An Introduction to the eTNC

Abstract

The equivariant Tamagawa Number Conjecture (eTNC) is a deep prediction connecting analytically defined objects with algebraically defined objects that are associated to ‘motives’ (whatever they are). Famous specialisations of this conjecture include the analytic class number formula (which is known), the BSD conjecture, and equivariant refinements of such. In the case of abelian extensions of number fields, the \((p\text{-part})\) of the eTNC admits a rather down-to-earth formulation. In this talk I will give a motivation towards the formulation of the eTNC. I will then introduce determinant modules and then give the formulation of the aforementioned specialisation. If time permits, I will then briefly outline the proof of one of the only unconditionally known cases: that of abelian extensions of \(\mathbb{Q}\) via the equivariant main conjecture.

1 Motivation

An overarching theme in algebraic number theory is the study of special values of \(L\)-functions (e.g the Riemann zeta function). By special values we usually mean values of \(L\)-functions at integer arguments or the leading term of the Taylor expansion of the \(L\)-function around an integer argument. Many spectacular results exist out there linking these special values to the algebraic world. For example:

1. The Herbrand-Ribet Theorem: \(p\) divides the class number of \(\mathbb{Q}(\zeta_p)\) if and only if \(p\) divides the numerator of some Bernoulli number \(B_n\) for some \(0 < n < p - 1\) (recall that the Bernoulli numbers occur as special values of the Riemann zeta function)

2. The Analytic Class Number Formula: Expresses the residue of the pole at \(s = 1\) of the Dedekind zeta function of a number field \(K\) in terms of classical algebraic invariants of \(K\) (regulator, discriminant etc.)

The theory of special values of \(L\)-functions of the ‘motive’ (whatever this is precisely is far beyond the scope of this talk but its okay to believe that a motive is some arithmeto-geometrically significant object) \(h^0(\text{Spec}(K))\) where \(K\) is an abelian number field is rich and, by now, fairly well-understood. Results for other motives are far more sparse and are only predicted in general as part of large conjectural frameworks.

An example of this is the Birch and Swinnerton-Dyer conjecture (a millenium prize problem) which is an elliptic curve version of the analytic class number formula and is still wide-open (although there exist proofs of special cases, see the Coates-Wiles Theorem which is a proof of the weak BSD for elliptic curves with CM of analytic rank at most 1).

The formulation of the TNC is an attempt to unify all the various ‘analytic class number formula’-esque conjectures under one banner. The eTNC is an attempt to do this while taking into account the action of some algebra on the ‘motive’ as well. For example, the motive \(h^0(\text{Spec}(K))\) comes equipped with the action of the algebra \(\mathbb{Z}[\text{Gal}(K/\mathbb{Q})]\).
2 Determinant Modules

Let $R$ be a connected commutative unital ring and $P$ a finitely generated projective $R$-module of rank $r$. We define

$$d_R(P) := \bigwedge_R^r P$$

to be the determinant module of $P$. It is an invertible $R$-module.

In general, we can write $1_R \in R$ as a sum of primitive idempotents $1_R = \sum_i e_i$ of $R$ and we then have a decomposition $R = \bigoplus_i e_i R$ where each $e_i R$ is connected and, similarly, a decomposition $P = \bigoplus_i e_i P$. Then each $e_i P$ has constant local rank $r_i$ and we define

$$d_R(P) := \bigoplus_i \bigwedge_{R_i}^{r_i} e_i P$$

Given a morphism of finitely generated projective $R$-modules $f : P_1 \to P_2$ we have an induced morphism of determinants

$$d_R(f) : d_R(P_1) \to d_R(P_2)$$

and so we get a functor from finitely generated projective $R$-modules to invertible $R$-modules. We shall refer to this functor as the determinant functor.

Remark. In fact the actual definition of determinant functor must take into account grading in order to avoid subtle sign issues. Since we will completely gloss over this in this talk (and it would only serve to complicate things), I will not discuss this.

An important property of this construction is the following: given a short exact sequence of finitely generated projective $R$-modules

$$0 \to P_1 \to P_2 \to P_3 \to 0$$

there is a canonical isomorphism of determinants

$$d_R(P_2) \cong d_R(P_1) \otimes_R d_R(P_3)$$

This functor descends to a functor on perfect complexes of $R$-modules (i.e. quasi-isomorphic to a bounded complex of finitely generated projective $R$-modules) by setting

$$d_R(C^\bullet) := \bigotimes_{i \in \mathbb{Z}} d_R(C^i)^{(-1)^i}$$

for an appropriately chosen representative $C^\bullet$ (one can check that this definition is independent of the choice of representative of $C^\bullet$ and, in fact, it is a Theorem of Deligne that this is the only appropriate way to extend the determinant functor to the derived category).

A key fact is the following: if $C^\bullet$ has the property that all of its cohomology modules are perfect (when regarded as complexes), one has a canonical isomorphism

$$d_R(C^\bullet) \cong \bigotimes_i d_R(H^i(C))^{(-1)^i}$$

Finally, if $R \to S$ is a homomorphism of rings then we also have a canonical base-change isomorphism

$$d_R(C^\bullet) \otimes_R S \cong d_S(C^\bullet \otimes_R^L S)$$
3 Étale cohomology complexes

We fix forever a prime \( p \), a number field \( K \) and a finite abelian extension \( E \) of \( K \). Set \( G = \text{Gal}(E/K) \). Let \( S \) be the set of places of \( K \) consisting of the archimedean places, the \( p \)-adic places, and all the primes that ramify in \( E \). Write furthermore \( \mathcal{O}_{E,S(E)} \) for ring of \( S(E) \)-integers of \( E \) (i.e. those that are integers away from the primes above those in \( S(E) \)). Now, we have a natural localisation morphism of étale cohomology complexes

\[
\mathbb{R}\Gamma(\mathcal{O}_{E,S(E)}, \mathbb{Z}_p) \to \bigoplus_{w/v \in S(E)} \mathbb{R}\Gamma(E_w, \mathbb{Z}_p)
\]

We define \( \mathbb{R}\Gamma_c(\mathcal{O}_{E,S(E)}, \mathbb{Z}_p) \) to be the mapping fibre of this morphism in \( D(\mathbb{Z}_p[G]) \) and we note that this a perfect complex of \( \mathbb{Z}_p[G] \)-modules.

Define now

\[
C_E := \mathbb{R}\text{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{E,S(E)}, \mathbb{Z}_p)[−2]
\]

A calculation involving Artin-Verdier duality gives a canonical isomorphism \( H^0(C_E) \cong \mathcal{O}_{E,S(E)}^\times \otimes_\mathbb{Z} \mathbb{Z}_p \) and an exact sequence

\[
0 \to \text{Cl}_{S(E)}(E) \to H^1(C_E) \to X_{E,S(E)} \to 0
\]

where \( X_{E,S(E)} \) is the kernel of the augmentation map \( Y_{E,S(E)} \to \mathbb{Z}_p \) and \( Y_{E,S(E)} := \bigoplus_{w/v \in S(E)} \mathbb{Z}_p \).

4 The eTNC for the extension \( E/K \)

Fix in this section an isomorphism \( \mathbb{C} \cong \mathbb{C}_p \). The Dirichlet regulator isomorphism \( \mathbb{R} \otimes_\mathbb{Z} \mathcal{O}_{E,S(E)} \to \mathbb{R} \otimes_\mathbb{Z} X_{E,S(E)} \) induces a canonical isomorphism

\[
\vartheta_E : d_{\mathbb{C}_p[G]}(C_E \otimes_\mathbb{Z} \mathbb{C}_p) \cong d_{\mathbb{C}_p[G]}(\mathcal{O}_{E,S(E)}^\times) \otimes_{\mathbb{C}_p[G]} d_{\mathbb{C}_p[G]}(X_{E,S(E)}) \\
\cong d_{\mathbb{C}_p[G]}(X_{E,S(E)}) \otimes_{\mathbb{C}_p[G]} d_{\mathbb{C}_p[G]}(X_{E,S(E)}) \\
\cong \mathbb{C}_p[G]
\]

Now, for each character \( \chi \in \hat{G} \), we have the the \( S(E) \)-truncated Artin \( L \)-function

\[
L_{E,S(E)}(\chi, s) := \prod_{v \notin S(E)} \left( 1 - \chi(\text{Fr}_v)nu^{-s} \right)^{-1}
\]

and let \( L_{E,S(E)}(\chi, 0) \) be its leading term of its Taylor expansion at \( s = 0 \). We then have an equivariant leading term

\[
\theta_{E,S(E)}(0) := \sum_{\chi \in \hat{G}} L_{E,S(E)}(\chi^{-1}, 0)e_\chi \in \mathbb{R}[G]^\times
\]

We may regard \( \theta_{E,S(E)}(0) \in \mathbb{C}_p[G]^\times \). We define the zeta element \( \zeta_E \) to be the inverse image of \( \Theta_{E,S(E)}(0) \) under the isomorphism \( \vartheta_E \). We may now formulate (the \( p \)-part of) the eTNC for the extension \( E/K \):

**Conjecture 1.** One has an equality of free rank one \( \mathbb{Z}_p[G] \)-modules

\[
d_{\mathbb{Z}_p[G]}(C_E) = \langle \zeta_E \rangle_{\mathbb{Z}_p[G]}
\]
Remark. There are a wide variety of consequences of the above conjecture:

- The Stark Conjectures and several integral refinements thereof (Rubin-Stark, Emmons-Popescu, etc.)
- The Mazur-Rubin-Sano Conjecture on Rubin-Stark elements
- The Brumer-Stark Conjecture
- Various results concerning Fitting ideals of Class Groups

Remark. The eTNC is known in the following cases:

- $K = \mathbb{Q}$
- $K$ is an imaginary quadratic field and $p$ is a prime that is completely split in $K/\mathbb{Q}$ and does not divide the class number of $K$.
- $K$ is a function field
- Various results in the non-abelian setting for particular extensions.

5 The eTNC for abelian extensions of $\mathbb{Q}$

The above conjecture is valid when $K = \mathbb{Q}$ and $E/K$ is any abelian extension by the work of Burns and Greither. In this section I will (very briefly) outline the proof of this.

To do this, we first observe that class field theory (together with functoriality properties of the eTNC) means that it suffices to prove the conjecture for cyclotomic fields. To this end, we fix a non-negative integer $m$ (and for simplicity lets assume that $m$ is coprime to $p$). Set $E_m = \mathbb{Q}(\zeta_m)^+ \}$ (here we take totally real parts also for simplicity). Set $G_m := \text{Gal}(E_m/\mathbb{Q})$ and $S(m) := S(E_m)$. Define the Iwasawa algebra $\Lambda_m := \lim_{\leftarrow n} \mathbb{Z}[G_m]$ and define a perfect complex of $\Lambda_m$-modules $C_\infty := \lim_{\leftarrow n} C_{E_{mp^n}}$.

Then there is a canonical isomorphism $H^0(C_m) \cong \lim_{\leftarrow n} \mathcal{O}_{E_{mp^n}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p$ and a short exact sequence

$0 \to \text{Cl}_m^\infty \to H^1(C_m) \to X_m^\infty \to 0$

where $\text{Cl}_m^\infty := \lim_{\leftarrow n} \text{Cl}_{S(mp^n)}(E_{mp^n})$ and similarly for $X_m^\infty$.

It is a classically known result that $\text{Cl}_m^\infty$ is a torsion $\Lambda_m$-module. Hence if $Q_m$ denotes the total quotient ring of $\Lambda_m$ (obtained by inverting all the non-zero-divisors) one has $Q_m \otimes_{\Lambda_m} \text{Cl}_m^\infty = 0$. Since $\mathbb{Z}_p$ is (trivially) a torsion $\Lambda_m$-module, it then follows that there is an isomorphism

$d_{Q_m(\chi)}(Q_m(\chi) \otimes_{\Lambda_m} C_m^\infty) \cong (Q_m(\chi) \otimes_{\Lambda_m} \mathcal{O}_m^\infty) \otimes Q_m(\chi) (Q_m(\chi) \otimes_{\Lambda_m} (Y_m^\infty)^*)$

where for each $\chi \in \hat{G}_m$ which is $\mathbb{Q}_p$-rational, $Q_m(\chi)$ is the field corresponding to $\chi$ in the Artin-Wedderburn decomposition $Q_m = \prod_{\chi} Q_m(\chi)$. Now define

$\eta_m := ((1 - \zeta_{mp^n})(1 - \zeta_{mp^{-1}}))_{n \in \mathbb{N}} \in \mathcal{O}_m^\infty$

$\tau_m := (\sigma_{mp^n})_{n \in \mathbb{N}} \in Y_m^\infty$

$L_m := \eta_m \otimes \tau_m^{-1}$
where $\sigma_{mp^n}$ is the place corresponding to the inclusion $E_{mp^n} \subseteq \mathbb{C}$.

We can now formulate the following equivariant Iwasawa main conjecture:

**Theorem 2** (Burns-Greither, Flach). There is an equality of invertible $\Lambda_m$-modules

$$d_{\Lambda_m}(C_m^\infty) = \langle L_m \rangle_{\Lambda_m}$$

**Proof.** (Very sketchy) $\Lambda_m$ is a Cohen-Macaulay ring so it suffices to check the equality at each height one prime ideal of $\Lambda_m$. Call such an ideal $p$ regular if $p \not\in p$. Then one uses the Euler system distribution relation to show that the required equality after localising at $p$ is equivalent to the classical Iwasawa main conjecture.

If $p$ contains $p$ then the required equality follows from the vanishing of the Iwasawa $\mu$-invariant of the $\Lambda_m$-module $\mathcal{O}_m^\infty / \langle \eta_m \rangle$.

Burns and Greither then use this to prove the eTNC as follows: There is a canonical ring map $\Lambda_m \to \mathbb{Z}_p[G_m]$ and a natural codescent isomorphism $C_m^\infty \otimes_{\Lambda_m} \mathbb{Z}_p[G_m] \cong C_{E_m}$ and so there is a canonical codescent isomorphism

$$d_{\Lambda_m}(C_m^\infty) \otimes_{\Lambda_m} \mathbb{Z}_p[G_m] \cong d_{\mathbb{Z}_p[G_m]}(C_{E_m})$$

The image of $L_m$ under this isomorphism is a $\mathbb{Z}_p[G_m]$-basis of $d_{\mathbb{Z}_p[G_m]}(C_{E_m})$ and so it suffices to show that its image coincides with the zeta element $\zeta_{E_m}$. This is done by a lengthy computation involving so-called Bockstein homomorphisms.