Project : Solitons and Vortices

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Helper Set 1

Recommended reference : *Numerical Recipes* (Flannery, Teukolsky, Press, Vetterling), .*Classical Solutions to Quantum Field Theory* (Erick Weinberg)

The equation you want to solve is the Klein-Gordon equation

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + \frac{dV(\phi)}{d\phi} = 0 \tag{1}$$

where the scalar field $\phi(t,\mathbf{x})$ is a function of both space $\mathbf{x}$ and time t, and $V(\phi)$ is the potential.

1. Solve analytically the "free-field" equation for $V(\phi) = 0$, in 1 space x and 1 time t dimensions in a box $0 < x \leq L$. Find the definite solutions for the following cases

- Initial conditions $\phi(x,t=0) = 0$, $\dot{\phi}(x,t=0) = 1$ and Dirichlet Boundary conditions $\phi(L,t) = \phi(0,t) = 0$.
- Initial conditions $\phi(x,t=0) = 1$, $\dot{\phi}(x,t=0) = 0$ and periodic boundary conditions $\phi(L,t) = \phi(0,t)$, and $\dot{\phi}(L,t) = \dot{\phi}(0,t)$.

Repeat the above for the "massive" KG equation

$$V = \frac{1}{2}m^2\phi^2 \tag{2}$$

where the mass $m > 0$.

Key words : Klein-Gordon equation, Boundary conditions, Initial conditions, Dirichlet B.C.

2. Consider the following potential

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{3}$$

where $\lambda > 0$ and $v > 0$. Plot this potential, and show that the minima of this potential is at $\phi_{min} = \pm v$.

Consider *static* solutions, where $\dot{\phi} = \ddot{\phi} = 0$. The KG in this limit is then

$$\frac{\partial^2\phi}{\partial x^2} - \lambda(\phi^2 - v^2)\phi = 0. \tag{4}$$

- Show that the following constant solutions $\phi(x,t) = \pm v$ are solutions to the static equation Eq. (4). These are known as *vacuum* solutions. What are the appropriate boundary conditions for this case?
- Show that the following is also a solution to Eq. (4)

$$\phi^+(x) = v\tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right] \tag{5}$$

where $m = \sqrt{\lambda}v$ and x_0 is some constant. This is known as the *kink* solution. Again, what is the appropriate boundary conditions?

- Show that Eq. (4) is invariant under the *parity* transform $x \to -x$, and hence argue that the *anti-kink*

$$\phi^-(x) = v\tanh\left[\frac{m}{\sqrt{2}}(-x + x_0)\right] \tag{6}$$

is also a solution. As you should be familiar now : what is the appropriate boundary condition for this case?

- Plot the vacuum, kink and anti-kink solutions.

The energy density of such solutions is defined by

$$\rho(x,t) \equiv \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi). \tag{7}$$

Plot the energy densities for the vacuum, kink and anti-kink solutions. Can you see why the vacuum solutions are named as such?

Key words : Kink solutions.

3. Read about Finite Differencing methods. Some suggested references : Wikipedia page on Finite Difference, *Numerical Recipes* Chap 5.

Key words : Forward/Backward differencing, Stencils, Stability, Courant Condition, Runge-Kutta Methods

4. Write a Finite-difference code to numerical solve the problem you considered in Q1 of this problem set. Use Runge-Kutta 2nd order evolution scheme. Plot out the result, and compare it with your analytical solutions.

5. Using your code above, check that the static solutions you found in Q2 are *static*.