Four decades of black hole uniqueness theorems

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Abstract

Research conducted over almost forty years into uniqueness theorems for equilibrium black holes is surveyed. Results obtained from the 1960s until 2004 are discussed decade by decade. This paper is based on a talk given at the Kerr Fest: Black Holes in Astrophysics, General Relativity & Quantum Gravity, Christchurch, August 2004. It appeared in *The Kerr Spacetime: Rotating Black Holes in General Relativity, pp115-143,* eds. D L Wiltshire, M Visser & S M Scott, (Cambridge University Press, 2009).

A brief postscript added in 2012 lists a small selection of more recent reviews of black hole uniqueness theorems and tests.

1 Introduction

It is approaching forty years since Werner Israel announced the first black hole uniqueness theorem at a meeting at King's College London [1]. He had investigated an interesting class of static asymptotically flat solutions of Einstein's vacuum field equations. The solutions had regular event horizons, and obeyed the type of regularity conditions that a broad class of non-rotating equilibrium black hole metrics might plausibly be expected to satisfy. His striking conclusion was that the class was exhausted by the positive mass Schwarzschild family of metrics. This result initiated research on black hole uniqueness theorems which continues today. Israel's investigations and all immediately subsequent work on uniqueness theorems were carried out, explicitly or implicitly, within the astrophysical context of gravitational collapse. In the early years attention was centred mainly on four dimensional static or stationary black holes that were either purely gravitational or minimimally coupled to an electromagnetic field. More recently, developments in string theory and cosmology have encouraged studies of uniqueness theorems for higher dimensional black holes and black holes in the presence of numerous new matter field combinations.

In an elegant article Israel has described the background and influences which led him to formulate his theorem and the immediate reactions, including his own, to his result [2]. Historically flavoured accounts, which include discussions of the evolution of research on black holes and the uniqueness theorems, have also been written by Kip Thorne and Brandon Carter [3, 4]. In the 1960s observational results such as the discovery of quasars and the microwave background radiation stimulated a new interest in relativistic astrophysics. There was increased activity, and more sophistication, in the modelling of equilibrium end states of stellar systems and gravitational collapse [5]. The pioneering work on spherically symmetric gravitational collapse carried out in the 1930s [6] was extended to non-spherical collapse, see e.g. [7, 8]. Significant strides were made in the use of new mathematical tools to study general relativity. Especially notable amongst these, as far as the early theory of black holes was concerned, were the constructions of the analytic extensions of the Schwarzschild and Reissner-Nordström solutions [9, 10, 11], the analyses of congruences of null geodesics and the optics of null rays, the precise formulation of the notion of asymptotic flatness in terms of a conformal boundary [12], and the introduction of trapped surfaces. In addition, novel approaches to exploring Einstein's equations, such as the Newman-Penrose formalism [13], were leading to new insights into exact solutions and their structure.

In 1963, by using a null tetrad formalism in a search for algebraically special solutions of Einstein's vacuum field equations, Roy Kerr found an asymptotically flat and stationary family of solutions, metrics of a type that had eluded discovery for many years [14]. He identified each member of the family as the exterior metric of a spinning object with mass m and angular momentum per unit mass a. The final sentence of his brief paper announcing these solutions begins: 'It would be desirable to calculate an interior solution to get more insight into this' [the multipole moment structure]. Completely satisfying model interiors to the Kerr metric have not yet been constructed, but the importance of these metrics does not reside in the fact that they might model the exterior of some rather particular stellar source. The Kerr family of metrics are the most physically significant solutions of Einstein's vacuum field equations because they contain the Schwarzschild family in the limit of zero angular momentum and because they are believed to constitute, when $a^2 \leq m^2$, the unique family of asymptotically flat and stationary black hole solutions. Within a few years of Kerr's discovery the maximal analytic extension of the Kerr solution was constructed and many of its global properties elucidated. [15, 16, 17].

In current astrophysics the equilibrium vacuum black hole solutions are regarded as being the stationary exact solutions of primary relevance, with accreting matter or other exterior dynamical processes treated as small pertubations. However, exact black hole solutions with non-zero energy momentum tensors have always been studied, despite the fact that direct experimental or observational support for the gravitational field equations of the systems is often weak or non-existent. In particular, results obtained for the vacuum space-times are often paralleled by similar results for electrovac systems describing the coupling of gravity and the source-free Maxwell field. In 1965 Ted Newman and graduate students in his general relativity class at the University of Pittsburgh published a family of electrovac solutions containing three parameters m, a and total charge e, [18]. It was found by considering a complexification of a null tetrad for the Reissner - Nordström solution, making a complex coordinate transformation, and then imposing a reality condition to recover a real metric. The Kerr-Newman metrics reduce to the Reissner-Nordström solutions when $a = 0$ and to the Kerr family when $e = 0$ and are asymptotically flat and stationary. Their Weyl tensors are algebraically special, of Petrov type D, and the metrics are of Kerr-Schild type so they can be written as the sum of a flat metric and the tensor product of a null vector with itself. When $0 < e^2 + a^2 \leqq m^2$ they represent rotating, charged, asymptotically flat and stationary black holes. Later it was realized that this family could easily be extended to a four parameter family by adding a magnetic charge p. The general theory of the equilibrium states of asymptotically flat black holes is based on concepts and structures which were first noted in investigations of the Kerr and Kerr-Newman families. In 1968 Israel extended his vacuum uniqueness theorem to static asymptotically flat electrovac space-times [19]. He showed that the unique black hole metrics, in the class he considered, were members of the Reissner-Nordström family of solutions with charge e and $e^2 < m^2$. This result, while not unexpected, physically or mathematically, required ingenious extensions of the calculations in his proof of the first theorem.

Until 1972 the only known asymptotically flat stationary, but not static, solutions of Einstein's vacuum equations with positive mass were members of the Kerr family. In that year further stationary vacuum solutions, which also happened to be axisymmetric, were published [20]. Nevertheless, sufficient was known about the Kerr solution by the time Israel published his uniqueness theorem for him to be able to ask, towards the end of his paper, if in the time independent but rotating case a similar uniqueness result might hold for Kerr-Newman metrics. This question developed into what for a while came to be referred to as the 'Carter-Israel conjecture'. This proposed that the Kerr-Newman solutions with $a^2 + e^2 + p^2 \leqq m^2$ were the only stationary and asymptotically flat electrovac solutions of Einstein's equations that were well-behaved from infinity to a regular black hole event horizon. More broadly it was conjectured that, irrespective of a wide range of initial conditions, the vacuum space-time outside a sufficiently massive collapsed object must settle down so that asymptotically in time its metric is well approximated by a member of the Kerr (or Kerr-Newman) family. The emergence of these conjectures was significantly influenced by John Wheeler with his 'black holes have no hair' conjecture [21], and by Roger Penrose and the wide ranging paper he published in 1969 [22]. Amongst the topics Penrose discussed in this paper was the question of whether or not singularities that form as a result of gravitational collapse are always hidden behind an event horizon. He raised the question of the existence of a 'cosmic censor' that would forbid the appearance of 'naked' singularities unclothed by an event horizon. Subsequently there have been many investigations of what has become termed 'the weak cosmic censorship hypothesis' [23]. Roughly speaking this says that, generically, naked singularities visible to distant observers do not arise in gravitational collapse. Although numerous models providing examples and possible counter-examples have been studied and formal theorems proven, the extent of the validity of the hypothesis is still not settled. Nevertheless, in the proofs of the uniqueness theorems it is always assumed that there are no singularities exterior to the event horizon.

In this article a broad introduction to the way in which uniqueness theorems for equilibrium black holes have evolved over the decades will be presented. Obviously the point of view and selection of topics is personal and incomplete. Fortunately more detail and other perspectives are available in various review papers. These will be referred to throughout this article. Comprehensive overviews of the four dimensional black hole uniqueness theorems, and related research such as hair and no hair investigations, can be found in a monograph and subsequent electronic journal article by Markus Heusler [24, 25]. Here attention will be centred on classical (bosonic) physics and the uniqueness theorems for static and stationary black hole solutions of Einstein's equations in dimensions $d \geq 4$. So supersymmetric black holes and models in manifolds with dimension less than four, for instance, will not be discussed.

In the next section Israel's first theorem will be reviewed and some of the issues raised by it will be noted. These issues will be addressed in subsequent sections, decade by decade. The third section deals with the 1970s when the foundations of the general theory and the basic uniqueness results for static and stationary black hole space-times were established. This is followed in the next section by a discussion of the progress made in the 1980s. During that period novel approaches to the uniqueness problems for rotating and non-rotating black holes were introduced and new theorems were proven. In addition that decade saw the construction of black hole solutions of Einstein's equations in higher dimensions and the investigation of systems with more complicated matter configurations. The motivation for much of this research came from various approaches to the problem of unifying gravity with the other fundamental forces, rather than from astrophysical considerations. The fifth section considers developments in the 1990s. Once again there were two rather distinct strands of activity. There was a rigorous re-consideration of the mathematical foundations of the theory of four dimensional equilibrium black holes laid down in the 1970s. As a result of this research a number of gaps in the early work have now been filled and mathematically more complete theorems have been established. There was also a vigorous continuation and extension of the research on black holes and new matter field combinations that draws much of its inspiration from the study of gauge theories, thermodynamics and string theory. Uniqueness theorems for higher dimensional black holes and black holes in the presence of a non-zero cosmological constant are discussed in the sixth section. This work, stimulated by string theory and cosmology, has shown how changing the space-time dimension, or the structure of the field equations and the boundary conditions, affects uniqueness theorems. The most notable new result has been the recent demonstration that in five dimensions the higher dimensional Kerr black holes are not the only stationary rotating vacuum black hole solutions.

It is a pleasure to acknowledge the contributions made by Roy Kerr to general relativity. The metrics bearing the names of Kerr, Newman, Nordström, Reissner and Schwarzschild have been central to the study of black hole space-times. One looks forward to the time when all the theoretical studies will be tested in detail by observations and experiments. There are compelling questions to be answered. What is the relationship between the observed astrophysical black holes and the Kerr and Schwarzschild black hole solutions [26]? More speculatively, what, if anything, will the Large Hadron Collider (LHC) being constructed at Cern reveal about black holes [27]?

The sign conventions of reference [28] are followed, and $c = G = 1$. Unless it is explicitly stated otherwise it will be assumed that the cosmological constant is zero and the space-times considered are asymptotically flat and four dimensional.

2 Israel's 1967 theorem and issues raised by it

In his paper *Event Horizons in Static Vacuum Space-Times,* published in 1967, Israel investigated four dimensional space-times, satisfying Einstein's vacuum field equations [1]. The space-times are static; that is, there exists a time-like hypersurface orthogonal Killing vector field, k^α,

$$k^\alpha k_\alpha < 0;\ k_{[\alpha}\nabla_\beta k_{\gamma]} = 0, \tag{1}$$

and an adapted coordinate system (t, x^a), such that $k^\alpha = (1, 0, 0, 0)$, and the four dimensional line element is

$$ds^2 = -v^2 dt^2 + g_{ab} dx^a dx^b. \tag{2}$$

Here v and the Riemannian 3-metric g_{ab} are independent of t. In this coordinate system the vacuum field equations take the form

$$\begin{aligned}
{}^{(4)}R_{tt} &\equiv v D^a D_a v = 0 \\
{}^{(4)}R_{ta} &\equiv 0 \\
{}^{(4)}R_{ab} &\equiv R_{ab} - v^{-1} D_a D_b v = 0.
\end{aligned} \tag{3}$$

where R_{ab} and D_a denote, respectively, the Ricci tensor and covariant derivative corresponding to g_{ab}. The class of static space-times considered by Israel is required to satisfy the following conditions.

On any $t = const.$ space-like hypersurface, Σ, maximally consistent with $k^\alpha k_\alpha < 0$:

(a) Σ is regular, empty, non-compact and 'asymptotically Euclidean', with the Killing vector k^α normalized so that $k^\alpha k_\alpha \to -1$ asymptotically.

(b) The invariant ${}^{(4)}R_{\alpha\beta\gamma\delta}.{}^{(4)}R^{\alpha\beta\gamma\delta}$ formed from the four dimensional Riemann tensor is bounded on Σ.

(c) If v has a vanishing lower bound on Σ, the intrinsic geometry (characterized by ${}^{(2)}R$) of the 2-spaces $v = c$ is assumed to approach a limit as $c \to 0^+$ corresponding to a closed regular two-space of finite area.

(d) The equipotential surfaces in Σ, v =const.> 0, are regular, simply connected closed 2-spaces.

Conditions (a) to (c) aim to enforce asymptotic flatness and geometrical regularity on and outside the boundary of the black hole given by $v \to 0$. The latter is also assumed to be a connected, compact 2-surface with spherical topology, so a single black hole is being considered. Condition (d), the assumption that the equipotential surfaces of v do not bifurcate, and hence have spherical topology, implies the absence of points where the gravitational force acting on a test particle is zero. The status of this assumption is different from the others and its significance and implications were unclear. It was of central technical importance in the proof of the theorem because it allowed Σ to be covered by a single coordinate system with v as one of the coordinates. Using this coordinate system Israel constructed a number of identities from which he was able to deduce that the only static four dimensional vacuum space-time satisfying (a), (b), (c) and (d) is Schwarzschild's spherically symmetric vacuum solution,

$$ds^2 = -v^2 dt^2 + v^{-2} dr^2 + r^2 d\Omega_2^2,$$
$$v^2 = (1 - 2mr^{-1}),\ 0 < 2m < r < \infty. \tag{4}$$

In 1968 Israel published the proof of a similar theorem for static electrovac space-times [19]. By making similar assumptions and taking a similar approach, but extending the calculations of his vacuum proof in a non-trivial and ingenious way, he obtained an analogous uniqueness theorem for the Reissner-Nordström black hole solutions with $e^2 < m^{2.}$.

Israel's theorem prompted a number of questions. Some arose immediately. Others became of interest later, mainly through the influence of string theory and cosmology. They include the following.

- What is the appropriate global four dimensional formulation of black hole space-times? What are the possible topologies of the two dimensional surface of the black hole? In the equilibrium case what is the relationship between the Lorentzian four geometry and the 'reduced Riemannian' uniqueness problem of the type studied by Israel?

- Are the Kerr and Kerr-Newman families the unique equilibrium vacuum and electrovac black hole solutions when rotation is permitted?

- What is the significance of the equipotential condition (d) in Israel's two theorems and how restrictive is it?

- Could an equilibrium, static or stationary, space-time contain more than one black hole? In other words, could the assumption that the equilibrium black hole horizon has only one connected component be dropped and uniqueness theorems still be proven?

- How mathematically rigorous could the uniqueness theorems be made?

- Would uniqueness theorems still hold when matter fields other than the electromagnetic field were considered?

- What would be the effect of changing the dimension of space-time or the field equations by, say, introducing a non-zero cosmological constant?

Some of these questions continue to be addressed today.

3 The 1970s - laying the foundations

During the 1970s the basic framework and theorems which have shaped or influenced all subsequent research on black hole uniqueness theorems were formulated or established.

A paper by Stephen Hawking, published in 1972, initiated the detailed global analysis of four dimensional, asymptotically flat, stationary black hole systems [29]. In this paper he drew on previous work on the global structure of space-time, primarily by Penrose, Robert Geroch and himself, to describe the causal structure exterior to black holes. His lectures at the influential 1972 Les Houches summer school also dealt with these investigations [30]. These results were presented in more detail in the 1973 monograph *The Large Scale Structure of Space-Time* [31]. In these works asymptotic flatness is imposed by using Penrose's definition of weakly asymptotically simple space-times [32, 33]. In Hawking's paper it is assumed that the space-time M can be conformally embedded in a manifold $\widetilde{M}$ with boundaries, future and past null infinity $\mathfrak{I}^+$ and $\mathfrak{I}^-$, providing end points for null geodesics that propagate to asymptotically large distances to the future or from the past. The boundary of the region from which particles or photons can escape to future null infinity, that is the boundary of the set of events in the causal past of future null infinity, defines the future event horizon H^+. In the general setting this is not assumed to be connected, allowing for the possibility of systems with more than one black hole. The event horizon, generated by null geodesic segments, forms the boundary between the black holes region and the asymptotically flat region exterior to the black holes.The two dimensional surface formed by the intersection of a connected component of H^+ and a suitable space-like hypersurface, defining a moment in time, corresponds to (the surface of) the black hole at that time. By changing the time orientation a past event horizon, H^- and white hole may be similarly defined. The manifold M is required to satisfy a condition, asymptotic predictability, which ensures that there are no naked singularities.

In the equilibrium case, it is assumed that the space-time is stationary. This means, in the black hole context, that there exists a one parameter group of isometries generated by a Killing vector field, k^α, that in the asymptotically flat region approaches a unit time-like vector field at infinity. When a time-like Killing vector field is hypersurface orthogonal the space-time is not only stationary but is also static. Henceforth in this article attention will be mainly confined to the equilibrium situation and the domain of outer communications $\langle\langle M \rangle\rangle$. This is the set of events from which there exist both future and past directed curves extending to arbitrary large asymptotic distances. The Killing vector field k^α cannot be assumed to be time-like in all of $\langle\langle M \rangle\rangle$ as this would disallow ergo-regions, as in Kerr-Newman black holes, where k^α is space-like. All the uniqueness theorems apply to $\langle\langle M \rangle\rangle$.

Hawking used this framework to show, justifying one of Israel's assumptions, that the topology of the two-surface of an equilibrium black hole is spherical. More precisely, he used the Gauss-Bonnet theorem to establish that, when the dominant energy condition is satisfied, the two dimensional spatial cross-section of each connected component of the horizon (in this article often just called the boundary surface or horizon of a black hole) must have spherical or toroidal topology. He then provided an additional argument aimed at eliminating the possibility of toroidal topology.

Hawking also introduced the strong rigidity theorem, for analytic manifolds and metrics, when matter in the space-time is assumed to satisfy the energy condition and well behaved hyperbolic equations. This theorem relates the teleologically defined event horizon to the more locally defined concept of a Killing horizon [34, 35, 36]. A null hypersurface whose null generators coincide with the orbits of a one parameter group of isometries is called a Killing horizon. According to the strong rigidity theorem the event horizon of stationary black hole is the Killing horizon of a Killing vector l^α. The horizon is called rotating if this Killing vector field does not coincide with k^α. When the horizon is rotating Hawking concluded that there must exist a second Killing vector field m^α. He then argued that the domain of outer communications of a rotating black hole had to be axially symmetric, with the axial symmetry generated by a Killing vector field m^α.

The relation between the appropriately normalized Killing vector fields can be written, $l^\alpha = k^\alpha + \Omega m^\alpha$, where the non-zero constant Ω is the angular velocity of the horizon. When Ω is zero (and m^α is undefined) so that the event horizon is a Killing horizon for the asymptotically time-like Killing vector k^a, it was argued that the domain of outer communications had to be static. In other words, according to this staticity argument an asymptotically flat and stationary black hole which is not rotating must have a static domain of outer communications and therefore k^α must be time-like and hypersurface orthogonal in $\langle\langle M \rangle\rangle$ [37]. Part of the proof of this result was based on unsatisfactory heuristic considerations and it was not until the 1990's that the staticity theorem was firmly established. This theorem will be briefly discussed again later.

Hawking's calculations employed the assumption that the space-times, horizons and metrics considered were analytic and analytic continuation arguments were used. A theorem proven by Henning Müller zum Hagen, based on the elliptic nature of the relevant equations, provides the justification for the assumption of analyticity locally in stationary systems [38]. However analytic continuations are not necessarily unique. The complete elimination of certain analyticity assumptions, probably more of mathematical than physical significance but still desirable, remains to be effected, see for example [39].

The strong rigidity and staticity theorems are important because they permit the equilibrium uniqueness problems to be reduced from problems in four dimensional Lorentzian geometry to two distinct types of lower dimensional Riemannian boundary value problems. In the rotating case the system may be taken to be stationary and axially symmetric and hence the uniqueness problem may be reduced to a two dimensional Riemannian problem. In the non-rotating case the system may be taken to be static and hence the problem may be reduced to a three dimensional Riemannian problem. In the remainder of this article most attention will be focused on these dimensionally reduced uniqueness problems.

On each connected component of the horizon of an equilibrium black hole the normal Killing vector field l^α satisfies the equation $\nabla_\alpha(l^\beta l_\beta) = -2\kappa l_\alpha$. According to the first law of black hole mechanics κ, the surface gravity, is constant there. A connected component of the horizon is called non-degenerate if the surface gravity is non-zero there and degenerate otherwise. The connected component of a non-degenerate future horizon can be regarded, in a precisely defined sense, as comprising a branch of a bifurcate Killing horizon. This is a pair of Killing horizons, for the same Killing vector field, which intersect on a compact space-like bifurcation two-surface where the Killing vector vanishes. Old arguments for this technically important result were superseded by better ones in the 1990s [40]. The early uniqueness theorems applied only to black holes with non-degenerate horizons, satisfying the bifurcation property, as in the non-extremal Kerr-Newman black holes. The first attacks on these reduced Riemannian problems also assumed that the horizon was connected so that there was only one black hole. Subsequently, uniqueness theorems for static systems with non-connected horizons have been proven, and comparatively recently theorems that rigorously include the possibility of degenerate horizons have also been constructed. While the physical plausibility of stable equilibrium systems of more than one black hole may be questionable, and the realizability in nature of degenerate horizons is moot, dealing with them mathematically has brought its own rewards. Re-considerations of the agenda setting global analyses of equilibrium black hole space-times will be discussed in a later section.

Carter also presented a series of important lectures at the 1972 Les Houches summer school [37]. In his lecture notes he collected together and extended results he had obtained over a number of years and presented a systematic analysis of asymptotically flat, stationary and axi-symmetric black holes. Subsequently he has reconsidered and extended this material in a number of reviews and lecture series [41, 42, 43]. A major topic in his lectures was the reduction of the uniqueness problems for stationary, axisymmetric vacuum and Einstein-Maxwell space-times to two dimensional boundary value problems. It was well known that locally, in coordinates adapted to the symmetries, certain of the Einstein and other field equations for such systems may be reduced to a small number of non-linear elliptic equations with a small number of metric and field components as dependent variables. The remaining field and metric components are then derivable from these variables by quadratures [24, 28]. Carter showed that this could be done globally on the domain of outer communications with the regularity and black hole boundary conditions formulated in a comparatively simple way. He dealt with domains of outer communication for which each connected component of the future boundary H^+ of $\langle\langle M\rangle\rangle$ is non-degenerate and, by Hawking's theorem, topologically $R \times S^2$. He also made a natural causality requirement, that off the axi-symmetry axis $X = m^\alpha m_\alpha > 0$ in $\langle\langle M\rangle\rangle$. For vacuum and electrovac systems, in particular, he demonstrated that, apart from the axis of symmetry where X is zero, a simply connected domain of outer communications could be covered by a single coordinate system (t, x, y, ϕ) in which the metric took a Papapetrou form. In these coordinates, $k^\alpha = (1, 0, 0, 0)$ and $m^\alpha = (0, 0, 0, 1)$. He showed that the axi-symmetric stationary black hole metric on $\langle\langle M\rangle\rangle$ may be written in the form

$$\begin{aligned} ds^2 &= -Vdt^2 + 2Wdtd\varphi + Xd\varphi^2 + \Xi d\sigma^2, \\ XV + W^2 &= (x^2 - c^2)(1 - y^2), \end{aligned} \tag{5}$$

where for a single black hole, $0 < c < x$, $-1 < y < 1$, $c = \frac{\kappa A}{4\pi}$, and A is the black hole area. Carter then reduced the vacuum and Einstein-Maxwell uniqueness problems for single black holes to boundary value problems for systems of elliptic partial differential equations on a two dimensional manifold

D with global prolate spheroidal coordinates (x, y) and metric

$$d\sigma^2 = \frac{dx^2}{x^2 - c^2} + \frac{dy^2}{1 - y^2}. \tag{6}$$

In the vacuum case, to which attention will now be confined, $c = m - 2\Omega J$, where m is the mass and J is the angular momentum about the axi-symmetry axis. The relevant Ernst-like vacuum field equations on D can be conveniently written in terms of a single complex field $E = X + iY$, where Y is a potential for W, and derived from a Lagrangian density

$$L = \frac{\nabla E.\nabla \overline{E}}{(E + \overline{E})^2}, \tag{7}$$

where ∇ denotes the covariant derivative with respect to the two-metric. The complex field equation is

$$\nabla\left(\frac{\rho \nabla E}{(E + \overline{E})^2}\right) + \frac{2\rho \nabla E.\nabla \overline{E}}{(E + \overline{E})^3} = 0. \tag{8}$$

All the metric components are uniquely determined by E and the boundary conditions. (When the metric is not only axi-symmetric but also static, $Y = W = 0$, and the field equation reduces to the linear equation $\nabla(\rho \nabla \ln X) = 0$.) For a black hole solution E is required to be regular when $x > c > 0$, and $-1 < y < 1$. Boundary conditions on E and its derivatives ensure regularity on the axis of symmetry as $y \to \pm 1$ and regularity of the horizon as $x \to c > 0$. The conditions, $x^{-2}X = (1 - y^2) + O(x^{-1})$, $Y = 2Jy(3 - y^2) + O(x^{-1})$ as $x \to \infty$, ensure asymptotic flatness.

In 1971 Carter was able to prove, within this framework, that stationary axisymmetric vacuum black hole solutions must fall into discrete sets of continuous families, each depending on at least one and and most two parameters [44]. The unique family admitting the possiblity of zero angular momentum is the Kerr family with $a^2 < m^2$. This was a highly suggestive but not conclusive result. Since the theorem was deduced by considering equations and solutions linearized about solutions of Eq.(8) it was not at all clear if, or how, the full non-linear theory could be handled. However, in 1975 I constructed a proof of the uniqueness of the Kerr black hole by using Carter's general framework [45]. Two possible black hole solutions E_1 and E_2 were used to construct a non-trivial generalized Green's identity of the form *divergence = positive terms mod field equations.* This was integrated over the two dimensional manifold D. Stokes' theorem and the boundary conditions were then used to show that the integral of the left hand side was zero. Consequently each of the postive terms on the right hand side had to be zero and this implied that $E_1 = E_2$. Hence, Kerr black holes, with metrics on the domain of outer communication given, in Boyer-Lindquist coordinates, by

$$ds^2 = -Vdt^2 + 2Wdtd\varphi + Xd\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \tag{9}$$

where $0 \leq a^2 < m^2$; $r_+ = m + (m^2 - a^2)^{1/2} < r < \infty$, and

$$\begin{aligned}
&V = (\frac{\Delta - a^2\sin^2\theta}{\Sigma}),\ W = -(\frac{a\sin^2\theta(r^2 + a^2 - \Delta)}{\Sigma}),\\
&X = \frac{(r^2 + a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta,\\
&\Sigma = r^2 + a^2\cos^2\theta,\ \Delta = r^2 + a^2 - 2mr,
\end{aligned} \tag{10}$$

are the only stationary, axially symmetric, vacuum black hole solutions with non-degenerate connected horizons. According to Hawking's rigidity theorem, 'axially symmetric' can be removed from the previous sentence.

In a separate development in the early 1970s Israel's theorems for static black holes were reconsidered by Müller zum Hagen, Hans Jurgen Seifert and myself. First we looked at static, single black hole vacuum space-times [46]. In this case the event horizon is connected and, by the generalized Smarr formula [37], necessarily non-degenerate. In a somewhat technical paper we were able to avoid using both Israel's assumption (d) about the equipotential surfaces of v, and his assumption about the spherical topology of the horizon. Our extension of Israel's theorem made use of the fact that the spacial part of the Schwarzschild metric, g_{ab}, is conformally flat. Indeed all asymptotically Euclidean, spherically symmetric three-metrics are locally conformally flat. Now a three-metric is conformally flat if and only if the Cotton tensor

$$R_{abc} \equiv D_b(R_{ac} - \frac{1}{4}Rg_{ac}) - D_c(R_{ab} - \frac{1}{4}Rg_{ab}). \tag{11}$$

is zero. By using a three dimensionally covariant approach we were able to show that the Cotton tensor had to vanish and thence to conclude that the only static vacuum black holes in four dimensions, with connected horizons, were Schwarzschild black holes. I was soon able to simplify and improve this proof [47]. To illustrate this approach an outline, based mainly on the latter paper but also containing results from [46], is presented in the appendix.

Using similar techniques we also extended Israel's static electrovac theorem to prove uniqueness of the Reissner-Nordström black hole when the horizon was again assumed to be connected [48]. The Smarr formula does not imply that the horizon is non-degenerate in this case, and satisfactory rigorous progress with degenerate electrovac horizons was not made until the late 1990s. In this paper we also noted that solutions of the Einstein-Maxwell equations might exist for which the metric was static but the Maxwell field was time dependent. We identified the form of these Maxwell fields, and the reduced equations they had to satisfy. However we were only able to construct a plausibility argument against such black hole solutions. Subsequently it has been shown that Einstein-Maxwell solutions of this type, albeit not asymptotically flat solutions since they are cylindrically symmetric, do exist [49]. Further investigation of this type of non-inherited symmetry for other fields might be of interest. I also managed to generalize Carter's no-bifurcation result from the vacuum case considered by him to stationary Einstein-Maxwell space-times [50]. I showed that black hole solutions, with connected non-degenerate horizons, formed discrete continuous families, each depending on at most four parameters (effectively the mass m, angular momentum/unit mass a, electric charge e and magnetic charge p). Furthermore, of these only the four parameter Kerr-Newman family contained members with zero angular momentum.

Investigations of Weyl metrics corresponding to static, axially-symmetric, multi-black hole configurations, with non-degenerate horizons, were undertaken by Müller zum Hagen and Seifert, and independently by Gary Gibbons [51, 52]. The type of method that Hermann Bondi [53] had used to tackle the static, axially symmetric two body problem was employed. It was shown that the condition of elementary flatness failed to hold everywhere on the axis of axial symmetry. Hence it was concluded that static, axially symmetric configurations of more than one black hole in vacuum, or of black holes and massive bodies which do not surround or partially surround a black hole, do not exist. Jim Hartle and Hawking appreciated that things were different when the black holes were charged [54]. They showed that completed Majumdar-Papapetrou electrovac solutions [55, 56], derivable from a potential with discrete point sources, could be interpreted as static, charged multi-black hole solutions. Each of the black holes has a degenerate horizon and a charge with magnitude equal to its mass. The electrostatic repulsion balances the gravitational attraction and the system is in neutral equilibrium. The single black hole solution is the $e^2 = m^2$ Reissner -Nordström solution. While these multi-black hole solutions are physically artificial, their existence showed that mathematically complete uniqueness theorems for electrovac systems had to take into account both the Kerr-Newman and the Majumdar-Papapetrou solutions and systems with horizons that need not be connected and could be degenerate. When static axisymmetric electrovac space-times were considered, and each black hole was assumed to have the same mass to charge ratio, Gibbons concluded that the solutions had to be Majumdar-Papapetrou black holes [57].

Studies of black holes with other fields, such as scalar fields, were also initiated. Working within the same framework as Israel, J.E. Chase showed that the only black hole solution of the static Einstein-scalar field equations, when the massless scalar field was minimally coupled, was the Schwarzschild solution [58]. In other words the scalar field had to be constant. A similar conclusion was reached by Hawking when he considered stationary Brans-Dicke black holes [59]. His calculation was a very simple one using, in a mathematically standard way, just the linear scalar field equation in the Einstein gauge. Interestingly this calculation, and a similar one by Jacob Bekenstein, did not depend heavily on all the detailed properties of the horizon.

Wheeler's 'black holes have no hair' conjecture inspired a number of investigations of matter in equilibrium black hole systems. According to the original no hair conjecture collapse leads to equilibrium black holes determined uniquely by their mass, angular momentum and charge (electric and/or magnetic), asymptotically measurable conserved quantities subject to a Gauss law, and have no other independent characteristics (hair) [21, 60]. The linear stability analyses, see e.g [61], and Richard Price's observation of a late time power law decay in pertubations of the Schwarzschild black hole [62], provided support for both the weak cosmic censorship hypothesis and the no hair conjecture. Other early investigations also supported the no hair conjecture. For instance, Bekenstein showed that the domains of outer communication of static and stationary black holes could not support minimally coupled massive or massless scalar fields, massive spin 1 or Proca fields, nor massive spin 2 fields [63, 64]. He was able to draw his conclusions without using the Einstein equations so details of the gravitational coupling were not used, only the linear matter field equations and boundary conditions were needed. Bekenstein also studied a black hole solution, with a conformally coupled scalar field, that had scalar hair [65, 66]. It turned out that this solution has unsatisfactory features, the scalar field diverges on the horizon and the solution is unstable. Nevertheless such work was the forerunner of many later hair and no-hair investigations.

By the mid 1970s the uniqueness theorems for static and stationary black hole systems discussed above had been constructed and the main thrust of theoretical interest in black holes had turned to the investigation of quantum effects. While not all of the results obtained in this decade, and discussed above, were totally satisfactory or complete [4] they provided the foundations and reference points for all subsequent investigations. At the end of the decade the main gap in the uniqueness theorems appeared to be the lack of a proof of the uniqueness of a single charged stationary black hole. It seemed clear that the uniqueness proof for the Kerr solution was extendable to a proof of Kerr-Newman uniqueness. However the technical detail of my electrovac no-bifurcation result was sufficiently complicated to make the prospect of trying to construct a proof rather unpalatable, unless a more systematic way of attacking the problem could be found.

4 The 1980s - systematization and new beginnings

The 1980s saw both the introduction of new techniques for dealing with the original stationary and static black hole uniqueness problems and the investigation of new systems of black holes. The interest in the latter was grounded not so much in astrophysical considerations as in renewed attempts to develop quantum theories that incorporated gravity. It included the construction of higher dimensional black hole solutions and the investigation of systems such as Einstein-dilaton-Yang Mills black holes.

The uniqueness problem for stationary, axially symmetric electrovac black hole space-times was independently reconsidered, within the general framework set up by Carter, by Gary Bunting and Pawel Mazur. The reduced two dimensional electrovac uniqueness problem is formally similar to the vacuum problem outlined above, but there are four equations and dependent variables instead of two, so the system of equations is more complicated. It had long been realized that the Lagrangian formulation of these equations might play an important role in the proof of the uniqueness theorems. In fact I had used the Lagrangian for the vacuum equations given by Eq.(7), which is positive and quadratic in the derivatives, in a reformulation of Carter's no-bifurcation result [67]. However there are more productive interpretations of the Lagrangian formalism. It had been known since the mid 1970s that the Euler-Lagrange equations corresponding, as in Eq.(8), to the basic Einstein equations for stationary axi-symmetric metrics, could be interpreted as harmonic map equations [68]. In addition, in the 1970s there was a growth of interest in generalized sigma models; that is, in the study of harmonic maps from a Riemannian space M to a Riemannian coset space $N = G/H$, where G is a connected Lie group and H is a closed sub-group of G. Influenced by these developments Bunting and Mazur used these interpretations of the Lagrangian structure of the equations. Bunting's approach was more geometrically based, and in fact applied to a general class of harmonic mappings between Riemannian manifolds. He constructed an identity which implied that the harmonic map was unique when the sectional curvature of the target manifold was non-positive [69, 70]. Mazur on the other hand focused on a non-linear sigma model interpretation of the equations, with the target space N a Riemannian symmetric space. Exploiting the symmetries of the field equations, he constructed generalized Green's identities when $N = SU(p,q)/S(U(p) \times U(q))$. When $p = 1$, $q = 2$ he obtained the identity needed to prove the uniqueness of the Kerr-Newman black holes. This is a generalization of the identity used in the proof of the uniqueness of the Kerr black hole which corresponds to the choice $N = SU(1,1)/U(1)$ [71, 72, 73]. Bunting and Mazur's systematic approaches provided computational rationales lacking in the earlier calculations, and enabled further generalizations to be explored within well-understood frameworks. In summary, Bunting and Mazur succeeded in proving that stationary axi-symmetric black hole solutions of the Einstein-Maxwell electrovac equations, with non-degenerate connected event horizons, are necessarily members of the Kerr-Newman family with, if magnetic charge p is included, $a^2 + e^2 + p^2 < m^2$.

In another interesting development Bunting and Masood-ul-Alam constructed a new approach to the static vacuum black hole uniqueness problem [74]. They used results from the positive mass theorem, published in 1979, to show, without the simplifying assumption of axial symmetry used in earlier multi-black hole calculations, that a non-degenerate event horizon of a static black hole had to be connected. In other words, there could not be more than one such vacuum black hole in static equilibrium. The thrust of their proof was to show, again, that the three metric g_{ab} was conformally flat. However their novel method of proving conformal flatness did not make use of the Cotton tensor and so was not so tied to use in only three dimensions.

In addition their approach relied much less on the details of the field equations. Consequently it subsequently proved much easier to apply it to other systems. Their proof that the constant time three-manifold Σ with 3-metric g_{ab} must be conformally flat proceeded along the following lines. Starting with $(\Sigma,\ g_{ab})$, as in Eq.(2), they constructed an asymptotically Euclidean, complete Riemannian three-manifold (N, γ_{ab}) with zero scalar curvature and zero mass. This was done by first conformally transforming the metric,

$$g_{ab} \rightarrow \mp\gamma_{ab} = \frac{1}{16}(1 \mp v)^4 g_{ab} \tag{12}$$

on two copies of (Σ, g_{ab}) so that $(\Sigma, +\gamma_{ab})$ was asymptotically Euclidean with mass $m = 0$, and $(\Sigma, -\gamma_{ab})$ "compactified the infinity". Then the 2 copies of Σ were pasted along their boundaries to form the complete three-manifold (N, γ_{ab}). They then utilised the following corollary to the positive mass theorem proven in 1979 [75, 76].

Consider a complete oriented Riemannian three-manifold which is asymptotically Euclidean. If the scalar curvature of the three-metric is non-negative and the mass is zero, then the Riemannian manifold is isometric to Euclidean three-space with the standard Euclidean metric.

From this result it follows that (N, γ_{ab}) has to be flat and therefore (Σ, g) is conformally flat. Thus, as in the earlier uniqueness proofs, the metric must be spherically symmetric. Therefore the exterior Schwarzschild spacetime exhausts the class of maximally extended static vacuum, asymptotically flat space-times with non-degenerate, but not necessarily connected, horizons.

Further uniqueness theorems for static electrovac black holes were also proven [77]. In particular, Bunting and Masood-ul-Alam's type of approach was used to construct a theorem showing that a non-degenerate horizon of a static electrovac black hole had to be connected, and hence the horizon of a Reissner-Nordström black hole with $e^2 < m^2$ [78].

In this decade new exact stationary black hole solutions, some with more complicated matter field configurations than had been considered in the past, were increasingly studied. These studies were often undertaken as contributions to ambitious programmes for unifying gravity and other fundamental forces. They did shed new light on the no hair conjecture and the extent to which black hole uniqueness theorems might apply. For example, generalizations of the four dimensional Einstein-Maxwell equations, which typically arise from Kaluza-Klein theories, and stationary black hole solutions were studied and uniqueness theorems constructed [79]. Investigation of a model naturally arising in the low energy limit of N = 4 supergravity led Gibbons to find a family of static, spherically symmetric Einstein-Maxwell-dilaton black hole solutions in four dimensions [80, 81]. These have scalar hair but carry no independent dilatonic charge. In 1989 static spherically symmetric non-abelian SU(2) Einstein-Yang-Mills black holes, with vanishing Yang-Mills charges and therefore asymptotically indistinguishable from the Schwarzschild black hole, were found. The solutions form an infinite discrete family and are labelled by the number of radial nodes of the Yang-Mills potential exterior to a horizon of given size. Hence there is not a unique static black hole solution within this Einstein-Yang-Mills class [82, 83, 84]. Although these latter solutions proved unstable, such failures of the no-hair conjecture and uniqueness encouraged the subsequent investigation of numerous black hole solutions with new matter configurations.

Interest in higher dimensional black holes also started to increase. Higher dimensional versions of the Schwarzschild and Reissner-Nordström solutions had been found in the 1960s [85] and in 1987 Robert Myers found the higher dimensional analogue of the static Majumdar-Papapetrou family [86]. The metric for the Schwarzschild black hole with mass m in $d > 3$ dimensions is

$$ds^2 = -v^2 dt^2 + v^{-2} dr^2 + r^2 d\Omega^2_{d-2} \tag{13}$$

where $v^2 = (1 - \frac{C}{r^{d-3}}) > 0$, $d\Omega^2_{d-2}$ is the metric on the $(d-2)-$dimensional unit radius sphere which has area A_{d-2} and $C = \frac{16\pi m}{A_{d-2}(d-2)}$. As the form of this metric suggests, these static higher dimensional black holes have similar properties to the four dimensional solutions. Myers and Malcolm Perry found and studied the the d dimensional generalizations of the Kerr metrics [87]. In general the Myers-Perry metrics are characterized by $[(d-1)/2]$ angular momentum invariants and the mass. The family of metrics with a single spin parameter J is given by

$$\begin{aligned} ds^2 &= -dt^2 + \Delta(dt + a\sin^2\theta d\phi)^2 + \Psi dr^2 + \\ &\quad + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\Omega^2_{(d-4)}. \\ \rho^2 &= r^2 + a^2\cos^2\theta, \quad \Delta = \frac{\mu}{r^{d-5}\rho^2}, \Psi = \frac{r^{d-5}\rho^2}{r^{d-5}(r^2+a^2) - \mu}, \\ m &= \frac{(d-2)A_{(d-2)}}{16\pi}\mu, \quad J = \frac{2ma}{(d-2)}. \end{aligned} \tag{14}$$

When $a = 0$ the metric reduces to the metric given in Eq. (13) and when $d = 4$ the metric reduces to the Kerr metric. When $d > 4$, there are three

Killing vector fields. If $d = 5$ there is a horizon if $\mu > a^2$ and no horizon if $\mu \leq a^2$. If $d > 5$ a horizon exists for arbitrarily large spin. Further interesting properties are discussed in their paper.

5 The 1990s - rigour and exotic fields

During the last decade of the twentieth century two rather different lines of research on uniqueness theorems were actively pursued. On the one hand there was a renewed effort to improve and extend the scope and rigour of the uniqueness theorems for four dimensional black holes. Here the approach was more mathematical in nature and emphasized rigorous geometrical analysis. On the other hand activity in theoretical physics related to string theory, quantum gravity and thermodynamics encouraged the continued, less formal, investigations of black holes with new exterior matter fields.

First it should be noted that further progress was made on eliminating the possibility of static multi-black hole space-times in a number of four dimensional systems. Bunting and Masood-ul-Alam's approach to proving conformal flatness, which does not require the assumption that the horizon is connected, was used in a new proof that the exterior Reissner-Nordström solutions with $e^2 < m^2$ are the only static, asymptotically flat electrovac space-times with non-degenerate horizons [88]. Proofs of the uniqueness of the family of static Einstein-Maxwell-dilaton metrics, originally found by Gibbons [80, 81], were also constructed by using the same general approach [89, 90]. Stationary axially symmetric black holes with non-degenerate horizons that are not connected were also studied, but no definitive conclusions have yet been reached [91, 92, 93]. Whether regular stationary black hole space-times exist in which repulsive spin-spin forces between black holes are strong enough to balance the attractive gravitational forces remains unknown. To date, no uniqueness theorems dealing with stationary, but not static, black holes that may possess degenerate horizons, such as the extreme Kerr and the extreme Kerr-Newman horizons, have been proven.

Since the early 1990s significant progress has been made, particularly by Bob Wald, Piotr Chruściel and their collaborators, in tidying up and improving the global framework, erected in the 1970s, on which the uniqueness theorems rest. Mathematical shortcomings in the earlier work, of varying degrees of importance, were highlighted by Chruściel in a 1994 review, challengingly entitled *"No Hair" theorems: Folklore, Conjectures, Results* [94].

Statements, definitions and theorems from the foundational work have, where necessary, been corrected, sharpened and extended, and this line of rigorous mathematics has now been incorporated into a programme of classification of static and stationary solutions of Einstein's equations [95, 96, 97]. Mention can be made of the more important results. Chruściel and Wald obtained improved topological results by employing the topological censorship theorem [98]. For a globally hyperbolic and asymptotically flat space-time satisfying the null energy condition the topological censorship theorem states that every causal curve from $\mathfrak{I}^-$ to $\mathfrak{I}^+$ is homotopic to a topologically trivial curve from $\mathfrak{I}^-$ to $\mathfrak{I}^+$ [99]. Chruściel and Wald showed that when it applied, the domain of outer communications had to be simply connected. They also gave a more complete proof of the spherical topology of the surface of stationary black holes. Basically, if the horizon topology is not spherical there could be causal curves, outside the horizon but linking it, that were not deformable to infinity, thus violating the topological censorship theorem [100]. An improved version of the rigidity theorem for analytic space-times, with horizons that are analytic submanifolds but not necessarily connected or non-degenerate, was constructed by Chruściel. A more powerful and satisfactory proof of the staticity theorem, that non-rotating stationary black holes with a bifurcate Killing horizon must be static, was constructed by Daniel Sudarsky and Wald [101, 102]. The new proof made the justifiable use of a slicing by a maximal hypersurface, and supersedes earlier proofs which had unsatisfactory features. Mention should also be made of the establishment, by István Rácz and Wald, of the technically important result, referred to earlier, concerning bifurcate horizons [40]. These authors considered non-degenerate event (and Killing) horizons with compact cross-sections, in globally hyperbolic space-times containing black holes but not white holes. This is the appropriate setting within which to consider the equililibrium end state of gravitational collapse. They showed that such a space-time could be globally extended so that the image of the horizon in the enlarged space-time is a proper subset of a regular bifurcate Killing horizon. They also found the conditions under which matter fields could be extended to the enlarged space-time, thus providing justification for hypotheses made, explicitly or implicitly, in the earlier uniqueness theorems. In the late 1990s Chruściel extended the method of Bunting and Masood-al-Alam and the proof of the uniqueness theorem for static vacuum space-times [103]. He considered horizons that may not be connected and may have degenerate components on which the surface gravity vanishes, and constructed the most complete black hole theorem to date.

The statement of his main theorem, which applies to black holes solutions with asymptotically flat regions (ends) in four dimensions, is the following.

Let (M, g) be a static solution of the vacuum Einstein equations with defining Killing vector $k.^{\alpha}$ Suppose that M contains a connected space-like hypersurface Σ the closure $\overline{\Sigma}$ of which is the union of a finite number of asymptotically flat ends and of a compact interior, such that: 1. $g_{\mu\nu}k^{\mu}k^{v} < 0$ on Σ. 2. The topological boundary $\partial\Sigma = \overline{\Sigma}\backslash\Sigma$ of Σ is a non-empty topological manifold with $g_{\mu\nu}k^{\mu}k^{v} = 0$ on $\partial\Sigma$. Then Σ is diffeomorphic to R^3 minus a ball, so that it is simply connected, it has only one asymptotically flat end, and its boundary $\partial\Sigma$ is connected. Further there exists a neighbourhood of Σ in M which is isometrically diffeomorphic to an open subset of the Schwarzschild space-time.

An analogous, although less complete, theorem for static electrovac space-times that included the possibility of non-connected, degenerate horizons was also constructed [104, 105]. It was shown that if the horizon is connected, then the space-time is a Reissner-Nordström solution with $e^2 \leqq m^{2\cdot}$. If the horizon is not connected, and all the degenerate connected components of the horizon with non-zero charge have charges of the same sign, then the space-time is a Majumdar-Papapetrou black hole solution.

In the work more oriented towards the study of black holes and high energy physics, there was a proliferation of research into 'exotic' matter field configurations such as dilatons, Skyrmions and sphalerons, into various types of non-minimal scalar field couplings and into fields arising in low energy limits of string theory. This type of research continues today. The immediate physical relevance of the Lagrangian systems considered is often of less importance than the contribution their study makes to deciding the extent to which black hole solution spaces can be parametrized by small numbers of global charges, or to deciding whether or not a class of systems admits stable solutions with non-trivial hair. Gravitating non-abelian gauge theories and gravity coupled scalar fields have featured prominently in this research. It has been shown, for example, that black holes in non abelian gauge theories, and in theories with appropriately coupled scalar fields, can have very different hair properties from black holes in the originally studied Einstein-Maxwell or minimally coupled scalar field theories. Such research has also provided models that demonstrate the effect of varying the assumptions made in the early uniqueness theorems. It effectively includes many constructive proofs of existence and/or non-uniqueness. For instance, the existence of Einstein-Yang-Mills black holes that have zero angular momentum but need not be static has been established [106], and static black holes that need not be axially symmetric, let alone spherically symmetric, have been shown to exist [107]. Uniqueness theorems for self-gravitating harmonic mappings and discussions of Einstein-Skyrme systems can be found in reference [24], and further information about black holes in the presence of matter fields can be found, for example, in references [25, 108, 109, 110].

6 The 2000s - higher dimensions and the cosmological constant

The important role of black holes in string theory, and recent conjectures that black hole production may occur and be observable in high energy experiments (TeV gravity) at the LHC [27], have stimulated investigations of black holes in higher dimensional space-times. In addition observational results in cosmology, and theoretical speculations in string theory, have encouraged the continued development of earlier work on black hole solutions of Einstein's equations with a non-zero cosmological constant Λ.

Uniqueness theorems for asymptotically flat black holes with static exteriors have, not unexpectedly, been extended to higher dimensions. In fact Seungsu Hwang showed in 1998 that the Schwarzshild-Tangherlini family, Eq.(13), is the unique family of static vacuum black hole metrics with non-degenerate horizons [111]. Subsequently other four dimensional uniqueness theorems for static black holes with non-degenerate horizons have been extended to dimension $d > 4$ [112, 113, 114, 115, 116, 117]. All these calculations deal with the relevant reduced $(d-1)$ dimensional Riemannian problem. They all follow the approach introduced in four dimensions by Bunting and Massod ul Alam, and need higher dimensional positive energy theorems (a topic still being explored) to show that the the exterior $(d-1)$ dimensional Riemannian metric must be conformally flat. Appropriate arguments are then employed to show that the conformally flat Riemannian metrics, and the space-time metrics and fields, must be spherically symmetric and members of the relevant known family of solutions. The stability of certain static higher dimensional black holes, such as the Schwarzschild family, has also been investigated and confirmed [118].

It obviously follows from the uniqueness theorems above that those black hole space-times have horizons that are topologically S^{d-2}, as do the Myers-

Perry black holes. However the general methods used to restrict horizon topologies in four dimensions cannot be used in the same way in higher dimensions. Although, unlike the Gauss-Bonnet theorem, a version of topological censorship holds in any dimension it does not restrict the horizon topology as much when $d > 4$ [119]. Furthermore it is clear that a rigidity theorem in higher dimensions would not by itself imply the existence of sufficient isometries to allow the construction of generalizations of harmonic map or sigma model formulations of the equations governing stationary black hole exterior geometries. These differences were highlighted in 2002 when it was shown that in five dimensions, in addition to the Myers-Perry black hole family with rotation in a single plane, there is another asymptotically flat, stationary, vacuum black hole family characterized by its mass m and spin J. This black ring family, as it was termed by its discovers Roberto Emparan and Harvey Reall, also has three Killing vector fields [120, 121]. However its horizon is topologically $S^1 \times S^2$ whereas the Myers-Perry black holes have S^3 horizon topology. Moreover there is a range of values for its mass and spin for which there exist two black ring solutions as well as a Myers-Perry black hole. Hence there is not a unique family of stationary black hole vacuum solutions in five dimensions, and the global parameters m and J do not identify a unique rotating black hole. The Emparan-Reall family has many interesting properties and there are charged and supersymmetric analogues. It suffices to note here that it contains no static and spherically symmetric limit black hole. Furthermore, analysis of pertubations off the spherically symmetric vacuum solution suggests that the Myers-Perry solutions are the only regular black holes near the static limit. The full discussion of this remark and more details about stability, including the cases where Λ is non-zero, can be found in references [118, 122].

It is natural to ask if uniqueness theorems can be constructed when the class of solutions considered is restricted by further conditions. A couple of results have shown that this is possible in five dimensions at least [123, 124]. When it is assumed that there are two commuting rotational Killing vectors, in addition to the stationary Killing vector field, and that the horizon is topologically S^3, it has been shown that vacuum black holes with non-degenerate horizons, must be members of the Myers-Perry family. The additional assumptions enable the appropriate extensions of the four dimensional uniqueness proofs for stationary black holes to be constructed. In the vacuum case, for example, the uniqueness problem is formulated as a $N = SL(3, R)/SO(3)$ non-linear sigma model boundary value problem and the corresponding Mazur identity is constructed. However, as is pointed out in reference [123] this approach does not appear to be extendable to higher dimensions. In six dimensions, for instance, the Myers-Perry black hole has only two commuting space-like Killing vector fields. However the direct generalization of the sigma model formulation used in four and five dimensions requires the six dimensional space-time to admit three such Killing vector fields.

When $d > 4$, the full global context has not, by 2004, been explored in the same depth as it has been in four dimensions. Differences from the four dimensional case, another example being the failure of conformal null infinity to exist for radiating systems in odd dimensions [125], suggest further failures of uniqueness. Indeed Reall has conjectured that when $d > 4$, in addition to the known solutions, there exist stationary asymptotically flat vacuum solutions with only two Killing vector fields [126].

In conclusion, a brief comment should be made about black hole solutions of Einstein's equations with a non-zero cosmological constant Λ. The Kerr-Newman family of metrics admits generalizations which include a cosmological constant, and these provide useful black hole reference models [37, 127]. Both (locally) asymptotically de Sitter ($\Lambda > 0$) and asymptotically anti-de Sitter ($\Lambda < 0$) black hole models have been studied quite extensively, mainly since the 1990s. Topological and hair results may change when Λ is non-zero; examples of papers which include general overviews of investigations of these topics are cited in references [122, 128, 129, 130]. Not so much is known about uniqueness theorems when Λ is non-zero. There are non-existence [131] and uniqueness results for static black holes solutions with $\Lambda < 0$. Broadly stated, it has been shown that a static asymptotically AdS single black hole solution with a non-degenerate horizon must be a Schwarzschild-AdS black hole solution if it has a certain C^2 conformal spatial completion [132].

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A A simple proof of the uniqueness of the Schwarzschild black hole

Consider the static metric and vacuum field equations given by Eqs.(1-3). The conditions which isolated single black hole solutions must satisfy are formulated on a regular hypersurface Σ, $t = const$, where $0 < v < 1$. They are:

(a) asymptotic flatness which here is formulated on Σ as the requirement of asymptotically Euclidean topology with the usual boundary conditions, given in asymptotically Euclidean coordinates, x^a, by

$$g_{ab} = (1 + 2mr^{-1})\delta_{ab} + h_{ab};\; v = 1 - mr^{-1} + \mu;\; m \text{ const.}; \tag{15}$$

where h_{ab} and μ are all $O(r^{-2})$ with first derivatives $O(r^{-3})$ as $r = (\delta_{ab}x^a x^b)^{1/2} \to \infty$.

(b) Regularity of the horizon of the single black hole which is formulated here as the requirement that the intersection of the future and past horizons constitute a regular compact, connected, two dimensional boundary B to Σ as $v \to 0$. It can be shown that the extrinsic curvature of B in $\overline{\Sigma}$, with respect to g_{ab}, must vanish, and also that the function

$$w \equiv -\frac{1}{2}\nabla_{[\alpha}k_{\beta]}\nabla^{\alpha}k^{\beta} = g^{ab}v_{,a}\, v_{,b} \tag{16}$$

is constant on B. The latter constant, denoted, w_0, is the square of the surface gravity. It is necessarily non-zero (that is the horizon is necessarily non-degenerate) since the horizon is assumed connected.

Using the vacuum field equations, Eqs.(3), the following identities can be constructed

$$D_a(vD^a v) = D_a v D^a v, \tag{17}$$

$$D_a(v^{-1}D^a w) = 2vR_{ab}R^{ab}. \tag{18}$$

By integrating Eq.(17) over Σ and using the boundary conditions, it can be seen that the mass m is non-negative, and zero if and only if v is constant and g_{ab} and the four-metric are flat. In a similar way integration of the first of Eqs.(3) leads to the recovery, in this framework, of the generalized Smarr formula

$$4\pi m = w_0^{1/2} S_0, \tag{19}$$

Here S_0 is the area of B. Integration of Eq.(18) and the use of the Gauss-Bonnet theorem on B gives

$$w_0^{1/2} \int_B ({}^{(2)}R) dS = 8\pi w_0^{1/2}(1-p) \geqq 0, \tag{20}$$

with equality if and only if the three metric g_{ab} has zero Ricci tensor and is therefore flat. It follows that the genus p must be zero and the topology of B must be spherical. Now by using the field equations to evaluate the Cotton tensor R_{abc}, given in Eq.(11), it can be shown that

$$R_{abc}R^{abc} = 4v^{-4} w D_a D^a w - 4v^{-5} w D^a w D_a v - 3v^{-4} D_a w D^a w. \tag{21}$$

Therefore at critical points of the harmonic function v on Σ, where $w = 0$, the Cotton tensor and the gradient of w must vanish. This expression can be used to construct the identities

$$\begin{aligned} D_a(pv^{-1}D^a w + qwD^a v) = &+\frac{3}{4}v^{-1}w^{-1}p[D_a w + 8wv(D_a v)(1-v^2)^{-1}]^2 + \\ &+ \frac{p}{4}v^3 w^{-1} R_{abc}R^{abc}, \end{aligned} \tag{22}$$

where $p(v) = (cv^2+d)(1-v^2)^{-3}$ and $q(v) = -2c(1-v^2)^{-3}+6(cv^2+d)(1-v^2)^{-4}$ and c and d are real numbers.

It follows from Eqs. (21) and (22) that the divergence on the left hand side of Eq. (22), which is overtly regular everywhere on Σ, is non-negative on Σ when c and d are chosen so that p is non-negative. Making two such sets of choices in Eq.(22), $c = -1, d = +1$ and $c = 1, d = 0$, integrating over Σ, and using the boundary conditions and Gauss's theorem then gives the two inequalities

$$\begin{aligned} w_0 S_0 &\leqq \pi, \\ w_0^{3/2} S_0 &\geqq \frac{\pi}{4m}. \end{aligned} \tag{23}$$

It is straightforward to see that these inequalities and Eq.(19) are compatible if and only if equality holds in Eq.(23). For this to be the case the right hand side of Eq.(22) must vanish. Hence R_{abc} must be zero, so the three-geometry must be conformally flat and $w = w_0(1-v^2)^4$. Since w has no zeroes on Σ coordinates (v, x^A) like those used by Israel can now be introduced on Σ. The three-metric on Σ then takes the form

$$ds^2 = w_0^{-1}(1-v^2)^{-4}dv^2 + g_{AB}dx^A dx^B. \tag{24}$$

The conformal flatness of this metric can be shown to imply that the level surfaces of v are umbilically embedded in Σ [46]. It now follows quickly from the field equations that the four-metric, Eq.(2), is the Schwarzschild metric.

B Postscript

Since this article was completed in early 2005 research on black holes and uniqueness theorems has continued apace. The review of black hole theorems, [25], has been updated by new authors, [133]. This concentrates on four dimensional stationary space-times. Uniqueness theorems for black holes in higher dimensional space-times have recently been reviewed in [134]. The very important matter of astrophysical tests of the uniqueness of the Kerr family of black hole solutions has received increasing attention. A brief review of research on this topic is contained in [135].

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