Addendum to Generalized forms, Chern-Simons and Einstein-Yang-Mills theory

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March 27, 2010

Abstract
This technical report is an addendum to [1] where polychains and a Stokes' theorem for generalized forms were introduced. Here these concepts are studied further using type $N = 2$ generalized forms. The second Chern class and first Pontrjagin class and their variations are also considered.

1 Introduction

In [1] polychains (poly-chains) and a Stokes' theorem for generalized forms were introduced and used in the construction of generalized characteristic classes and actions for Einstein-Yang-Mills theory. Whereas an ordinary characteristic class may be obtained by integrating an ordinary differential form over an ordinary chain, generalized characteristic classes are obtained by integrating generalized forms over polychains. The focus in this addendum is on type $N = 2$ generalized forms and its primary purpose is to exhibit their use in the construction of generalized second Chern classes, generalized first Pontrjagin classes and Chern-Simons three-forms. Connections and fields are defined on a manifold M, of dimension n, which is not assumed to be metric. Integrals are computed over $3-$ and $4-$ polychains, of type $N = 2$, in M. The latter are ordered sets of ordinary (real, singular) chains. Metric structures on the chains, if they exist, are defined by the pull-backs of the generalized connections and their group structure. There can be singularities in the metric geometry, or no metric geometry at all, on the polychains.

A brief review of the properties of type $N = 2$ generalized forms and integration over type $N = 2$ polychains is given in section two. As in [1] two cases, corresponding to two classes of exterior derivatives, are carried along. In the third section generalized connections, the generalized second Chern class and first Pontrjagin class, and generalized Chern-Simons three forms are discussed using type $N = 2$ forms.

The notation employed in this report is the same as in [1]. Bold-face Roman letters again denote generalized forms. Ordinary forms are again denoted by Greek letters. Where it is useful the degree of a form is indicated above it, for example $\overset{p}{\mathbf{a}}$. When the degree is obvious from the context it will not be indicated so explicitly. The exterior product of any two forms, α and β, is written $\alpha\beta$. As usual any ordinary form $\overset{q}{\alpha}$, with q either negative or greater than n is zero. Hence type $N = 2$ generalized forms, $\overset{p}{\mathbf{a}}$, with p less than minus two or greater than n are zero. This paper develops results which can be found primarily in [1] and [2]. References to other work on generalized forms can be found in [2] and [1].

2 Type $N = 2$ generalized forms

This section contains a brief review of type $N = 2$ generalized forms on an n dimensional manifold M. These are elements of the $\Lambda_{(2)}(M) = \oplus_{p=-2}^{p=n}\Lambda^p_{(2)}(M)$, where $\Lambda^p_{(2)}(M)$ denotes the type $N = 2$ generalized $p-$forms on M. A generalized $p-$form, $\overset{p}{\mathbf{a}} \in \Lambda^p_{(2)}(M)$, may be expressed as

$$\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}_1\mathbf{m}^1 + \overset{p+1}{\alpha}_2\mathbf{m}^2 + \overset{p+2}{\alpha}\mathbf{m}^1\mathbf{m}^2, \tag{1}$$

where $\mathbf{m}^1$ and $\mathbf{m}^2$ are two linearly independent degree minus one$-$forms, with non-zero exterior product $\mathbf{m}^1\mathbf{m}^2$. The degrees of the ordinary differential forms $\overset{p}{\alpha}$, $\overset{p+1}{\alpha}_1$, $\overset{p+1}{\alpha}_2$, $\overset{p+2}{\alpha}$ are p, $p+1$ and $p+2$. For non-zero, type $N = 2$, generalized forms p can take integer values from -2 to n. All the forms obey the usual rules of exterior multiplication. If,

$$\overset{q}{\mathbf{b}} = \overset{q}{\beta} + \overset{q+1}{\beta}_1\mathbf{m}^1 + \overset{q+1}{\beta}_2\mathbf{m}^2 + \overset{q+2}{\beta}\mathbf{m}^1\mathbf{m}^2 \tag{2}$$

is a generalized $q-$form of type $N = 2$, the exterior product, $\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}$, is given by the degree $(p+q)$ generalized form $\overset{p+q}{\mathbf{c}} \in \Lambda^{p+q}_{(2)}(M)$ where

$$\begin{aligned}
\overset{p+q}{\mathbf{c}} &= \overset{p+q}{\gamma} + \overset{p+q+1}{\gamma_1}\mathbf{m}^1 + \overset{p+q+1}{\gamma_2}\mathbf{m}^2 + \overset{p+q+2}{\gamma}\mathbf{m}^1\mathbf{m}^2, \text{ where} \qquad (3)\\
\overset{p+q}{\gamma} &= \overset{p}{\alpha}\overset{q}{\beta},\\
\overset{p+q+1}{\gamma_1} &= \overset{p}{\alpha}\overset{q+1}{\beta}_1 + (-1)^q\overset{p+1}{\alpha}_1\overset{q}{\beta}, \quad \overset{p+q+1}{\gamma_2} = \overset{p}{\alpha}\overset{q+1}{\beta_2} + (-1)^q\overset{p+1}{\alpha_2}\overset{q}{\beta},\\
\overset{p+q+2}{\gamma} &= \overset{p}{\alpha}\overset{q+2}{\beta} + (-1)^{q+1}\overset{p+1}{\alpha}_1\overset{q+1}{\beta_2} + (-1)^q\overset{p+1}{\alpha}_2\overset{q+1}{\beta}_1 + \overset{p+2}{\alpha}\overset{q}{\beta}.
\end{aligned}$$

As in the case of ordinary forms the exterior product for generalized forms is bilinear, associative and anticommutative

$$\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}} = (-1)^{pq}\overset{q}{\mathbf{b}}\overset{p}{\mathbf{a}}. \tag{4}$$

If φ is a smooth map $\varphi : M \to N$, then the induced map of generalized forms, $\varphi^* : \Lambda^p_{(2)}(N) \to \Lambda^p_{(2)}(M)$, is the linear map defined by using the standard pull-back map for ordinary forms

$$\varphi^*(\overset{p}{\mathbf{a}}) = \varphi^*(\overset{p}{\alpha}) + \varphi^*(\overset{p+1}{\alpha}_1)\mathbf{m}^1 + \varphi^*(\overset{p+1}{\alpha}_2)\mathbf{m}^2 + \varphi^*(\overset{p+2}{\alpha})\mathbf{m}^1\mathbf{m}^2, \tag{5}$$

Then $\varphi^*(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = \varphi^*(\overset{p}{\mathbf{a}})\varphi^*(\overset{q}{\mathbf{b}})$.

For ordinary forms there is a unique exterior derivative $d : \Lambda^p_{(0)}(M) \rightarrow \Lambda^{p+1}_{(0)}(M)$ satisfying the four conditions, [3],

(i) $d(\alpha + \beta) = d\alpha + d\beta$.

(ii) for $f \in f \in \Lambda^0_{(0)}(M)$, df has its usual meaning as the differential of f,

(iii) $d \circ d = 0$,

(iv) $d(\alpha\beta) = d\alpha\beta + (-1)^p \alpha d\beta$.

Exterior derivatives, $d : \Lambda^p_{(N)}(M) \rightarrow \Lambda^{p+1}_{(N)}(M)$, for generalized forms of type N greater than zero, also satisfy these conditions but they are not uniquely determied by them. As in [2] and [1] two classes of exterior derivative will be considered and carried along. These correspond to the choices

$$d\mathbf{m}^i = \delta^i_1 \epsilon, \tag{6}$$

with i, j, k summing and ranging over one to two. In the first case $\epsilon = 0$. In the second case ϵ is a non-zero constant. It can be convenient to scale the minus one-form $\mathbf{m}^1$ so that $\epsilon = 1$ but that will not be done here.

The exterior derivative of a generalized $p-$form $\overset{p}{\mathbf{a}}_{(2)}$is the generalized $(p+1)-$form

$$d\overset{p}{\mathbf{a}}_{(2)} = [d\overset{p}{\alpha} + (-1)^{p+1}\epsilon \overset{p+1}{\alpha}{}_1] + d\overset{p+1}{\alpha}{}_1 \mathbf{m}^1 + [d\overset{p+1}{\alpha}{}_2 + (-1)^p \epsilon \overset{p+2}{\alpha}]\mathbf{m}^2 + d\overset{p+2}{\alpha}\mathbf{m}^1\mathbf{m}^2, \tag{7}$$

and clearly satisfies $\varphi^*(d\overset{p}{\mathbf{a}}_{(N)}) = d(\varphi^* \overset{p}{\mathbf{a}}_{(N)})$. A type $N = 2$ form $\overset{p}{\mathbf{a}}$ is closed if and only if

$$\begin{aligned} d\overset{p}{\alpha} + (-1)^{p+1}\epsilon \overset{p+1}{\alpha}{}_1 &= 0, \\ d\overset{p+1}{\alpha}{}_1 &= 0, \\ d\overset{p+1}{\alpha}{}_2 + (-1)^p \epsilon \overset{p+2}{\alpha} &= 0, \\ d\overset{p+2}{\alpha} &= 0. \end{aligned} \tag{8}$$

Hence in case (i), where $\epsilon = 0$, $\overset{p}{\mathbf{a}}$ is closed if and only if all the ordinary forms defining it are closed. On the other hand in case (ii), where ϵ is non-zero, $\overset{p}{\mathbf{a}}$ is closed if and only if it is exact. In the latter case

$$\begin{aligned} \overset{p}{\mathbf{a}} &= \overset{p}{\alpha} + \epsilon^{-1}(-1)^p d\overset{p}{\alpha}\mathbf{m}^1 + \overset{p+1}{\alpha}{}_2 \mathbf{m}^2 + \epsilon^{-1}(-1)^{p+1} d\overset{p+1}{\alpha}_2 \mathbf{m}^1\mathbf{m}^2 \\ &= d[(-1)^p \epsilon^{-1}\overset{p}{\alpha}\mathbf{m}^1 + (-1)^{p+1}\epsilon^{-1}\overset{p+1}{\alpha}_2 \mathbf{m}^1\mathbf{m}^2]. \end{aligned} \tag{9}$$

Integration of generalized forms is defined using polychains. A $p-$polychain of type $N=2$ in M is an ordered quadruple of ordinary (real, singular) chains in M

$$\mathbf{c}_p^{(2)} = (c_p, c_{p+1}^1, c_{p+1}^2, c_{p+2}), \tag{10}$$

where c_p is an ordinary $p-$chain, c_{p+1}^1 and c_{p+1}^2 are ordinary $p+1-$chains and c_{p+2} is an ordinary $p+2-$chain, [1]. The ordinary chains have respective boundaries $\partial c_p, \partial c_{p+1}^1, \partial c_{p+1}^2$ and ∂c_{p+2}. The boundary of the polychain $\mathbf{c}_p^{(2)}$ is the $(p-1)-$polychain

$$\partial \mathbf{c}_p^{(2)} = (\partial c_p, \partial c_{p+1}^1 + (-1)^p \epsilon c_p, \partial c_{p+1}^2, \partial c_{p+2} + (-1)^{p+1} \epsilon c_{p+1}^2), \tag{11}$$

and $\partial^2 \mathbf{c}_p^{(2)} = 0$. The integral of a generalized form $\overset{p}{\mathbf{a}}_{(2)}$ over a polychain $\mathbf{c}_p^{(2)}$ is

$$\int_{\mathbf{c}_p^{(2)}} \overset{p}{\mathbf{a}}_{(2)} = \int_{c_p} \overset{p}{\alpha} + \int_{c_{p+1}^1} \overset{p+1}{\alpha}_1 + \int_{c_{p+1}^2} \overset{p+1}{\alpha}_2 + \int_{c_{p+2}} \overset{p+2}{\alpha}. \tag{12}$$

There is a Stokes' theorem for generalized forms and polychains. When $N=2$

$$\int_{\mathbf{c}_p^{(2)}} d\overset{p-1}{\mathbf{a}}_{(2)} = \int_{\partial \mathbf{c}_p^{(2)}} \overset{p-1}{\mathbf{a}}_{(2)}. \tag{13}$$

3 N=2 connections, the generalized Chern-Pontrjagin four-forms and Chern-Simons integrals

Let M be a smooth manifold of dimension n greater than or equal to six. Consider now a $\mathfrak{g}-$valued type $N=2$ connection one-form $\mathbf{A}$ on M, where $\mathfrak{g}$ is the Lie algebra of a (unimodular) matrix Lie group G,

$$\mathbf{A} = \omega + \Theta \mathbf{m}^1 + \Sigma \mathbf{m}^2 + \Pi \mathbf{m}^1 \mathbf{m}^2. \tag{14}$$

Here ω is a $\mathfrak{g}-$valued one-form, Θ and Σ are a pair of $\mathfrak{g}-$valued two-forms, and Π is a $\mathfrak{g}-$valued three-form and matrix representations are used. The covariant exterior derivative of a type 2 generalized (square matrix-valued) $p-$form P is

$$D_{(2)} \mathbf{P} = d\mathbf{P} + \mathbf{A}\mathbf{P} + (-1)^{p+1} \mathbf{P}\mathbf{A}. \tag{15}$$

The curvature two form of $\mathbf{A}$ is given by

$$\mathbf{F} = d\mathbf{A} + \mathbf{A}\mathbf{A} \tag{16}$$

and

$$D^2_{(2)}\mathbf{P} = \mathbf{F}\mathbf{P} - \mathbf{P}\mathbf{F} \tag{17}$$

Furthermore

$$\mathbf{F} = (\Omega + \epsilon\Theta) + D\Theta\mathbf{m}^1 + (D\Sigma - \epsilon\Pi)\mathbf{m}^2 + (D\Pi + \Theta\Sigma - \Sigma\Theta)\mathbf{m}^1\mathbf{m}^2, \tag{18}$$

where $\Omega = d\omega + \omega\omega$.

Here D denotes the covariant exterior derivative with respect to ω and satisfies the two relations which will be useful below,

$$\begin{aligned} D\overset{p}{\alpha} &= d\overset{p}{\alpha} + \omega\overset{p}{\alpha} + (-1)^{p+1}\overset{p}{\alpha}\omega, \\ D^2\overset{p}{\alpha} &= \Omega\overset{p}{\alpha} - \overset{p}{\alpha}\Omega. \end{aligned} \tag{19}$$

Under a gauge transformation

$$\begin{aligned} \mathbf{A} &\longmapsto (\gamma)^{-1}d\gamma + \gamma^{-1}\mathbf{A}\gamma, \\ \mathbf{F} &\longmapsto (\gamma)^{-1}\mathbf{F}\gamma, \end{aligned} \tag{20}$$

where γ is a $G-$valued function on M. A global generalized connection, on a $G-$bundle over M can be constructed from these local expressions in the usual way, for instance by first constructing a coordinate bundle, [4].

The generalized connection is flat, that is the generalized curvature $\mathbf{F} = 0$, if and only if

$$\Omega = -\epsilon\Theta,\ D\Theta = 0,\ D\Sigma = \epsilon\Pi,\ D\Pi + \Theta\Sigma - \Sigma\Theta = 0. \tag{21}$$

In case (i) where $\epsilon = 0$, $\mathbf{F} = 0$ if and only if

$$\Omega = 0,\ D\Theta = 0,\ D\Sigma = 0,\ D\Pi + \Theta\Sigma - \Sigma\Theta = 0. \tag{22}$$

Then a $G-$gauge can be chosen so that, in a contractible open set at least,

$$\omega = 0,\ \Theta = d\vartheta,\ \Sigma = d\sigma,\ \Pi = \sigma d\vartheta - d\vartheta\sigma + d\pi,$$

where ϑ, σ and π are respectively two $\mathfrak{g}-$valued one-forms and a $\mathfrak{g}$-valued two-form. The latter three forms are not unique of course (with the freedom $\vartheta \to \vartheta + d\overset{0}{\chi}_1$, $\sigma \to \sigma + d\overset{0}{\chi}_2$, $\pi \to \pi + d\vartheta\overset{0}{\chi}_2 - \overset{0}{\chi}_2 d\vartheta + d\overset{1}{\chi}$, where $\overset{0}{\chi}_1$and $\overset{0}{\chi}_2$ are arbitrary $\mathfrak{g}$-valued functions and $\overset{1}{\chi}$ is a $\mathfrak{g}$-valued one-form). Thus, when $\epsilon = 0$, the flat generalized potential can be written, in a general gauge, as

$$\mathbf{A} = (\gamma)^{-1}d\gamma + \gamma^{-1}[d\vartheta\mathbf{m}^1 + d\sigma\mathbf{m}^2 + (\sigma d\vartheta - d\vartheta\sigma + d\pi)\mathbf{m}^1\mathbf{m}^2]\gamma, \qquad (23)$$

where γ is a $G-$valued function on M.

In case (ii) where ϵ is non-zero, $\mathbf{F} = 0$, if and only if

$$\Theta = -\epsilon^{-1}\Omega, \Pi = \epsilon^{-1}D\Sigma, \qquad (24)$$

and hence

$$\mathbf{A} = \omega - \epsilon^{-1}\Omega\mathbf{m}^1 + \Sigma\mathbf{m}^2 + \epsilon^{-1}D\Sigma\mathbf{m}^1\mathbf{m}^2. \qquad (25)$$

As was noted in previous papers, e.g. [2], a "generalized gauge transformation" can be associated with the matrix group G and its Lie algebra $\mathfrak{g}$. This is given by

$$\begin{aligned} \mathbf{A} &\longmapsto (\mathbf{g}_{(2)})^{-1}d\mathbf{g}_{(2)} + (\mathbf{g}_{(2)})^{-1}\mathbf{A}\mathbf{g}_{(2)} \\ \mathbf{F} &\longmapsto (\mathbf{g}_{(2)})^{-1}\mathbf{F}\mathbf{g}_{(2)}, \end{aligned} \qquad (26)$$

where $\mathbf{g}_{(2)}$ belongs to $\mathbf{G}_{(2)}$, the group of type $N = 2$ zero-forms on M. Any element of $\mathbf{G}_{(2)}$ has the form

$$\mathbf{g}_{(2)} = [1 + \overset{1}{\gamma}_1\mathbf{m}^1 + \overset{1}{\gamma}_2\mathbf{m}^2 + (\overset{2}{\gamma} + \overset{1}{\gamma}_2\overset{1}{\gamma}_1)\mathbf{m}^1\mathbf{m}^2]\overset{0}{\gamma}, \qquad (27)$$

where $\overset{0}{\gamma}$ is an ordinary G-valued zero-form, and $\overset{1}{\gamma}_1, \overset{1}{\gamma}_2, \overset{2}{\gamma}$ are two ordinary $\mathfrak{g}$-valued one-forms and a $\mathfrak{g}$-valued two-form. The curvature $\mathbf{F}$ is zero if and only if the connection one form is given by

$$\mathbf{A} = (\mathbf{g}_{(2)})^{-1}d\mathbf{g}_{(2)}, \qquad (28)$$

for some, non-unique, $\mathbf{g}_{(2)}$ belonging to $\mathbf{G}_{(2)}$.

In case (i) where $\epsilon = 0$, when $\mathbf{F} = 0$ a choice of $\mathbf{g}_{(2)}$, in a contractible open region in M, is

$$\mathbf{g}_{(2)} = [1 + \vartheta\mathbf{m}^1 + \sigma\mathbf{m}^2 + \{\widetilde{\pi} + \sigma\vartheta\}\mathbf{m}^1\mathbf{m}^2]\gamma. \qquad (29)$$

Here the $\mathfrak{g}-$valued two-form $\widetilde{\pi} = \pi - (\sigma\vartheta + \vartheta\sigma)$ and γ is a $G-$valued function as above.

In case (ii) where ϵ is non-zero, when $\mathbf{F} = 0$ a choice of $\mathbf{g}_{(2)}$ is

$$\mathbf{g}_{(2)} = [1 - \epsilon^{-1}\omega\mathbf{m}^1 + \epsilon^{-1}\Sigma\mathbf{m}^1\mathbf{m}^2]. \tag{30}$$

Since $Tr(\mathbf{F}) = \mathbf{0}$, the four-forms corresponding to the generalized second Chern class or generalized first Pontrjagin class on M are equal to

$$\begin{aligned}\mathbf{CP} =& kTr(\mathbf{FF}) \\ &= kTr\{\Omega\Omega + 2\epsilon\Omega\Theta + \epsilon^2\Theta\Theta + \\ &+ d(2\Omega\Theta + \epsilon\Theta\Theta)\mathbf{m}^1 + [2d(\Omega\Sigma) + 2\epsilon(\Theta D\Sigma - \Omega\Pi - \epsilon\Theta\Pi)]\mathbf{m}^2 \\ &+ 2d(\Omega\Pi + \epsilon\Theta\Pi - \Theta D\Sigma)\mathbf{m}^{12}\},\end{aligned} \tag{31}$$

when the constant $k = \frac{\kappa}{8\pi^2}$, $\kappa = \pm 1$. The generalized four-form, $\mathbf{CP}$, is of course invariant under both the gauge transformations and the generalized gauge transformations above.

Using the results of section two with $p = 4$, together with Stoke's theorem, the generalized second Chern-Pontrjagin class integral for a polychain $\mathbf{c}_4^{(2)} = (c_4, c_5^1, c_5^2, c_6)$ is given by the integral

$$\begin{aligned}\int_{\mathbf{c}_4^{(2)}} \mathbf{CP} =& k\int_{\mathbf{c}_4^{(2)}} Tr(\mathbf{FF}) \\ &= k[\int_{c_4} Tr(\Omega\Omega + 2\epsilon\Omega\Theta + \epsilon^2\Theta\Theta] + \\ &+ \int_{\partial c_5^1} Tr(2\Omega\Theta + \epsilon\Theta\Theta) + 2\int_{\partial c_5^2} Tr(\Omega\Sigma) + 2\epsilon\int_{c_5^2} Tr(\Theta D\Sigma - \Omega\Pi - \epsilon\Theta\Pi) \\ &+ 2\int_{\partial c_6^2} Tr(\Omega\Pi + \epsilon\Theta\Pi - \Theta D\Sigma)].\end{aligned} \tag{32}$$

The generalized four-form, $\mathbf{CP}$, is the exterior derivative of a generalized Chern-Simons three-form, $\mathbf{CS}$, $\mathbf{CP} = d(\mathbf{CS})$, where

$$\begin{aligned}\mathbf{CS} &= kTr(\mathbf{AF} - \frac{1}{3}\mathbf{AAA}) \\ &= kTr\{\omega\Omega - \frac{1}{3}\omega\omega\omega + \epsilon\omega\Theta + [2\Omega\Theta + \epsilon\Theta\Theta - d(\omega\Theta)]\mathbf{m}^1 \\ &+ [2\Omega\Sigma + \epsilon\Sigma\Theta - d(\omega\Sigma) - \epsilon\omega\Pi]\mathbf{m}^2 \\ &+ [\Sigma D\Theta - \Theta D\Sigma + 2\Pi\Omega + 2\epsilon\Pi\Theta - d(\omega\Pi)]\mathbf{m}^1\mathbf{m}^2\}.\end{aligned} \tag{33}$$

The generalized Chern-Simons integral for a polychain $\mathbf{c}_3^{(2)} = (c_3, c_4^1, c_4^2, c_5)$ is, after using Stoke's theorem, given by

$$
\begin{aligned}
\int_{\mathbf{c}_3^{(2)}} \mathbf{CS} = k\{ & \int_{c_3} Tr(\omega\Omega - \frac{1}{3}\omega\omega\omega + \epsilon\omega\Theta) + \\
& \int_{c_4^1} Tr(2\Omega\Theta + \epsilon\Theta\Theta) - \int_{\partial c_4^1} Tr(\omega\Theta) + \\
& + \int_{c_4^2} Tr(2\Omega\Sigma + \epsilon\Sigma\Theta - \epsilon\omega\Pi) - \int_{\partial c_4^2} Tr(\omega\Sigma) + \\
& \int_{c_5} Tr(\Sigma D\Theta - \Theta D\Sigma + 2\Pi\Omega + 2\epsilon\Pi\Theta) - \int_{\partial c_5} Tr(\omega\Pi)\}.
\end{aligned} \tag{34}
$$

When the polychain $\mathbf{c}_3^{(2)}$ is the boundary of a polychain $\widetilde{\mathbf{c}}_4^{(2)} = (\widetilde{c}_4, \widetilde{c}_5^1, \widetilde{c}_5^2, \widetilde{c}_6)$, so that

$$
\begin{aligned}
\mathbf{c}_3^{(2)} = \partial\widetilde{\mathbf{c}}_4^{(2)} = (\partial\widetilde{c}_4, \partial\widetilde{c}_5^1 + \epsilon\widetilde{c}_4, \partial\widetilde{c}_5^2, \partial\widetilde{c}_6 - \epsilon\widetilde{c}_5^2),\ \text{that is} \\
c_3 = \partial\widetilde{c}_4,\ c_4^1 = \partial\widetilde{c}_5^1 + \epsilon\widetilde{c}_4,\ c_4^2 = \partial\widetilde{c}_5^2,\ c_5 = \partial\widetilde{c}_6 - \epsilon\widetilde{c}_5^2,
\end{aligned} \tag{35}
$$

then

$$
\begin{aligned}
\int_{\partial\widetilde{\mathbf{c}}_4^{(2)}} \mathbf{CS} = k\{ & \int_{\partial\widetilde{c}_4} Tr(\omega\Omega - \frac{1}{3}\omega\omega\omega) + \int_{\partial\widetilde{c}_5^1 + \epsilon\widetilde{c}_4} Tr(2\Omega\Theta + \epsilon\Theta\Theta) + \\
& + \int_{\partial\widetilde{c}_5^2} Tr(2\Omega\Sigma + \epsilon\Sigma\Theta - \epsilon\omega\Pi) + \\
& \int_{\partial\widetilde{c}_6 - \epsilon\widetilde{c}_5^2} Tr(\Sigma D\Theta - \Theta D\Sigma + 2\Pi\Omega + 2\epsilon\Pi\Theta) + \int_{\epsilon\partial\widetilde{c}_5^2} Tr(\omega\Pi)\}.
\end{aligned} \tag{36}
$$

By Stoke's theorem for generalized forms

$$
\int_{\partial\widetilde{\mathbf{c}}_4^{(2)}} \mathbf{CS} = \int_{\widetilde{\mathbf{c}}_4^{(2)}} \mathbf{CH}. \tag{37}
$$

Under the gauge transformation given by Eq.(20)

$$
\begin{aligned}
& \mathbf{CP} \rightarrow \mathbf{CP}, \\
& \mathbf{CS} \rightarrow \mathbf{CS} - kd\{Tr[(d\gamma)\gamma^{-1}\mathbf{A}]\} - \frac{k}{3} Tr[(\gamma^{-1}d\gamma)^3].
\end{aligned} \tag{38}
$$

The last (winding number) term is closed so when $\mathbf{c}_3^{(2)} = \partial\widetilde{\mathbf{c}}_4^{(2)}$

$$\int_{\mathbf{c}_3^{(2)}} \mathbf{CS} \to \int_{\mathbf{c}_3^{(2)}} \mathbf{CS}. \tag{39}$$

When generalized gauge transformations, Eq.(26), are considered, the generalized winding number term is again closed and both Eq.(38) and Eq.(39) hold when γ is replaced by $\mathbf{g}_{(2)}$.

Next consider variations of these forms and invariants. Analogously to the result for ordinary forms the variation of a generalized Chern-Simons three-form is given by

$$\delta\mathbf{CS} = k2Tr[(\delta\mathbf{A})\mathbf{F}] + kd\{Tr[(\delta\mathbf{A})\mathbf{A}]\}. \tag{40}$$

and hence

$$\delta\int_{\mathbf{c}_3^{(2)}} \mathbf{CS} = k\{2\int_{\mathbf{c}_3^{(2)}} Tr[(\delta\mathbf{A})\mathbf{F}] + \int_{\partial\mathbf{c}_3^{(2)}} Tr[(\delta\mathbf{A})\mathbf{A}]\}. \tag{41}$$

For type $N = 2$ connections

$$\delta\mathbf{A} = \delta\omega + \delta\Theta\mathbf{m}^1 + \delta\Sigma\mathbf{m}^2 + \delta\Pi\mathbf{m}^1\mathbf{m}^2, \tag{42}$$

where $\delta\omega$, $\delta\Theta$, $\delta\Sigma$and $\delta\Pi$ denote variations of ordinary forms. The terms entering Eq.(41) are therefore the traces of

$$\begin{aligned}(\delta\mathbf{A})\mathbf{F} = {} & \delta\omega(\Omega + \epsilon\Theta) + [\delta\omega D\Theta + \delta\Theta(\Omega + \epsilon\Theta)]\mathbf{m}^1 \\ & + [\delta\omega(D\Sigma - \epsilon\Pi) + \delta\Sigma(\Omega + \epsilon\Theta)]\mathbf{m}^2 \\ & + [\delta\omega(D\Pi + \Theta\Sigma - \Sigma\Theta) + \delta\Theta(\epsilon\Pi - D\Sigma) + \\ & + \delta\Sigma D\Theta + \delta\Pi(\Omega + \epsilon\Theta)]\mathbf{m}^1\mathbf{m}^2,\end{aligned} \tag{43}$$

and

$$\begin{aligned}(\delta\mathbf{A})\mathbf{A} = {} & \delta\omega\omega + (\delta\omega\Theta - \delta\Theta\omega)\mathbf{m}^1 + (\delta\omega\Sigma - \delta\Sigma\omega)\mathbf{m}^2 \\ & + (\delta\omega\Pi + \delta\Pi\omega + \delta\Theta\Sigma - \delta\Sigma\Theta)\mathbf{m}^1\mathbf{m}^2.\end{aligned} \tag{44}$$

When, as in Eq.(35), $\mathbf{c}_3^{(2)} = \partial \widetilde{\mathbf{c}}_4^{(2)}$ it follows that

$$\delta \int_{\partial \mathbf{c}_4^{(2)}} \mathbf{CS} = k\{\int_{\partial \widetilde{c}_4} \delta\omega(\Omega + \epsilon\Theta) + \int_{\partial \widetilde{c}_5^1 + \epsilon \widetilde{c}_4} [\delta\omega D\Theta + \delta\Theta(\Omega + \epsilon\Theta)] \tag{45}$$
$$+ \int_{\partial \widetilde{c}_5^2} [\delta\omega(D\Sigma - \epsilon\Pi) + \delta\Sigma(\Omega + \epsilon\Theta)]$$
$$+ \int_{\partial \widetilde{c}_6 - \epsilon \widetilde{c}_5^2} [\delta\omega(D\Pi + \Theta\Sigma - \Sigma\Theta) + \delta\Theta(\epsilon\Pi - D\Sigma) +$$
$$+ \delta\Sigma D\Theta + \delta\Pi(\Omega + \epsilon\Theta)]\}.$$

Finally it should be recalled, [1], that if $\mathbf{a}$ is a closed generalized zero-form then

$$\mathbf{aCP} = d(\mathbf{aCS}). \tag{46}$$

For $N = 2$ forms, if the closed zero-form is given by

$$\mathbf{a} = \overset{0}{\alpha} + \overset{1}{\alpha}_1 \mathbf{m}^1 + \overset{1}{\alpha}_2 \mathbf{m}^2 + \overset{2}{\alpha} \mathbf{m}^1 \mathbf{m}^2, \tag{47}$$

then it follows from Eqs.(8) and (9) that, in the first case where $\epsilon = 0$ the ordinary forms $\overset{0}{\alpha}$, $\overset{1}{\alpha}_1$, $\overset{1}{\alpha}_2$ and $\overset{2}{\alpha}$ are all closed and in the second case where ϵ is non-zero

$$\mathbf{a} = \alpha + \epsilon^{-1} d\alpha \mathbf{m}^1 + \beta \mathbf{m}^2 - \epsilon^{-1}(d\beta)\mathbf{m}^1\mathbf{m}^2 \tag{48}$$
$$= \epsilon^{-1} d[\alpha \mathbf{m}^1 - \beta \mathbf{m}^1 \mathbf{m}^2],$$

where $\alpha = \overset{0}{\alpha}$ and $\beta = \overset{1}{\alpha}_2$.

References

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