Generalized exterior forms, geometry and space-time

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Abstract

The properties of generalised p-forms, first introduced by Sparling, are discussed and developed. Generalised Cartan structure equations for generalised affine connections are introduced. A new representation of Einstein's equations, using genneralised forms is given.

In this letter a development of a *generalized* exterior algebras and calculi of p-forms, will be presented. This type of extension of the ordinary calculus and algebra of differential forms was first introduced by Sparling in order to associate an abstract twistor structure with any real analytic Einstein space-time, [1-4]. However it is clear that it is a tool which can be employed in more general physical and geometrical contexts. Here the aim is to show how the formalism can be further developed by constructing certain generalized connections and using them to provide a formulation of Einstein's vacuum equations.

A generalized p-form, $\overset{p}{\mathbf{a}}$, is defined to be an ordered pair of ordinary p- and p+1-forms, that is

$$\overset{p}{\mathbf{a}} \equiv (\overset{p}{\alpha}, \overset{p+1}{\alpha}) \in \Lambda^p \times \Lambda^{p+1}, \tag{1}$$

where Λ^p denotes the module of p-forms on a differentiable manifold M of dimension n. By defining a minus one-form to be an ordered pair

$$\overset{-1}{\mathbf{a}} = (0, \overset{0}{\alpha}), \tag{2}$$

where $\overset{0}{\alpha}$ is a function on M, the range of p can be taken to be $-1 \leq p \leq n$. The manifold and forms may be real or complex but here n is taken to be the real dimension of M. The module of generalised p-forms will be denoted by Λ^p_G and the formal sum $\sum^p_{-1\leq p\leq n} \Lambda^p_G$ will be denoted by Λ_G. The letters over the forms indicate the degrees of the forms. Whenever these degrees are obvious they will be omitted. In the following bold Latin letters will be used for generalized forms and normal Greek letters for ordinary forms. A generalized form given by a pair $(\overset{p}{\alpha}, 0)$ will be identified with the ordinary p-form $\overset{p}{\alpha}$. Hence, for example, a function on M will be identified with the generalized 0-form $(\overset{0}{\alpha}, 0)$ while the pair $(0, \overset{0}{\alpha})$ defines a generalized minus one-form.

If $\overset{p}{\mathbf{a}} \equiv (\overset{p}{\alpha}, \overset{p+1}{\alpha})$ and $\overset{q}{\mathbf{b}} \equiv (\overset{q}{\beta}, \overset{q+1}{\beta})$, then the *(left)* generalized exterior product, $\wedge : \Lambda^p_G \times \Lambda^q_G \to \Lambda^{p+q}_G$, and the *(left)* generalized exterior derivative, $d : \Lambda^p_G \to \Lambda^{p+1}_G$, are defined to be:

$$\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}} \equiv (\overset{p}{\alpha} \wedge \overset{q}{\beta}, \overset{p}{\alpha} \wedge \overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha} \wedge \overset{q}{\beta}) \tag{3}$$

and

$$d\overset{p}{\mathbf{a}} \equiv (d\overset{p}{\alpha} + (-1)^{p+1} k \overset{p+1}{\alpha}, d\overset{p+1}{\alpha}), \tag{4}$$

where k is a constant which, in the following, is assumed to be non-zero. It should be noted that it follows that

$$\overset{-1}{\mathbf{a}} \wedge \overset{-1}{\mathbf{b}} = (0,0). \tag{5}$$

These exterior products and derivatives of generalised forms can easily be shown to satisfy the standard rules of exterior algebra and calculus. This exterior product is associative and distributive. If $\mathbf{a}$ and $\mathbf{b}$ are, respectively generalized p-forms and q-forms then $\mathbf{a} \wedge \mathbf{b} = (-1)^{pq}\mathbf{b} \wedge \mathbf{a}$. The exterior derivative is an anti-derivation from p-forms to (p+1)-forms, that is $d(\mathbf{a} \wedge \mathbf{b}) = d\mathbf{a} \wedge \mathbf{b} + (-1)^p \mathbf{a} \wedge d\mathbf{b}$. Furthermore, $d^2 = 0$.

The above rules for the exterior algebra and calculus follow immediately if, following Sparling in references [1-3] , the existence of a form ς, of degree *minus one,* is assumed to exist and satisfy all the basic rules of exterior algebra and calculus, together with the condition that $d\varsigma = k$. Tthe rules for exterior multiplication and exterior differentiation, given above, follow when the generalized p-form $\overset{p}{\mathbf{a}} = (\overset{p}{\alpha}, \overset{p+1}{\alpha})$, is identified with $\overset{p}{\alpha} + \overset{p+1}{\alpha} \wedge \varsigma$ and then added, multiplied and differentiated according to the ordinary rules of exterior algebra and calculus.[1]

The following generalized Poincaré lemma holds. Let $\overset{p}{\mathbf{a}} \equiv (\overset{p}{\alpha}, \overset{p+1}{\alpha})$ be a closed generalized p-form, so that $d\overset{p}{\mathbf{a}} = 0$. Then,

(a) $d\overset{-1}{\mathbf{a}} = 0$ if and only if $\overset{-1}{\mathbf{a}} = 0$;

(b) in any simply connected neighbourhood of any point of M,

(i) $d\overset{0}{\mathbf{a}} = 0$ if and only if there exist ordinary 0-forms $\overset{0}{\beta}$ such that $\overset{0}{\mathbf{a}} = (\overset{0}{\beta}, k^{-1}d\overset{0}{\beta})$ or $\overset{0}{\mathbf{a}} = d\overset{-1}{\mathbf{b}}$,

where $\overset{-1}{\mathbf{b}} = (0, k^{-1}\overset{0}{\beta})$.

(ii) $d\overset{p}{\mathbf{a}} = 0$, $1 \leq p \leq n$, if and only if there exist ordinary (p-1)- and p-forms $\overset{p-1}{\beta}$ and $\overset{p}{\beta}$ such that

$\overset{p}{\mathbf{a}} = (d\overset{p-1}{\beta} + (-1)^p k\overset{p}{\beta}, d\overset{p}{\beta})$. Hence $\overset{p}{\mathbf{a}} = d\overset{p-1}{\mathbf{b}}$, where $\overset{p-1}{\mathbf{b}} = (\overset{p-1}{\beta}, \overset{p}{\beta})$.

Next consider Lie groups and Lie algebras and let G= Gl(n) or one of its sub-groups. In the present context it is natural to associate with G the

[1] Similarly rules for *right* exterior multiplication and *right* exterior derivatives can be obtained by identifying a generalised p-form $\overset{p}{\mathbf{a}}$ is identified with $\overset{p}{\alpha} + \varsigma \wedge \overset{p+1}{\alpha}$. The resulting *right* algebra and calculus is isomorphic to the *left* exterior algebra and calculus and it is the latter which is always used in this paper.

semi-direct product of G and the Lie algebra of G (viewed as an additive abelian group). Define the (associated) Lie group $\mathbf{G}$ by

$$\mathbf{G} = \{\mathbf{a} \mid \mathbf{a} = \alpha(1, A)\}, \qquad (6)$$
$$\alpha(1, A) \equiv (\alpha, 0) \wedge (1, A) = (\alpha, \alpha A),$$

where $\mathbf{a}$ is a generalized zero form, α belongs to the Lie group G (GL(n) or a subgroup), with identity 1, and A is an ordinary 1-form with values in the Lie algebra of G. The product of two elements of $\mathbf{G}$, $\mathbf{a} = \alpha(1, A)$ and $\mathbf{b} = \beta(1, B)$ is given by the above rules for left exterior multiplication, and is $\mathbf{ab} = \alpha\beta(1, B + \beta^{-1}A\beta)$. The inverse of $\mathbf{a}$ is $\mathbf{a}^{-1} = \alpha^{-1}(1, -\alpha A \alpha^{-1})$ and the identity is $(1, 0)$, where 0 the zero 1-form. Right fundamental 1-forms $\mathbf{r}$ are formally defined to be forms of the type

$$d\mathbf{a} \wedge \mathbf{a}^{-1} = (d\alpha \wedge \alpha^{-1} - k\alpha A \alpha^{-1}, \alpha(dA + kA \wedge A)\alpha^{-1}), \qquad (7)$$

and satisfy the Maurer-Cartan equation

$$d\mathbf{r} - \mathbf{r} \wedge \mathbf{r} = 0. \qquad (8)$$

Similarly left fundamental 1-forms $\mathbf{l}$ are formally defined by $\mathbf{a}^{-1} \wedge d\mathbf{a}$. When k is non-zero $\mathbf{l}$ can be neatly written in the form

$$\mathbf{l} = (\lambda, -k^{-1}(d\lambda + \lambda \wedge \lambda)),$$
$$\lambda = \alpha^{-1}d\alpha - kA, \qquad (9)$$

and $\mathbf{l}$ satisfies the Maurer-Cartan equation

$$d\mathbf{l} + \mathbf{l} \wedge \mathbf{l} = 0. \qquad (10)$$

Next, in order to construct generalised Cartan structure equations for generalized connection and curvature forms on M, a generalised moving coframe of 1-forms, $\mathbf{e}^a = (\theta^a, -\Theta^a)$, and a generalized Lie algebra valued 1-form $\mathbf{\Gamma}^a_b = (\omega^a_b, -\Omega^a_b)$ are introduced. When the aim is to identify the latter as a generalized affine connection, as it will be initially, the Lie algebra corresponding to the generalized structure group, $\mathbf{G}$, is gl(n) or a sub-algebra and the lower case Latin indices range and sum over 1 to n. It will be convenient to use covariant exterior derivatives and the generalized covariant exterior derivative is denoted here by $\mathbf{D}$. The first generalized Cartan structure equation is given by

$$\mathbf{T}^a = \mathbf{D}\mathbf{e}^a \equiv d\mathbf{e}^a - \mathbf{e}^b \wedge \mathbf{\Gamma}^a_b. \qquad (11)$$

The generalized torsion, the 2-form $\mathbf{T}^a$, is in fact given by

$$\mathbf{T}^a = (d\theta^a - \theta^b \wedge \omega^a_b - k\Theta^a, -D\Theta^a + \theta^b \wedge \Omega^a_b), \text{ where}$$
$$D\Theta^a = d\Theta^a + \Theta^b \wedge \omega^a_b. \tag{12}$$

Here D denotes an ordinary exterior covariant derivative. The second generalized Cartan structure equation is given by

$$\mathbf{F}^a_b = d\mathbf{\Gamma}^a_b + \mathbf{\Gamma}^a_c \wedge \mathbf{\Gamma}^c_b, \tag{13}$$

and $\mathbf{F}^a_b$ is the generalized curvature of $\mathbf{\Gamma}^a_b$. Here

$$\mathbf{F}^a_b = (d\omega^a_b + \omega^a_c \wedge \omega^c_b - k\Omega^a_b, -D\Omega^a_b), \text{ where}$$
$$D\Omega^a_b = d\Omega^a_b + \Omega^c_b \wedge \omega^a_c - \Omega^a_c \wedge \omega^c_b. \tag{14}$$

When θ^a is a co-frame on M, and ω^a_b are connection 1-forms with torsion $k\Theta^a$ and curvature $k\Omega^a_b$, then the ordinary Cartan structure equations

$$d\theta^a - \theta^b \wedge \omega^a_b = k\Theta^a,$$
$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = k\Omega^a_b, \tag{15}$$

and the ordinary Bianchi identities

$$D\Theta^a = \theta^b \wedge \Omega^a_b,$$
$$D\Omega^a_b = 0, \tag{16}$$

are satisfied if and only if the generalized affine connection is flat. That is equations (15) and (16) are equivalent to

$$\mathbf{De}^a = 0,$$
$$\mathbf{F}^a_b = 0. \tag{17}$$

Hence when the ordinary Cartan structure equations are satisfied

$$\mathbf{e}^a = (\mathbf{a}^{-1})^a_b d\mathbf{x}^b,$$
$$\mathbf{\Gamma}^a_b = (\mathbf{a}^{-1})^a_c d(\mathbf{a})^c_{b,} \tag{18}$$

where $\mathbf{a}^a_b = \alpha^a_c(\delta^c_b, A^a_b)$ is a generalized 0-form with values in the Lie group $\mathbf{G}$. Here α^a_b has values in GL(n) (or the appropriate sub-group) and A^a_b has values in the corresponding Lie algebra.

Gauge transformations are determined by generalized functions on M with values in the Lie group **G,** as above. The gauge transformations determined by an element of **G,** again written $\mathbf{a}^a_b = \alpha^a_c(\delta^c_b, A^a_b)$, are given by

$$\begin{aligned}
\mathbf{e}^a &\rightarrow (\mathbf{a}^{-1})^a_b \mathbf{e}^b, \\
\mathbf{\Gamma}^a_b &\rightarrow (\mathbf{a}^{-1})^a_c d\mathbf{a}^c_b + (\mathbf{a}^{-1})^a_c \mathbf{\Gamma}^c_d \mathbf{a}^d_b, \\
\mathbf{De}^a &\rightarrow (\mathbf{a}^{-1})^a_b \mathbf{De}^b, \\
\mathbf{F}^a_b &\rightarrow (\mathbf{a}^{-1})^a_c \mathbf{F}^c_d \mathbf{a}^d_b
\end{aligned} \tag{19}$$

These transformations are equivalent to the following

$$\begin{aligned}
&\theta^a \rightarrow (\alpha^{-1})^a_b \theta^b, \\
&\omega^a_b \rightarrow (\alpha^{-1})^a_c d\alpha^c_b + (\alpha^{-1})^a_c \omega^c_d \alpha^d_b - kA^a_b \equiv \varpi^a_b - kA^a_b, \\
&\Theta^a \rightarrow (\alpha^{-1})^a_b [\Theta^b - \alpha^b_c A^c_d (\alpha^{-1})^d_e \wedge \theta^e], \\
&\Omega^a_b \rightarrow (\alpha^{-1})^a_c \Omega^c_d \alpha^d_b - D_\varpi A^a_b + kA^a_c \wedge A^c_b,
\end{aligned} \tag{20}$$

where D_ϖ denotes the covariant exterior derivative with respect to ϖ^a_b.

These formulae encode the affine structure and the formalism provides a unifying framework for different connections. A simple example of this is provided by the following result.

Proposition : Let a metric on M have line element

$$ds^2 = \eta_{ab} \theta^a \otimes \theta^b, \tag{21}$$

where η_{ab}=η_{ba}, are the components of the metric with respect to the co-frame θ^a, are constant. Let $\omega^a_b = \omega^a_{bc}\theta^c$ be a general connection with torsion $\Theta^a = \frac{1}{2}\Theta^a_{bc}\theta^b \wedge \theta^c, \Theta^a_{bc} = -\Theta^a_{cb}$, and let $A_{ab} = A_{abc}\theta^c$. Then if in the above generalized gauge transformations $\alpha^a_b = \delta^a_b$ and

(i) if $A_{(ab)} = k^{-1}\omega_{(ab)}$ the transformed connection is metric;

(ii) if, furthermore, $A_{[ab]} = \{1/2[\Theta_{cab} - \Theta_{bca} - \Theta_{abc}] + k^{-1}\omega_{(ac)b} - k^{-1}\omega_{(bc)a}\}\theta^c$, the transformed connection is also torsion free and hence is the Levi-Civita connection of the metric.

Similar results can be derived for generalised connections on principal and associated bundles. Rather than pursue that line here, an illustration of the use of formalism above will be given by employing it to provide a simple formulation of Einstein's vacuum field equations in four dimensions. In this example it is assumed that k is non-zero and not equal to one. Upper case Latin indices sum and range over 0-1 and are two-component spinor indices,[5]. Consider the 4-metric on a four dimensional manifold M given by

$$ds^2 = \alpha_A \otimes \beta^A + \beta^A \otimes \alpha_A, \tag{22}$$

where, α^A and β^A are spinor-valued 1-forms on M. Define the two generalized spinor valued 1-forms

$$\begin{aligned} \mathbf{r}^A &\equiv (\alpha^A, -k^{-1}\mu^A), \\ \mathbf{s}^A &\equiv (\beta^A, k^{-1}\upsilon^A), \end{aligned} \tag{23}$$

and the generalized connection 1-form

$$\begin{aligned} \mathbf{\Gamma}^A_B &= (\omega^A_B, -\Omega^A_B), \text{ where} \\ \Omega^A_B &= d\omega^A_B + \omega^A_C \wedge \omega^C_B, \end{aligned} \tag{24}$$

and ω^A_B is a sl(2,C)-valued 1-form. Then it can be seen directly from reference [6], (see also [7]), that the metric is Ricci flat if and only if the generalized spinor valued 1-forms have vanishing exterior derivatives and their generalized exterior product is an ordinary 2-form. That is the metric is Ricci flat if and only if $\mathbf{r}^{(A} \wedge \mathbf{s}^{B)}$ are ordinary 2-forms and

$$\begin{aligned} \mathbf{Dr}^A &= 0, \\ \mathbf{Ds}^A &= 0, \end{aligned} \tag{25}$$

where $\mathbf{D}$ is the generalized exterior covariant derivative determined by $\mathbf{\Gamma}^A_B$. These conditions encode the Ricci flatness of the metric and ensure that ω^A_B is the anti-self dual part of the Levi-Civita spin connection. Equations (25) are formally similar to the first order equations used by Plebanski, [8], in his analysis of half-flat geometries.

In conclusion it should be noted that the concept of a generalized p-form discussed above is a special case of a broader generalisation in which generalized p-forms are represented by n+1-tuples of ordinary forms. Using the type of notation introduced by Sparling and mentioned above on an n dimensional manifold, assume that n minus one-forms ς_a exist and satisfy the conditions ,

$$\begin{aligned} \zeta_1 \wedge \zeta_2 ..... \wedge \zeta_n &\neq 0, \\ d\zeta_a &= k_a, \\ k_a \; constants, a &= 1...n. \end{aligned} \tag{26}$$

Now define a generalized p-form to be

$$\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}{}^{a_1} \wedge \zeta_{a_1} + ...\overset{n}{\alpha}{}^{a_1 a_2 ... a_{n-p}} \wedge \varsigma_{a_1} \wedge \varsigma_{a_2} .... \wedge \varsigma_{a_{n-p}} \tag{27}$$

or, including zeros, the equivalent (n+1) tuple. Here $\overset{k}{\alpha}{}^{a_1 .....a_{k-p}} = \overset{k}{\alpha}{}^{[a_1 .....a_{k-p}]}$, $k = p$ to n, are ordinary k-forms (all sub-scripted indices ranging and summing over 1 to n). The rules for exterior multiplication and exterior derivative can be computed immediately from the last two equations. Extensions to include super-symmetry and the infinite dimensional case appear to pose no major formal problems. Further developments of the above formalisms, including the definitions of Lie derivatives, general generalized connection and metric geometries, generalized Hodge duality, co-differentials or adjoints, inner products, Laplacians, co-homology, and physical aplications, will be presented elsewhere.

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