A very personal introduction to the equivariant Tamagawa number conjecture

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Abstract This is a strange document. It is basically the kind of document I wished I had had at my hands when I first started to read about the Rubin-Stark conjecture and, eventually, the equivariant Tamagawa number conjecture (eTNC) – a mainly motivating guide to these topics focusing on the underlying ideas. This is therefore my, certainly insufficient, attempt on showing how the subject fits together.

1 Introduction

Prologue One of the horror stories mathematicians tell each other about physicists is that those seriously believe in the validity of the ridiculous equation

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}. \tag{1}$$

It is in fact true, that (1) is used in Physics as a regularisation in the mathematical description of the Casimir effect but I daresay that no serious physicist actually takes that equation at face value. Lurking in the background is namely the Riemann ζ-function which, for a complex variable s of real part greater than 1, is given by the expression

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{2}$$

and admits a holomorphic continuation to $\mathbb{C} \setminus \{1\}$. As a matter of fact, this continuation satisfies $\zeta(-1) = \frac{-1}{12}$ and if you put on physicist's hat, you therefore arrive at (1) by illegally evaluating (2) at $s = -1$. Moral of the story is that $\zeta(-1)$ shows up in physics. The Riemann ζ-function in turn is the easiest example of an L-function.

The rise of L-functions in mathematics It is always very important for me to see how questions in current mathematical research relate back to classical questions. We shall therefore undertake a quick and terribly one-sided historical tour illustrating why we care about special values of L-functions. For a more extensive treatment of the historic

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background the reader may wish to consult the excellent introduction of [Hof18].

The success story of L-functions in mathematics begins with the following Theorem of Dirichlet on primes in arithmetic progressions:

1.1 Theorem (Prime Number Theorem, 1837)**.** Let a and m two coprime integers. Then there are infinitely many prime numbers p such that

$$p \equiv a \mod m.$$

For the proof of this theorem Dirichlet invented an analytic gadget we nowadays call a ***Dirichlet L-function***[1]. For a complex-valued Dirichlet character χ mod m this is the analytic function defined by

$$L(\chi, s) = \sum_{n=0}^{\infty} \frac{\chi(n)}{n^s} \quad \text{for Re } s > 1$$

and holomorphically continued to $\mathbb{C}$ if $\chi \neq 1$. For $\chi = 1$ we recover the Riemann ζ-function discussed above. Dirichlet showed in [Dir37] that $L(\chi, 1) \neq 0$ if $\chi \neq 1$ and then deduced Theorem 1.1 from this.

A similar story can be told about the proof of the famous

1.2 Theorem (Gauss Conjecture)**.** Let $d > 0$ be a square-free integer. The field $\mathbb{Q}(\sqrt{-d})$ has class number one if and only if $d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$.

A first proof of this theorem by Heegner [Hee52] was incomplete, the first accepted proofs were due to Baker [Bak67] and, independently around the same time, Stark [Sta67]. The crucial idea in Stark's proof is the evaluation of a certain L-function (associated to a quadratic form) at $s = 1$ in two ways. Stark therefore wondered if it is possible to evaluate a general Artin L-series at $s = 1$ and subsequently developed a conjecture concerning these in a series of seminal papers ([Sta71], [Sta75], [Sta76] and [Sta80]) – Stark's conjectures were born (see later). He also realised that easier, but equivalent by the functional equation, is determining the leading term at $s = 0$ and so Stark's conjecture is now usually stated as a conjecture concerning the leading term at $s = 0$.[2]

Summarising the above, we have seen that being able to determine special values of L-functions can lead to proofs of classical results. There is however also interest in these values themselves from a theoretical point of view, namely their deep connection to arithmetic. A good starting point for this investigation is the analytic class number formula which we shall scrutinise in the next section.

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[1] The name stems from the fact that Dirichlet originally used the letter L to denote these functions. An interesting discussion about why he might have chosen that letter can be found at: `https://mathoverflow.net/questions/21375/why-are-they-called-l-functions`

[2] For a detailed account on Stark's motivation see [Sta11].

Notation If $L|K$ is a field extension and S a set of places of K, we will denote by S_L the set of places of L lying over the places in S. In order to lighten the notation, we will however drop the subscript L when no confusion can arise.
For a $\mathbb{Z}$-module A and R a ring, we will furthermore shorten the notation $R \otimes_{\mathbb{Z}} A$ to $R \cdot A$ or even RA.

Simplification In this introduction we will only be concerned with the abelian versions of all mentioned conjectures. By Stark's conjecture, for instance, we will therefore always mean the Stark's abelian conjecture. Similarly, we shall only consider the relevant special case of the eTNC for finite abelian extensions of number fields.

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2 The analytic class number formula

Let K be a number field and $\mathcal{O}_K$ its ring of integers. For the purpose of this section we denote by r the rank of its unit group $\mathcal{O}_K^\times$ and by $S_\infty = \{v_0, \dots, v_r\}$ the set of infinite places of K. The natural generalisation of (2) is the ***Dedekind ζ-function***, which is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \mathrm{N}\mathfrak{a}^{-s} \qquad \text{for } \mathrm{Re}\, s > 1,$$

where the sum ranges over all non-zero ideals $\mathfrak{a} \subseteq \mathcal{O}_K$. This function can be holomorphically continued to $\mathbb{C} \setminus \{1\}$ and the analytic class formula describes the residue at 1 or, equivalent by the functional equation it satisfies, the leading term at 0. In order to be able to state the analytic class number formula, we recall the following classical

2.1 Definition. Let $\varepsilon_1, \dots, \varepsilon_r$ be a $\mathbb{Z}$-basis of the free part $(\mathcal{O}_K^\times)_{\mathrm{tf}}$ of $\mathcal{O}_K^\times$. The ***regulator*** R_K is the modulus of any $(r \times r)$-minor of the $(r \times (r+1))$-matrix

$$\begin{pmatrix} \log|\varepsilon_1|_{v_0} & \dots & \log|\varepsilon_r|_{v_0} \\ \vdots & & \vdots \\ \log|\varepsilon_1|_{v_r} & \dots & \log|\varepsilon_r|_{v_r} \end{pmatrix}.$$

If $K = \mathbb{Q}$ or K is an imaginary quadratic field, we set $R_K = 1$.

Although the regulator does not depend on the choice of fundamental units or $(r \times r)$-minor (because of the product formula), the necessity of such a choice does nevertheless seem unsatisfactory. As a fellow PhD student was taught in his undergraduate: "Every time you make a unnecessary choice, an angel dies." Slightly less dramatic one could say that this shows there is something else going on beneath the surface we are not able to see in our current formulation. In order to arrive at a better interpretation of the regulator, we will now first introduce a bit more notation.

Let $Y_{K,S_\infty} = \bigoplus_{v \in S_\infty} v\mathbb{Z}$ be the free abelian group on S_∞. We define the subgroup $X_{K,S_\infty} \subseteq Y_{K,S_\infty}$ by means of the exact sequence

$$0 \longrightarrow X_{K,S_\infty} \longrightarrow Y_{K,S_\infty} \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

where deg denotes the augmentation map $\sum_{v\in S_\infty} a_v v \mapsto \sum_{v\in S_\infty} a_v$. The ***Dirichlet regulator map*** is now defined as

$$\lambda_{K,S_\infty}\colon \mathcal{O}_K^\times \to \mathbb{R}Y_{S_\infty}, \quad a \mapsto -\sum_{v\in S_\infty} \log|a|_v \cdot v.$$

Due to the product formula the image of λ_{K,S_∞} is in fact contained in $\mathbb{R}X_{K,S_\infty}$: Let $a \in \mathcal{O}_K^\times$ and pick any $v_0 \in S_\infty$, then

$$\log|a|_{v_0} = -\sum_{v\neq v_0} \log|a|_v \quad \Rightarrow \quad \lambda_{K,S_\infty}(a) = -\sum_{v\neq v_0} \log|a|_v(v - v_0).$$

This also shows that the regulator R_K is exactly the determinant of $\lambda_{K,S_\infty}\colon \mathbb{R}\mathcal{O}_K^\times \to \mathbb{R}X_{K,S_\infty}$ and the choice of minor appearing in definition 2.1 corresponds to a choice of basis for X_{K,S_∞}. We could therefore give an alternative definition using the connection between exterior powers and determinants. The reader needing a refreshment of the basic theory of exterior powers is kindly refered to the the excellent expositions [Conb] and [Cona].

2.2 Definition. The ***regulator*** R_K of K is the unique positive real number such that the induced map

$$\lambda_{K,S_\infty}\colon \mathbb{R}\bigwedge_{\mathbb{Z}}^r \mathcal{O}_K^\times \to \mathbb{R}\bigwedge_{\mathbb{Z}}^r X_{S_\infty}$$

on top exterior powers satisifies $\lambda_{K,S_\infty}(\bigwedge_{\mathbb{Z}}^r(\mathcal{O}_K^\times)_{\mathrm{tf}}) = R_K \cdot \bigwedge_{\mathbb{Z}}^r X_{S_\infty}$.

In particular, if $\varepsilon_1,\dots,\varepsilon_r \in (\mathcal{O}_K^\times)_{\mathrm{tf}}$ and $w_1,\dots,w_r \in X_{K,S_\infty}$ denote $\mathbb{Z}$-bases, then

$$\lambda_{S_\infty}(\varepsilon_1 \wedge \dots \wedge \varepsilon_r) = R_K \cdot (w_1 \wedge \dots \wedge w_r).$$

2.3 Remark. The Dirichlet unit theorem states that the regulator map $\lambda_{K,S_\infty}\colon \mathbb{R}\mathcal{O}_K^\times \to \mathbb{R}X_{K,S_\infty}$ is an isomorphism and so $(\mathcal{O}_K^\times)_{\mathrm{tf}}$ is a lattice in $\mathbb{R}^r$. The fundamental domain of this lattice has volume $\sqrt{r+1}\cdot R_K$ which means that the regulator R_K can be thought of as a “density” of units. The smaller the regulator is, the “more” units does K have. The regulator is therefore an important arithmetic invariant.

2.4 Theorem (analytic class number formula)**.** The Dedekind zeta function ζ_K vanishes to order r at $s = 0$ with leading term

$$\tfrac{1}{r!}\zeta_K^{(r)}(0) = -\frac{h_K}{w_K} R_K,$$

where h_K denotes the class number of K and $w_K = |\mu(K)|$ the number of roots of unity contained in K.

In our alternative interpretation of the regulator the analytic class number formula then becomes the equality

$$\tfrac{1}{r!}\zeta_K^{(r)}(0) \cdot \bigwedge_{\mathbb{Z}}^r X_{K,S_\infty} = \tfrac{h_K}{w_K} \cdot \lambda_{K,S_\infty}\left(\bigwedge_{\mathbb{Z}}^r (\mathcal{O}_K^\times)_{\mathrm{tf}}\right). \tag{3}$$

2.5 Remark. The analytical class formula connects two very different realms: The Dedekind ζ-function admits a decomposition called an ***Euler product***, i. e.

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - \mathrm{N}\mathfrak{p}^{-s})^{-1} \quad \text{for } \operatorname{Re} s > 1,$$

where $\mathfrak{p}$ ranges over all prime ideals of $\mathcal{O}_K$. On the left hand side of (3) we therefore have the primarily analytical object $\frac{1}{r!}\zeta_K^{(r)}$ which is computed from purely local data, namely the Euler factors at the primes of K, and on the right hand side consists of arithmetic invariants of K that are intrinsically global. This is just one of many reasons for the significance of the analytical class number formula.

3 Stark's conjecture

We now move from the Dedekind zeta function to the next easiest case of an Artin L-function, namely the Dirichlet L-function of a finite abelian extension $L|K$. Let G be the Galois group of $L|K$ and $\chi\colon G \to \mathbb{C}^\times$ a character. Then we set

$$L(\chi, s) = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1} \quad \text{for } \operatorname{Re} s > 1, \tag{4}$$

where the product ranges over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ and

$$\chi(\mathfrak{p}) = \begin{cases} \chi(\mathrm{Frob}_{\mathfrak{p}}) & \text{if } \mathfrak{p} \text{ is unramified in } L|K, \\ 0 & \text{if } \mathfrak{p} \text{ ramifies in } L|K. \end{cases} \tag{5}$$

This L-function can be meromorphically continued to $\mathbb{C}$ and satisfies a functional equation. The functional equation is best described in terms of the completed L-function that is obtained by adding a bunch of "fudge" factors. Tate in his thesis [Tat67] gave a conceptual proof of the functional equation that includes a interpretation of these fudge factors as Euler factors at infinity. One should therefore view (4) as a completed L-function the Euler factors at the infinite places of which have been removed. Since there is no reason why we should the infinite places differently than the finite places, we shall also allow the removal of Euler factors at finite primes. Observe that our definition (5) exactly amounts to the removal of the Euler factors at the ramified primes.

Hypotheses. Let S be a finite set of places of K satisfying the following hypotheses:

- **(H1)** S contains S_∞,
- **(H2)** S contains the places ramifying in $L|K$.

3.1 Definition. The ***S-imprimitive L-function*** is given by

$$L_S(\chi, s) = \prod_{\mathfrak{p} \notin S} (1 - \chi(\mathrm{Frob}_{\mathfrak{p}})\mathrm{N}\mathfrak{p}^{-s})^{-1} \quad \text{for } \operatorname{Re} s > 1.$$

This expression converges for $\operatorname{Re}(s) > 1$ and can be holomorphically continued to the whole complex plane $\mathbb{C}$ if $\chi \neq 1$. In the case of $\chi = 1$, the S-Dedekind ζ-function $\zeta_{K,S}(s) = L_S(1, s)$ admits a holomorphic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$. We also have an analogous analytic class number formula.

3.2 Theorem (S-imprimitive analytic class number formula)**.** The S-imprimitive Dedekind ζ-function $\zeta_{K,S}$ vanishes to order $|S|-1$ at $s=0$ with leading term

$$\zeta^*_{K,S}(0) = -\frac{h_{K,S}}{w_K} R_{K,S},$$

where $h_{K,S}$ denotes the S-class number of K and the S-regulator $R_{K,S}$ can be defined as the obvious generalisation of Definition 2.2.

3.3 Remark. Recall that the group $\mathcal{O}^\times_{K,S}$ of S-units of K is of rank $|S|-1$ and that the prime ideals of $\mathcal{O}_{K,S}$ correspond to the prime ideals of $\mathcal{O}_K$ not contained in S. The S-imprimitive ζ-function $\zeta_{K,S}$ can therefore be viewed as the zeta function associated to $\mathcal{O}_{K,S}$. For a non-trivial character χ, in turn, the S-imprimitive L-function should be viewed as associated to the "χ-component" of $\mathcal{O}_{L,S}$ (see Proposition 3.4).

The first step towards determining the leading term of $L_S(\chi,s)$ at $s=0$ is the determination of the order of vanishing at $s=0$. Luckily, this number is well-understood.

3.4 Proposition. Let $r_S(\chi)$ denote the order of vanishing of $L_S(\chi,s)$ at $s=0$. Then

$$r_S(\chi) = \begin{cases} |S|-1 & \text{if } \chi = 1, \\ |\{v \in S \mid \chi(G_v) = 1\}| & \text{if } \chi \neq 1, \end{cases}$$

where $G_v \subseteq G_K$ denotes a choice of decomposition group at v. If S_L denotes the set of places of L lying above the places in S and $\mathcal{O}^\times_{L,S}$ the group of S_L-units of L, then one also has that

$$r_S(\chi) = \dim_{\mathbb{C}} e_\chi \mathbb{C}\mathcal{O}^\times_{L,S}, \tag{6}$$

where $e_\chi = \frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1}$ is the usual idempotent (see [Tat84, Proposition I.3.4]).

The next stage on our journey to a (conjectural) description of leading terms is therefore the investigation of the derivatives of $L_S(\chi,s)$ at $s=0$ which we will start by examining the first derivative.

Suppose that S contains at least one place that splits completely in $L|K$ and that $|S| \geq 2$. These conditions ensure that the L-function has a zero of order at least 1 at $s=0$.

3.5 Conjecture (Stark)**.** Let $v_0 \in S$ be a place that splits completely in $L|K$ and fix a place w_0 of L lying above v_0. There exists an S-unit $\varepsilon \in \mathcal{O}^\times_{L,S}$ such that

$$L'_S(\chi,0) = -\tfrac{1}{w_L}\sum_{\sigma\in G}\chi(\sigma)\log|\sigma\varepsilon|_{w_0}.$$

3.6 Remark. Stark's principal conjecture (as Tate calls it in [Tat84]) gives a description of the leading term of $L(\chi,s)$ at $s=0$ up to a rational factor. In turn, Conjecture 3.5 gives a precise formula for $L'(\chi,0)$ and hence of the leading term in the case that $L(\chi,s)$ vanishes to order 1 at $s=0$. Thus, Conjecture 3.5 can be considered an integral refinement of Stark's principal conjecture. It was proposed by Stark in [Sta80].

3.7 Example. Let us consider the case where $K = \mathbb{Q}$, $L = K$ is a real quadratic number field and $S = S_\infty \cup S_{\mathrm{ram}}$. Let $\chi \in \widehat{G}$ be the non-trivial character. Since K is totally real, the infinite place of $\mathbb{Q}$ is split in $K|\mathbb{Q}$ and so $L_S(\chi, s)$ vanishes to order 1 at $s = 0$ by Proposition 3.4. The equation

$$\zeta_L(s) = \zeta_{\mathbb{Q}}(s) \cdot L_S(\chi, s) \quad \text{ for all } s \in \mathbb{C} \setminus \{1\}$$

therefore implies that

$$\zeta_L^*(0) = \zeta_{\mathbb{Q}}^*(0) \cdot L_S'(\chi, 0)$$

by comparing leading coefficients. Combining this with the analytic class number formula, we get that

$$L_S'(\chi, 0) = 2 \cdot \frac{h_L}{2} R_L = -h_L \log|\epsilon|,$$

where ϵ is a fundamental unit of $\mathcal{O}_L$. Note that $|\epsilon| = |\sigma\epsilon|^{-1}$ for the non-trivial element $\sigma \in G$. Setting $\varepsilon = \epsilon^{h_L}$ we then have

$$L_S'(\chi, 0) = -\log|\varepsilon| = -\tfrac{1}{2}(\log|\varepsilon| - \log|\sigma\varepsilon|) = -\tfrac{1}{w_L} \sum_{\sigma \in G} \chi(\sigma) \log|\sigma\varepsilon|.$$

The next proposition gives a reformulation of this conjecture in the spirit of the last section.

3.8 Proposition (Tate)**.** Conjecture 3.5 is equivalent to

$$L_S'(\chi, 0) \cdot e_{\overline{\chi}} X_{L,S} \subseteq \tfrac{1}{w_L} e_{\overline{\chi}} \lambda_{L,S}(\mathcal{O}_{L,S}^\times),$$

where the equality takes place in $\mathbb{C}X_{L,S}$.

Proof. Suppose the stated inclusion is true. Let $w_1 \in S_L \setminus \{w_0\}$, then there exists an element $\varepsilon \in \mathcal{O}_{L,S}^\times$ such that

$$L_S'(\chi, 0) \cdot e_{\overline{\chi}}(w_1 - w_0) = -\tfrac{1}{w_L} e_{\overline{\chi}} \lambda_{L,S}(\varepsilon).$$

Let us first further investigate the right hand side:

$$-|G| \cdot e_{\overline{\chi}} \lambda_{L,S}(\varepsilon) = \sum_{\sigma \in G} \chi(\sigma)\sigma \sum_{w \in S_L} \log|\varepsilon|_w w = \sum_{\sigma \in G} \chi(\sigma) \sum_{w \in S_L} \sum_{\sigma w = w'} \log|\varepsilon|_{\sigma^{-1}w'} w'.$$

Since v_0 splits completely, the coefficient in front of w_0 in this sum is

$$\sum_{\sigma \in G} \sum_{w | v_0} \chi(\sigma) \log|\sigma\varepsilon|_{w_0} w_0.$$

Comparing coefficients (in $\mathbb{C}Y_{L,S}$) therefore gives

$$L_S'(\chi, 0) = -\frac{1}{w_L} \sum_{\sigma \in G} \chi(\sigma) \log|\sigma\varepsilon|_{w_0}.$$

For the converse it suffices to show that the map

$$w_0^* \colon L_S'(\chi, 0) e_{\overline{\chi}} \mathbb{C}X_{L,S} \to \mathbb{C}, \quad \sum_{w \in S_L} a_w w \mapsto a_{w_0}$$

is injective as then all the steps performed above are reversible and due to the fact (see [Rub96, Lemma 2.6 (ii)]) that $L'_S(\chi,0)e_{\overline{\chi}}X_{L,S} = L'_S(\chi,0)e_{\overline{\chi}}\mathbb{Z}[G](w_1 - w_0)$. We may assume that $r(\chi) = 1$, because otherwise $L'_S(\chi,0) = 0$ and the statement is trivially true. Hence we have $\dim_{\mathbb{C}} L'_S(\chi,0)e_{\overline{\chi}}\mathbb{C}X_{L,S} = 1$ by Proposition 3.4 and it is enough to check that the map above is non-zero. This is however clear from

$$w_0^*(L'_S(\chi,0)e_{\overline{\chi}}(w_1 - w_0)) = -\frac{1}{|G|}L'_S(\chi,0) \neq 0. \qquad \square$$

The proposition also provides an easy and elegant way to package conjecture 3.5 for all characters:

3.9 Corollary. The following statements are equivalent:

(a) Conjecture 3.5 is true for each character $\chi \in \widehat{G}$,

(b) we have

$$\theta'_{L|K,S}(0) \cdot X_{L,S} \subseteq \tfrac{1}{w_L}\lambda_{L,S}(\mathcal{O}_{L,S}^\times)$$

as subsets of $\mathbb{C}X_{L,S}$, where $\theta'_{L|K,S}(0) = \sum_{\chi\in\widehat{G}} L'_S(\chi,0)e_{\overline{\chi}}$.

We therefore define a $\mathbb{C}[G]$-valued meromorphic function of a complex variable s by setting

$$\theta_{L|K,S}(s) = \sum_{\chi\in\widehat{G}} L_S(\chi,s)e_{\overline{\chi}},$$

which encodes the information on the individual L-functions in a convenient way. For example we can single out each $L_S(\chi,s)$ by multiplying $\theta_{L|K,S}(s)$ by $e_{\overline{\chi}}$. Moreover, $\theta_{L|K,S}(s)$ also knows about the relationship between the individual L-series:

For every integer $r \geq 0$ we define the r-th order ***Stickelberger element*** by

$$\theta^{(r)}_{L|K,S}(0) = \sum_{\chi\in\widehat{G}} (\lim_{s\to 0} s^{-r}L_S(\chi,s))e_{\overline{\chi}}.$$

Since the complex conjugate of $L_S^{(r)}(\chi,0)$ for $\chi \neq 1$ is precisely $L_S^{(r)}(\overline{\chi},0)$, we see that the Stickelberger element has real coefficients. We can also easily construct a multiplicative inverse $\sum_{\chi\in\widehat{G}} a_\chi e_\chi$ by setting

$$a_\chi = \begin{cases} r! \cdot L_S^{(r)}(\overline{\chi},s)) \cdot |L_S^{(r)}(\chi,s)|^{-2} & \text{if } L_S^{(r)}(s,0) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that

$$\theta^{(r)}_{L|K,S}(0) = 0 \quad \text{or} \quad \theta^{(r)}_{L|K,S}(0) \in \mathbb{R}[G]^\times. \tag{7}$$

Let us now consider a generalisation of Conjecture 3.5 to higher orders of vanishing.

Hypotheses. In order to be sure that the L-function has a zero of order at least r at $s = 0$ we assume that

(H3) S contains a set V of r places which split completely in $L|K$,

(H4) $|S| \geq r + 1$.

To ensure order of vanishing $\geq r$ one could also get by with the subtler assumption that S is a so-called ***r*-cover**. This leads to a conjecture by Emmons and Popescu (see e. g. [Val18, Section 4]).

Recall that we were working solely with the torsion-free part in (3). More amenable (from a technical point of view) is to remove torsion by introducing a finite auxiliary set T of primes such that $S \cap T = \varnothing$. We set

$$\mathcal{O}_{L,S,T}^{\times} = \{x \in \mathcal{O}_{L,S}^{\times} \mid x \equiv 1 \mod w \text{ for all } w \in T_K\}.$$

Note that we have an exact sequence

$$1 \longrightarrow \mathcal{O}_{L,S,T}^{*} \longrightarrow \mathcal{O}_{L,S}^{\times} \longrightarrow \bigoplus_{w \in T_L} \mathbb{F}_w^{\times} \longrightarrow \mathrm{Cl}_{S,T}(L) \longrightarrow \mathrm{Cl}_S(L) \longrightarrow 1,$$

where $\mathbb{F}_w$ denotes the residue field of L at w and $\mathrm{Cl}_{S,T}(L)$ is the S_L-ray class group modulo T_L. In particular, the index $(\mathcal{O}_{L,S}^{\times} : \mathcal{O}_{L,S,T}^{\times})$ is finite. If T is big enough, the resulting group $\mathcal{O}_{L,S,T}^{\times}$ will however be torsion-free. More precisely:

3.10 Lemma. If T contains two primes of different residue characteristic or a prime of large enough norm, then $\mathcal{O}_{L,S,T}^{\times}$ is torsion-free.

Proof. Let $\xi \in \mathcal{O}_{L,S,T}^{\times}$ be a root of unity and take $\mathfrak{p} \in T_L$. For the first part we show that if $\xi \equiv 1 \mod \mathfrak{p}$, then the order of ξ is a power of p, where p is the characteristic of the residue field of $\mathfrak{p}$.

Let $L_{\mathfrak{p}}$ be the completion of L at $\mathfrak{p}$ and denote by $U_{\mathfrak{p}}^{(n)}$ the principal units of level n of $L_{\mathfrak{p}}$. For n big enough, the group $U_{\mathfrak{p}}^{(n)}$ is torsion-free and so the tautological exact sequence

$$1 \longrightarrow U_{\mathfrak{p}}^{(n)} \longrightarrow U_{\mathfrak{p}}^{(1)} \longrightarrow U_{\mathfrak{p}}^{(1)}/U_{\mathfrak{p}}^{(n)} \longrightarrow 1$$

implies that the torsion subgroup of $U_{\mathfrak{p}}^{(1)}$ is isomorphic to $U_{\mathfrak{p}}^{(1)}/U_{\mathfrak{p}}^{(n)}$, hence is a p-group.

For the second part of the Lemma suppose that $\mathrm{N}\mathfrak{p} > p^{\frac{[L_{\mathfrak{p}}:\mathbb{Q}_p]}{p-1}}$. Denote by $f_{\mathfrak{p}}$ and $e_{\mathfrak{p}}$ the residue and ramification degree of the extension $L_{\mathfrak{p}}|\mathbb{Q}_p$, respectively. Then this condition implies that

$$f_{\mathfrak{p}} > \frac{[L_{\mathfrak{p}} : \mathbb{Q}_p]}{p-1} \quad \Leftrightarrow \quad 1 > \frac{e_{\mathfrak{p}}}{p-1}.$$

Hence the p-adic logarithm yields an isomorphism $U_{\mathfrak{p}}^{(1)} \cong \mathfrak{p}\mathcal{O}_{L_{\mathfrak{p}}}$. In particular, $U_{\mathfrak{p}}^{(1)}$ is torsion-free. $\square$

Hypothesis. In light of Lemma 3.10 we will from now on assume that

(H5) $\mathcal{O}_{L,S,T}^{\times}$ is torsion-free

is satisfied.

Since we modified the group of units, we also have to modify the corresponding L-function. We therefore define the S-truncated and T-modified L-function by

$$L_{S,T}(\chi, s) = L_S(\chi, s) \cdot \prod_{\mathfrak{p} \in T} (1 - \chi(\mathfrak{p}) \mathrm{N}\mathfrak{p}^{1-s}).$$

It follows from a deep theorem by Deligne and Ribet [DR80, Theorem 0.4] that the associated Stickelberger element

$$\theta_{L|K,S,T}^{(r)}(0) = \sum_{\chi \in \widehat{G}} (\lim_{s \to 0} s^{-r} L_{S,T}(s, \chi)) e_{\overline{\chi}}$$

is actually contained in $\mathbb{Z}[G]$.

3.11 Conjecture. Assume that S, T and r satisfy hypotheses **(H1)** to **(H5)**. Then

$$\theta_{L|K,S,T}^{(r)}(0) \cdot \bigwedge_{\mathbb{Z}[G]}^{r} X_{L,S} \subseteq \mathbb{Q}\lambda_{L,S}(\bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times}). \tag{8}$$

3.12 Remark. Conjecture 3.11 was first stated by Rubin in [Rub96, Conjecture A]. It is shown in [Rub96, Proposition 2.3] that this Conjecture is equivalent to Stark's principal conjecture for the characters $\chi \in \widehat{G}$ such that $r_S(\chi) = r$ (in Tate's formulation, see [Tat84, Conjecture I.5.1]). Rubin also proposed an integral refinement of Conjecture 3.11 which we shall see later.

3.13 Remark. Let us briefly daydream about what a conjecture concerning leading terms (instead of r-th derivatives) might look like in light of Conjecture 3.11. Denote

$$\theta_{L|K,S,T}^{*}(0) = \sum_{\chi \in \widehat{G}} L_{S,T}^{*}(\chi, 0) e_{\overline{\chi}} \quad \in \mathbb{R}[G]^{\times},$$

where $L_{S,T}^{*}(\chi, 0)$ denotes the leading term of the Taylor expansion of $L_{S,T}(\chi, s)$ at $s = 0$. Note that

$$\theta_{L|K,S,T}^{*}(0) e_{\overline{\chi}} = \theta_{L|K,S,T}^{(r_S(\chi))}(0) e_{\overline{\chi}}.$$

That is, Conjecture 3.11 only contains non-trivial information about the χ-components of $\theta_{L|K,S,T}^{*}$ for the characters χ such that $r_S(\chi) = r$ because of hypothesis (H3). Let us however pretend for the moment that Conjecture 3.11 could be simultaneously meaningful for all r appearing as a order of vanishing $r_S(\chi)$. Then we would want to take the "χ-part" of (8) for $r = r_S(\chi)$ for every character χ and combine these into a single equation. The result would have a sum of exterior powers of different degrees on each side, and (6) implies that these algebraic objects are exactly the determinant modules of $\mathbb{Q}\mathcal{O}_{L,S,T}^{\times}$ and $\mathbb{Q}X_{L,S}$. Thus, a conjecture concerning leading terms should be formulated using determinant modules instead of r-th exterior powers. A good introduction to the basic theory of determinant modules can be found in [Pop11, Section 1.3] or [Bur11, Lecture 1].

4 The Rubin-Stark Conjecture

For every $v \in S$ choose a place w of L lying over v. We also fix an ordering $V = \{v_1, \dots, v_r\}$ and pick an element $v_0 \in S \setminus V$.

Unlike the rank 1 abelian Conjecture 3.5 does Conjecture 3.11 not predict the existence of particular units but only of an element $\varepsilon \in \mathbb{Q} \bigwedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{L,S,T}$ such that

$$\theta^{(r)}_{L|K,S,T}(0) \cdot \bigwedge_{v \in V} (w - w_0) = \lambda_{L,S}(\varepsilon).$$

That is, the element $\theta^{(r)}_{L|K,S,T}(0)$ is, up to a rational factor, given by a sum of determinants of logarithms of units. The existence of that element ε can also be viewed slightly differently: The Dirichlet regulator induces an isomorphism

$$\lambda_{L,S} \colon \mathbb{R} \bigwedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{L,S,T} \xrightarrow{\simeq} \mathbb{R} \bigwedge^r_{\mathbb{Z}[G]} X_{L,S}$$

and so there exists a unique element $\epsilon^V_{L|K,S,T} \in \mathbb{R} \bigwedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{L,S,T}$ called the r-th order ***Rubin-Stark element*** such that

$$\lambda_{L,S}(\epsilon^V_{L|K,S,T}) = \theta^{(r)}_{L|K,S,T} \cdot \bigwedge_{v \in V} (w - w_0).$$

4.1 Lemma. The Rubin-Stark element $\epsilon^V_{L|K,S,T}$ does not depend on the choice of $v_0 \in S \setminus V$.

Proof. If $|S| = r + 1$, there is nothing to show, so suppose $|S| > r + 1$. Let $v'_0 \in S \setminus V$ and take w'_0 lying above v'_0 in L. It suffices to show that

$$\theta^{(r)}_{L|K,S,T} \cdot \bigwedge_{v \in V} (w - w_0) = \theta^{(r)}_{L|K,S,T} \cdot \bigwedge_{v \in V} (w - w'_0)$$

as elements of $\mathbb{C}[G]$. Clearly, we have

$$(w - w_0) = (w - w'_0) + (w'_0 - w_0)$$

and hence an equality

$$\bigwedge_{v \in V} (w - w_0) = \bigwedge_{v \in V} (w - w'_0) + \sum_x x \quad \text{in } \mathbb{C}Y_{L,S},$$

where in the sum on the right only summands $x = x_1 \wedge \cdots \wedge x_r$ such that $x_i \in \{w_0, w'_0\}$ for one i appear. If $\chi \in \widehat{G}$ is a character, then

$$e_\chi \theta^{(r)}_{L|K,S,T} w_0 = 0 = e_\chi \theta^{(r)}_{L|K,S,T} w'_0,$$

by [Rub96, Lemma 2.6 (i)]. As a consequence,

$$e_\chi \theta^{(r)}_{L|K,S,T} \bigwedge_{v\in V}(w - w_0) = e_\chi \theta^{(r)}_{L|K,S,T} \bigwedge_{v\in V}(w - w_0')$$

for every character χ. Piecing the χ-parts together then gives the claim. □

4.2 Remark. The element $\epsilon^V_{L|K,S,T}$ could still depend on the choice of places lying above the places in V or the ordering of V. The Conjecture we about to make does however not.

If Conjecture 3.11 holds true, then $\varepsilon = \epsilon^V_{L|K,S,T}$ by uniqueness. Hence Conjecture 3.11 should not be viewed as predicting the existence of a certain element but rather as claiming that the Rubin-Stark element is contained in the rational subspace $\mathbb{Q}\bigwedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{L,S,T}$ of $\mathbb{R}\bigwedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{L,S,T}$. In order to obtain an integral refinement of Conjecture 3.11 we shall now give an *integral* sublattice the Rubin-Stark element should be contained in.

The most obvious guess would of course be that in fact already $\epsilon^V_{L|K,S,T} \in \bigwedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{L,S,T}$. This is however not true in general, as shown in [Rub96, §4.1]. The expectation is nevertheless that $\epsilon^V_{L|K,S,T}$ should not be too far from being integral, in a sense to be made precise.

Algebraic Preliminaries Let M be a G-module and $f \in \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$, then f also induces a map

$$\bigwedge^r_{\mathbb{Z}[G]} M \to \bigwedge^{r-1}_{\mathbb{Z}[G]} M$$

for all $r \geq 1$, defined by

$$m_1 \wedge \cdots \wedge m_r \mapsto \sum_{i=1}^r (-1)^{i-1} f(m_i) \cdot m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_r.$$

This morphism is, sometimes maybe confusingly, also denoted by f. Iterating this construction yields a morphism

$$\bigwedge^s_{\mathbb{Z}[G]} \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G]) \to \mathrm{Hom}_{\mathbb{Z}[G]}\Big(\bigwedge^r_{\mathbb{Z}[G]} M, \bigwedge^{r-s}_{\mathbb{Z}[G]} M\Big)$$

for all $r, s \geq 0$ such that $r \geq s$, defined by

$$f_1 \wedge \cdots \wedge f_s \mapsto \{m \mapsto (f_s \circ \cdots \circ f_1)(m)\}.$$

By using this homomorphism we will regard an element of $\bigwedge^r_{\mathbb{Z}[G]} \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$ as an element of $\mathrm{Hom}_{\mathbb{Z}[G]}(\bigwedge^r_{\mathbb{Z}[G]} M, \mathbb{Z}[G])$.

4.3 Definition. Let M be a finitely generated G-module and $r \geq 0$ an integer. We define the r-th ***Rubin lattice*** to be

$$\bigcap_{\mathbb{Z}[G]}^{r} M = \{a \in \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} M \mid \Phi(a) \in \mathbb{Z}[G] \text{ for all } \Phi \in \bigwedge_{\mathbb{Z}[G]}^{r} \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])\}.$$

Note in particular that the notation "$\bigcap$" does in this case not refer to an intersection.

4.4 Remark. Note that the canonical map

$$\bigwedge_{\mathbb{Z}[G]}^{r} M \to \bigcap_{\mathbb{Z}[G]}^{r} M$$

is in general neither injective nor surjective. Its kernel consists of those elements in $\bigwedge_{\mathbb{Z}[G]}^{r} M$ which are annihilated by non-zero-divisors of $\mathbb{Z}[G]$. We will denote the image of this map by $\overline{\bigwedge_{\mathbb{Z}[G]}^{r} M}$.

4.5 Proposition (Properties of Rubin lattices)**.** Let M be a finitely generated and $\mathbb{Z}$-free G-module.

(a) We have

$$\bigcap_{\mathbb{Z}[G]}^{0} M = \mathbb{Z}[G] \qquad \text{and} \qquad \bigcap_{\mathbb{Z}[G]}^{1} M = M.$$

(b) The index $(\bigcap_{\mathbb{Z}[G]}^{r} M : \overline{\bigwedge_{\mathbb{Z}[G]}^{r} M})$ is finite and

$$\mathbb{Z}[\tfrac{1}{|G|}] \cdot \bigcap_{\mathbb{Z}[G]}^{r} M = \mathbb{Z}[\tfrac{1}{|G|}] \cdot \overline{\bigwedge_{\mathbb{Z}[G]}^{r} M}.$$

(c) If M is $\mathbb{Z}[G]$-projective, then $\bigcap_{\mathbb{Z}[G]}^{r} M = \bigwedge_{\mathbb{Z}[G]}^{r} M$.

Proof. See [Rub96, Proposition 1.2]. □

4.6 Example. Let M be a finitely generated and $\mathbb{Z}$-free G-module. Since $\mathbb{Z}[G]$ is cohomologically trivial, we have $\mathbb{Z}[G]^G = N_G \mathbb{Z}[G]$ and hence

$$\mathrm{Hom}_{\mathbb{Z}[G]}(M^G, \mathbb{Z}[G]) = \mathrm{Hom}_{\mathbb{Z}[G]}(M^G, N_G \mathbb{Z}[G]).$$

Thus, if $m_1, \ldots, m_r \in M^G$ and $\varphi_1, \ldots, \varphi_r \in \mathrm{Hom}_{\mathbb{Z}[G]}(M^G, \mathbb{Z}[G])$ for an integer $r \geq 1$, then

$$(\varphi_1 \wedge \cdots \wedge \varphi_r)(m_1 \wedge \cdots \wedge m_r) \in N_G^r \mathbb{Z}[G] = |G|^{r-1} N_G \mathbb{Z}[G].$$

This shows that $|G|^{1-r} \overline{\bigwedge_{\mathbb{Z}[G]}^{r} M^G} \subseteq \bigcap_{\mathbb{Z}[G]}^{r} M$. One can check that in fact

$$\bigcap_{\mathbb{Z}[G]}^{r} M^G = |G|^{\max\{0, 1-r\}} \overline{\bigwedge_{\mathbb{Z}[G]}^{r} M^G}$$

for all integers $r \geq 0$. In particular, if $1 < r \leq n$, then $\bigcap_{\mathbb{Z}[G]}^{r} M^G$ is strictly larger than $\bigwedge_{\mathbb{Z}[G]}^{r} M^G$.

Armed with enough algebraic background we can now return to our arithmetic setting.

4.7 Conjecture (Rubin-Stark)**.** If S, T and r satisfy the hypotheses **(H1)** to **(H5)**, then

$$\varepsilon^V_{L,S,T} \in \bigcap_{\mathbb{Z}[G]}^{r} \mathcal{O}^\times_{L,S,T}.$$

Equivalently, we have

$$\theta^{(r)}_{L|K,S,T}(0) \cdot \bigwedge_{\mathbb{Z}[G]}^{r} X_{L,S} \subseteq \lambda_{L,S}(\bigcap_{\mathbb{Z}[G]}^{r} \mathcal{O}^\times_{L,S,T}).$$

For now the choice of the Rubin lattice as the right object to use might seem a bit arbitrary. We will later see another interpretation of the Rubin lattice that makes it seem completely natural (see Remark 6.3 (a)).

5 A special case of the eTNC

We have seen in Remark 3.13 that the use of determinant modules seems natural for the purpose of formulating leading term conjectures. Let us therefore immerse a little in the formalism of determinant modules.
We will drop hypotheses (H3) - (H5) for now and return to considering S-units instead of (S, T)-units. We will keep the hypotheses (H1) and (H2), though.

The Dirichlet regulator induces an isomorphism

$$\lambda_{L,S} \colon \mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}\mathcal{O}^\times_{L,S}) \xrightarrow{\simeq} \mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}X_{L,S})$$

and so composition of this isomorphism with the evaluation morphism gives a map

$$\mathrm{ev} \circ (\lambda \otimes \mathrm{id}) \colon \mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}\mathcal{O}^\times_{L,S}) \otimes_{\mathbb{R}[G]} \Big(\mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}X_{L,S})\Big)^{-1} \xrightarrow{\simeq} \mathbb{R}[G]. \tag{9}$$

5.1 Conjecture. Assume that S satisfies hypotheses **(H1)** and **(H2)**. Then (9) restricts to an isomorphism

$$\mathrm{Det}_{\mathbb{Q}[G]}(\mathbb{Q}\mathcal{O}^\times_{L,S}) \otimes_{\mathbb{Q}[G]} \Big(\mathrm{Det}_{\mathbb{Q}[G]}(\mathbb{Q}X_{L,S})\Big)^{-1} \cong \theta^*_{L|K,S}(0) \cdot \mathbb{Q}[G].$$

5.2 Remark. (a) Conjecture 5.1 is a version of Stark's principal conjecture that has been mentioned several times before. In particular, if S, T and r also satisfy hypotheses (H3) - (H5), then taking χ-components for those characters satisfying $r_S(\chi) = r$ gives back Conjecture 3.11.

(b) The module $\mathrm{Det}_{\mathbb{Q}[G]}(\mathbb{Q}\mathcal{O}_{L,S}^{\times}) \otimes_{\mathbb{Q}[G]} \left(\mathrm{Det}_{\mathbb{Q}[G]}(\mathbb{Q}X_{L,S})\right)^{-1}$ is identified with a rational subspace of the left hand side of (9) via extension of scalars: Let M and N be $\mathbb{Q}[G]$-modules, then

$$\begin{aligned}\mathbb{R}\left(\mathrm{Det}_{\mathbb{Q}[G]}(M) \otimes_{\mathbb{Q}[G]} \mathrm{Det}_{\mathbb{Q}[G]}(N)\right) &= \left(\mathbb{R}\mathrm{Det}_{\mathbb{Q}[G]}(M)\right) \otimes_{\mathbb{Q}[G]} \mathrm{Det}_{\mathbb{Q}[G]}(N) \\ &\cong \mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}M) \otimes_{\mathbb{Q}[G]} \mathrm{Det}_{\mathbb{Q}[G]}(N) \\ &\cong \mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}M) \otimes_{\mathbb{R}[G]} \mathbb{R}[G] \otimes_{\mathbb{Q}[G]} \mathrm{Det}_{\mathbb{Q}[G]}(N) \\ &\cong \mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}M) \otimes_{\mathbb{R}[G]} (\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}[G]) \otimes_{\mathbb{Q}[G]} \mathrm{Det}_{\mathbb{Q}[G]}(N) \\ &\cong \mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}M) \otimes_{\mathbb{R}[G]} \mathbb{R}\mathrm{Det}_{\mathbb{Q}[G]}(N) \\ &\cong \mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}M) \otimes_{\mathbb{R}[G]} \mathrm{Det}_{\mathbb{R}[G]}(N).\end{aligned}$$

In line with the philosophy of earlier we shall give an integral refinement of Conjecture 5.1. That is, give an integral sublattice we expect to be equal to the inverse image of $\theta^{*}_{L|K,S}$ under the isomorphism (9). A first impulse would certainly be to construct this lattice by simply taking the $\mathbb{Z}[G]$-determinants of $\mathcal{O}_{L,S}^{\times}$ and $X_{L,S}$. This is however not possible since these two modules are in general not cohomologically trivial and hence do not admit a well-defined determinant. Nevertheless, we can "approximate" these two modules by cohomologically trivial modules.

5.3 Theorem. Suppose that $\mathrm{Cl}_S(L) = 1$. Then there exists an exact sequence, called a ***Tate sequence***, of the form

$$0 \longrightarrow \mathcal{O}_{L,S,T}^{\times} \longrightarrow E_1 \longrightarrow E_0 \longrightarrow X_{L,S} \longrightarrow 0, \tag{10}$$

where E_0 and E_1 are finitely generated $\mathbb{Z}[G]$-modules such that E_0 is projective and E_1 is cohomologically trivial.

5.4 Remark. A sequence of the form (10) was first constructed by Tate via a clever usage of the compatibility of local and global class field theory (see e. g. [Tat84, Chapter II, Theorem 5.1] for the construction). In a little more detail, (10) represents a canonical extension class $c_{L|K,S} \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{L,S}, \mathcal{O}_{L,S}^{\times})$ that is obtained from the local and global fundamental classes.

Dealing with Tate sequences in their current incarnation as 2-extension turns out to be quite cumbersome. For instance, constructing a Tate sequence in the case of non-vanishing class group is possible (see [RW96]) but requires considerable effort. It appears much more practical to interpret (10) as a complex

$$C^{\bullet}\colon \cdots \to 0 \to E_1 \to E_0 \to 0 \to \dots$$

that is concentrated in degrees 0 and 1. Since the cohomology of this complex is

$$H^q(C^{\bullet}) = \begin{cases} \mathcal{O}_{L,S}^{\times} & \text{if } q = 0, \\ X_{L,S} & \text{if } q = 1, \\ 0 & \text{otherwise,} \end{cases}$$

we can easily get back the sequence (10) from the complex $C^\bullet$. It is then possible to construct the complex $C^\bullet$ using (modified) étale cohomology with compact support, see [BKS16, Proposition 2.9].

The sequence (10) yields an isomorphism

$$\begin{aligned}
\mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}\mathcal{O}_{L,S}^\times) \otimes_{\mathbb{R}[G]} \Big(\mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}X_{L,S})\Big)^{-1} &\cong \mathrm{Det}_{\mathbb{R}[G]}(E_1) \otimes_{\mathbb{R}[G]} \Big(\mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}E_1)\Big)^{-1} \\
&= \mathbb{R}\mathrm{Det}_{\mathbb{Z}[G]}(E_1) \otimes_{\mathbb{R}[G]} \mathbb{R}\Big(\mathrm{Det}_{\mathbb{Z}[G]}(E_1)\Big)^{-1} \\
&= \mathbb{R}\mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet),
\end{aligned}$$

so composing with (9) gives an isomorphism

$$\vartheta_{\lambda_{L,S}} \colon \mathbb{R}\mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet) \xrightarrow{\simeq} \mathbb{R}[G].$$

5.5 Definition. We define the ***determinant lattice*** to be

$$\Xi(L,S) = \vartheta_{\lambda_{L,S}}\Big(\mathrm{Det}_{\mathbb{Z}[G]} C^\bullet\Big) \quad \subseteq \mathbb{R}[G].$$

Recall that quasi-isomorphic (perfect) complexes have the same determinant and one can show that the determinant lattice is also well-defined modulo quasi-isomorphism (see [Bur11, Proposition 1.11 and Remark 1.12]). Since avoiding choices was a ubiquitous theme so far, we should really use the derived category $\mathrm{D}(\mathbb{Z}[G])$ of $\mathbb{Z}[G]$-modules to formulate our leading term conjecture. We will however resist this impulse for now.

5.6 Conjecture (eTNC)**.** Assume that S satisfies the hypotheses **(H1)** and **(H2)**. Then we have

$$\Xi(L,S) = \theta^*_{L|K,S}(0) \cdot \mathbb{Z}[G],$$

where the equality takes place in $\mathbb{R}[G]$.

The eTNC is known to be valid in the following (non-comprehensive) list of cases:

- L is a finite abelian extension of $\mathbb{Q}$,
- there exists an imaginary quadratic field F which has class number one and is such that $F \subseteq K$, $L|F$ is finite abelian and $[L:K]$ is both odd and divisible only by primes which split completely in $F|\mathbb{Q}$,
- $L|K$ is quadratic.

A more comprehensive list of proven cases of the eTNC (including references) can be found, for example, in [JN16, Section 4.3].

6 The eTNC implies the Rubin-Stark Conjecture

Another piece of motivation for the eTNC is that it unites – and at the same time also refines – numerous Conjectures that have been made previously. It has been proven in

[Bur07] that among these conjectures the eTNC implies is also the Rubin-Stark Conjecture 4.7. The aim of this section is to sketch the easier proof of this implication that is given in [BKS16].

T-modification Since the Rubin-Stark Conjecture uses the T-modified S-units and L-function, we will first also T-modify Conjecture 5.6. For this purpose we will use the perfect complex

$$D^\bullet_{L,S,T} = \mathrm{R}\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m) \quad \in \mathrm{D}(\mathbb{Z}[G])$$

that is constructed in [BKS16, Proposition 2.4]. By [BKS16, Remark 2.7], the cohomology of this complex is

$$H^q(D^\bullet_{L,S,T}) = \begin{cases} \mathcal{O}^\times_{L,S,T} & \text{if } q = 0, \\ \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L}) & \text{if } q = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the appearing group $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L})$ sits in an exact sequence

$$0 \longrightarrow \mathrm{Cl}_{S,T}(K) \longrightarrow \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L}) \longrightarrow X_{L,S} \longrightarrow 0. \tag{11}$$

6.1 Proposition. Let S satisfy hypotheses **(H1)** and **(H2)**, and let T be any finite set of places of K that is disjoint from S. Then following conditions on $L|K$ are equivalent:

(a) Conjecture 5.6 is valid,

(b) in $\mathbb{R}[G]$ one has an equality

$$\vartheta_{\lambda_{L,S}}(\mathrm{Det}_{\mathbb{Z}[G]}(D^\bullet_{L,S,T}) = \theta^*_{L|K,S,T}(0) \cdot \mathbb{Z}[G].$$

Proof. See [BKS16, Proposition 3.4]. □

An explicit resolution The next step in the proof is to choose a convenient representative of the complex $D^\bullet_{L,S,T}$. Let $S = \{v_0, \ldots, v_n\}$ and for every $i \in \{0, \ldots, n\}$ fix a place w_i of L lying above v_i. It is described in [BKS16, Section 5.4] that we can represent $D^\bullet_{L,S,T}$ by an exact sequence of the form

$$0 \longrightarrow \mathcal{O}^\times_{L,S,T} \longrightarrow P \xrightarrow{\psi} F \xrightarrow{\pi} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L}) \longrightarrow 0, \tag{12}$$

where F is a $\mathbb{Z}[G]$-free module of rank $d > n$, the module P is cohomologically trivial and the map π has the following property: There is a $\mathbb{Z}[G]$-basis $b_1, \ldots, b_d$ of F such that the composition

$$F \xrightarrow{\pi} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L}) \longrightarrow X_{L,S},$$

where the right arrow is the map from 11, sends $b_i \mapsto w_i - w_0$ for all $i \in \{1, \ldots, n\}$. If we assume (H5), that is $\mathcal{O}^\times_{L,S,T}$ is torsion-free, then P also has to be torsion-free. Hence, P is $\mathbb{Z}[G]$-projective. Note that the complex

$$\cdots \to 0 \to P \xrightarrow{\psi} F \to 0 \to \ldots,$$

where P is places in degree 0, is quasi-isomorphic to $D^\bullet_{L,S,T}$. Hence we have an isomorphism

$$\mathrm{Det}_{\mathbb{Z}[G]}(D^\bullet_{L,S,T}) \cong \mathrm{Det}_{\mathbb{Z}[G]}(P) \otimes_{\mathbb{Z}[G]} \Big(\mathrm{Det}_{\mathbb{Z}[G]}(F)\Big)^{-1}$$

For each $1 \le i \le d$, we define

$$\psi_i = b_i^* \circ \psi \quad \in \mathrm{Hom}_{\mathbb{Z}[G]}(P, \mathbb{Z}[G]),$$

where $b_1^*, \ldots, b_d^* \in \mathrm{Hom}_{\mathbb{Z}[G]}(F, \mathbb{Z}[G])$ is the dual basis of $b_1, \ldots, b_d \in F$.

The zeta element Let $z_{L|K,S,T} \in \mathbb{R}\mathrm{Det}_{\mathbb{Z}[G]}(D^\bullet_{L,S,T})$ denote the preimage of $\theta^*_{L|K,S,T}$ under $\vartheta_{\lambda_{L,S}}$ and assume that the eTNC holds for $L|K$. Then we have

$$z_{L|K,S,T} \cdot \mathbb{Z}[G] = \mathrm{Det}_{\mathbb{Z}[G]}(D^\bullet_{L,S,T}) \cong \mathrm{Det}_{\mathbb{Z}[G]}(P) \otimes_{\mathbb{Z}[G]} \Big(\mathrm{Det}_{\mathbb{Z}[G]}(F)\Big)^{-1}.$$

Since F is a free $\mathbb{Z}[G]$-module, we have $(\mathrm{Det}_{\mathbb{Z}[G]}F)^{-1} \cong \mathbb{Z}[G]$ and so the isomorphism above implies that $\det_{\mathbb{Z}[G]} P \cong \mathbb{Z}[G]$. By Swan's theorem, there exists an integer s and an ideal $\mathfrak{a} \subseteq \mathbb{Z}[G]$ of finite index such that

$$P \cong \mathbb{Z}[G]^s \oplus \mathfrak{a}.$$

Hence we have

$$\mathrm{Det}_{\mathbb{Z}[G]}(P) \cong \mathrm{Det}_{\mathbb{Z}[G]}(\mathfrak{a}) \otimes_{\mathbb{Z}[G]} (\mathrm{Det}_{\mathbb{Z}[G]}(\mathbb{Z}[G]^s))^{-1} \cong \mathfrak{a} \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G] \cong \mathfrak{a}$$

since $\mathfrak{a}$ has constant local rank 1. Combining this with the isomorphism above, we see that $\mathfrak{a}$ is $\mathbb{Z}[G]$-free and so P is $\mathbb{Z}[G]$-free itself. From (12) we get that

$$\begin{aligned}
\mathrm{rk}_{\mathbb{Z}}\, P &= \mathrm{rk}_{\mathbb{Z}}\, \mathcal{O}^\times_{L,S,T} + \mathrm{rk}_{\mathbb{Z}}\, F - \mathrm{rk}_{\mathbb{Z}}\, \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L}) = \\
&= (|S| - 1) + d \cdot |G| - \mathrm{rk}_{\mathbb{Z}}\, X_{L,S} = \\
&= (|S| - 1) + d \cdot |G| - (|S| - 1) = \\
&= d \cdot |G|,
\end{aligned}$$

where the second equality uses the exact sequence (11). Thus, we must have that P is $\mathbb{Z}[G]$-free of rank d and so we have an isomorphism

$$\mathrm{Det}_{\mathbb{Z}[G]}(P) \otimes_{\mathbb{Z}[G]} \Big(\mathrm{Det}_{\mathbb{Z}[G]}(F)\Big)^{-1} \cong \bigwedge_{\mathbb{Z}[G]}^{d} P \otimes_{\mathbb{Z}[G]} \bigwedge_{\mathbb{Z}[G]}^{d} \mathrm{Hom}_{\mathbb{Z}[G]}(F, \mathbb{Z}[G]).$$

We define $z_b \in \mathrm{Det}_{\mathbb{Z}[G]}(D^\bullet_{L,S,T})$ to be the element that is mapped to $z_{L|K,S,T}$ via the isomorphism

$$\bigwedge_{\mathbb{Z}[G]}^{d} P \xrightarrow{\simeq} \bigwedge_{\mathbb{Z}[G]}^{d} P \otimes_{\mathbb{Z}[G]} \bigwedge_{\mathbb{Z}[G]}^{d} \mathrm{Hom}_{\mathbb{Z}[G]}(F, \mathbb{Z}[G]) \cong \mathrm{Det}_{\mathbb{Z}[G]}(D^\bullet_{L,S,T}),$$

where the first isomorphism is defined by

$$a \mapsto a \otimes (b_1^* \wedge \cdots \wedge b_d^*).$$

6.2 Theorem ([BKS16], 5.14)**.** Assume that the eTNC for $L|K$ holds and that S, T, and r satisfy hypotheses **(H1)** to **(H5)**.

(a) Regarding $\mathcal{O}_{L,S,T}^{\times}$ as a submodule of P, one has that

$$(\psi_{r+1} \wedge \cdots \wedge \psi_d)(z_b) \in \bigcap_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times} \quad \left(\subseteq \bigwedge_{\mathbb{Z}[G]}^{r} P \right).$$

(b) Moreover,

$$\lambda_{L,S}\left((\psi_{r+1} \wedge \cdots \wedge \psi_d)(z_b)\right) = (-1)^{r(d-r)} \cdot \theta_{L|K,S,T}^{(r)}(0) \cdot \bigwedge_{i=1}^{r} (w_i - w_0).$$

6.3 Remark. The proof of Theorem 6.2 will make use of the following:

(a) The injection $\mathcal{O}_{L,S,T}^{\times} \to P$ induces an injection

$$\bigcap_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times} \to \bigcap_{\mathbb{Z}[G]}^{r} P = \bigwedge_{\mathbb{Z}[G]}^{r} P.$$

If we regard this injection as an inclusion, we have

$$\bigcap_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times} = (\mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times}) \cap \bigwedge_{\mathbb{Z}[G]}^{r} P.$$

This is proved in [BKS16, Lemma 4.7] and retrospectively provides justification for the definition of the Rubin lattice.

(b) We will need to use the representation theory of $\mathbb{Q}[G]$. All characters ψ of G defined over $\mathbb{Q}$ are of the form

$$\psi = \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)|\mathbb{Q})} \chi^{\sigma},$$

were $\chi \in \widehat{G}$ is a (complex) character and $\mathbb{Q}(\chi)$ denotes the extension of $\mathbb{Q}$ generated by the values of χ. In other words, the rational characters of G can be thought of equivalence classes of complex characters with respect to the action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on $\widehat{G}$. See [Pop11, Example 1.3.3] for more details.

Proof. Take any character $\chi \in \widehat{G}$. Recall that

$$r_S(\chi) = \dim_{\mathbb{C}} e_{\chi} \mathbb{C} X_{L,S} = \dim_{\mathbb{C}} e_{\chi} \mathbb{C} \mathcal{O}_{L,S,T}^{\times}$$

and this doesn't change if we take a class $[\chi] \in \widehat{G}/\sim$, where $\sim$ denotes the action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on $\widehat{G}$:

$$r_S(\chi) = \dim_{\mathbb{Q}(\chi)} e_{[\chi]} \mathbb{Q} X_{L,S} = \dim_{\mathbb{Q}(\chi)} e_{[\chi]} \mathbb{Q} \mathcal{O}_{L,S,T}^{\times}$$

since $\mathbb{Q} X_{L,S} \cong \mathbb{Q} \mathcal{O}_{L,S,T}^{\times}$. Here $\mathbb{Q}(\chi)$ is the smallest extension of $\mathbb{Q}$ containing all values of χ and

$$e_{[\chi]} = \sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}(\chi)|\mathbb{Q})} e_{\chi^\sigma} \quad \in \mathbb{Q}[G].$$

Note that $e_{[\chi]}\mathbb{Q}[G] \cong \mathbb{Q}(\chi)$. Consider the map

$$\Psi = \bigoplus_{i=r+1}^{d} \psi_i \colon e_{[\chi]}\mathbb{Q}P \to e_{[\chi]}\mathbb{Q}[G]^{\oplus(d-r)}.$$

If $r_S(\chi) = r$, then $\dim_{\mathbb{Q}(\chi)} e_{[\chi]}\mathbb{Q}X_{L,S} = r$, so $\{e_{[\chi]}(w_i - w_0)\}_{1\le i\le r}$ is a $\mathbb{Q}(\chi)$-basis of $e_{[\chi]}\mathbb{Q}X_{L,S}$ since $w_1,\dots,w_r$ split completely. In particular, $e_{[\chi]}(w_i - w_0) = 0$ for $i \in \{r+1,\dots,n\}$. This means that $\bigoplus_{i=r+1}^d e_{[\chi]}\mathbb{Q}b_i \subseteq e_{[\chi]}\mathbb{Q}\ker\pi$ since $\mathbb{Q}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L}) \cong \mathbb{Q}X_{L,S}$ by (11). Comparing dimensions, this inclusion has to be an equality. Hence

$$e_{[\chi]}\mathbb{Q}\,\mathrm{im}\,\psi = e_{[\chi]}\mathbb{Q}\ker\pi = \bigoplus_{i=r+1}^{d} e_{[\chi]}\mathbb{Q}b_i.$$

This implies that $b_i^*(e_{[\chi]}\mathbb{Q}\,\mathrm{im}\,\psi) = e_{[\chi]}\cdot\mathbb{Q}$ for all $i \in \{r+1,\dots,d\}$ and so Ψ is surjective.
On the contrary, if $r_S(\chi) > r$, then

$$\begin{aligned}\dim_{\mathbb{Q}(\chi)} e_{[\chi]}\mathbb{Q}\,\mathrm{im}\,\psi = \dim_{\mathbb{Q}(\chi)} e_{[\chi]}\mathbb{Q}\ker\pi &= \dim_{\mathbb{Q}(\chi)} e_{[\chi]}\mathbb{Q}F - \dim_{\mathbb{Q}(\chi)} e_{[\chi]}\mathbb{Q}X_{L,S}\\ &= d - r_S(\chi) < d - r,\end{aligned}$$

so Ψ can not possibly be surjective.

Applying [BKS16, Lemma 4.2], we have that

$$e_{[\chi]}\cdot(\bigwedge_{i=r+1}^{d}\psi_i)(z_b) \in \begin{cases} e_{[\chi]}\mathbb{Q}\bigwedge^r_{\mathbb{Z}[G]}\mathcal{O}^\times_{L,S,T} & \text{if } r_S(\chi) = r,\\ 0 & \text{if } r_S(\chi) > r.\end{cases}$$

Let $e_r = \sum_{r_S(\chi)=r} e_{[\chi]}$, then this implies that

$$e_r\cdot(\bigwedge_{i=r+1}^{d}\psi_i)(z_b) = (\bigwedge_{i=r+1}^{d}\psi_i)(z_b) \quad \text{belongs to} \quad (\mathbb{Q}\bigwedge^r_{\mathbb{Z}[G]}\mathcal{O}^\times_{L,S,T}) \cap \bigwedge^r_{\mathbb{Z}[G]} P = \bigcap^r_{\mathbb{Z}[G]}\mathcal{O}_{L,S,T}.$$

For convenience we now change to e_χ instead of $e_{[\chi]}$. [BKS16, Lemma 4.3] says that the map

$$\begin{aligned}&\bigwedge^{d}_{e_\chi\mathbb{C}[G]} e_\chi\mathbb{C}P \otimes_{e_\chi\mathbb{C}[G]} \bigwedge^{d}_{e_\chi\mathbb{C}[G]} \mathrm{Hom}_{e_\chi\mathbb{C}[G]}(e_\chi\mathbb{C}P, e_\chi\mathbb{C}[G])\\ &\quad\longrightarrow \bigwedge^{r_S(\chi)}_{e_\chi\mathbb{C}[G]} e_\chi\mathbb{C}\mathcal{O}^\times_{L,S,T} \otimes_{e_\chi\mathbb{C}[G]} \bigwedge^{r_S(\chi)}_{e_\chi\mathbb{C}[G]} \mathrm{Hom}_{e_\chi\mathbb{C}[G]}(e_\chi\mathbb{C}X_{L,S}, e_\chi\mathbb{C}[G])\end{aligned}$$

sends $e_\chi(z_b \otimes \bigwedge_{i=1}^d b_i^*)$ to

$$e_\chi \cdot \left((-1)^{r_S(\chi)(d-r_S(\chi))} (\bigwedge_{i=r_S(\chi)+1}^{d} \psi_i)(z_b) \otimes \bigwedge_{i=1}^{r_S(\chi)} (w_i - w_0)^* \right).$$

Piecing the χ-parts together, this means that the map

$$\begin{aligned} e_r\mathbb{C}\mathrm{Det}_{\mathbb{Z}[G]}(D^\bullet_{L,S,T}) &\cong \mathrm{Det}_{e_r\mathbb{C}[G]} e_r\mathbb{C}P \otimes_{e_r\mathbb{C}[G]} (\mathrm{Det}_{e_r\mathbb{C}[G]} e_r\mathbb{C}F)^{-1} \\ &\cong \mathrm{Det}_{e_r\mathbb{C}[G]} e_r\mathbb{C}\mathcal{O}^\times_{L,S,T} \otimes_{e_r\mathbb{C}[G]} (\mathrm{Det}_{e_r\mathbb{C}[G]} e_r\mathbb{C}X_{L,S})^{-1} \end{aligned}$$

sends the element $z_b \otimes \bigwedge_{i=1}^d b_i^*$ to

$$\begin{aligned} e_r \cdot \left((-1)^{r(d-r)} (\bigwedge_{i=r+1}^{d} \psi_i)(z_b) \otimes \bigwedge_{i=1}^{r} (w_i - w_0)^* \right) &= \\ = (-1)^{r(d-r)} (\bigwedge_{i=r+1}^{d} \psi_i)(z_b) \otimes \bigwedge_{i=1}^{r} (w_i - w_0)^*. \end{aligned}$$

By definition of the zeta element, the map

$$\begin{aligned} \mathbb{C}\mathrm{Det}_{\mathbb{Z}[G]}(D^\bullet_{K,S,T}) &\cong \mathrm{Det}_{\mathbb{C}[G]} \mathbb{C}\mathcal{O}^\times_{L,S,T} \otimes_{\mathbb{C}[G]} (\det_{\mathbb{C}[G]} \mathbb{C}X_{L,S})^{-1} \\ &\cong \mathbb{C}[G] \end{aligned}$$

sends $z_{L|K,S,T}$ to $\theta^*_{L|K,S,T}(0)$. Therefore we must have that

$$\begin{aligned} \lambda_{L,S}\left(e_r(-1)^{r(d-r)} (\bigwedge_{i=r+1}^{d} \psi_i)(z_b) \right) &= e_r\theta^*_{L|K,S,T}(0) \cdot \bigwedge_{i=1}^{r} (w_i - w_0) \\ &= \theta^{(r)}_{L|K,S,T}(0) \cdot \bigwedge_{i=1}^{r} (w_i - w_0). \end{aligned}$$

□

6.4 Corollary. Assume that S, T, and r satisfy the hypoteses **(H1)** to **(H5)**. Then the eTNC for $L|K$ implies the Rubin-Stark Conjecture for the data $(L|K, S, T, V)$.

Proof. By definition the Rubin-Stark element $\varepsilon^V_{L|K,S,T}$ is the unique element satisfying

$$\lambda_{L,S}(\varepsilon^V_{L|K,S,T}) = \theta^{(r)}_{L|K,S,T} \cdot \bigwedge_{i=1}^{r} (w_i - w_0).$$

Thus, Theorem 6.2 (b) directly implies that

$$\varepsilon^V_{L|K,S,T} = (-1)^{r(d-r)} \cdot (\bigwedge_{i=r+1}^{d} \psi_i)(z_b)$$

and 6.2 (a) gives $\varepsilon^V_{L|K,S,T} \in \bigcap^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{L,S,T}$, which is exactly the Rubin-Stark Conjecture for the data $(L|K, S, T, V)$. □

References

[Bak67] A. Baker. "Linear forms in the logarithms of algebraic numbers. I, II, III". In: ***Mathematika 13 (1966), 204-216; ibid. 14 (1967), 102-107; ibid.*** 14 (1967), pp. 220–228.

[BKS16] David Burns, Masato Kurihara and Takamichi Sano. "On zeta elements for $\mathbb{G}_m$". In: ***Doc. Math.*** 21 (2016), pp. 555–626.

[Bur07] David Burns. "Congruences between derivatives of abelian L-functions at $s = 0$". In: ***Invent. Math.*** 169.3 (2007), pp. 451–499.

[Bur11] David Burns. "An introduction to the equivariant Tamagawa number conjecture: the relation to Stark's conjecture". In: ***Arithmetic of L-functions***. Vol. 18. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2011, pp. 125–152.

[Cona] Keith Conrad. ***Base Extension and Exterior Powers***. URL: `https://kconrad.math.uconn.edu/blurbs/linmultialg/extmodbaseextn.pdf`.

[Conb] Keith Conrad. ***Exterior Powers***. URL: `https://kconrad.math.uconn.edu/blurbs/linmultialg/extmod.pdf`.

[Dir37] P. G. L. Dirichlet. "Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält". In: ***Abhandlungen der Königlichen Preußischen Akademie der Wissenschaften zu Berlin*** 48 (1837).

[DR80] Pierre Deligne and Kenneth A. Ribet. "Values of abelian L-functions at negative integers over totally real fields". In: ***Invent. Math.*** 59.3 (1980), pp. 227–286.

[Hee52] Kurt Heegner. "Diophantische Analysis und Modulfunktionen". In: ***Math. Z.*** 56 (1952), pp. 227–253.

[Hof18] Martin Hofer. "Elliptic $\mathfrak{p}$-units and the equivariant Tamagawa Number Conjecture". PhD thesis. Ludwig-Maximilians-Universität München, 2018.

[JN16] Henri Johnston and Andreas Nickel. "On the equivariant Tamagawa number conjecture for Tate motives and unconditional annihilation results". In: ***Trans. Amer. Math. Soc.*** 368.9 (2016), pp. 6539–6574. ISSN: 0002-9947.

[Pop11] Cristian D. Popescu. "Integral and p-adic refinements of the abelian Stark conjecture". In: ***Arithmetic of L-functions***. Vol. 18. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2011, pp. 45–101.

[Rub96] Karl Rubin. "A Stark conjecture "over **Z**" for abelian L-functions with multiple zeros". In: ***Ann. Inst. Fourier (Grenoble)*** 46.1 (1996), pp. 33–62.

[RW96] Jürgen Ritter and Alfred Weiss. "A Tate sequence for global units". In: ***Compositio Math.*** 102.2 (1996), pp. 147–178.

[Sta11] Harold M. Stark. "The origin of the "Stark conjectures"". In: ***Arithmetic of L-functions***. Vol. 18. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2011, pp. 33–44.

[Sta67] Harold M. Stark. "A complete determination of the complex quadratic fields of class-number one". In: ***Michigan Math. J.*** 14 (1967), pp. 1–27.

[Sta71] Harold M. Stark. "Values of L-functions at $s = 1$. I. L-functions for quadratic forms". In: ***Advances in Math.*** 7 (1971), 301–343 (1971).

[Sta75] Harold M. Stark. "L-functions at $s = 1$. II. Artin L-functions with rational characters". In: ***Advances in Math.*** 17.1 (1975), pp. 60–92.

[Sta76] Harold M. Stark. "L-functions at $s = 1$. III. Totally real fields and Hilbert's twelfth problem". In: ***Advances in Math.*** 22.1 (1976), pp. 64–84.

[Sta80] Harold M. Stark. "L-functions at $s = 1$. IV. First derivatives at $s = 0$". In: ***Adv. in Math.*** 35.3 (1980), pp. 197–235.

[Tat67] John T. Tate. "Fourier analysis in number fields, and Hecke's zeta-functions". In: ***Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)***. Thompson, Washington, D.C., 1967, pp. 305–347.

[Tat84] John T. Tate. ***Les conjectures de Stark sur les fonctions*** L ***d'Artin en*** $s = 0$. Vol. 47. Progress in Mathematics. Lecture notes edited by Dominique Bernardi and Norbert Schappacher. Birkhäuser Boston, Inc., Boston, MA, 1984.

[Val18] Daniel Vallières. "The equivariant Tamagawa number conjecture and the extended abelian Stark conjecture". In: ***J. Reine Angew. Math.*** 734 (2018), pp. 1–58.