Lorentzian metrics from holomorphic metrics

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Abstract: The relationship between real and holomorphic 4-metrics is considered. Methods of constructing Lorentzian 4-metrics from holomorphic 4-metrics are outlined. The computation of Lorentzian solutions of Einstein’s equations is briefly discussed.
Holomorphic 4-metrics have been extensively studied since the initial work of Newman, Penrose and Plebański in the 1970’s, [1-3]. The aim of this paper is two-fold. First the real geometry associated with such metrics will be re-considered. This discussion will include a review of some known results, [4-7], as well as new ones. The second, and related aim, is to outline ways in which real Lorentzian 4-metrics can be obtained from holomorphic 4-metrics, and to consider the curvature properties of the Lorentzian metrics.

Let $M$ be a real, oriented eight dimensional manifold equipped with an integrable complex structure $I$. Let $g$ be a symmetric tensor of rank $(0,2)$ on $M$, which is holomorphic, annihilates all $(0,1)$ vectors and is non-degenerate in the sense that if $g(X, Y) - ig(IX, Y) = 0$, for all real vector fields $Y$ tangent to $M$, then the real vector field $X$ must vanish. Then $g$ is a holomorphic metric on the oriented complex 4-manifold and $g(IX, IY) = -g(X, Y)$. A holomorphic Levi-Civita connection and curvature tensor for $g$ can be computed using the standard formulae. Since $g$ is a 4-metric, associated with it there are two holomorphic almost quaternionic metric structures $(M, g, +J^i)$ and $(M, g, -J^i)$, $i = 1, 2, 3$. Here the $(\frac{1}{1})$ tensors $+J^i$ and $-J^i$ are determined by $g$ and bases of holomorphic self-dual, (+), and anti-self-dual, (-), 2-forms. Each such triple satisfies the algebraic relations (i) $g(J^iU, J^iV) = g(U, V)$, for all $(1,0)$ vectors $U$ and $V$, and (ii) $J^iJ^j = -\delta^{ij}I + \epsilon^{ijk}J^k$, where $I$ is a unit tensor and $\epsilon^{ijk}$ is the totally skew Levi-Civita symbol. [Here, and where possible elsewhere, the superscripts, $+$, have been suppressed.]

If $g = h + i k$, its real and imaginary parts, $h$ and $k$, are real 8-metrics of signature $(4,4)$. Furthermore they are anti-Hermitian with respect to the almost complex structure $I$, that is $h(IX, IY) = -h(X, Y)$, and similarly for $k$. Furthermore, $k(X, Y) = -h(IX, Y)$, and $h$ and $k$ are anti-Kählerian since $\nabla_h I = \nabla_k I = 0$, where the covariant derivatives are the Levi-Civita covariant derivatives of the metrics. (Conversely, by starting from these real structures, the holomorphic metric can be constructed.) The holomorphic tensors $+J^i$ and $-J^i$ can be extended, by taking the direct sums with their complex conjugates, to the real $(\frac{1}{1})$ tensors $+J^i = +J^i \oplus +\bar{J}^i$; and similarly for $-J^i$. It follows immediately, from the holomorphic results in the previous paragraph, that each of the (four) triples $(M, h, J^i)$ and $(M, k, J^i)$ constitutes a (real) almost quaternionic metric structure. When the products $IJ^i$ are considered further interesting structures emerge.

When the holomorphic metric, $g$, is half-flat, one of the two triples of tensors, $J^i$, associated with it as above, can be chosen to be covariantly constant.
with respect to the holomorphic Levi-Civita connection. It immediately follows that the corresponding triple of real tensors $J^i$ is covariantly constant with respect to the Levi-Civita connections of both $h$ and $k$. Consequently, in the half-flat case, the triples $(M,h,J^i)$ and $(M,k,J^i)$ are each hyper-Kähler structures. By using a known construction [8], these latter structures can be encoded in, and recovered from, a complex structure on a complex five manifold, the twistor space, over the eight dimensional real manifold $M$. Recovery of the half-flat holomorphic metric $g$ includes a further step, the extraction of the complex structure $I$ on $M$.

There are a number of ways in which Lorentzian 4-metrics on real four manifolds can be obtained from holomorphic metrics; examples are included in references [9-13]. For instance, let a holomorphic half-flat 4-metric on $M$, $g$, be expressed in terms of Plebański’s “second” system of coordinates, $z^a = x^a + iy^a$, $(a = 1 - 4)$, [3]. Then a real Lorentzian metric may be obtained by pulling back the real part, of $g$, $h$, to the real four dimensional sub-manifold $N$ given by $y^a = 0$. The imaginary part of $g$, $k$, is degenerate on $N$. Alternatively, Lorentzian 4-metrics can be constructed by superposing complex co-frames and their complex conjugates. One particular way to pursue this latter approach is to first construct on $M$ real $p$-forms, $2 \leq p \leq 8$. A natural construction, using co-frames which satisfy Cartan’s structure equations for a half-flat holomorphic metric $g$, gives real $p$-forms which also satisfy geometrically interesting equations. In particular these latter $p$-forms have vanishing covariant exterior derivative with respect to an $so(1,3)$-valued connection. That connection is determined by the non-flat, holomorphic, $sl(2,C)$-valued spin connection (and its complex conjugate) of the half-flat metric, $g$. For $p$ equal to 2 and 3, real co-frames, which satisfy the real Cartan structure equations for Lorentzian 4-metrics on a real four dimensional submanifold $N$, have been constructed by using the pull-backs of these real forms to $N$.

Investigation of the curvature properties of the Lorentzian 4-metrics on $N$, constructed using the methods outlined in the previous paragraph, is continuing. Explicit examples of classes of solutions of Einstein’s vacuum field equations, and solutions corresponding to pure radiation sources, have been obtained from metrics constructed in this way. The curvature of the linearized form of these vacuum solutions can be directly identified with the superposition of complex conjugate solutions of the zero rest-mass, spin two field equations constructed from Hertz potentials in Minkowski space-time, [14]. Further details and references are contained in reference [13].
References


