eq. (3.37) as promised

$$\frac{d^2x^\rho}{ds^2}+\Gamma^\rho_{\nu\sigma}\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds}=0, \tag{A.11}$$

where the Christoffel symbols $\Gamma^\rho_{\nu\sigma}$ are given by the Levi-Civita connection Eq. (3.32).

Appendix B

**Lie Derivatives and Isometries

In this appendix, we expand upon some of the more hand-wavy aspects of symmetries we used in the lectures in more precise language. Cool cats read these.

B.1 Lie Derivatives

In section 3.1.1 we introduced the covariant derivative, which tells us how to take a tensor at point $p \in \mathcal{M}$ to a neighbouring point, along some vector field $\mathbf{V}$ which tells us the direction of the motion. But *how* the tensor gets mapped to its neighbouring point (i.e. "moved") is completely up to us to decide – so we can define the connection Eq. (3.3) that tells us exactly how to move the tensor. Although, if further we have a metric on $\mathcal{M}$, there exists the "special" Levi-Civita connection which we use in GR.

However, it will be nice if we can define a "connection-independent" and "metric-independent" notion of taking derivative. One possibility is to, instead of defining the connection to tell us how to move the tensor, we can use the *partial derivative of* $\mathbf{V}$ to tell us how to move the tensor. Such a derivative is called the **Lie Derivative**, and we use the symbol $\mathcal{L}_{\mathbf{V}}$. There is an underlying mathematical structure that allows us to define Lie derivatives, which we will derive in B.3.1 below, but let's ignore that for the moment.

The Lie derivative acting a rank-$(1,0)$ tensor (i.e. a vector) U^μ is given by

$$\boxed{\mathcal{L}_{\mathbf{V}} U^\mu = V^\nu \partial_\nu U^\mu - \partial_\nu V^\mu U^\nu}. \tag{B.1}$$

Comparing this to the covariant derivative of a vector Eq. (3.9) in component form, we see that the connection $\Gamma^\mu_{\nu\sigma}$ has been replaced by the derivative of the vector $\mathbf{V}$, and that the first term is now "dotted" with V^ν. The fact that we can write the covariant derivative in the form Eq. (3.9) without using the underlying vector $\mathbf{V}$ tells us that $\mathbf{V}$ plays no role in the derivative besides telling us where we are taking the derivative towards. But for the Lie derivative, the underlying vector $\mathbf{V}$ plays a crucial role – different vector fields $\mathbf{V}$ will give us different Lie derivatives.

The Lie derivative of a co-vector W_μ is similarly given by

$$\boxed{\mathcal{L}_{\mathbf{V}} W_\mu = V^\nu \partial_\nu W_\mu + \partial_\mu V^\nu W_\nu}. \tag{B.2}$$

The presence of the partial derivatives in Eq. (B.1) and Eq. (B.2) make the equations look non-covariant, but in fact you can show that if you replace $\partial_\mu \to \nabla_\mu$ and use a torsion-free connection (like the Levi-Civita connection), you can show that the equations *are* covariant[1] despite appearances.

[1]**The statement is actually even stronger. Eq. (B.1) and Eq. (B.2), and their generalization Eq. (B.3) holds for *any* derivative operator – see Robert Wald *General Relativity* Appendix C for details.

In general, the Lie derivative of any rank-(p,q) tensor is given by

$$\mathcal{L}_{\mathbf{V}} T^{\mu_1\mu_2\dots}{}_{\nu_1\nu_2\dots} = V^\rho \partial_\rho T^{\mu_1\mu_2\dots}{}_{\nu_1\nu_2\dots} - (\partial_\rho V^{\mu_1}) T^{\rho\mu_2\dots}{}_{\nu_1\nu_2\dots} - \dots + (\partial_{\nu_1} V^\rho) T^{\mu_1\mu_2\dots}{}_{\rho\nu_2\dots} + \dots, \tag{B.3}$$

which is the same as

$$\boxed{\mathcal{L}_{\mathbf{V}} T^{\mu_1\mu_2\dots}{}_{\nu_1\nu_2\dots} = V^\rho \nabla_\rho T^{\mu_1\mu_2\dots}{}_{\nu_1\nu_2\dots} - (\nabla_\rho V^{\mu_1}) T^{\rho\mu_2\dots}{}_{\nu_1\nu_2\dots} - \dots + (\nabla_{\nu_1} V^\rho) T^{\mu_1\mu_2\dots}{}_{\rho\nu_2\dots} + \dots}, \tag{B.4}$$

as you can check for yourself.

The Lie derivative is a linear derivative, so it obeys Leibniz's rule etc.

(Homework) Show that Eq. (B.1) is actually the commutator of the two vector fields $\mathcal{L}_{\mathbf{V}} U^\mu = [\mathbf{V}, \mathbf{U}]^\mu$. This is called the **Lie Bracket**.

B.2 Isometries and Killing Vector Fields

The fact that the Lie derivative doesn't require the metric nor a connection means that we can use it to say things about the metric tensor. In particular, it allows us to precisely define what it means for a metric to be **symmetric**, as follows.

Let $\mathbf{K}$ be a vector field on the manifold $\mathcal{M}$. Suppose now the metric tensor $\bar{\mathbf{g}}$ is *invariant* under the Lie derivative with respect to $\mathbf{K}$,

$$\mathcal{L}_{\mathbf{K}} \bar{\mathbf{g}} = 0. \tag{B.5}$$

Eq. (B.5) says that the metric $\bar{\mathbf{g}}$ is invariant when you move it along the vector field $\mathbf{K}$ – in other words $\mathbf{K}$ defines a *symmetry* of $\bar{\mathbf{g}}$, called an **isometry**. The vector field $\mathbf{K}$ is called a **Killing vector field**.

If the connection is the Levi-Civita connection as it is in GR, we can write Eq. (B.5) in component form using Eq. (B.4)

$$\begin{aligned} \mathcal{L}_{\mathbf{K}} g_{\mu\nu} &= K^\rho \nabla_\rho g_{\mu\nu} + (\nabla_\mu K^\rho) g_{\rho\nu} + (\nabla_\nu K^\rho) g_{\mu\rho} \\ &= \nabla_\mu K_\nu + \nabla_\nu K_\mu, \end{aligned} \tag{B.6}$$

where we have used the fact that the metric is covariantly conserved (or metric compatible) in the 2nd line as we discussed in Page 60. Eq. (B.5) then becomes the **Killing's Equation**

$$\boxed{\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0}. \tag{B.7}$$

You might have seen Eq. (B.7) in some GR literature as a definition for the Killing vector field – though we emphasise that isometries are metric and connection independent concepts, and Eq. (B.7) is only true if we have used the Levi-Civita connection.

For any given metric spacetime, solutions of Eq. (B.7) are isometries of the spacetime. Of course, metrics can possess none, one, or more isometries. An n-dimensional spacetime with the maximum number of isometries is called **maximally symmetric** spacetimes as we discussed in the section 4.4 on Cosmology. It is in general quite hard to solve Eq. (B.7) given a metric, but given any vector field $\mathbf{V}$ it is easy to check whether it is a Killing vector field or not.

Given a Killing vector field, we can quickly compute the conserved quantities associated with this isometry using the equation

$$\boxed{K_\mu \frac{dx^\mu}{d\tau} = \text{constant}} \tag{B.8}$$

so if $p^\mu = m dx^\mu/d\tau$, then

$$K_\mu p^\mu = \text{constant}. \tag{B.9}$$

To make contact with the conserved quantities of the Schwarzschild metric we studied in section 5.1.3, the timelike Killing vector of Schwarzschild is $\mathbf{K} = \partial_t = K^\mu \partial_\mu$ so in component form $K^\mu = (1,0,0,0)$, and plugging this into Eq. (B.9) immediately gets us Eq. (5.13). The three Killing vectors associated with the rotation isometry are (in coordinate basis of Schwarzschild coordinates)

$$\begin{aligned}
R^\mu &= (0,0,0,1) \\
S^\mu &= (0,0,\cos\phi,-\cot\theta\sin\phi) \\
T^\mu &= (0,0,-\sin\phi,-\cot\theta\cos\phi).
\end{aligned} \tag{B.10}$$

B.3 Maps of Manifolds : Diffeomorphisms

This section is really here just here for completion, and is written rather densely. If the appendix is a mathematical justification for the lectures' loose use of "symmetries", this section is the justification for the justification. A formal development of symmetries would have started here. For a more thorough description, please look at S. Carroll *Spacetime and Geometry* Appendix A or R. Wald *G*eneral Relativity Appendix C.

Let $\mathcal{M}$ and $\mathcal{N}$ be two manifolds, then we can define a *smooth* and bijective (i.e. onto and 1-to-1) map ϕ from $\mathcal{M}$ to $\mathcal{N}$,

$$\phi : \mathcal{M} \to \mathcal{N}. \tag{B.11}$$

Such a map is called a **diffeomorphism**. Since ϕ is smooth and bijective, its inverse map $\phi^{-1} : \mathcal{N} \to \mathcal{M}$ is also smooth. See Figure B.1. Diffeomorphisms are often affectionately shortened to "diffs" or "diffeos".

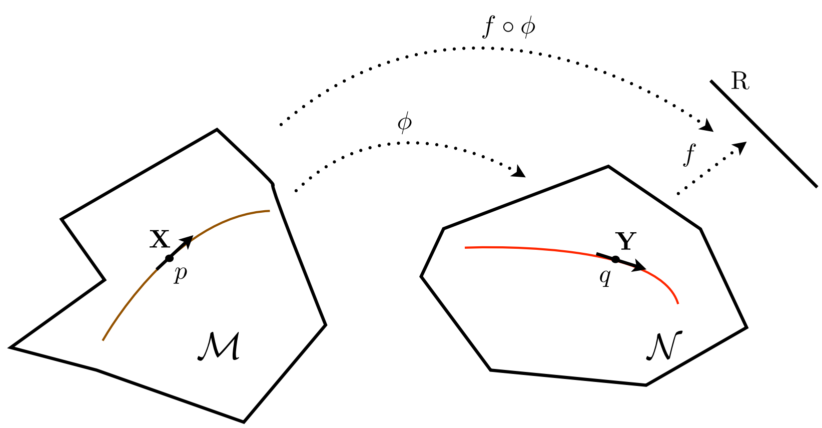


Figure B.1: A diffeomorphism ϕ is a smooth bijective map from $\mathcal{M}$ to $\mathcal{N}$. If f is a function on $\mathcal{N}$ then the composite map $f \circ \phi \equiv \phi^*(f)$ is called the **pullback**, which "pulls" the function from $\mathcal{N}$ to $\mathcal{M}$. Since ϕ is smooth and bijective, it also maps a smooth curve (red line) from $\mathcal{M}$ to $\mathcal{N}$. The tangent vector $\mathbf{X}$ at $p \in \mathcal{M}$ is then mapped by the **pushforward** $\phi_*(\mathbf{X}) = \mathbf{Y}$ into the tangent vector $\mathbf{Y}$ at $\phi(p) = q \in \mathcal{N}$.

Suppose now f is a function that lives on $\mathcal{N}$, hence $f : \mathcal{N} \to \mathbb{R}$. Then from any point $p \in \mathcal{M}$ we can first map it to some other point $q \in \mathcal{N}$ using ϕ (i.e. $\phi : p \in \mathcal{M} \mapsto q \in \mathcal{N}$), and then $f(q)$ then give us some real number. The composite map $f \circ \phi$ is called a "**pullback** of f by ϕ" and written as

$$\phi^*(f) = f \circ \phi, \tag{B.12}$$

since we can think of ϕ^* as "pulling back f from $\mathcal{N}$ to $\mathcal{M}$. Unfortunately, we cannot "push forward" a function g from $\mathcal{M}$ to $\mathcal{N}$ using ϕ (though we can always use ϕ^{-1}) – try it! However, we can define push forward a vector in the following way.

Let $p \in \mathcal{M}$ and $\mathbf{X} \in T_pM$, i.e. a vector from the vector space T_pM. Then $\phi(p) \mapsto q \in \mathcal{N}$, and at this point there is a vector space T_qN. We can then use ϕ define a map ϕ_*, or **pushforward**, that takes $\mathbf{X}$

into a vector in T_qN, i.e.

$$\phi_*(\mathbf{X}) = \mathbf{Y} \in T_qN \text{ s.t. } \mathbf{Y}(f) = \mathbf{X}(\phi^* f). \tag{B.13}$$

Notice the placement of the asterisks – upper $*$ means pullback and lower $*$ means pushforward. Eq. (B.13) while obtuse, actually says something quite simple. Since ϕ maps $\mathcal{M}$ to $\mathcal{N}$, then we can use ϕ to map a curve (say parameterized by λ) to a curve in $\mathcal{N}$. As we have studied in section 2.2 on vectors, tangents of any curve is a vector, so the pushforward ϕ_* is a map from tangents on the curve in $\mathcal{M}$ to the tangents of the "push forwarded" curve in $\mathcal{N}$.

Again, we cannot pullback a vector. But, since co-vectors map vectors to functions, it won't surprise you that you can pullback a co-vector. Say $\bar{\mathbf{U}}$ is a co-vector on $\mathcal{N}$, and $\mathbf{V}$ is a vector on $\mathcal{M}$, we can define the pullback of $\bar{\mathbf{U}}$ to be equal to the pushforward of $\mathbf{V}$, i.e.

$$\phi^*(\bar{\mathbf{U}})(\mathbf{V}) = \bar{\mathbf{U}}(\phi_*\mathbf{V}). \tag{B.14}$$

We all like components, so let's now see how these maps operationally work. Let $x^{\mu'}$ and y^μ be coordinates on $\mathcal{M}$ and $\mathcal{N}$ respectively, then Eq. (B.13) in component form of their respective coordinate bases is

$$\begin{aligned} \mathbf{Y}(f) = \mathbf{Y}^{\mu'}\partial_{\mu'}f = (\phi_*(\mathbf{X}))^{\mu'}\partial_{\mu'}f &\equiv X^\mu\partial_\mu(\phi^*f) \\ &= X^{\mu'}\frac{\partial y^{\mu'}}{\partial x^\mu}\partial_{\mu'}f \end{aligned} \tag{B.15}$$

or using $(\phi_*(\mathbf{X}))^{\mu'} \equiv (\phi_*)^{\mu'}{}_\mu X^\mu$ we get

$$(\phi_*)^{\mu'}{}_\mu = \frac{\partial y^\mu}{\partial x^{\mu'}}. \tag{B.16}$$

Similarly, you can use Eq. (B.14) to show that for $(\phi^*\bar{\mathbf{U}})_\mu = (\phi^*)_\mu{}^{\mu'}U_{\mu'}$

$$(\phi^*)_\mu{}^{\mu'} = \frac{\partial x^\mu}{\partial y^{\mu'}}. \tag{B.17}$$

Hence, both Eq. (B.16) and Eq. (B.17) are simply coordinate transforms once we lay down coordinates on $\mathcal{M}$ and $\mathcal{N}$. In particular, when we pushforward a general tensor $T^{\mu_1\cdots}{}_{\nu_1\ldots}$ at p with coordinates x^μ to another point $q = \phi(p)$ with coordinates $y^{\mu'}$ this is

$$(\phi_*T)^{\mu_1'\cdots}{}_{\nu_1'\cdots}\Big|_{q=\phi(p)} = \left[\left(\frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\right) \times \cdots \times \left(\frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}}\right) \times \ldots\right]_p (T^{\mu_1\cdots}{}_{\nu_1\ldots})_p\,. \tag{B.18}$$

The fact that diffeomorphisms can be cast as coordinate transformations mean something both profound and prosaic when we consider physics: *Manifolds which are related by diffeomorphisms describe exactly the same physics.*

The prosaic bit of the statement is simply that "physics is invariant under coordinate transforms", and diffeomorphisms are just a super-highbrow word for it. However, the profound bit of the statement is that since GR is described by the dynamics of tensor fields living on a differential manifolds, the fact that there exist diffeomorphisms mean that *there is a redundancy in the description of physics.* In words, there exist a complete set of solutions from GR which describe exactly the same physics. You have seen something like this before – in Electrodynamics, there exist a redundancy in its description by Maxwell equations by, i.e. solutions to the equations with the vector potential $\mathbf{A} \to \mathbf{A} + \nabla\alpha$ is completely equivalent. Such redundancy in the description of physics is called a **Gauge Symmetry**.

B.3.1 Lie Derivative via Diffeomorphisms

In general, diffeomorphisms allow us to map between two different manifolds. But, they can also be used to map a manifold onto itself $\phi : \mathcal{M} \to \mathcal{M}$. While in general, they can be any kind of map you can think of, there is very special kind of map which *maps a point on a curve $\mathcal{C}$ to another point on another curve*, as follows.

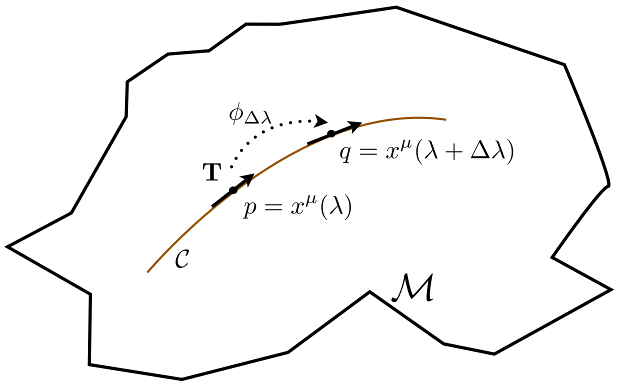


Figure B.2: A diffeomorphism ϕ is a smooth bijective map from $\mathcal{M}$ to itself, along the curve $\mathcal{C}$. This diffeomorphism allows us to move a tensor from q to p, so that we can compare them, i.e. do a derivative. This derivative is called a **Lie Derivative** along the curve.

Let's lay down coordinates x^μ on $\mathcal{M}$, and consider a curve $x^\mu(\lambda)$ parameterized by λ on $\mathcal{M}$. Let the diffeomorphism $\phi_{\Delta\lambda}$ be the map which takes any point $p = x^\mu(\lambda)$ to another point $q = x^\mu(\lambda + \Delta\lambda)$ on the same curv. We can now pushforward any vector $\mathbf{T}$ from p to the point q by using $(\phi_{\Delta\lambda})_*(\mathbf{T}) \in T_q M$. We can also pushforward any vector from q to p by using the inverse $\phi_{-\Delta\lambda}$.

We can then define the **Lie Derivative of T at** p along this curve by pushforwarding a vector from q back to p and then subtracting it with the vector at p, i.e.

$$\mathcal{L}_{\mathbf{V}}\mathbf{T}\big|_p = \lim_{\Delta\lambda\to 0} \frac{(\phi_{-\Delta\lambda})_*(\mathbf{T}) - \mathbf{T}}{\Delta\lambda}\bigg|_p, \tag{B.19}$$

where $\mathbf{V}$ is simply the tangent vector of the curve at p,

$$\mathbf{V}_p \equiv \frac{d}{d\lambda}\bigg|_p = \frac{\partial x^\mu}{\partial\lambda}\bigg|_p \partial_\mu. \tag{B.20}$$

Since we can pushforward any tensor using ϕ_*, Eq. (B.20) is also the definition for the Lie derivative of any rank-(p, q) tensor $\mathbf{T}$.

To obtain the final formula Eq. (B.3) in general for all coordinate systems is a lot of algebra (for the details see R. Wald *General Relativity* Appendix C). The general formula is

$$\mathcal{L}_{\mathbf{V}} T^{\mu\cdots}{}_{\nu\ldots}\big|_p = \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[((\phi_{-\Delta t})_* T)^{\mu\cdots}{}_{\nu\ldots} - T^{\mu\cdots}{}_{\nu\ldots}\right]_p \tag{B.21}$$

where the first term of the RHS is given by the Eq. (B.18).

We cheat a little here by doing it for a specific coordinate system, where the curve $\mathcal{C}$ is parameterized by the time coordinate t and hence the tangent vector $\mathbf{V} = \partial_t$. Hence what $\phi_{\Delta t}$ does is to send a point with $x^\mu = (t, x, y, z)$ to a point q with coordinates $y^\mu = (t + \Delta t, x, y, z)$. Then using the component form of the pushforward map Eq. (B.17), we have

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu \tag{B.22}$$

where, remember that we have used $(\phi_{-\Delta t})_*$ not $(\phi_{\Delta t})_*$ so the y and x is swapped. This means that using Eq. (B.18), we have

$$((\phi_{-\Delta t})_* T)^{\mu\cdots}{}_{\nu\ldots}|_p = T^{\mu\cdots}{}_{\nu\ldots}|_{q=\phi(p)} \,. \tag{B.23}$$

So in *this coordinate system* only, the Lie derivative Eq. (B.20) becomes the simple

$$\begin{aligned}\mathcal{L}_{\mathbf{V}} T^{\mu\cdots}{}_{\nu\ldots}|_p &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[T^{\mu\cdots}{}_{\nu\ldots}(t+\Delta t, x^i) - T^{\mu\cdots}{}_{\nu\ldots}(t, x^i)\right] \\ &= \frac{\partial}{\partial t} T^{\mu\cdots}{}_{\nu\ldots}(t, x^i)\end{aligned} \tag{B.24}$$

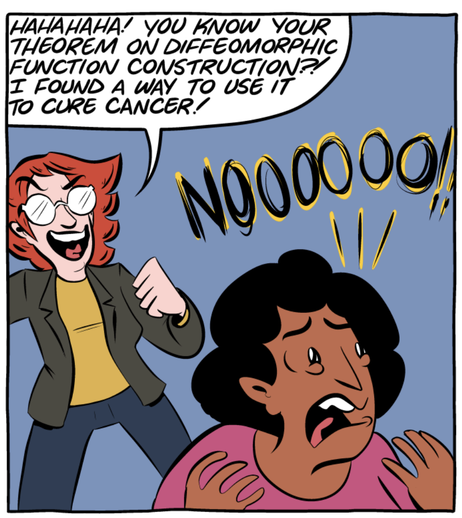
i.e. it's the partial derivative w.r.t. t. Comparing this to Eq. (B.3), this Eq. (B.24) only possess the first term on the RHS (since the rest vanish by coordinate choice and Eq. (B.22)). The important point here is that, since in one particular coordinate system the Lie derivative is simply a partial derivative, it must obey Leibniz's Rule in *all* other coordinate systems.

Eq. (B.20) can be constructed for *any* given vector *field* $\mathbf{V}$ (not just a curve).

B.3.2 Active and Passive Forms

Notice the form of the Lie derivative looks a lot like coordinate transformations – this is the reason why isometries (i.e. Killing vector fields) where we discussed in the section on Cosmology section 4.4 can be seen as a symmetry in terms of coordinate transforms. In these lectures, we have consistently taken the "active" viewpoint, where isometries means "invariance of the metric when physically moving from point p to point q". There is an equivalent "passive" viewpoint, where isometries are cast as a symmetry of the coordinate transforms at point p. Mathematically, what this means is that when we define the Lie derivative, instead of pushing forward the vector $\mathbf{X}$, we *pullback the coordinate system* (which are simply functions). The two pictures are equivalent, but personally I like to push things around. (See discussion in H. Reall's lecture notes (http://www.damtp.cam.ac.uk/user/hsr1000/lecturenotes_2012.pdf Chapter 7.2).

Finally, there is one last super technical bit. We have defined a covariant derivative in section 3.1.1. We need a way to "pushforward" the covariant derivative – again for details see H. Reall's lecture notes above.



Funtime Activity:
Forcibly converting pure mathematicians
into applied mathematicians.

Figure B.3: Credit : SMBC comic http://www.smbc-comics.com/