Exact Lexicographic Scheduling and Approximate Rescheduling

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Abstract

In industrial resource allocation problems, an initial planning stage may solve a nominal problem instance and a subsequent recovery stage may intervene to repair inefficiencies and infeasibilities due to uncertainty, e.g. machine failures and job processing time variations. In this context, we investigate the minimum makespan scheduling problem, a.k.a. $P||C_{\max}$, under uncertainty. We propose a two-stage robust scheduling approach where first-stage decisions are computed with exact lexicographic scheduling and second-stage decisions are derived using approximate rescheduling. We explore recovery strategies accounting for planning decisions and constrained by limited permitted deviations from the original schedule. Our approach is substantiated analytically, with a price of robustness characterization parameterized by the degree of uncertainty, and numerically. This analysis is based on optimal substructure imposed by lexicographic optimality. Thus, lexicographic optimization enables more efficient rescheduling. Further, we revisit state-of-the-art exact lexicographic optimization methods and propose a lexicographic branch-and-bound algorithm whose performance is validated computationally.

Keywords: Scheduling, Lexicographic Optimization, Exact MILP Methods, Robust Optimization, Price of Robustness

1. Introduction

Motivated by industrial resource allocation problems, we consider scheduling under uncertainty, e.g. a machine may unexpectedly fail, a client may suddenly

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cancel a job, or jobs are completed earlier than expected. We focus on robust scheduling, i.e. hedge against worst-case realizations of imprecise parameter values, such as job processing times and number of available machines, lying in well-defined uncertainty sets [5, 8, 26, 56]. Because static robust optimization may produce conservative solutions compared to ones obtained with perfect knowledge [54], we investigate two-stage robust optimization with recovery [6, 9, 10, 31, 36]. As shown in Figure 1, (i) an initial planning stage computes a solution with nominal parameter values and (ii) a subsequent recovery stage modifies the solution once the uncertainty is realized, i.e. after the final parameter values become known.

We elaborate on the fundamental makespan scheduling problem, a.k.a. $P||C_{\max}$ [15, 27, 35]. With perfect knowledge, an instance I of the problem posits a set $\mathcal{J}$ of jobs, each one associated with processing time p_j, a set $\mathcal{M}$ of parallel identical machines and the objective is to construct a non-preemptive schedule S of minimum makespan $C_{\max} = \max_{i \in \mathcal{M}}\{C_i\}$, i.e. maximum machine completion time. In a two-stage setting under uncertainty, the planning stage solves a nominal instance I_{init} producing a solution S_{init} and the recovery stage transforms schedule S_{init} to a new schedule S_{new} for I_{new} by repairing inefficiencies and infeasibilities, e.g. due to job processing time variations and machine failures, as illustrated in Figure 2.

In the extreme case, the recovery stage may solve I_{new} from scratch without accounting for the first-stage decisions in S_{init}. This level of flexibility may sacrifice benefits due to planning and can be resource-consuming. For example, significantly modifying machine schedules may incur substantial communication costs in distributed computing [57]. To mitigate this overhead, we only allow a bounded number of modifications to S_{init}. Technically, we distinguish between *binding* and *free* optimization decisions. Binding decisions are variable evaluations determined from the initial solution after uncertainty realization. Free decisions are variable evaluations that cannot be determined from the initial solution, but are essential to ensure feasibility. For instance, scheduling a job with a modified processing time is a binding decision because the planning stage already specifies an assignment. Assigning a new job after uncertainty realization is a free decision because no assignment is given in the planning stage. Further, we study rescheduling with limited binding decision modifications and thereby stay close to the initial solution.

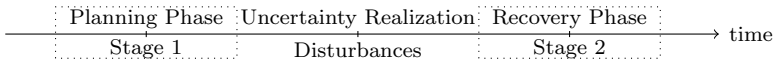


Figure 1: Recoverable robustness setting

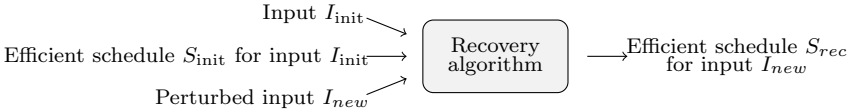


Figure 2: Makespan recovery problem

By allowing few modifications, first-stage decisions remain critical.

A two-stage robust optimization method should specify (i) a way of producing the initial solution S_{init}, and (ii) a recovery strategy for restoring S_{init} and deriving the recovered solution S_{rec}, after uncertainty realization. Analyzing a two-stage robust optimization method requires defining (i) the uncertainty set of the problem, and (ii) the investigated performance guarantee.

Uncertainty Set. The uncertainty set of a robust optimization problem specifies a range of possible values for the uncertain parameters [33]. We consider a generalization of well-known Γ-uncertainty sets, where the final parameter values $\hat{p}_j$ vary in an interval $[p_j^L, p_j^U]$ and at most k parameters can deviate from their nominal values [11]. Here, the uncertainty set is defined by a pair (k, f), where k is the maximum number of unstable parameters with respect to perturbation factor $f > 1$. A parameter $p_j > 0$ is stable if $p_j/f \leq \hat{p}_j \leq fp_j$ and unstable, otherwise.

Performance Guarantee. Theoretical performance guarantees are useful for determining when robust optimization methods are efficient [8, 26]. Denote by $C(I_{new})$ the cost, e.g. makespan, of a solution obtained by some robust optimization method and by $C^*(I_{new})$ the cost of an optimal solution obtained with perfect knowledge [40]. We consider the so-called *price of robustness* which is defined as the ratio between the two and seek a tight, worst-case performance guarantee $\rho = \max_{I_{new} \in \mathcal{I}}(C(I_{new})/C^*(I_{new}))$ within the set $\mathcal{I}$ of all problem instances [12].

Related Work. Prior literature shows that scheduling problems become computationally harder after incorporating uncertainty [28, 29, 56]. Kasperski & Zielinski [32] survey techniques and negative results for robust scheduling, including $P||C_{\max}$ with uncertain job processing times. Typically, robustness is achieved by optimizing the worst-case (i) cost or (ii) distance from the achievable optimum with perfect knowledge, over all scenarios in the uncertainty set. These robust counterparts of standard deterministic optimization problems are often referred to as *minmax* and *minmax regret*, respectively [33].

With perfect knowledge, $P||C_{\max}$ is strongly $\mathcal{NP}$-hard, but admits greedy constant-factor approximation algorithms and polynomial-time approximation schemes (PTASs) [15, 35]. When the number of machines is constant, $Pm||C_{\max}$ is weakly $\mathcal{NP}$-hard and has fully polynomial-time approximation schemes. With budgeted uncertainty, where B bounds the deviation of the sum of processing times from their nominal values, minmax $P||C_{\max}$ admits a PTAS when B is constant and a 3-approximation algorithm for arbitrary B [13].

To our knowledge, no prior work analyzes the price of robustness for $P||C_{\max}$ under uncertainty. Determining the price of robustness for fundamental combinatorial optimization problems has been repeatedly posed as an open question by domain experts [8, 12, 26]. The current manuscript shows that such an analysis provides useful structural properties and quantifies the effect of uncertainty in robust solutions for $P||C_{\max}$.

Lexicographic Optimization. LexOpt is at the core of our two-stage scheduling approach. LexOpt is a subclass of multiobjective optimization minimizing m objective functions $F_1, \ldots, F_m : \mathcal{S} \to \mathbb{R}_0^+$, in decreasing priority order [21, 45]. In other words, LexOpt optimizes the highest-rank objective F_1, then the second most important objective F_2, then the third F_3, etc.:

$$\operatorname{lex\,min}\{F_1(S), \ldots, F_m(S) : S \in \mathcal{S}\}. \tag{LexOpt}$$

There are indications that LexOpt is useful in optimization under uncertainty. LexOpt maintains a good approximate schedule when jobs are added and deleted dynamically [48, 53]. LexOpt is also useful for cryptographic systems against attacks [58]. We consider the LexOpt scheduling problem $\operatorname{lex\,min}\{C_1(S), \ldots, C_m(S) :$

$S \in \mathcal{S}\}$ of computing a schedule S with lexicographically minimal machine completion times and show that it enables more efficient two-stage robust scheduling. That is, we identify robust scheduling as a new LexOpt application.

Apart from optimization under uncertainty, designing efficient LexOpt methods is motivated by LexOpt applications: equitable allocation of a divisible resource [25, 37], fairness [14], and exploiting opponent mistakes in game theory [42, 50]. Solution strategies include sequential, weighting, and highest-rank objective methods [16, 19, 21, 44, 45, 51, 52]. There is work characterizing the convex hull of LexOpt problems [1, 30, 41]. Logic-based methods are also applicable [38].

Contributions and Paper Organization. Our main contribution is a two-stage robust scheduling approach for $P||C_{\max}$ under uncertainty, where first-stage decisions are computed with mixed-integer linear programming and lexicographic optimization, while second-stage decisions are derived using approximation algorithms. Despite the relevant literature on two-stage robust optimization for various applications [6, 17, 18, 36], we are not aware of any work on cornerstone scheduling problems, such as $P||C_{\max}$, combined with a price of robustness characterization.

The manuscript proceeds as follows. Section 2 formally defines $P||C_{\max}$, LexOpt scheduling, and the considered perturbation types. Section 3 develops a branch-and-bound algorithm for the LexOpt scheduling problem. Section 4 proposes a recovery strategy and analyzes the performance of the overall two-stage approach theoretically. Section 5 substantiates the branch-and-bound method with respect to state-of-the-art LexOpt approaches adapted to LexOpt scheduling. Further, Section 5 validates our two-stage method empirically. Section 6 concludes.

After proving that the makespan recovery problem is strongly $\mathcal{NP}$-hard, at least as hard as solving the problem with full input knowledge, we elaborate on performance guarantees for two-stage $P||C_{\max}$ under uncertainty. Technically, we investigate a basic recovery strategy that enforces all available binding decisions and performs only essential actions to regain feasibility. On the negative side, every recovered solution is a weak approximation if planning produces an arbitrary nominal optimal solution. Specifically, every recovered solution attains an $\Omega(m)$ price of robustness, even in the case of a single perturbation. On the positive side, we obtain significantly better performance guarantees if the initial solution is LexOpt.

For a single perturbation, planning using LexOpt ensures a price of robustness equal to 2. For multiple perturbations, the initial solution can be weakly reoptimizable with a high-degree of uncertainty. However, we show an asymptotically tight $O(f(1+\frac{k}{m-k})(f+k)(1+\frac{\delta}{m}))$ price of robustness, where k is the number of unstable jobs with respect to a perturbation factor f and δ/m is the fraction of additional machines, after uncertainty realization. This result exploits our uncertainty set structure. Therefore, when k, f and δ/m are constant, our approach achieves an $O(1)$ price of robustness.

The main paper includes the proof of Theorem 3 and part of the proofs for Theorems 4-5, which bound the price of robustness of our two-stage robust scheduling approach for $P||C_{max}$ under uncertainty in the case of single and multiple perturbations, respectively, by exploiting the optimal substructure imposed by LexOpt. All remaining proofs are provided in the supplementary material.

2. Problem Definitions

This section defines the $P||C_{\max}$ problem (Section 2.1), the LexOpt scheduling problem (Section 2.2), and describes the investigated perturbations (Section 2.3).

2.1. Makespan Scheduling Problem

An instance I of the makespan scheduling problem, a.k.a. $P||C_{\max}$, is a pair $(m, \mathcal{J})$, where $\mathcal{J} = \{J_1, \ldots, J_n\}$ is a set of n jobs, with processing times $p_1, \ldots, p_n$, to be executed by a set $\mathcal{M} = \{M_1, \ldots, M_m\}$ of m parallel identical machines. Job $J_j \in \mathcal{J}$ must be processed by exactly one machine $M_i \in \mathcal{M}$ for p_j units of time non-preemptively, i.e. in a single continuous interval without interruptions. Each machine processes at most one job per time. The objective is to minimize the last machine completion time. Given a schedule S, let $C_{\max}(S)$ and $C_i(S)$ be the makespan and the completion time of machine $M_i \in \mathcal{M}$, respectively, in S. In the following mixed-integer linear programming (MILP) formulation, binary variable $x_{i,j}$ is 1 if job $J_j \in \mathcal{J}$ is executed by machine $M_i \in \mathcal{M}$ and 0, otherwise.

$$\min_{C_{\max}, C_i, x_{i,j}} \quad C_{\max} \tag{1a}$$

$$C_{\max} \geq C_i \qquad M_i \in \mathcal{M} \tag{1b}$$

$$C_i = \sum_{j=1}^{n} x_{i,j} \cdot p_j \qquad M_i \in \mathcal{M} \tag{1c}$$

$$\sum_{i=1}^{m} x_{i,j} = 1 \qquad J_j \in \mathcal{J} \tag{1d}$$

$$x_{i,j} \in \{0,1\} \qquad J_j \in \mathcal{J}, M_i \in \mathcal{M}. \tag{1e}$$

Expression (1a) minimizes makespan. Constraints (1b) enforce that $C_{\max} = \max_{1 \leq i \leq m}\{C_i\}$. Constraints (1c) ensure that a machine executes at most one job per time. Constraints (1d) impose that each job is assigned to exactly one machine.

2.2. LexOpt Scheduling Problem

The problem $\operatorname{lex}\min\{F_1(S), \ldots, F_m(S) : S \in \mathcal{S}\}$ minimizes m objective functions $F_1, \ldots, F_m : \mathcal{S} \to \mathbb{R}_0^+$ over a set $\mathcal{S}$ of feasible solutions. The functions are sorted in decreasing priority order, i.e. F_i is more important than $F_{i'}$, for $i < i'$. In a LexOpt solution S^*, $F_1(S^*) = v_1^* = \min\{F_1(S) : S \in \mathcal{S}\}$ and $F_i(S^*) = v_i^* = \min\{F_i(S) : S \in \mathcal{S}, F_1(S) = v_1^*, \ldots, F_{i-1}(S) = v_{i-1}^*\}$, for $i = 2, \ldots, m$.

Consider two solutions S and S' to the above LexOpt problem. S and S' are *lexicographically distinct* if there is at least one $q \in \{1, \ldots, m\}$ such that $F_q(S) \neq F_q(S')$. Further, S is *lexicographically smaller* than S', i.e. $S <_{\text{lex}} S'$ or $\vec{F}(S) <_{\text{lex}} \vec{F}(S')$, if (i) S and S' are lexicographically distinct and (ii) $F_q(S) < F_q(S')$, where q is the smallest component in which they differ, i.e. $q = \min\{i : F_i(S) \neq F_i(S'), 1 \leq i \leq m\}$. S is *lexicographically not greater* than S', i.e. $S \leq_{\text{lex}} S'$ or $\vec{F}(S) \leq_{\text{lex}} \vec{F}(S')$, if either S and S' are lexicographically equal, i.e. not lexicographically distinct, or $S <_{\text{lex}} S'$. The LexOpt problem $\operatorname{lex}\min\{F_1(S), \ldots, F_m(S) : S \in \mathcal{S}\}$ computes a solution S^* such that $\vec{F}(S^*) \leq_{\text{lex}} \vec{F}(S)$, for all $S \in \mathcal{S}$.

An optimal solution $S = (\vec{x}, \vec{C})$ to an instance $I = (m, \mathcal{J})$ of the LexOpt scheduling problem minimizes m objective functions $F_1, \ldots, F_m$ lexicographically, where F_q is the distinct q-th greatest machine completion time, for $q = 1, \ldots, m$. Lemma 1 provides an ordering of the machine completion times in a LexOpt schedule and states valid inequalities.

Lemma 1. *In an optimal solution to the LexOpt scheduling problem:*

1. $C_i \geq C_{i+1}$, *for* $i = 1, \ldots, m-1$,

2. $i \cdot C_i + \left[\sum_{q=i+1}^{m} C_q\right] \leq \sum_{j=1}^{n} p_j \leq \left[\sum_{q=1}^{i-1} C_q\right] + (m-i+1) \cdot C_i, \forall\, i = 1, \ldots, m.$

Equations (2a) - (2g) formulate LexOpt scheduling using Lemma 1.

$$\operatorname*{lex\,min}_{C_i, x_{i,j}} \quad C_1, \ldots, C_m \tag{2a}$$

$$C_i \geq C_{i+1} \qquad M_i \in \mathcal{M} \setminus \{M_m\} \tag{2b}$$

$$\sum_{q=1}^{i-1} C_q + (m-i+1) \cdot C_i \geq \sum_{j=1}^{n} p_j \qquad M_i \in \mathcal{M} \tag{2c}$$

$$i \cdot C_i + \sum_{q=i+1}^{m} C_q \leq \sum_{j=1}^{n} p_j \qquad M_i \in \mathcal{M} \tag{2d}$$

$$C_i = \sum_{j=1}^{n} x_{i,j} \cdot p_j \qquad M_i \in \mathcal{M} \tag{2e}$$

$$\sum_{i=1}^{m} x_{i,j} = 1 \qquad J_j \in \mathcal{J} \tag{2f}$$

$$x_{i,j} \in \{0,1\} \qquad J_j \in \mathcal{J}, M_i \in \mathcal{M}. \tag{2g}$$

2.3. Perturbations

A two-stage makespan scheduling problem is specified by an initial instance $I_{init} = (m, \mathcal{J})$ and a perturbed instance $I_{new} = (\hat{m}, \hat{\mathcal{J}})$ of $P||C_{\max}$. Let $\mathcal{M}$ and $\hat{\mathcal{M}}$ be the set of machines in I_{init} and I_{new}, respectively. We similarly define the sets $\mathcal{J}$ and $\hat{\mathcal{J}}$, denoting by p_j and $\hat{p}_j$ the corresponding processing times in I_{init} and I_{new} for each job $J_j \in \mathcal{J} \cap \hat{\mathcal{J}}$. With uncertainty realization, instance I_{init} is transformed to I_{new}. This manuscript investigates the two-stage makespan problem in the case of (i) a single perturbation, and (ii) multiple perturbations. In the former case, the effect of uncertainty realization is one of the following perturbations:

1. *[Processing time reduction]* The processing time p_j of job $J_j \in \mathcal{J}$ is decreased and becomes $\hat{p}_j = p_j / f_j$, for some $f_j > 1$.
2. *[Processing time augmentation]* The processing time p_j of job $J_j \in \mathcal{J}$ is increased and becomes $\hat{p}_j = f_j p_j$, for some $f_j > 1$.
3. *[Job cancellation]* Job $J_j \in \mathcal{J}$ is removed, i.e. $\hat{\mathcal{J}} = \mathcal{J} \setminus \{J_j\}$.
4. *[Job arrival]* New job $J_j \notin \mathcal{J}$ arrives, i.e. $\hat{\mathcal{J}} = \mathcal{J} \cup \{J_j\}$.

5. *[Machine failure]* Machine $M_i \in \mathcal{M}$ fails, i.e. $\hat{\mathcal{M}} = \mathcal{M} \setminus \{M_i\}$.
6. *[Machine activation]* New machine $M_i \notin \mathcal{M}$ is added, i.e. $\hat{\mathcal{M}} = \mathcal{M} \cup \{M_i\}$.

These perturbations are frequently encountered in practice and investigated in the literature [32]. In the case of multiple perturbations, I_{new} is obtained from I_{init} by applying a series of perturbations. Certain perturbations can be considered as equivalent. Specifically, in some proofs: (i) cancelling job $J_j \in \mathcal{J}$ is identical to reducing p_j to zero, i.e. $f_j \to \infty$, (ii) failure of machine $M_i \in \mathcal{M}$ is equivalent to new arrivals of the jobs in $\mathcal{J}_i$, where $\mathcal{J}_i$ is the set of jobs assigned to machine M_i in schedule S_{init}, (iii) job arrivals are treated similarly to processing time augmentations. Let f_j be the *perturbation factor* of job $J_j \in \mathcal{J}$. In our uncertainty set, f is the $(k+1)$-th greatest f_j, $k = |\{J_{j'} \in \mathcal{J} : f_{j'} > f\}|$ is the number of *unstable jobs* and $\delta = \max\{\hat{m} - m, 0\}$ is the number of *surplus machines* after uncertainty realization.

3. Exact LexOpt Branch-and-Bound Algorithm (Stage 1)

This section introduces a LexOpt branch-and-bound algorithm. The supplementary material describes the sequential [19, 16], weighting [51, 52], and highest-rank objective [44] methods adapted to LexOpt scheduling. The branch-and-bound algorithm uses vectorial bounds to eliminate subtrees that cannot lexicographically dominate the incumbent, i.e. the best solution found thus far, by extending ideas for computing *ideal points* in multiobjective optimization [21]. In LexOpt scheduling, we may derive vectorial lower and upper bounds by approximating a multiprocessor scheduling problem with rejections that generalizes $P||C_{\max}$. So, we propose packing-based algorithms for computing vectorial bounds. Next, we describe the branch-and-bound algorithm, our bounding approach and show their correctness.

Definition 1 (Vectorial Bound). *Suppose that $\vec{C}(S) = (C_1(S), \ldots, C_m(S))$ is the non-increasing vector of machine completion times in a feasible schedule S of the LexOpt scheduling problem. Vector $\vec{L} = (L_1, \ldots, L_m)$ is a* **vectorial lower bound** *of S if $L_i \leq C_i(S)$, for each $1 \leq i \leq m$. A* **vectorial upper bound** *$\vec{U} = (U_1, \ldots, U_m)$ of S has $U_i \geq C_i(S)$, for each $1 \leq i \leq m$.*

3.1. Branch-and-Bound Description

Initially, we sort jobs in non-increasing processing times, i.e. $p_1 \geq \ldots \geq p_n$. The search space is a tree with $n+1$ levels. The root node appears at level 0. The leaves are the set $\mathcal{S}$ of all possible m^n possible schedules, i.e. job-to-machine assignments. Each non-leaf node v at level $\ell \in \{0, 1, \ldots, n-1\}$ of the tree represents a fixed assignment of jobs $J_1, \ldots, J_\ell$ to the m machines and jobs $J_{\ell+1}, \ldots, J_n$ remain to be assigned. In addition, node v has m children corresponding to every possible assignment of job $J_{\ell+1}$ to the m machines.

Denote by $\mathcal{S}(v)$ the set of all schedules in the subtree rooted at node v. The branch-and-bound algorithm computes a vectorial lower bound $\vec{L}$ on the lexicographically smallest schedule $S^* \in \mathcal{S}(v)$ below node v. Moreover, the primal heuristic applied in each node is longest processing time first (LPT) [27]. In each schedule S obtained by LPT, the branch-and-bound algorithm reorders the machines so that $C_1(S) \geq \ldots \geq C_m(S)$. Note that this lexicographic ordering may not hold for the partial schedule of jobs $J_1, \ldots, J_\ell$ associated with node v.

Using the above components, the branch-and-bound algorithm traverses the search tree via *depth-first search*. Stack Q stores the set of visited nodes that remain to be explored. Variable I stores the *incumbent*, i.e. the current lexicographically smallest solution. In every step, the algorithm picks the node u on top of Q and explores its m children. At each $v \in \mathit{children}(u)$, if LPT finds a solution S such that $\vec{C}(S) <_{\text{lex}} \vec{C}(I)$, then I is updated. If v is not a leaf, Algorithm 1 computes a vectorial lower bound $\vec{L}$ of the lexicographically best solution in $\mathcal{S}(v)$. When $\vec{C}(I) \leq_{\text{lex}} \vec{L}$, the set $\mathcal{S}(v)$ does not contain any solution lexicographically better than I and the subtree rooted at v is *fathomed*. Otherwise, v is pushed onto stack Q. Upon termination, the incumbent is optimal because every other solution has been rejected as not lexicographically smaller than the incumbent.

3.2. Vectorial Bound Computation

Next, we describe the computation of a vectorial lower bound $\vec{L} = (L_1, \ldots, L_m)$ and a vectorial upper bound $\vec{U} = (U_1, \ldots, U_m)$ at a node v in the ℓ-th level of the search tree. Correctness proofs are provided in the supplementary material. The algorithm performs m iterations. In iteration $i \in \{1, \ldots, m\}$, it calculates a lower bound L_i (Algorithm 1) and an upper bound U_i (Algorithm 2) on the i-th machine

Algorithm 1 Computation of the i-th vectorial lower bound component

1: Select the job index $\min\{h : \sum_{j=\ell+1}^{h} p_j \geq \sum_{q=1}^{i-1}(U_q - t_q)\}$.
2: Compute the remaining load $\lambda = \sum_{j=h+1}^{n} p_j$.
3: Set $\tau = \max_{i \leq q \leq m}\{t_q\}$.
4: Return the maximum among:

- $\min_{i \leq q \leq m}\{t_q\} + p_{h+1}$, and
- $\max_{i \leq q \leq m}\{t_q\} + \max\left\{\frac{1}{m-i+1}\left(\lambda - \sum_{q=i+1}^{m}(\tau - t_q)\right), 0\right\}$.

completion time using bounds $U_1, \ldots, U_{i-1}$ and $L_1, \ldots, L_{i-1}$, respectively. Recall that $p_1 \geq \ldots \geq p_n$. W.l.o.g., each machine executes all jobs with index $\leq \ell$ before any job with index $> \ell$. So, for each schedule in $\mathcal{S}(v)$, a unique vector $\vec{t} = (t_1, \ldots, t_m)$ specifies the machine completion times by considering only jobs $J_1, \ldots, J_\ell$ and ignoring the remaining ones. Further, no job J_j with $\ell + 1 \leq j \leq n$ is executed before time t_q on machine M_q, for $1 \leq q \leq m$.

Vectorial lower bound component L_i. This computation is equivalent to constructing a *pseudo-schedule* $\widetilde{S}$ where some jobs are scheduled fractionally, i.e. fragmented across machines. Initially, Algorithm 1 fractionally assigns jobs $J_{\ell+1}, \ldots, J_h$ to machines $M_1, \ldots, M_{i-1}$, where h is the smallest index such that $\sum_{j=\ell+1}^{h} p_j \geq \sum_{q=1}^{i-1}(U_q - t_q)$. For each $q = 1 \ldots i-1$, machine M_q is assigned sufficiently large job pieces so that its completion time is greater than or equal to U_q. Next, Algorithm 1 fractionally assigns the remaining load $\lambda = \sum_{j=h+1}^{n} p_j$ of jobs $J_{h+1}, \ldots, J_n$ to machines $M_i, \ldots, M_m$. This assignment minimizes the i-th greatest completion time in $\widetilde{S}$. Assuming that $p_{n+1} = 0$, the value L_i is the maximum among $\min_{i \leq q \leq m}\{t_q\} + p_{h+1}$ and $\max_{i \leq q \leq m}\{t_q\} + \max\left\{\frac{1}{m-i+1}\left(\lambda - \sum_{q=i+1}^{m}(\tau - t_q)\right), 0\right\}$, where $\tau = \max_{i \leq q \leq m}\{t_q\}$.

Lemma 2. *Consider a node v in the ℓ-th level of the search tree and a machine index $i \in \{1, \ldots, m\}$. Algorithm 1 produces a value $L_i \leq C_i(S)$ for each feasible schedule $S \in \mathcal{S}(v)$ below v such that $C_q(S) \leq U_q$, $\forall\ q = 1, \ldots, i-1$.*

Vectorial upper bound component U_i. Like L_i, the computation of U_i can be interpreted as constructing a fractional pseudo-schedule $\widetilde{S}$. Additionally, Algorithm 2

Algorithm 2 Computation of the i-th vectorial upper bound component

1: Compute the remaining load $\lambda = \sum_{j=\ell}^{n} p_j - \sum_{q=1}^{i-1}(L_q - t_q)$.
2: Sort machines $M_i, \ldots, M_m$ so that $t_i \leq \ldots \leq t_m$.
3: Select the machine index $\min\left\{\mu : \frac{1}{\mu-i+1}\left(\sum_{q=i}^{\mu} t_q + \lambda\right) \leq t_{\mu+1},\ \ i \leq \mu \leq m\right\}$.
4: Return the minimum among $\max\left\{\frac{1}{\mu-i+1}\left(\sum_{q=i}^{\mu} t_q + \lambda\right) + p_\ell,\ \ t_m\right\}$ and $C_i(I)$.

uses the incumbent I. Schedule $\widetilde{S}$ combines the partial schedule for jobs $J_1, \ldots, J_\ell$ at node v with a pseudo-schedule for the remaining jobs $J_{\ell+1}, \ldots, J_n$ computed by Algorithm 2. Initially, Algorithm 2 assigns a total load $\sum_{q=1}^{i-1}(L_q - t_q)$ of the smallest jobs to machines $M_1, \ldots, M_{i-1}$ so that the completion time of M_q becomes exactly equal to L_q, for $q = 1, \ldots, i-1$. That is, a piece $\tilde{p}_h$ of job J_h and jobs $J_{h+1}, J_{h+2}, \ldots, J_n$ are assigned fractionally to machines $M_1, \ldots, M_{i-1}$ so that $\tilde{p}_h + \sum_{j=h+1}^{n} p_j = \sum_{q=1}^{i-1}(L_q - t_q)$. Next, Algorithm 2 assigns the remaining load $\lambda = \sum_{j=\ell+1}^{h-1} p_j + (p_h - \tilde{p}_h)$ of jobs $J_{\ell+1}, \ldots, J_h$ fractionally and uniformly to the least loaded machines among $M_i, \ldots, M_m$ as follows. Firstly, the partial completion times are sorted so that $t_i \leq \ldots \leq t_m$. This sorting occurs only for computing the vectorial upper bound and does not modify any partial schedule of a node in the search tree. Let μ be the minimum machine index such that (i) the remaining load λ can be fractionally scheduled to machines $M_i, \ldots, M_\mu$ so that they end up with a common completion time $\tau = \frac{1}{\mu-i+1}\left(\sum_{q=i}^{\mu} t_q + \lambda\right)$, and (ii) the partial completion time t_q of any other machine among $M_{\mu+1}, \ldots, M_m$ is at least τ, i.e. $t_{\mu+1} \geq \tau$. Then, U_i is set as the minimum among $\max\{\tau + p_\ell, t_m\}$ and $C_i(I)$.

Lemma 3. *Consider a node v in the ℓ-th level of the search tree and a machine index $i \in \{1, \ldots, m\}$. Algorithm 2 produces a value $U_i \geq C_i(S)$ for each feasible schedule $S \in \mathcal{S}(v)$ below v such that $C_q(S) \geq L_q$, $\forall\ q = 1, \ldots, i-1$.*

Theorem 1 states the correctness of our branch-and-bound algorithm.

Theorem 1. *The branch-and-bound algorithm computes a LexOpt solution.*

4. Approximate Recovery Algorithm with Binding Decisions (Stage 2)

This section presents our recovery (reoptimization) strategy (Section 4.1) and analyzes the price of robustness for our two-stage approach in the case of single

Algorithm 3 Recovery Strategy

1: Perform all binding decisions (job assignments) with respect to schedule S_{init}.
2: Schedule free (unassigned) jobs using Longest Processing Time first (LPT).

(Section 4.2) and multiple (Section 4.3) perturbations.

4.1. Recovery Algorithm Description

A reoptimization strategy transforms the initial schedule S_{init} to a new schedule S_{new} for the perturbed instance I_{new}. Theorem 2 shows that optimally solving this problem is already $\mathcal{NP}$-hard.

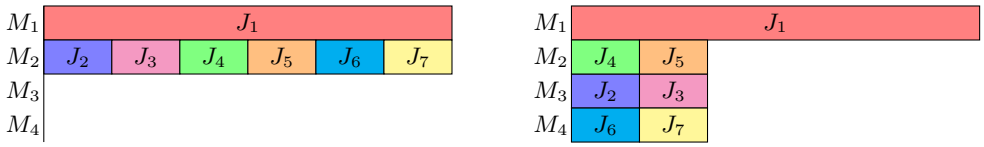
Theorem 2. *The makespan recovery problem is strongly $\mathcal{NP}$-hard, even in the case of a single perturbation.*

To describe our recovery strategy, Definition 2 distinguishes between *binding* and *free* optimization decisions. Binding decisions are job assignments in S_{init} which remain valid for the perturbed instance I_{new}. Free decisions are assignments of new jobs or jobs originally assigned to machines which failed due to uncertainty.

Definition 2. *Consider a makespan recovery problem instance $(I_{init}, S_{init}, I_{new})$ with $I_{init} = (\mathcal{M}, \mathcal{J})$ and $I_{new} = (\hat{\mathcal{M}}, \hat{\mathcal{J}})$.*

- ***Binding decisions*** *$\{x_{i,j} : (x_{i,j}(S_{init}) = 1) \wedge (i \in \hat{\mathcal{M}} \cap \mathcal{M}) \wedge (j \in \hat{\mathcal{J}} \cap \mathcal{J})\}$ are variable evaluations attainable from S_{init} in the recovery process.*
- ***Free decisions*** *$\{x_{i,j} : (j \in \hat{\mathcal{J}}) \wedge (\nexists i' \in \mathcal{M} \cap \hat{\mathcal{M}} : x_{i',j}(S_{init}) = 1)\}$ are variable evaluations that cannot be determined from S_{init} but are needed to recover feasibility.*

Our recovery strategy (Algorithm 3) maintains all binding decisions and makes free decisions using LPT [27]. Theoretically, enforcing the binding decisions exploits all relevant information in S_{init} for solving the perturbed instance I_{new}, thus quantifies the benefit of staying close to S_{init}. Practically, modifying S_{init} may incur transformation costs and our reoptimization algorithm mitigates this overhead. The supplementary material presents a more flexible recovery strategy with a bounded number of binding decision modifications.



(a) Weakly recoverable optimal S_{init}. (b) Efficiently recoverable LexOpt S_{init}.

Figure 3: Illustration of the benefit obtained by LexOpt schedules.

4.2. Single Perturbation

This section analyzes our two-stage approach in the case of a single perturbation. With an arbitrary optimal initial solution, Theorem 3 shows that the recovery strategy results in a non-constant price of robustness. When the initial solution is LexOpt, Theorem 4 provides a significantly better performance guarantee. Figure 3 illustrates a degenerate instance for deriving Theorem 3, highlighting the significance of LexOpt for $P||C_{\max}$ under uncertainty.

Theorem 3. *For the makespan recovery problem with a single perturbation, Algorithm 3 achieves an $\Omega(m)$ price of robustness with an arbitrary optimal initial schedule S_{init}.*

Proof:

Consider an instance I_{init} with m machines and $n+1$ jobs, where $n = k \cdot m$, for some integer $k \in \mathbb{Z}^+$, $p_1 = np$ and $p_j = p$, for $j = 2, \ldots, n+1$. The schedule S_{init} that assigns job J_1 to machine M_1, jobs $J_2, \ldots, J_n, J_{n+1}$ to machine M_2 and keeps the remaining machines $M_3, \ldots, M_m$ idle, is optimal for I_{init}. Suppose that I_{init} is perturbed because job J_1 is cancelled and let I_{new} be the new instance (we may alternatively consider a large reduction of processing time p_1). Then, Algorithm 3 produces a schedule S_{rec} with makespan $C_{\max}(S_{rec}) = \sum_{j=1}^{n} p_j$. However, an optimal schedule S_{new} for I_{new} has makespan $C_{\max}(S_{new}) = \frac{1}{m}\sum_{j=1}^{n} p_j$. Figure 3 illustrates such two schedules, where $\frac{C_{\max}(S_{rec})}{C_{\max}(S_{new})} = \Omega(m)$. ∎

Before proving Theorem 4, we state Lemma 4 which relates the optimal makespan of two instances with a different number of machines and set of jobs. Denote by $C^*_{\max}(m, \mathcal{J})$ the optimal makespan for instance $(m, \mathcal{J})$. Lemma 4 highlights the

importance of a LexOpt schedule: consider a LexOpt schedule S^* for $(m, \mathcal{J})$ and an arbitrary subset $\mathcal{M}' \subseteq \mathcal{M}$ of $m-1$ machines, i.e. $\mathcal{M}' = \mathcal{M} \setminus \{M_\ell\}$. Also, let $\mathcal{J}'$ be the subset of jobs assigned to the machines in $\mathcal{M}'$ by S^*. Then, the subschedule of S^* on $\mathcal{M}'$ is optimal for $(m-1, \mathcal{J}')$.

Lemma 4. *Consider a makespan problem instance $(m, \mathcal{J})$ and let S be LexOpt schedule. Given an arbitrary machine $M_\ell \in \mathcal{M}$, denote by $\mathcal{J}'$ the subset of all jobs assigned to the machines in $\mathcal{M} \setminus \{M_\ell\}$ by S. Then, it holds that:*

1. $\max_{M_i \in \mathcal{M} \setminus \{M_\ell\}} \{C_i(S)\} = C^*_{\max}(m-1, \mathcal{J}')$, *and*
2. $C^*_{\max}(m-1, \mathcal{J}) \leq 2 \cdot C^*_{\max}(m, \mathcal{J})$.

Theorem 4. *For the makespan recovery problem with a single perturbation, Algorithm 3 achieves a tight price of robustness equal to 2, if S_{init} is LexOpt.*

Proof:

The sequel proves the theorem in the case of a job reduction. For other perturbations described in Section 2, the proof is presented in the supplementary material. The supplementary material also shows that the obtained price of robustness is tight for every perturbation that we consider.

Consider a LexOpt schedule S_{init} for instance I_{init} and suppose that the processing time of job J_j decreases by $\delta \in (0, p_j]$, i.e. $p_j \leftarrow p_j - \delta$. Cancelling job $J_j \in \mathcal{J}$ is equivalent to reducing p_j to zero. Suppose that machine M_ℓ executes J_j in S_{init}. W.l.o.g., job J_j completes last in M_ℓ. Algorithm 3 returns the recovered schedule S_{rec} which keeps the job assignments in S_{init}, but decreases p_j and $C_\ell(S_{init})$ by δ. Let S_{new} be an optimal schedule for the perturbed instance I_{new}. We distinguish two cases depending on whether M_ℓ completes last in S_{rec}, or not.

First, suppose $C_\ell(S_{rec}) < C_{\max}(S_{rec})$ and let $\mathcal{J}' \subseteq \mathcal{J}$ be the subset of jobs executed by the machines in $\mathcal{M} \setminus \{M_\ell\}$. Then,

$$\begin{aligned}
C_{\max}(S_{rec}) &= C^*_{\max}(m-1, \mathcal{J}') && \text{[Lemma 4.1]},\\
&\leq C^*_{\max}(m-1, \mathcal{J} \setminus \{J_j\}) && [\mathcal{J}' \subseteq \mathcal{J} \setminus \{J_j\}],\\
&\leq 2 \cdot C^*_{\max}(m, \mathcal{J} \setminus \{J_j\}) && \text{[Lemma 4.2]},\\
&\leq 2 \cdot C_{\max}(S_{new}) && \text{[Definition]}.
\end{aligned}$$

Subsequently, consider that $C_\ell(S_{rec}) = C_{\max}(S_{rec})$, i.e. $C_{\max}(S_{rec}) = C_{\max}(S_{init}) - \delta$. We claim that S_{rec} is optimal for I_{new}. Assume for contradiction that an optimal schedule S_{new} for I_{new} satisfies $C_{\max}(S_{new}) < C_{\max}(S_{init}) - \delta$. By adding δ

extra units of time on job J_j, we derive a feasible schedule $\widetilde{S}$ for I_{init} from S_{new}, such that $C_{\max}(\widetilde{S}) < C_{\max}(S_{init})$. This contradicts the optimality of S_{init} for I_{init}. ∎

4.3. Multiple Perturbations

Two-stage robust optimization can be viewed as a two-player game where (i) we solve an initial instance, (ii) a malicious adversary generates perturbations, and (iii) we transform the initial solution into an efficient solution for a new instance. Adversarial strategies with multiple perturbations can render the initial solution weakly reoptimizable. But LexOpt can manage a bounded degree of uncertainty. For this case, we show that Algorithm 3 produces solutions with a positive performance guarantee parameterized by the uncertainty set size. Definition 3 describes our uncertainty set $\mathcal{U}(f, k, \delta)$ with three parameters: (i) the factor f indicating the boundary between *stable* and *unstable* job perturbations, (ii) the number k of unstable jobs, and (iii) the number δ of surplus machines. We assume that the number k of unstable jobs is bounded by the number of machines m, i.e. $k < m$.

Definition 3. *For a makespan problem instance $(m, \mathcal{J})$ with processing times p_1, ..., p_n, the* ***uncertainty set*** *$\mathcal{U}(f, k, \delta)$ contains every instance $(\hat{m}, \hat{\mathcal{J}})$ with processing times $\hat{p}_1, \ldots, \hat{p}_n$ satisfying the following properties:*

- ***Stability/instability boundary.*** *$\hat{\mathcal{J}}$ can be partitioned into the set $\hat{\mathcal{J}}^s$ of stable jobs and the set $\hat{\mathcal{J}}^u$ of unstable jobs, where $p_j/f \leq \hat{p}_j \leq p_j \cdot f \;\forall\; J_j \in \hat{\mathcal{J}}^s$.*
- ***Bounded number of unstable jobs.*** *$|\hat{\mathcal{J}}^u| \leq k$, assuming that $k < m$.*
- ***Bounded number of surplus machines.*** *$\max\{\hat{m} - m, 0\} \leq \delta$.*

Suppose $C^*_{\max}(m, \mathcal{J})$ is the optimal makespan for the $P||C_{\max}$ instance $(m, \mathcal{J})$. Lemma 5 (i) formalizes the optimal substructure imposed by LexOpt, (ii) bounds pairwise machine completion time differences in LexOpt schedules, (iii) quantifies the sensitivity of the optimal makespan w.r.t. the number of machines, and (iv) quantifies sensitivity of the optimal makespan w.r.t. processing times.

Lemma 5. *Let $(m, \mathcal{J})$ be a makespan problem instance with a LexOpt schedule S.*

1. *If the subset $\mathcal{J}' \subseteq \mathcal{J}$ of jobs is executed by the subset $\mathcal{M}' \subseteq \mathcal{M}$ of machines in S, where $|\mathcal{M}'| = m'$, then the sub-schedule of S on $\mathcal{M}'$ is optimal for $(m', \mathcal{J}')$, i.e. $\max_{M_i \in \mathcal{M}'}\{C_i(S)\} = C^*_{\max}(m', \mathcal{J}')$.*

Table 1: Performance guarantees of our two-stage approach for different perturbations, parameterized by the (i) perturbation factor f, (ii) number $k < m$ of unstable jobs, and (iii) number δ of surplus machines. The term ρ is the product of the performance guarantees obtained for Types 1-3.

Type	Perturbation type	Performance guarantee
Type 1	Job cancellations, Processing time reductions	$2f \cdot (1 + \lceil \frac{k}{m-k} \rceil)$
Type 2	Processing time augmentations	$f + k$
Type 3	Machine activations	$(1 + \lceil \delta/m \rceil)$
Type 4	Job arrivals, Machine failures	$\max\{2, \rho\}$

2. *Assuming that $M_i, M_\ell \in \mathcal{M}$ are two different machines such that job $J_j \in \mathcal{J}$ is assigned to M_i in S, then $C_\ell(S) \geq C_i(S) - p_j$.*
3. *It holds that $C^*_{\max}(m-\ell, \mathcal{J}) \leq \left(1 + \left\lceil \frac{\ell}{m-\ell} \right\rceil\right) \cdot C^*_{\max}(m, \mathcal{J}) \; \forall \; \ell \in \{1, \ldots, m-1\}$.*
4. *Let $(m, \hat{\mathcal{J}})$ be a makespan problem instance s.t. $\mathcal{J} = \hat{\mathcal{J}}$ and $\frac{1}{f} \cdot \hat{p}_j \leq p_j \leq \hat{p}_j$ for each J_j, where p_j and $\hat{p}_j$ is the processing time of J_j in $\mathcal{J}$ and $\hat{\mathcal{J}}$, respectively. Then, $\frac{1}{f} \cdot C^*_{\max}(m, \hat{\mathcal{J}}) \leq C^*_{\max}(m, \mathcal{J}) \leq C^*_{\max}(m, \hat{\mathcal{J}})$.*

Table 1 lists performance guarantees for our two-stage approach, obtained by individually analyzing each type of perturbation. Despite distinguishing the arguments for each type of perturbation, we obtain a global price of robustness for all perturbations simultaneously by propagating the solution degradation with respect to the order of Table 1. Considering perturbations in this order is only for analysis purposes and does not restrict our uncertainty model. Theoretically, LexOpt is essential only for bounding the solution degradation due to job removals and processing time reductions. But practically, the optimal substructure imposed by LexOpt is beneficial in an integrated setting with all possible perturbations. Section 5 complements the theoretical analysis with numerical experiments highlighting the significance of LexOpt in the recovered solution quality. Theorem 5 quantifies the price of robustness for our two-stage approach.

Theorem 5. *For the two-stage robust makespan scheduling problem with $\mathcal{U}(f, k, \delta)$ uncertainty and $k < m$, our LexOpt-based approach achieves a price of robustness:*

$$2f \cdot \left(1 + \left\lceil \frac{k}{m-k} \right\rceil\right) \cdot (f + k) \cdot \left(1 + \left\lceil \frac{\delta}{m} \right\rceil\right).$$

Proof:

Next, we analyze the recovered solution after job cancellations and processing time reductions (Type 1). The supplementary material completes the theorem's proof for the other perturbations (Types 2-4). Further, the supplementary material shows that the obtained price of robustness is asymptotically tight.

Processing time reductions are only recovered using binding decisions. A job cancellation is equivalent to reducing the processing time to zero. Given the recovered schedule S_{rec}, we partition the machines $\mathcal{M}$ into the sets $\mathcal{M}^s$ of *stable machines*, which are not assigned unstable jobs, and $\mathcal{M}^u$ of *unstable machines*, which are assigned unstable jobs. That is, $C_i(S_{rec}) \geq \frac{1}{f} \cdot C_i(S_{init})$, for $M_i \in \mathcal{M}^s$, and $m^s = |\mathcal{M}^s| \geq m - k$. Also, $\mathcal{M}^u = \mathcal{M} \setminus \mathcal{M}^s$ and $m^u = |\mathcal{M}^u| \leq k$. Machine $M_i \in \mathcal{M}$ is *critical*, if it completes last in schedule S_{rec}, i.e. $C_i(S_{rec}) = C_{\max}(S_{rec})$. We distinguish two cases based on whether $\mathcal{M}^s$ contains a critical machine, or not.

Case 1: $\mathcal{M}^s$ contains a critical machine. Let $\mathcal{J}^s_{new} \subseteq \mathcal{J}$ be the jobs assigned to machines $\mathcal{M}^s$ by S_{rec}. Each job in $\mathcal{J}^s_{new}$ is perturbed by a factor of at most f. Let $\mathcal{J}^s_{init}$ denote the same jobs before uncertainty realization. Jobs in $\mathcal{J}^s_{init}$ are executed on $\mathcal{M}^s$ in S_{init} and appear in $\mathcal{J}^s_{new}$ with smaller processing times. Then,

$$
\begin{aligned}
C_{\max}(S_{rec}) &= \max_{M_i \in \mathcal{M}^s} \{C_i(S_{rec})\} && [\mathcal{M}^s \text{ contains a critical machine}],\\
&\leq \max_{M_i \in \mathcal{M}^s} \{C_i(S_{init})\} && [\text{Processing time reduction}],\\
&= C^*_{\max}(m^s, \mathcal{J}^s_{init}) && [\text{Lemma 5.1}],\\
&\leq f \cdot C^*_{\max}(m^s, \mathcal{J}^s_{new}) && [\text{Lemma 5.4}],\\
&\leq f \cdot C^*_{\max}(m^s, \mathcal{J}_{new}) && [\mathcal{J}^s_{new} \subseteq \mathcal{J}_{new}],\\
&= f \cdot C^*_{\max}(m - m^u, \mathcal{J}_{new}) && [m^s = m - m^u],\\
&\leq f \cdot \left(1 + \left\lceil \frac{m^u}{m - m^u} \right\rceil\right) \cdot C^*_{\max}(m, \mathcal{J}_{new}) && [\text{Lemma 5.3}],\\
&\leq f \cdot \left(1 + \left\lceil \frac{k}{m - k} \right\rceil\right) \cdot C_{\max}(S_{new}). && [m^u \leq k]
\end{aligned}
$$

Case 2: Only $\mathcal{M}^u$ contains critical machines. Consider an unstable critical machine $M_i \in \mathcal{M}^u$ in S_{rec}, i.e. $C_{\max}(S_{rec}) = C_i(S_{rec})$. If only one job is assigned to

M_i, then schedule S_{rec} is optimal. Now, assume that at least two jobs are assigned to M_i in S_{rec} . Since processing times are only reduced, $C_i(S_{rec}) \leq C_i(S_{init})$. Because $k < m$, there exists a machine $M_\ell \in \mathcal{M}^s$. Furthermore, since S_{init} is LexOpt, Lemma 5.2 ensures that $C_\ell(S_{init}) \geq C_i(S_{init}) - p_j$, for each $J_j \in \mathcal{J}$ assigned to M_i by S_{init}. As S_{init} contains at least two jobs, there exists a job J_j assigned to M_i by S_{init} such that $p_j \leq \frac{1}{2} \cdot C_i(S_{init})$. Hence, $C_i(S_{init}) \leq 2 \cdot C_\ell(S_{init})$. We conclude that $C_{\max}(S_{rec}) \leq 2 \cdot C_\ell(S_{init})$. Because $M_\ell \in \mathcal{M}^s$, similarly to Case 1, we get:

$$C_{\max}(S_{rec}) \leq 2f \cdot \left(1 + \left\lceil \frac{k}{m-k} \right\rceil\right) \cdot C_{\max}(S_{new}).$$

■

5. Numerical Results

Section 5.1 describes our system specifications and the generation of benchmark $P||C_{\max}$ instances. Section 5.2 evaluates the LexOpt branch-and-bound algorithm. Section 5.3 presents the generation of perturbed instances, i.e. the effect of uncertainty realization. Section 5.4 evaluates the price of robustness of our two-stage robust scheduling approach.

5.1. System Specification and Benchmark Instances

We ran all computations on an Intel Core i7-4790 CPU 3.60GHz, 15.6 GB RAM machine running Ubuntu 14.04 64-bit. Using Python 2.7.6 and Pyomo 4.4.1, we solve MILP models with CPLEX 12.6.3 and Gurobi 6.5.2. The source code and test cases are available on GitHub [34]. We have randomly generated $P||C_{\max}$ instances. *Well-formed instances* admit an optimal schedule close to a *perfect solution* where all machine completion times are equal, i.e. $C_i = C_{i'}$ for $M_i, M_{i'} \in \mathcal{M}$. *Degenerate instances* have a less-balanced optimal schedule. This section investigates well-formed instances and we complete the analysis with degenerate instances in the supplementary material.

Well-formed instances depend on 3 parameters: (i) the number m of machines, (ii) the number n of jobs, and (iii) a processing time seed q. Using the parameter

Table 2: Well-formed Instances

Instances	**m**	**n**	**q**
Moderate	$3, 4, 5, 6$	$20, 30, 40, 50$	$100, 1000$
Intermediate	$10, 12, 14, 16$	$100, 200, 300, 400$	$10000, 100000$
Hard	$10, 15, 20, 25$	$200, 300, 400, 500$	$10000, 100000$

values in Table 2, we generated *moderate*, *intermediate* and *hard* well-formed instances. For each combination of m, n and q, we generate 3 instances based on 3 different distributions of processing times: *uniform distribution* $p_j \sim \mathcal{U}(\{1, \ldots, q\})$, *normal distribution* $p_j \sim \mathcal{N}(q, q/3)$ and a *symmetric of normal distribution* s.t. $p \sim \mathcal{N}(q, q/3)$ and $p_j = q - p$ if $p \in [0, q]$, or $p_j = 2q - (p - q)$ if $p_j \in (q, 2q]$. Each processing time is rounded to the nearest integer. Further, (symmetric) normal processing times outside $[0, 2q]$ are rounded to the nearest of 0 and $2q$.

5.2. LexOpt Branch-and-Bound Algorithm Evaluation

This section numerically evaluates our branch-and-bound algorithm. We complement our evaluation with the sequential, highest-rank objective, and weighting methods adapted to LexOpt scheduling. Our termination criteria for each MILP solving is: (i) 10^3 CPU seconds time limit, and (ii) 10^{-4} relative error tolerance, where the relative gap $(Ub - Lb)/Ub$ is computed using the best-found incumbent Ub and the lower bound Lb.

The *sequential method* solves m MILP instances with repeated CPLEX calls, using the CPLEX reoptimize feature in each call to exploit information obtained from previous calls. If 10^3 CPU seconds in total elapse, then the method terminates when the ongoing MILP run is completed. The *highest-rank objective method* solves the LexOpt scheduling problem using the CPLEX solution pool feature. Initially, the standard MILP model for $P||C_{\max}$ is solved. Next, the tree exploration goes on and populates a pool with 2000 solutions. The *weighting method* computes a schedule S of minimal weighted value $W(S) = \sum_{i=1}^{m} B^{m-i} \cdot C_i(S)$. We set $B = 2$ and solve the resulting MILP models with CPLEX and Gurobi.

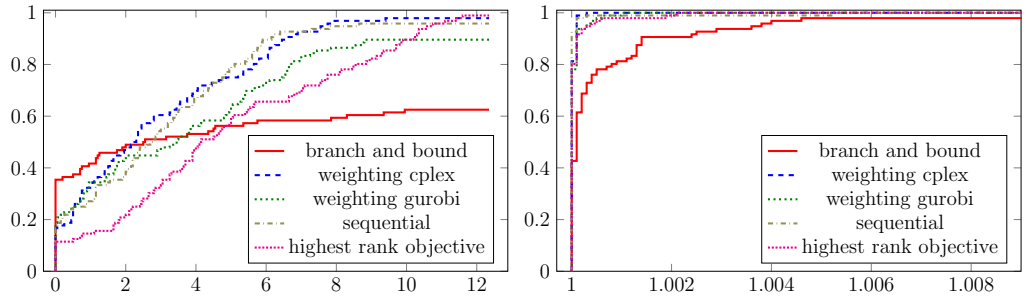
Figure 4 plots *performance profiles* of LexOpt methods on well-formed instances in order to compare running times and computed solutions [20]. The sequential and weighting methods perform similarly in terms of running time and number

of solved instances. But, the sequential method produces worse feasible solutions since lower-ranked objectives are not optimized in the case of a timeout. The highest-rank objective method has worse running times than the sequential and weighting methods on moderate instances, because populating the solution pool is a significant part of the overall running time. However, the highest-rank objective method attains significantly better running times than the sequential and weighting methods on intermediate and hard test cases, because populating the solution pool is a small fraction of the overall running time. The highest-rank objective method does not prove lexicographic optimality as it only generates 2000 candidate solutions. Nonetheless, it produces the best heuristic results for most test cases.

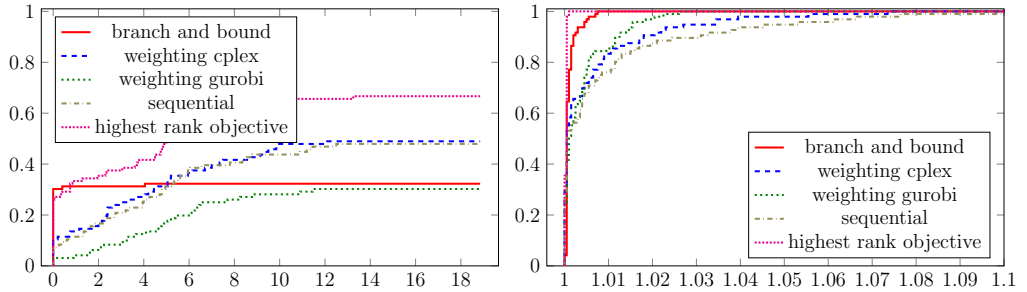
Figure 4 shows that our branch-and-bound algorithm proves global optimality faster than the other methods when it converges, i.e. when it does not timeout. Note that the branch-and-bound algorithm converges for $> 60\%$ of the moderate test cases and $> 30\%$ of the intermediate and hard instances. For intermediate and hard instances, the branch-and-bound algorithm consistently produces good heuristic solutions, i.e. better solutions than the sequential and weighting methods.

5.3. Generation of Perturbed Instances

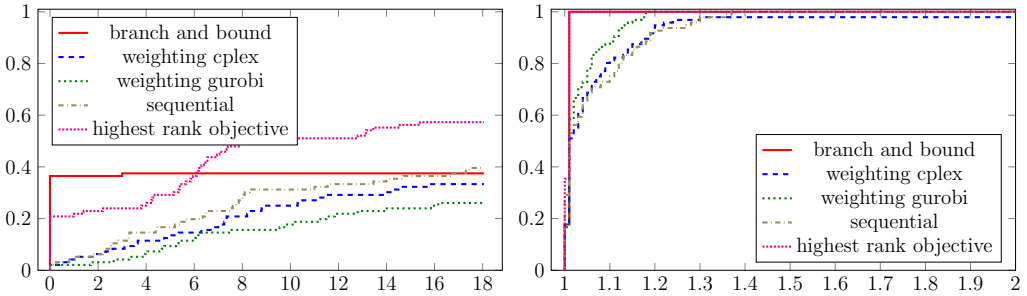
Recall that an instance of the makespan recovery problem is specified by: (i) an initial makespan problem instance I_{init}, (ii) an initial solution S_{init} to I_{init}, and (iii) a perturbed instance I_{new}. We generate the initial $P||C_{\max}$ instances according to Section 5.1. For each instance I_{init}, we generate a set $\mathcal{S}(I_{init})$ of 50 schedules using the CPLEX solution pool feature. In general, two different schedules $S_1, S_2 \in \mathcal{S}(I_{init})$ have weighted values $W(S_1) \neq W(S_2)$, where $W(S) = \sum_{i=1}^{m} B^{m-i} \cdot C_i(S)$. For each makespan problem instance I_{init}, we construct a perturbed instance I_{new} by generating random disturbances. A *job disturbance* is (i) a new job arrival, (ii) a job cancellation, (iii) a processing time augmentation, or (iv) a processing time reduction. A *machine disturbance* is (i) a new machine activation, or (ii) a machine failure. We randomly generate $d_n = \lceil 0.2 \cdot n \rceil$ job disturbances and $d_m = \lceil 0.2 \cdot m \rceil$ machine perturbations. The type of each *job disturbance* is chosen uniformly at random among the four options (i) - (iv). A new job $J_j \in \hat{\mathcal{J}}$ has processing time $\hat{p}_j \sim \mathcal{U}(\{1, \ldots, q\})$, using the parameter q that produced the original instance I_{init}. A job cancellation deletes an existing job chosen uniformly at random. To



(a) Moderate instances: time (s) on $\log_2$ scale (left), upper bounds on $[1, 1.009]$ (right).



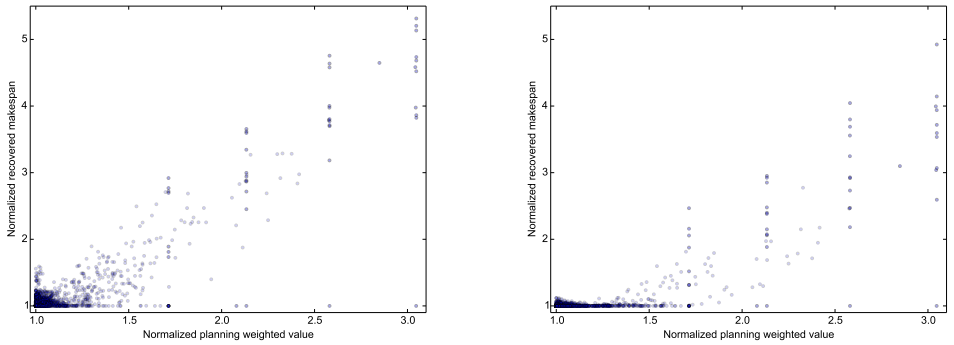
(b) Intermediate instances: time (s) on $\log_2$ scale (left), upper bounds on $[1, 1.1]$ (right).



(c) Hard instances: time (s) on $\log_2$ scale (left), upper bounds on $[1, 2]$ (right).

Figure 4: Performance profiles for the *well-formed test set* with 10^3 s timeout.

increase or decrease the processing time of job $J_j \in \mathcal{J}$, we randomly select $\hat{p}_j \sim \mathcal{U}(\{p_j + 1, \ldots, 2 \cdot q\})$ or $\hat{p}_j \sim \mathcal{U}(\{1, 2, \ldots p_j - 1\})$, respectively. The type of a *machine disturbance* is chosen uniformly at random among options (i)-(ii). A new machine activation increases the number of available machines by one. A machine cancellation deletes an existing machine chosen uniformly at random.



(a) Binding Recovery

(b) Flexible Recovery

Figure 5: Well-formed instances scatter plots illustrating the recovered solution makespan with respect to the initial solution weighted value.

5.4. Two-Stage Robust Scheduling Evaluation

In the two-stage robust makespan scheduling problem, solution S_{init} is transformed to a feasible solution S_{rec} for instance I_{new}. Figure 5 correlates the makespan of S_{rec} with the closeness of S_{init} to LexOpt. We quantify the closeness of S_{init} to LexOpt using the weighted value $W(S_{init}) = \sum_{i=1}^{m} B^{m-i} \cdot C_i(S_{init})$. The closest to LexOpt the schedule S_{init} is, the lowest the value $W(S_{init})$ we get. For each instance I_{init}, we recover every solution $S_{init} \in \mathcal{S}(I_{init})$ by applying our binding and flexible recovery strategies from Section 4.1. For the flexible recovery strategy, we set $g = 0.1n$, i.e. at most 10% of the binding decisions can be modified. Suppose that the normalized weighted value of an initial solution $S_{init} \in \mathcal{S}(I_{init})$ is $W^N(S_{init}) = \frac{W(S_{init})}{W^*(I_{init})}$, where $W^*(I_{init})$ is the best weighted value in the CPLEX solution pool for instance I_{init}. Similarly, assume that the normalized makespan of S_{rec} equal to $C^N(S_{rec}) = \frac{C_{\max}(S_{rec})}{C^*_{\max}(I_{new})}$, where $C^*_{\max}(I_{new})$ is the makespan of the best recovered schedule for instance I_{new}. Figure 5a shows that the makespan of solutions obtained with our binding recovery strategy tends to improve as the weighted value of the initial solution decreases. Figure 5b verifies this trend for the flexible recovery strategy. These results highlight the importance of LexOpt towards efficient two-stage robust scheduling. Our findings also motivate scheduling under uncertainty where the planning and recovery stages are investigated together.

6. Conclusion

Practical scheduling applications frequently require an initial, nominal schedule which is recovered after uncertainty realization. But significantly modifying the nominal schedule might not be desirable in domains such as distributed computing [57] and timetabling [46]. To this end, we use exact LexOpt scheduling for planning and approximate rescheduling for adaptability [2, 7, 17, 49].

We provide new insights on the combinatorial structure of robust scheduling. LexOpt handles highly-symmetric mixed-integer optimization problems [3, 23, 22, 47], but our results also highlight LexOpt benefits on scheduling under uncertainty. By exploiting optimal substructure imposed by LexOpt, we propose a two-stage robust makespan scheduling approach whose performance is substantiated with a price of robustness characterization. Numerical results with randomly generated instances demonstrate that the closest to LexOpt the initial solution is, the better the recovered solution quality we get. Beyond scheduling, extensions to uncertain min-max partitioning problems, e.g. facility location and network design, with generalized cost functions are possible [55].

Faced with the lack of strong lower bounding techniques for LexOpt scheduling, we develop a new branch-and-bound algorithm, based on vectorial bounds. The algorithm (i) avoids iterative MILP solving of sequential methods, (ii) bypasses precision issues of weighting methods, and (iii) reduces the symmetry of highest-rank objective methods. This approach is broadly relevant to LexOpt.

Acknowledgments

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Supplementary Material for "Exact Lexicographic Scheduling and Approximate Rescheduling"

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Contents. This document contains omitted parts of the manuscript *Exact Lexicographic Scheduling and Approximate Rescheduling*. The document is a companion to the original manuscript for readers interested in complementary technicalities which have been omitted to better convey our main message, i.e. the importance of LexOpt in scheduling under uncertainty and relevant challenges. These technicalities are essential for the completeness of the presented study. The manuscript itself and this supplementary document cover the topics in a similar order.

Appendix A provides omitted parts required for designing, analyzing, and evaluating the exact LexOpt methods. Appendix B shows that the makespan recovery problem is $\mathcal{NP}$-hard. Appendices C and D complete the robustness analysis of our two-stage approach in the case of a single and multiple perturbations, respectively. Appendix E presents a more flexible recovery strategy. Appendix F completes our numerical evaluation with degenerate instances. Finally, Appendix G provides a table with the notation used in both documents.

Appendix A Exact LexOpt Methods

Section A.1 proves valid inequalities and Section A.2 adapts the sequential, weighting, and highest-rank objective methods for the LexOpt scheduling problem. Section A.3 provides a pseudo-code, Section A.4 describes a primal heuristic, Section A.5 provides correctness proofs for the vectorial bounds and Section A.6 states a correctness proof for our branch-and-bound algorithm.

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A.1 LexOpt Scheduling Reformulation Lemma

Lemma 1. *In an optimal solution to the LexOpt scheduling problem:*

1. $C_i \geq C_{i+1}$, *for* $i = 1, \ldots, m-1$,
2. $i \cdot C_i + \left[\sum_{q=i+1}^{m} C_q\right] \leq \sum_{j=1}^{n} p_j \leq \left[\sum_{q=1}^{i-1} C_q\right] + (m-i+1) \cdot C_i$, $\forall\ i = 1, \ldots, m$.

Proof:

In any feasible schedule, the machines can be renumbered so as to satisfy the first property. For the second property, observe that $\sum_{i=1}^{m} C_i = \sum_{j=1}^{n} p_j$. Since $C_i \geq \ldots \geq C_m$, we get that $\sum_{q=1}^{i-1} C_q + (m-i+1) \cdot C_i \geq \sum_{j=1}^{n} p_j$. Similarly, given that $C_1 \geq \ldots \geq C_i$, we conclude that $i \cdot C_i + \sum_{q=i+1}^{m} C_q \leq \sum_{j=1}^{n} p_j$. ∎

A.2 State-of-the-Art LexOpt Methods

Sequential Method. This method (Algorithm 4) iteratively minimizes the objective functions $C_1, \ldots, C_n$ w.r.t. to their priority order, over the set $\mathcal{S}$ of feasible schedules [16, 19]. Let v_i^* be the value of C_i in a LexOpt solution. The i-th iteration computes v_i^* by solving MILP (1) with the extra constraint that the first $(i-1)$ objectives should be respectively equal to $v_1^*, \ldots, v_{i-1}^*$. Warm-starting iteration i with the solution at iteration $(i-1)$ improves the efficiency of the method.

Algorithm 4 Sequential Method

1: $v_1^* = \min\{C_1 : (\vec{x}, \vec{C}) \in \mathcal{S}\}$.
2: **for** $i = 2, \ldots, m$ **do**
3: $\quad v_i^* = \min\{C_i : x \in \mathcal{S}, C_1 = v_1^*, \ldots, C_{i\text{-}1} = v_{i\text{-}1}^*\}$
4: Return the solution computed in the last iteration.

Weighting Method. This method (Algorithm 5) minimizes a weighted sum $\sum_{i=1}^{m} w_i \cdot C_i$ of the objectives $C_1, \ldots, C_n$ [51]. Typically, $w_i = B^{m-i}$ for $i = 1, 2, \ldots, m$, where the *big-M parameter* $B > 1$ is a sufficiently large constant [52]. Note that the highest-rank objectives are associated with the largest weights. Further, this weighted sum can measure the distance of any solution from the LexOpt solution. For our numerical results, we set $B = 2$.

Algorithm 5 Weighting Method

Select big-M parameter $B = 2$.
for $i = 2, \ldots, m$ **do**
    Set machine weight $w_i = B^{m-i}$.
Solve $\min\{\sum_{i=1}^{m} w_i \cdot C_i : (\vec{x}, \vec{C}) \in \mathcal{S}\}$.

Highest-Rank Objective Method. This method (Algorithm 6) computes the pool $\mathcal{P}$ of all optimal solutions for the mono-objective problem $v_1^* = \min\{C_1 : (\vec{x}, \vec{C}) \in \mathcal{S}\}$ of minimizing the highest-rank objective function C_1, i.e. the makespan, and returns the lexicographically smallest solution $\operatorname{lex\,min}\{\vec{C}(S) : S \in \mathcal{P}\}$ in $\mathcal{P}$ [44]. A very large solution pool can be efficiently approximated with a smaller set of solutions using the CPLEX *solution pool* feature. In LexOpt, maintaining a single solution in the pool is sufficient, if the current solution is always replaced by a lexicographically smaller solution. A simple greedy lexicographic comparison algorithm checks when such an update is essential.

Algorithm 6 Highest-Rank Objective Method

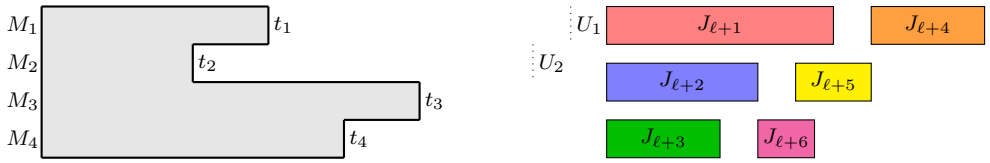
Solve $v_1^* = \min\{C_1 : (\vec{x}, \vec{C}) \in \mathcal{S}\}$.
Compute the solution pool $\mathcal{P} = \{(\vec{x}, \vec{C}) \in \mathcal{S} : C_1 = v_1^*\}$.
Return $\operatorname{lex\,min}\{\vec{C} : (\vec{x}, \vec{C}) \in \mathcal{P}\}$.

A.3 Branch-and-Bound Algorithm Pseudocode

Algorithm 7 LexOpt Branch-and-Bound Algorithm using Vectorial Bounds

Q: empty stack
r: root node
push(Q, r)
$I = \{+\infty\}^m$
while $Q \neq \emptyset$ **do**
    $u = \text{top}(Q)$
    for $v \in \text{children}(u)$ **do**
        if v is leaf **then**
            S: schedule of v
            $I = \operatorname{lex\,min}\{I, S\}$

```
11:     else
12:         S: heuristic schedule computed via LPT
13:         I = lex min{I, S}
14:         L⃗: vectorial lower bound of node v
15:         if L⃗ ≤lex C⃗(I) then
16:             push(Q, v)
```



(a) Partial schedule associated with node v. (b) Remaining jobs.

Figure 6: Computing vectorial lower bound component L_i at node v in the ℓ-th search tree level, by scheduling jobs $J_{\ell+1}, \ldots, J_n$ in the partial schedule of v. Jobs $J_{\ell+1}, \ldots, J_h$ are rejected in the intervals $[t_q, U_q]$, for $q = 1, \ldots, i-1$. L_i is computed by fractionally scheduling jobs $J_{h+1}, \ldots, J_m$ on machines $M_i, \ldots, M_m$ and lower bounding the completion time of machine M_i.

A.4 Longest Processing Time First Heuristic

The primal heuristic applied in each node v of the branch-and-bound tree is *Longest Processing Time First (LPT)* (Algorithm 8). LPT keeps the assignment of jobs $J_1, \ldots, J_\ell$, where ℓ is the level of node v, and greedily schedules jobs $J_{\ell+1}, \ldots, J_n$ with the order $p_{\ell+1} \geq \ldots \geq p_n$. In each step, LPT assigns the next job to the least-loaded machine, i.e. makes the lexicographically best decision.

Algorithm 8 Longest Processing Time First (LPT) at level ℓ

```
1: t⃗: Initial machine completion times
2: for j = (ℓ + 1), ..., n do
3:     i = arg min_{M_q ∈ M} {t_q}
4:     C_i ← t_i + p_j
5: Sort the machines so that C_1 ≥ ... ≥ C_m.
```

A.5 Correctness of Vectorial Bounds

This section proves Lemmas 2-3 and, thus, shows that Algorithms 1-2 correctly compute vectorial bounds.

Lemma 2. *Consider a node v of the search tree and a machine index $i \in \{1, \ldots, m\}$. Algorithm 1 produces a value $L_i \leq C_i(S)$ for each feasible schedule $S \in \mathcal{S}(v)$ below v such that $C_q(S) \leq U_q$, $\forall\ q = 1, \ldots, i-1$.*

Proof:

Schedule S and pseudo-schedule $\widetilde{S}$ (of Algorithm 1) assign jobs $J_1, \ldots, J_\ell$ to the same machines and the vector $\vec{t} = (t_1, \ldots, t_m)$ specifies machine completion times w.r.t. these jobs. All remaining jobs $\mathcal{R} = \{J_{\ell+1}, \ldots, J_n\}$ are scheduled differently in $\widetilde{S}$ and S. In $\widetilde{S}$, the jobs in $\widetilde{\mathcal{R}} = \{J_{\ell+1}, \ldots, J_h\}$ are fractionally assigned to machines $M_1, \ldots, M_{i-1}$ and the jobs in $\mathcal{R} \setminus \widetilde{\mathcal{R}} = \{J_{h+1}, \ldots, J_n\}$ to $M_i, \ldots, M_m$. Denote by $\mathcal{R}' \subseteq \mathcal{R}$ the corresponding subset of jobs assigned to machines $M_1, \ldots, M_{i-1}$, in S. That is, the jobs in $\mathcal{R} \setminus \mathcal{R}'$ are assigned to $M_i, \ldots, M_m$ in S.

Observe that $\sum_{J_j \in \mathcal{R}'} p_j = \sum_{q=1}^{i-1} (C_q(S) - t_q) \leq \sum_{q=1}^{i-1} (U_q - t_q) \leq \sum_{J_j \in \widetilde{\mathcal{R}}} p_j$, where the first equality holds by definition, the first inequality by the assumption $C_q(S) \leq U_q$, for $q = 1, \ldots, (i-1)$, and the second inequality because Algorithm 1 fits machines $M_1, \ldots, M_{i-1}$ at least up to their respective upper bounds. Moreover, we have that $\max_{J_j \in \mathcal{R} \setminus \mathcal{R}'}\{p_j\} \geq \max_{J_j \in \mathcal{R} \setminus \widetilde{\mathcal{R}}}\{p_j\} = p_{h+1}$. Otherwise, $\max_{J_j \in \mathcal{R} \setminus \mathcal{R}'}\{p_j\} < p_{h+1}$, which implies that $\mathcal{R}'$ contains all jobs $J_{\ell+1}, \ldots, J_{h+1}$. Hence, $\sum_{J_j \in \mathcal{R}'} p_j \geq \sum_{j=\ell+1}^{h+1} p_j > \sum_{J_j \in \widetilde{\mathcal{R}}} p_j$, i.e. a contradiction.

Since S assigns a job of processing time $\max_{J_j \in \mathcal{R} \setminus \mathcal{R}'}\{p_j\} \geq p_{h+1}$ to a machine in $M_i, \ldots, M_m$, $C_i(S) \geq \min_{i \leq q \leq m}\{t_q\} + \max_{J_j \in \mathcal{R} \setminus \mathcal{R}'}\{p_j\} \geq \min_{i \leq q \leq m}\{t_i\} + p_{h+1}$. Clearly, $C_i(S) \geq \max_{i \leq q \leq m}\{t_i\}$. Further, using a standard packing argument and the fact that $\sum_{J_j \in \mathcal{R} \setminus \mathcal{R}'} p_j \geq \sum_{J_j \in \mathcal{R} \setminus \widetilde{\mathcal{R}}} p_j$, if the quantity $\Lambda = \sum_{j=h+1}^{n} p_j - \sum_{q=i}^{m} (\tau - t_q)$ is positive, where $\tau = \max_{i \leq q \leq m}\{t_q\}$, then $C_i(S) \geq \max_{i \leq q \leq m}\{t_q\} + \frac{\Lambda}{m-i+1}$. We conclude that $L_i \leq C_i(S)$. ∎

Lemma 3. *Consider a node v of the search tree and a machine index $i \in \{1, \ldots, m\}$. Algorithm 2 produces a value $U_i \geq C_i(S)$ for each feasible schedule $S \in \mathcal{S}(v)$ below v such that $C_q(S) \geq L_q$, $\forall\ q = 1, \ldots, i-1$.*

Proof:

Recall that jobs $J_1, \ldots, J_\ell$ are identically assigned in schedule S and pseudo-schedule $\tilde{S}$ of Algorithm 2. Moreover, a vector $\vec{t} = (t_1, \ldots, t_m)$ specifies the machine completion times of S and $\tilde{S}$ w.r.t. these jobs. Let $\mathcal{R} = \{J_{\ell+1}, \ldots, J_n\}$ be

the set of remaining jobs. Denote by $\widetilde{\mathcal{R}} = \{J_{\ell+1}, \ldots, J_h\} \subseteq \mathcal{R}$ and $\mathcal{R}' \subseteq \mathcal{R}$ the subset of jobs assigned to machines $M_i, \ldots, M_m$ in $\widetilde{S}$ and S, respectively. By arguing similarly to the proof of Lemma 2, $\sum_{J_j \in \mathcal{R} \setminus \widetilde{\mathcal{R}}} p_j \geq \sum_{J_j \in \mathcal{R} \setminus \mathcal{R}'} p_j$. In addition, $\max_{J_j \in \mathcal{R} \setminus \widetilde{\mathcal{R}}} \{p_j\} \geq \max_{J_j \in \mathcal{R} \setminus \mathcal{R}'} \{p_j\}$.

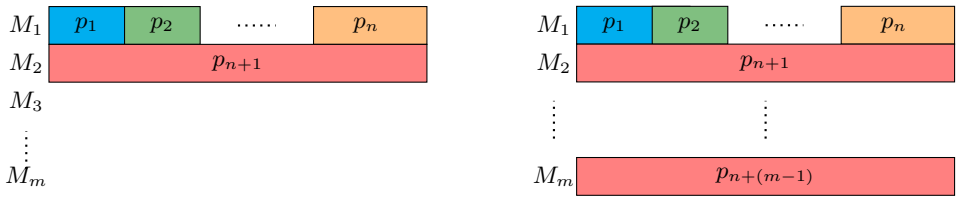
The total load of jobs $J_{\ell+1}, \ldots, J_n$ assigned to machines $M_i, \ldots, M_m$ in S is clearly $\sum_{J_j \in \mathcal{R} \setminus \mathcal{R}'} p_j \leq \lambda = \sum_{j=\ell+1}^{n} p_j - \sum_{q=1}^{i-1} (L_q - t_q)$. To compute U_i, Algorithm 2 assigns part of λ fractionally and uniformly to the least loaded machines among $M_i, \ldots, M_m$. In particular, it sorts these machines so that $t_i \leq \ldots \leq t_m$ and assigns λ units of processing time to machines $M_i, \ldots, M_\mu$ so that they end up having the same completion time $\tau = \frac{1}{\mu - i + 1} \left(\sum_{q=1}^{\mu} t_q + \lambda \right)$. Using a simple packing argument and the fact that $\max_{J_j \in \mathcal{R}} \{p_j\} = p_\ell$, we get $C_i(S) \leq \max\{\tau + p_\ell, t_m\}$. ∎

A.6 Optimality of Branch-and-Bound Algorithm

Theorem 1. *The branch-and-bound method computes a LexOpt solution.*

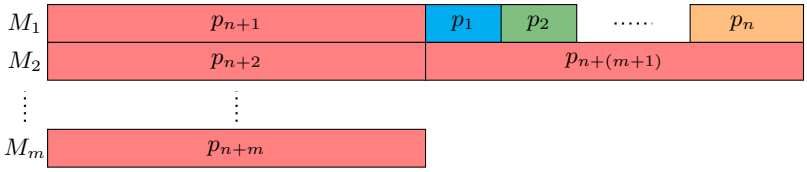
Proof:

Consider a tree node v. Let $\vec{L} = (L_1, \ldots, L_m)$ and I be the computed vectorial lower bound and the incumbent, when branch-and-bound Algorithm 7 explores v. We show the invariant that if node v is pruned, then $\vec{C}(S) \geq_{\text{lex}} \vec{C}(I)$, for every schedule $S \in \mathcal{S}(v)$. Node v is pruned when $L \geq_{\text{lex}} \vec{C}(I)$, i.e. (i) $L_1 > C_1(I)$, (ii) $L_q = C_q(I)$ $\forall\ q = 1, \ldots, i-1$ and $L_i > C_i(I)$, for some $i \in \{2, \ldots, m-1\}$, or (iii) $L_i = C_i(I)$ $\forall\ i = 1, \ldots, m$. In case (i), because $C_1(S) \geq L_1$, it holds that $\vec{C}(S) >_{\text{lex}} \vec{C}(I)$ $\forall\ S \in \mathcal{S}(v)$. In case (ii), either $C_1(S) > L_1$, or $C_1(S) = L_1\ \forall\ S \in \mathcal{S}(v)$. Let $\mathcal{S}_1(v) \subseteq \mathcal{S}(v)$ be the subset of schedules satisfying $C_1(S) = L_1 = C_1(I)$. Algorithm 2 computes $U_1 = C_1(I)$. By Lemma 2, either $C_2(S) > L_2$, or $C_2(S) = L_2$, for each $S \in \mathcal{S}_1(v)$. Let $\mathcal{S}_2(v) \subseteq \mathcal{S}_1(v)$ be the subset of schedules with $C_2(S) = L_2$. We define similarly all sets $\mathcal{S}_1(v), \ldots, \mathcal{S}_{i-1}(v)$. By Lemma 2, for any schedule in $\mathcal{S}_{i-1}(v)$, it holds that $C_q(S) = L_q = C_q(I)\ \forall\ q = 1, \ldots, i-1$ and $C_i(S) \geq L_i > C_i(I)$. Thus, for each $S \in \mathcal{S}(v)$, $\vec{C}(S) >_{\text{lex}} \vec{C}(I)$. Finally, in case (iii), for each $S \in \mathcal{S}_{m-1}(v)$, $C_q(S) = C_q(I)\ \forall\ q = 1, \ldots, m$ and $\vec{C}(S) = \vec{C}(I)$. The theorem follows. ∎



(a) Job cancellation, Processing time reduction.

(b) Job arrival, Processing time augmentation, Machine failure.



(c) Machine activation.

Figure 7: Schedule S_{init} in the $\mathcal{NP}$-hardness reduction of the makespan recovery problem from $P||C_{\max}$. Different perturbation types are considered individually. The original jobs $J_1, \ldots, J_n$ have processing times $p_1, \ldots, p_n$. Each dummy job $J_{n+1}, \ldots, J_{n+(m+1)}$ has processing time $\sum_{j=1}^{n} p_j$.

Appendix B $\mathcal{NP}$-Hardness of Makespan Recovery Problem

This section shows the $\mathcal{NP}$-hardness of the makespan recovery problem via a reduction from $P||C_{\max}$. That is, the makespan recovery problem is at least as hard as $P||C_{\max}$. Given a minimum makespan schedule S_{init} for an initial instance I_{init} of $P||C_{\max}$, the makespan recovery problem asks the existence of a feasible schedule with makespan T_{new} for a perturbed instance I_{new}. Thus, the knowledge of S_{init} does not mitigate the computational complexity for solving I_{new}.

Theorem 2. *The makespan recovery problem is strongly $\mathcal{NP}$-hard, even in the case of a single perturbation.*

Proof:

We prove the lemma for each type of perturbation of Section 2 individually. Given instance $I = (m, \mathcal{J})$ of $P||C_{\max}$ with target makespan T, we construct an instance $(I_{init}, S_{init}, I_{new})$ of the makespan recovery problem with target makespan T_{new} by adding dummy jobs. Let $p_1, \ldots, p_n$ be the processing times of the n jobs in $\mathcal{J}$. Figure 7 shows a construction for each perturbation type.

Job Removal, Processing Time Reduction. The initial instance I_{init} consists of m machines, the n original jobs and a dummy job of processing time $p_{n+1} = \sum_{j=1}^{n} p_j$. Optimal schedule S_{init} for I_{init} assigns all jobs $J_1, \ldots, J_n$ to machine M_1, job J_{n+1} to machine M_2 and leaves $M_3, \ldots, M_n$ empty. We obtain instance I_{new} from I_{init} by removing job J_{n+1} and setting $T_{new} = T$. Since I_{new} consists only of the jobs in I, I_{new} admits a feasible schedule of makespan T_{new} iff there exists a schedule of makespan T for I. The case of a processing time reduction can be treated similarly, by decreasing p_{n+1} down to 0 from $\sum_{j=1}^{n} p_j$.

Job Arrival, Processing Time Augmentation, Machine Failure. We construct an initial instance I_{init} with m machines, the n original jobs and $m-1$ dummy jobs $J_{n+1}, \ldots, J_{n+m-1}$ of processing time $p_\ell = \sum_{j=1}^{n} p_j$, for $\ell = n+1, \ldots, m-1$. The schedule S_{init} assigning jobs $J_1, \ldots, J_n$ to machine M_1 and a dummy job to every other machine is optimal for I_{init}. We perturb I_{init} by adding job J_{n+m} of processing time $p_{n+m} = \sum_{j=1}^{n} p_j$ and setting $T_{new} = \sum_{j=1}^{n} p_j + T$. In instance I_{new}, we ask the existence of a feasible schedule S_{new} of makespan T_{new}. Since $T < \sum_{j=1}^{n} p_j$, if such a schedule exists, every pair of dummy jobs must executed by different machines. Thus, I admits a schedule of makespan T iff there exists a schedule with makespan T_{new} for I_{new}. For a processing time augmentation and a machine removal, we use the same arguments, but different constructions. In the former case, we add a dummy job J_{n+m} in I_{init} with $p_{n+m} = 0$, which becomes $\sum_{j=1}^{n} p_j$ in I_{new}. In the latter case, we perturb I_{init} by removing M_1.

Machine Activation. We construct a initial instance I_{init} with m machines, all n original jobs, and $m+1$ dummy jobs $J_{n+1}, \ldots, J_{n+(m+1)}$ s.t. $p_\ell = \sum_{j=1}^{n} p_j$, for $\ell = n+1, \ldots, n+(m+1)$. The initial schedule S_{init} assigns a dummy job and all n original jobs on machine M_1, two dummy jobs on machine M_2 and one dummy job on each machine $M_3, \ldots, M_m$. Since any feasible schedule assigning at least two dummy jobs to one machine has makespan $\geq 2 \cdot \sum_{j=1}^{n} p_j$, S_{init} must be optimal for I_{init}. We perturb I_{init} by adding a new machine and setting $T_{new} = \sum_{j=1}^{n} p_j + T$. Because $T < \sum_{j=1}^{n} p_j$, any feasible schedule for I_{new} of length T_{new} must assign one dummy job to every machine. Thus, there exists a feasible schedule of makespan T_{new} for I_{new} iff I admits a feasible schedule of makespan T. ∎

Appendix C Robustness Analysis for a Single Perturbation

This section completes the proofs of Lemma 4 and Theorem 4 for analyzing the price of robustness of our two-stage approach in the case of a single perturbation.

Lemma 4. *Consider a makespan problem instance $(m, \mathcal{J})$ and let S be LexOpt schedule. Given a machine $M_\ell \in \mathcal{M}$, denote by $\mathcal{J}'$ the subset of all jobs assigned to the machines in $\mathcal{M} \setminus \{M_\ell\}$ by S. Then, it holds that:*

1. $\max_{M_i \in \mathcal{M}\setminus\{M_\ell\}}\{C_i(S)\} = C^*_{\max}(m-1, \mathcal{J}')$, *and*
2. $C^*_{\max}(m-1, \mathcal{J}) \leq 2 \cdot C^*_{\max}(m, \mathcal{J})$.

Proof:

Suppose that $\max_{M_i \in \mathcal{M}\setminus\{M_\ell\}}\{C_i(S)\} > C^*_{\max}(m-1, \mathcal{J}')$. Let S^* be a minimum makespan schedule for $(m-1, \mathcal{J}')$, i.e. $C_{\max}(S^*) = C^*_{\max}(m-1, \mathcal{J}')$. By scheduling the jobs in $\mathcal{J}'$ as in S^* and assigning the jobs in $\mathcal{J} \setminus \mathcal{J}'$ to M_ℓ, we obtain a feasible schedule $\widetilde{S}$ for $(m, \mathcal{J})$ s.t. $\widetilde{S} <_{\text{lex}} S$, which is a contradiction. Next, starting from an optimal schedule S^* for $(m, \mathcal{J})$, we produce a new schedule $\widetilde{S}$ by moving all jobs of machine M_m to machine M_{m-1}. Clearly, $\widetilde{S}$ is a feasible for $(m-1, \mathcal{J})$ and the makespan has at most doubled w.r.t. S^*. Hence, $C^*_{\max}(m-1, \mathcal{J}) \leq C_{\max}(\widetilde{S}) \leq 2 \cdot C_{\max}(S^*) = 2 \cdot C^*_{\max}(m, \mathcal{J})$. ∎

Theorem 4 (con't). *For the makespan recovery problem with a single perturbation, Algorithm 3 achieves a tight price of robustness equal to 2, if S_{init} is LexOpt.*

Proof:

The proof of the theorem for a processing time reduction or a job removal is presented in the main manuscript. Here, we proceed with the remaining perturbations of Section 2 and show the tightness of our analysis. Let $I_{init} = (m, \mathcal{J})$ be the initial instance with a LexOpt schedule S_{init}.

Job Arrival, Processing Time Augmentation. Suppose that I_{init} is perturbed with the arrival of job J_{n+1}. The recovered schedule S_{rec} maintains the assignments in S_{init} for $J_1, \ldots, J_n$ and assigns job J_{n+1} to a least loaded machine $M_\ell =$

$\arg\min_{M_i \in \mathcal{M}}\{C_i(S_{init})\}$. In an optimal schedule S_{new} for I_{new}, it clearly holds that $C_{\max}(S_{new}) \geq \max\{C_\ell(S_{new}), p_{n+1}\}$. Consider the auxiliary schedule $\widetilde{S}$ obtained from S_{new} by removing job J_{n+1}. Since $\widetilde{S}$ is feasible for I_{init}, $C_{\max}(S_{new}) \geq C_{\max}(\widetilde{S}) \geq C_{\max}(S_{init})$. Hence, $C_{\max}(S_{rec}) = \max\{C_\ell(S_{init}) + p_{n+1}, C_{\max}(S_{init})\} \leq C_{\max}(S_{init}) + p_{n+1} \leq 2 \cdot C_{\max}(S_{new})$. The case where I_{init} is perturbed by increasing p_j can be handled using the same arguments and treating the extra piece of J_j as a new job assigned to the machine executing J_j in S_{init}.

Machine Activation, Machine Failure. Consider the case where I_{init} is perturbed because machine M_m fails. Let $\mathcal{J}'$ be the subset of jobs assigned to M_m in S_{init}. Clearly, $\sum_{J_j \in \mathcal{J}'} p_j \leq C_{\max}(S_{init})$. The recovered schedule S_{rec} keeps the assignments in S_{init} for the jobs in $\mathcal{J} \setminus \mathcal{J}'$ and assigns the jobs in $\mathcal{J}'$ to $M_1, \ldots, M_{m-1}$ using LPT. Thus, $C_{\max}(S_{rec}) \leq C_{\max}(S_{init}) + \sum_{j \in \mathcal{J}'} p_j \leq 2 \cdot C_{\max}(S_{init})$. Let S_{new} be an optimal schedule for I_{new}. Since I_{new} has fewer machines than I_{init}, $C_{\max}(S_{init}) \leq C_{\max}(S_{new})$. Hence, $C_{\max}(S_{rec}) \leq 2 \cdot C_{\max}(S_{new})$. In the case where I_{init} is modified with the activation of a new machine M_{m+1}, S_{rec} has identical assignments with S_{init}, while M_{m+1} is left idle. By Lemma 4.2, $C_{\max}(S_{rec}) = C^*_{\max}(m, \mathcal{J}) \leq 2 \cdot C^*_{\max}(m+1, \mathcal{J}) = 2 \cdot C_{\max}(S_{new})$.

Tightness. Consider an instance $I_{init} = (m, \mathcal{J})$ with $n = m + 1$ jobs of equal processing time p. In a LexOpt schedule S_{init}, machine M_1 executes jobs J_1 and J_2, machine M_i processes job J_{i+1}, for $i = 2, \ldots, m$, and $C_{\max}(S_{init}) = 2p$. Assume that I_{init} is disturbed because (i) job J_n is removed, (ii) p_n is decreased down to zero, or (iii) M_{m+1} is activated. In every case, $C_{\max}(S_{rec}) = 2p$. However, an optimal schedule S_{new} for I_{new}, assigns exactly one job to each machine and $C_{\max}(S_{new}) = p$. Next, consider an instance $I_{init} = (m, \mathcal{J})$ with $n = 1 + (m-1) \cdot m$ jobs, where $p_1 = m$ and $p_j = 1$, for $j = 2, \ldots, n$. In a LexOpt schedule S_{init}, J_1 is assigned to machine M_1, exactly m unit jobs are processed by machine M_i, for $i = 1, \ldots, m$, and $C_{\max}(S_{init}) = m$. Suppose that I_{init} is perturbed because (i) job J_{n+1} with $p_{n+1} = m$ arrives, (ii) p_n is augmented and becomes $m + 1$, or (iii) machine M_1 fails. In each case, $C_{\max}(S_{rec}) = 2m$. But, in an optimal schedule S_{new} for I_{new}, a long job is assigned to the same machine with a unit job, and every other machine contains $m + 1$ unit jobs, i.e. $C_{\max}(S_{new}) = m + 1$. ∎

Appendix D Robustness Analysis for Multiple Perturbations

This section completes the proofs of Lemma 5 and Theorem 5 for analyzing the price of robustness of our two-stage approach in the case of multiple perturbations.

Lemma 5. *Let $(m, \mathcal{J})$ be a makespan problem instance with a LexOpt schedule S.*

1. *If the subset $\mathcal{J}' \subseteq \mathcal{J}$ of jobs is executed by the subset $\mathcal{M}' \subseteq \mathcal{M}$ of machines in S, where $|\mathcal{M}'| = m'$, then the sub-schedule of S on $\mathcal{M}'$ is optimal for $(m', \mathcal{J}')$, i.e. $\max_{M_i \in \mathcal{M}'}\{C_i(S)\} = C^*_{\max}(m', \mathcal{J}')$.*
2. *Assuming that $M_i, M_\ell \in \mathcal{M}$ are two different machines such that job $J_j \in \mathcal{J}$ is assigned to M_i in S, then $C_\ell(S) \geq C_i(S) - p_j$.*
3. *It holds that $C^*_{\max}(m-\ell, \mathcal{J}) \leq \left(1 + \left\lceil \frac{\ell}{m-\ell} \right\rceil\right) \cdot C^*_{\max}(m, \mathcal{J})\ \forall\ \ell \in \{1, \ldots, m-1\}$.*
4. *Let $(m, \hat{\mathcal{J}})$ be a makespan problem instance s.t. $\mathcal{J} = \hat{\mathcal{J}}$ and $\frac{1}{f} \cdot \hat{p}_j \leq p_j \leq \hat{p}_j$ for each J_j, where p_j and $\hat{p}_j$ is the processing time of J_j in $\mathcal{J}$ and $\hat{\mathcal{J}}$, respectively. Then, $\frac{1}{f} \cdot C^*_{\max}(m, \hat{\mathcal{J}}) \leq C^*_{\max}(m, \mathcal{J}) \leq C^*_{\max}(m, \hat{\mathcal{J}})$.*

Proof:

1. Assume for contradiction that $\max_{M_i \in \mathcal{M}'}\{C_i(S)\} > C^*_{\max}(m', \mathcal{J}')$. Starting from an optimal schedule S^* for $(m', \mathcal{J}')$, we construct a feasible schedule $\widetilde{S}$ for $(m, \mathcal{J})$ by assigning the jobs $\mathcal{J}'$ as in S^* and the jobs $\mathcal{J} \setminus \mathcal{J}'$ according to S. Then, $\widetilde{S} <_{\text{lex}} S$, which contradicts that S is a LexOpt schedule.

2. Assume for contradiction that $C_\ell(S) < C_i(S) - p_j$, i.e. $C_\ell(S) < \max\{C_\ell(S) + p_j, C_i(S) - p_j\} < C_i(S)$. Consider the schedule $\widetilde{S}$ obtained from S by moving J_j from M_i to M_ℓ. Then, $C_i(\widetilde{S}) = C_i(S) - p_j$, $C_\ell(\widetilde{S}) = C_\ell(S) + p_j$, and $C_{i'}(\widetilde{S}) = C_{i'}(S)$, for $M_{i'} \in \mathcal{M} \setminus \{M_i, M_\ell\}$. That is, $\widetilde{S} <_{lex} S$, which contradicts that S is LexOpt.

3. Starting from an optimal schedule S^* for $(m, \mathcal{J})$, we produce a schedule $\widetilde{S}$ by moving all jobs on machines $M_{m-\ell+1}, \ldots, M_m$ to the remaining machines via round-robin. For $i = 1, \ldots, \ell$, the jobs of $M_{m-\ell+i}$ are moved to machine $M_{i \bmod (m-\ell)}$, where $M_0 = M_{m-\ell}$. Machine $M_i \in \{M_1, \ldots, M_{m-\ell}\}$ receives jobs from at most $\lceil \ell/(m-\ell) \rceil$ machines. Schedule $\widetilde{S}$ uses $m - \ell$ machines and its makespan has increased by a factor at most $1 + \lceil \frac{\ell}{m-\ell} \rceil$ w.r.t. S^*. Hence, $C^*_{\max}(m - \ell, \mathcal{J}) \leq C_{\max}(\widetilde{S}) \leq \left(1 + \left\lceil \frac{\ell}{m-\ell} \right\rceil\right) \cdot C_{\max}(S^*) = \left(1 + \left\lceil \frac{\ell}{m-\ell} \right\rceil\right) \cdot C^*_{\max}(m, \mathcal{J})$.

4. Starting from an optimal schedule S^* for $(m, \mathcal{J})$, we construct a schedule $\hat{S}$ for $(m, \hat{\mathcal{J}})$ with identical assignments. If machine M_i executes a job of processing time p_j in S^*, then M_i executes a job of processing time $\hat{p}_j$ in $\hat{S}$. Since $\frac{1}{f} \cdot \hat{p}_j \leq p_j \leq \hat{p}_j$, we have that $\frac{1}{f} \cdot C_i(\hat{S}) \leq C_i(S^*) \leq C_i(\hat{S})$, for each machine M_i. ■

Theorem 5 (con't). *For the two-stage robust makespan scheduling problem with $\mathcal{U}(f, k, \delta)$ uncertainty and $k < m$, our LexOpt-based approach achieves a price of robustness:*

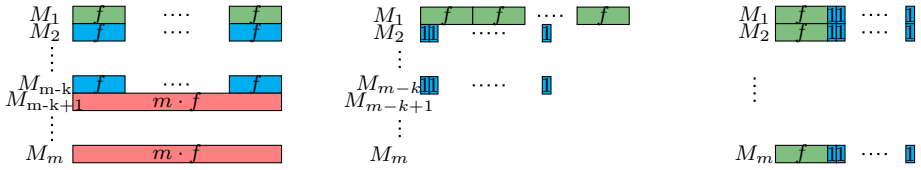
$$2f \cdot \left(1 + \left\lceil \frac{k}{m-k} \right\rceil\right) \cdot (f + k) \cdot \left(1 + \left\lceil \frac{\delta}{m} \right\rceil\right).$$

Proof:

The proof for a processing time reduction or a job removal is presented in the main manuscript. Here, we proceed with the remaining perturbations of Section 2 and show the tightness. For analysis purposes, we consider the perturbations in the order of Table 1. To propagate the solution degradation when analyzing each perturbation type, we consider that $C_{\max}(S_{init}) \leq \rho C_{\max}(S^*_{init})$. That is, the initial schedule S_{init} to be recovered is ρ-approximate for I_{init}, where $\rho \geq 1$ is arbitrary.

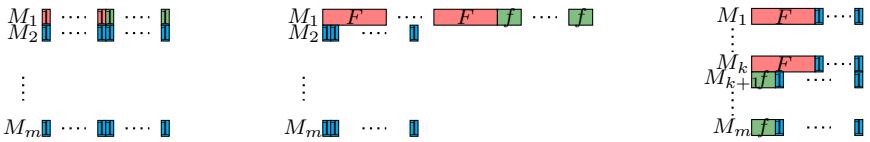
Job Cancellations, Processing Time Reductions (Type 1). Consider an instance I_{init} with m machines, $(m-k) \cdot m$ jobs of length f, and k jobs of length $m \cdot f$, where $f, k = o(m)$. Optimal schedule S_{init} assigns m jobs of length f on each of the first $m-k$ machines, one job of length $m \cdot f$ on each of the remaining k machines, and has makespan $C_{\max}(S_{init}) = m \cdot f$. We perturb I_{init} by decreasing the processing time of each job assigned to $M_2, \ldots, M_{m-k}$ down to 1, and cancelling the jobs assigned to the last k machines. The recovered schedule has makespan $C_{\max}(S_{rec}) = m \cdot f$. But an optimal schedule S_{new} assigns to each machine a job of length f and $m-k-1$ unit-length jobs, i.e. $C_{\max}(S_{new}) = (m-k-1)+f$. Figure 8 illustrates this tightness example, where $\frac{C_{\max}(S_{rec})}{C_{\max}(S_{new})} = \frac{m \cdot f}{(m-k-1)+f} = \frac{m \cdot f}{(m-k)\left(1+\frac{f-1}{m-k}\right)} = O((1 + \frac{k}{m-k})f)$.

Processing Time Augmentations (Type 2). Recall that job $J_j \in \mathcal{J}$ is *stable* if $\hat{p}_j \leq f p_j$ and *unstable* if not. Also, let $F = \max_{J_j \in \hat{\mathcal{J}}}\{\hat{p}_j\}$. Since $I_{new} \in \mathcal{U}(f, k, \delta)$, at most k processing times become equal to F and all remaining jobs are increased by a factor at most f. Given that S_{rec} is identical with S_{init}, except that some processing times are increased, $C_i(S_{rec}) \leq f \cdot C_i(S_{init}) + k \cdot F$ for each $M_i \in \mathcal{M}$. Given an



(a) LexOpt schedule S_{init}. (b) Recovered schedule S_{rec}. (c) Optimal schedule S_{new}.

Figure 8: Makespan recovery instance for which the $O(f \cdot (1 + \lceil \frac{k}{m-k} \rceil))$ factor is tight under job cancellations and processing time reductions.



(a) LexOpt schedule S_{init}. (b) Recovered schedule S_{rec}. (c) optimal schedule S_{new}.

Figure 9: Makespan recovery instance for which the $O(f + k)$ factor is tight under processing time augmentations.

optimal schedule S^*_{init} for I_{init} and that the processing times in I_{new} are one-to-one greater than or equal to the ones in I_{init}, $C_i(S_{init}) \leq \rho \cdot C_{\max}(S^*_{init}) \leq \rho \cdot C_{\max}(S_{new})$. In addition, $C_{\max}(S_{new}) \geq F$. Hence, $C_{\max}(S_{rec}) \leq (f + k)\rho \cdot C_{\max}(S_{new})$.

For the tightness, consider instance I_{init} (Figure 9) with m machines and $n = m^2$ unit-length jobs. In S_{init}, each machine executes m jobs and $C_{\max}(S_{init}) = m$. After uncertainty realization, k jobs assigned to M_1 get processing time F, every other job on M_1 gets processing time f, and all other processing times remain the same in I_{init} and I_{new}. Suppose that $F = f + m$, $F = \Theta(m)$, $f = o(m)$ and $k = o(m)$. Schedule S_{rec} performs identical assignments with S_{init}, i.e. $C_{\max}(S_{rec}) = k \cdot F + (m-k) \cdot f$. In an optimal schedule S_{new} for I_{new}, each machine M_i with $i \leq k$ processes a job of length F and k unit length jobs, while each machine M_i with $i > k$ executes a job of length f and $m+k$ unit length jobs, i.e. $C_{\max}(S_{new}) = F+m$. Thus, $\frac{C_{\max}(S_{rec})}{C_{\max}(S_{new})} = \frac{k \cdot F}{F+k} + \frac{(m-k) \cdot f}{f+m+k} = k \cdot \frac{1}{1+\frac{k}{F}} + f \cdot \frac{1}{1+\frac{f+2k}{m-k}} = O(f + k)$.

Machine Activations (Type 3). Denote by $\mathcal{M}^s$ the set of available machines in

S_{init} and by $\mathcal{M}^u$ the set of newly activated machines after uncertainty realization. Algorithm 3 keeps the schedule S_{init} for the machines in $\mathcal{M}^s$ and leaves the machines in $\mathcal{M}^u$ idle, i.e. $C_{\max}(S_{rec}) \leq \rho \cdot C^*_{\max}(m, \mathcal{J})$. By definition, $C_{\max}(S_{new}) = C^*_{\max}(m+k, \mathcal{J})$. Hence, by Lemma 5.3, we conclude that $C_{\max}(S_{rec}) \leq (1 + \lceil k/m \rceil)\rho \cdot C_{\max}(S_{new})$. For the tightness of this bound, consider an instance with m machines and $n = m \cdot (m+k)$ unit jobs. In S_{init}, each machine processes $m+k$ unit jobs and $C_{\max}(S_{init}) = m+k$. In S_{rec}, all newly activated machines are empty and $C_{\max}(S_{rec}) = m+k$. But an optimal schedule S_{new} for I_{new} assigns exactly m jobs on each machine, i.e. $C_{\max}(S_{new}) = m$. That is, $\frac{C_{\max}(S_{rec})}{C_{\max}(S_{init})} = 1 + \frac{k}{m}$.

Job Arrivals, Machine Failures (Type 4). Consider a set of *free jobs* arriving after uncertainty realization. Algorithm 3 schedules these jobs according to LPT. We partition the set $\mathcal{M}$ of machines into the set $\mathcal{M}^s$ of *stable machines*, not executing free jobs, and the set $\mathcal{M}^u$ of *unstable machines*, assigned free jobs, in S_{rec}. Since $\mathcal{J}_{init} \subset \mathcal{J}_{new}$ and $\mathcal{M}_{init} = \mathcal{M}_{new}$, if there is a machine $M_i \in \mathcal{M}^s$ with $C_i(S_{rec}) = C_{\max}(S_{rec})$, then $C_{\max}(S_{rec}) = C_{\max}(S_{init}) \leq \rho \cdot C^*_{\max}(I_{init}) \leq \rho \cdot C_{\max}(S_{new})$. If there is not such a machine in $\mathcal{M}^s$, we use the analysis for LPT by Graham [27]. Since the last completing job J_j begins at b_j, all machines are occupied until b_j in S_{rec}. Thus, $C_{\max}(S_{rec}) = b_j + p_j \leq \frac{1}{m}\sum_{J_{j'} \in \mathcal{J}_{new}} p_{j'} + p_j \leq 2 \cdot C_{\max}(S_{new})$. In both cases, $C_{\max}(S_{rec}) \leq \max\{2, \rho\} \cdot C_{\max}(S_{new})$. The case where a machine $M_i \in \mathcal{M}$ fails can be treated similarly by considering the jobs originally assigned to M_i in S_{init} as free. In the case where $\rho > 2$, we may design a makespan problem instance such that the arrival of a new job with a tiny processing time has the effect that the recovered schedule remains ρ-approximate. The tightness example for LPT implies that Algorithm 3 cannot result in a price of robustness better than 2. ∎

Appendix E Flexible Recovery Strategy

Next, we present a more flexible recovery strategy (than Algorithm 3) that modifies a bounded number of binding decisions [18, 43]. To this end, we formulate the makespan recovery problem as a MILP. Let $\mathcal{J}^B = \{J_j \in \mathcal{J}_{init} \cap \mathcal{J}_{new} : \exists i \text{ with } x_{i,j}(S_{\text{init}}) = 1\}$ be the binding decisions, i.e. the jobs appearing both in I_{init} and I_{new}. Algorithm 3 keeps the assignments in S_{init} for the *binding jobs* $\mathcal{J}^B$ and

greedily schedules the *free jobs* $\mathcal{J}^F = \mathcal{J} \setminus \mathcal{J}^B$ with LPT to produce S_{rec}. A more flexible recovery strategy migrates a bounded number g of binding jobs. These migrations produce better recovered solutions at the price of extra computational effort and higher transformation cost. Denote by $\mathcal{J}_i^B \subseteq \mathcal{J}^B$ and μ_j the subset of binding jobs assigned to machine M_i and the machine index which job $J_j \in \mathcal{J}^B$ is assigned to, respectively, in S_{init}. Our flexible recovery strategy solves MILP (1) with the additional constraint $\sum_{J_j \in \mathcal{J}^B} \sum_{M_i \in \mathcal{M} \setminus \{M_{\mu_j}\}} x_{i,j} \leq g$.

Appendix F Numerical Results with Degenerate Instances

This section complements our numerical results with degenerate instances of $P||C_{\max}$. Section F.1 describes the generation of these instances. Sections F.2-F.3 evaluate the LexOpt branch-and-bound algorithm and the robustness of our two-stage approach using the new instances.

F.1 Generation of Degenerate Instances

Degenerate instances have less balanced optimal solutions than *well-formed instances*. To produce degenerate instances, we sample integer processing times that can be encoded with b bits. Instances with small $\kappa = b/n$ values are easier to solve than instances with larger κ values [4]. The *phase transition* from "easy" to "hard" instances becomes sharper as n increases and occurs at the threshold value $\kappa^* = \frac{\log_2 m}{m-1}$. Instances with small κ admit exponentially many perfect solutions (where all machine completion times are equal). Instances with $\kappa > \kappa^*$ have less or no perfect solutions. Similar phase transitions occur for other fundamental combinatorial optimization problems, e.g. satisfiability [39] and the traveling salesman problem [24], where instances near the threshold value tend to be the most difficult.

We derive degenerate instances by varying two parameters: (i) the number m of machines and (ii) the number n of jobs. Further, we use a processing time seed $q = 2^{\lfloor \kappa(m) \cdot n \rfloor}$ with $\kappa(m) = (\log_2 m)/(m-1)$. Table 3 reports this information for *moderate* and *intermediate* degenerate instances. For each combination of m, n and the corresponding $q = 2^{\lfloor \kappa(m) \cdot n \rfloor}$ value, we generate 3 instances by sampling processing times from the set $\{1, \ldots, q\}$ using the uniform, normal and symmetric of normal distributions, similarly to the well-formed instances.

Table 3: Degenerate Instances

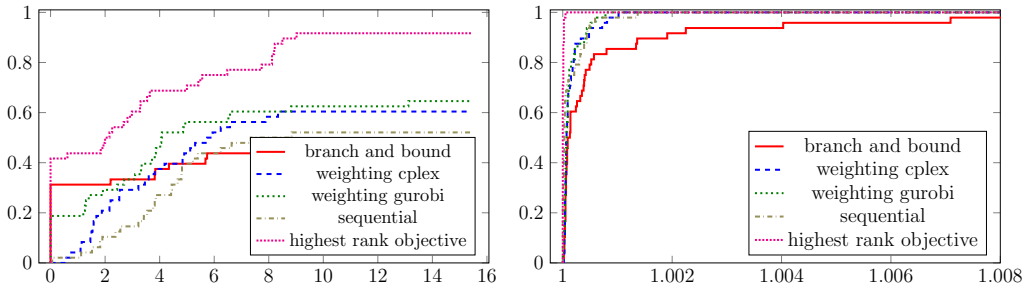
Instances	m	n	q
Moderate	3	$20, 25, 30, 35$	$2^{15}, 2^{19}, 2^{23}, 2^{27}$
	4	$25, 30, 35, 40$	$2^{16}, 2^{20}, 2^{23}, 2^{26}$
	5	$30, 35, 40, 45$	$2^{17}, 2^{20}, 2^{23}, 2^{26}$
	6	$35, 40, 45, 50$	$2^{18}, 2^{20}, 2^{23}, 2^{25}$
Intermediate	10	$40, 50, 60, 70$	$2^{14}, 2^{18}, 2^{22}, 2^{25}$
	12	$45, 55, 65, 75$	$2^{14}, 2^{17}, 2^{21}, 2^{24}$
	14	$55, 65, 75, 85$	$2^{16}, 2^{19}, 2^{21}, 2^{24}$
	16	$60, 70, 80, 90$	$2^{16}, 2^{18}, 2^{21}, 2^{24}$

F.2 LexOpt Branch-and-Bound Algorithm Evaluation

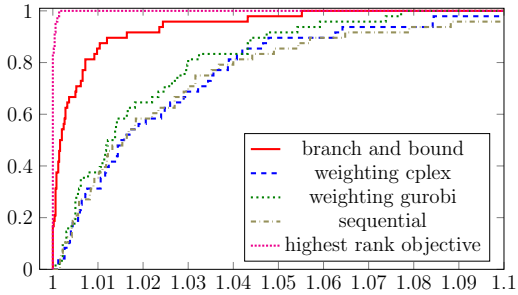
We evaluate our branch-and-bound algorithm on degenerate instances in comparison with the sequential, weighting and highest-rank objective methods. For MILP solving, we use (i) 10^3 CPU seconds time limit and (ii) 10^{-4} error tolerance, similarly to the numerical results obtained with well-formed instances. Figure 10 shows performance profiles for evaluating the running times and quality of computed solutions on degenerate instances. We observe that degenerate instances are significantly harder to solve than well-formed instances of identical size. For instance, no solver converges for any intermediate degenerate instance, while every solver converges for $> 30\%$ of the intermediate well-formed instances. In terms of solver comparison, we derive similar results to those obtained for well-formed instances. The sequential method performs similarly to the weighting method. The highest-rank objective method produces the best heuristic results. Our branch-and-bound method produces the second best heuristic result for intermediate degenerate instances and computes LexOpt solutions quickly when it terminates.

F.3 Two-Stage Robustness Assessment

Next, we investigate the impact of LexOpt to the quality of the recovered solution for degenerate instances. For each degenerate instance I_{init} derived according to Section F.1, we compute 50 diverse initial solutions using the CPLEX solution pool feature. To quantify the closeness of an initial solution to LexOpt, we use the weighted value $W(S) = \sum_{i=1}^{m} B^{m-i} \cdot C_i(S)$. Further, we obtain a perturbed instance I_{new} by generating random disturbances, similarly to well-formed instances. Then, we fix every initial solution by applying our binding and flexible recovery strategies in Section 4.1. As in the case of well-formed instances, Figures 11a and

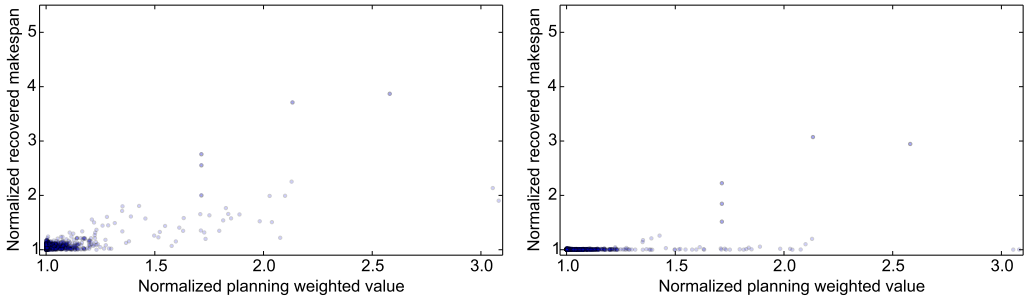


(a) Moderate instances: time (s) on $\log_2$ scale (left), upper bounds on $[1, 1.008]$ (right).



(b) Intermediate instances: upper bounds on $[1, 1.1]$. *No solver converges for any intermediate degenerate instance within the specified time limit.*

Figure 10: Performance profiles for the *degenerate test set* with 10^3 s timeout.



(a) Binding Recovery (b) Flexible Recovery

Figure 11: Degenerate instances scatter plots illustrating the recovered solution makespan with respect to the initial solution weighted value.

11b plot the normalized makespan $C^N(S_{rec})$ obtained by our recovery strategies and the normalized initial solution weighted value $W^N(S_{init})$ for every recovered solution. Clearly, the recovered solution improves if the initial solution weighted value decreases. Moreover, flexibility enables more efficient recovery. Interestingly, degenerate instances are recovered more efficiently than well-formed ones.

Appendix G Table of Notation

Table 4: Nomenclature

Name	Description
Makespan scheduling problem	
$I = (m, \mathcal{J})$	Instance
i, q, μ	Machine indices (q, μ typically used as auxiliary machine indices)
j, h, ℓ	Job indices (h, ℓ typically used as auxiliary job indices)
m	Number of machines
$M_i \in \mathcal{M}$	Machine M_i in the set $\mathcal{M} = \{M_1, \ldots, M_m\}$ of all machines
n	Number of jobs
$J_j \in \mathcal{J}$	Job J_j in the set $\mathcal{J} \in \{J_1, \ldots, J_n\}$ of all jobs
p_j	Processing time of job J_j
$C_{\max}$	Makespan
C_i	Completion time of machine M_i
$x_{i,j}$	Binary variable indicating an assignment of job J_j to machine M_i
$S = (\vec{y}, \vec{C}), S', \widetilde{S}$	Schedules ($S', \widetilde{S}$ typically used as auxiliary schedules)
S^*	LexOpt schedule
$\mathcal{S}$	Set of all feasible schedules
LexOpt scheduling problem	
$\leq_{\text{lex}}$	Operator for lexicographic comparison
F_i	i-th greatest completion time, i.e. i-th objective function
v_i^*	Value of F_i in a LexOpt schedule S^*
$\mathcal{T}_q$	Set of tuples $(i_1, \ldots, i_q)$ with q pairwise disjoint machine indices
w_i	Weight of objective function F_i (weighting method)
$\mathcal{P}$	Solution pool (highest-rank objective method)
Branch-and-bound algorithm	
Q	Stack of visited unexplored nodes
I	Incumbent, i.e. lexicographically best-found solution
u, v, r	Nodes (r is the root) in the branch-and-bound tree
$\mathcal{S}(u)$	Feasible solutions below node u in the branch-and-bound tree
ℓ	Node level, i.e. job index, in the branch-and-bound tree
t_i	Partial completion time of machine M_i in a branch-and-bound node
$\mathcal{R}$	Subset of jobs scheduled below a branch-and-bound node
$L_i, \vec{L}$	Component L_i of vectorial lower bound $\vec{L} = (L_1, \ldots, L_m)$
$U_i, \vec{U}$	Component U_i of vectorial upper bound $\vec{U} = (U_1, \ldots, U_m)$
τ	Time point
$\tilde{p}_j$	Piece of job J_j
λ, Λ	Amount of processing time load
Makespan recovery problem	
I_{init}	Initial instance $(m_{init}, \mathcal{J}_{init})$
I_{new}	Perturbed instance $(m_{new}, \mathcal{J}_{new})$
S_{init}	Initial optimal schedule for I_{init}
S_{rec}	Recovered schedule for I_{new}
S_{new}	Optimal schedule for I_{new}
ρ	Approximation ratio
Uncertainty modeling	

$\mathcal{U}(f,k,\delta)$	Uncertainty set
f	Perturbation factor
k	Number of unstable jobs
δ	Number of new machines
$C^*_{\max}(m,\mathcal{J})$	Optimal objective value of makespan problem instance $(m,\mathcal{J})$
Binding recovery strategy	
T, T_{new}	Target makespan for instance I and perturbed instance I_{new}
T_{new}	Target makespan for perturbed instance I_{new}
$\mathcal{M}'$	Subset of machines
m'	Number of machines in $\mathcal{M}'$
$\mathcal{J}'$	Subset of jobs
η	Reduction of p_j
$(\hat{m},\hat{\mathcal{J}})$	Neighboring instance of $(m,\mathcal{J})$
$\hat{p}_j$	Processing time of job J_j in $(\hat{m},\hat{\mathcal{J}})$
$\mathcal{M}^s,\mathcal{M}^u$	Stable machines $\mathcal{M}^s$ and unstable machines $\mathcal{M}^u=\mathcal{M}\setminus\mathcal{M}^s$
m^s,m^u	Number of stable (m^s) and unstable (m^u) machines
$\mathcal{J}^s_{init}$	Subset of stable jobs in I_{init}
$\mathcal{J}^s_{new}$	Subset of stable jobs in I_{new}
F	Maximum processing time in I_{new}
Flexible recovery strategy	
$\mathcal{J}^B,\mathcal{J}^F$	Subset of binding ($\mathcal{J}^B$) and free ($\mathcal{J}^F=\mathcal{J}\setminus\mathcal{J}^B$) jobs
$\mathcal{J}^B_i$	Binding jobs originally assigned to machine M_i
μ_j	Machine executing job J_j in S_{init}
g	Limit on migrations of binding job
Numerical results	
κ	Phase transition parameter
κ^*	Critical value of phase transition parameter
b	Number of bits for generating processing times
q	Processing time seed
$\mathcal{U}$	Discrete uniform distribution
$\mathcal{N}$	Normal distribution
W	Weighted value, i.e. weighted sum of objective functions
Ub, Lb	Best-found incumbent (Ub) and lower bound (Lb)
d_m, d_n	Number of machine (d_m) and job (d_n) disturbances
W^N	Normalized weighted value
W^*	Best computed weighted value
$C^N_{\max}$	Normalized makespan
$C^*_{\max}$	Best recovered makespan