# Project : Solitons and Vortices

Lecturer: Dr. Eugene A. Lim
2017 Year 3 Semester 2
Helper Set 3

(**Convergence Testing**). Suppose $u_e$ is the exact "true" result, and $u_h$ is a numerical result of some numerical method with spatial discretization of $h$. Then, in general, the difference between $u$ and $u_h$ can be written as

$$|u_h - u_e| \leq Ch^p \tag{1}$$

where $C$ is some $h$-independent number and $h^p$ is called the *convergence rate*. $p$ is then the *order* of your convergence.

1. Suppose you know the exact solution $u_e$, and want to determine the convergence rate of your numerical method. Show that

$$\frac{u_h - u_e}{u_{h/2} - u_e} = 2^p + \mathcal{O}(h) \tag{2}$$

and hence

$$\log_2 \left| \frac{u_h - u_e}{u_{h/2} - u_e} \right| = p + \mathcal{O}(h). \tag{3}$$

In other words, to check the convergence order of your code, run the simulations at spatial discretizations $h$ an $h/2$, and plot the Eq. (3) as a function of time. For $u$, pick a (physically important) point $x$ (i.e. don't pick a point where nothing happens in the simulations. Note that you must pick the *same* physical point for both $u_h$ and $u_{h/2}$. Compare the convergence order of your code to the (purported) order of your numerical scheme. How accurate is your code?

2. Instead of picking a point $x$ like you did in Q1 to compute $u_h$, you can also compute the *norm*

$$L_2(u_h) \equiv \sqrt{\int |u_h - u_e|^2 dx}. \tag{4}$$

This is the global error. Calculate the global error for your code, and compare it the order of the numerical scheme. The global convergence can sometimes be more useful if the physics of your simulations is spreaded over a bigger area. Compare the global error for simulations with $h$ and $h/2$. Compute also the global convergence order

$$\log_2 \left| \frac{L_2(u_h)}{L_2(u_{h/2})} \right| \tag{5}$$

and compare the results with the order of your numerical method.

3. In general however, we don't know the exact solution $u_e$. Nevetheless, we can still estimate the convergence order of your numerical scheme by comparing the results from *three* different resolutions: $u_h$, $u_{h/2}$ and $u_{h/4}$ in the following way. Show that

$$\frac{|u_h - u_{h/2}|}{|u_{h/2} - u_{h/4}|} = 2^{-p} + \mathcal{O}(h). \tag{6}$$

Pick the point of collision of your kink-antikink simulation, and compute the convergence $p$ as a function of time of your code using Eq. (6).