ON STARK ELEMENTS OF ARBITRARY WEIGHT AND THEIR p-ADIC FAMILIES

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ABSTRACT. We develop a detailed arithmetic theory related to special values at arbitrary integers of the Artin L-series of linear characters. To do so we define canonical generalized Stark elements of arbitrary 'rank' and 'weight', thereby extending the classical theory of Rubin-Stark elements. We then formulate an extension to arbitrary weight of the refined version of the Rubin-Stark Conjecture that we studied in an earlier article and also show that generalized Stark elements constitute a p-adic family by formulating precise conjectural congruence relations between elements of differing weights. We prove both of these conjectures in several important cases.

1. Introduction

1.1. The seminal conjecture of Stark predicts that canonical elements constructed (unconditionally) from the leading terms at zero of the Artin L-series of complex linear characters should belong to the rational vector spaces that are spanned by the r-th exterior powers of suitable groups of algebraic units, where r denotes the order of vanishing at zero of the relevant L-series.

It is believed that if Stark's conjecture is true, then these 'Stark elements' should constitute a higher rank Euler system for the multiplicative group $\mathbb{G}_m$ over number fields and so there is considerable interest in studying their detailed arithmetic properties, especially the integrality.

The basic integral properties of Stark elements were first studied by Stark himself in [26], and then by Tate in [27], for the case $r = 1$ and then subsequently by Rubin in [22] where the so-called 'Rubin-Stark Conjecture' was formulated in the setting of general order of vanishing.

More recently, we formulated a strong refinement of the Rubin-Stark Conjecture in [5, Conj. 7.3], obtained a natural interpretation of this conjecture in terms of a general theory of 'arithmetic zeta elements' that was motivated by an earlier approach of Kato to the formulation of generalized Iwasawa main conjectures and derived a series of consequences of our conjecture concerning the detailed algebraic properties of number fields.

In this article we shall use the leading terms at arbitrary integer points of the L-series of linear characters to unconditionally define for non-negative integers r and even integers w canonical '(generalized) Stark elements of rank r and weight w'. In this context the 'rank' relates to the exterior power of the arithmetic module in which the element is constructed and the 'weight' to the integer point $j_w := -w/2$ at which one takes the leading term of the L-series (so that w is the weight of the associated motive $h^0(\operatorname{Spec} L)(j_w)$). In particular, in weight 0, our construction recovers the classical theory of Rubin-Stark elements, and hence in weight 0 and rank 1 recovers the original constructions of Stark.

Our approach will show that generalized Stark elements of any fixed rank and weight should encode (in a very explicit way) detailed arithmetic information concerning the Galois structure of important étale cohomology groups.

In addition, our approach leads naturally to the simultaneous study of generalized Stark elements of differing weights and thereby introduces the perspective of p-adic families to the investigation. We remark that this philosophy of p-adic families has not hitherto been used in the setting of the Rubin-Stark Conjecture and is we feel worthy of further consideration.

To be a little more precise about the results proved here we fix an odd prime p and a finite abelian extension L/K of number fields. We assume that L/K is unramified outside a finite set of places S of K containing all places that are either archimedean or p-adic and we set $G := \mathrm{Gal}(L/K)$.

Then, as a first step, in Conjecture 3.5 we predict that generalized Stark elements of weight w (and of appropriate rank) over L explicitly determine the initial Fitting ideal of the étale cohomology group $H^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j_w))$, regarded as a $\mathbb{Z}_p[G]$-module in the natural way.

This conjecture constitutes a natural extension of [5, Conj. 7.3] from the case of weight 0 to the case of arbitrary weight. In addition, an interpretation of generalized Stark elements in terms of the theory of arithmetic zeta elements allows us to prove Conjecture 3.5 for all absolutely abelian fields and for the minus part of CM-extensions of totally real fields (see Theorems 3.10 and 4.3).

Next we write p^n for the number of p-power roots of unity in L and note that the Galois modules $H^2(\mathcal{O}_{L,S}, \mathbb{Z}/p^n(1-j))(j-k)$ and $H^2(\mathcal{O}_{L,S}, \mathbb{Z}/p^n(1-k))$ are isomorphic for any choice of integers j and k. In particular, since Conjecture 3.5 implies that Stark elements of weight w over L determine the initial Fitting ideal of the $(\mathbb{Z}/p^n\mathbb{Z})[G]$-module $H^2(\mathcal{O}_{L,S}, \mathbb{Z}/p^n(1-j_w))$, it suggests Stark elements of differing weights (and fixed rank) should be related by congruences modulo p^n.

In Conjecture 3.12 we use algebraic techniques developed in [5] to formulate, modulo the assumed validity of a weak version of Conjecture 3.5, a precise and explicit family of congruence relations between Stark elements of differing weights over arbitrary number fields L.

We show that this very general family of conjectural congruences recovers upon appropriate specialization a wide variety of results in the literature ranging from the classical congruences of Kummer concerning Bernoulli numbers to the results of Beilinson and Hüber-Wildeshaus concerning the cyclotomic elements of Deligne-Soulé and a more recent conjecture of Solomon concerning certain 'explicit reciprocity laws' for Rubin-Stark elements. These various connections allow us, in particular, to derive strong evidence, both theoretical and numerical, in support of Conjecture 3.12 (see Theorem 3.13 and Remark 3.14).

The main contents of this article is as follows. In §2 we give the definition of generalized Stark elements and then in §3 we formulate our central conjectures concerning these elements (in Conjectures 3.5 and 3.12). In §4 we give a natural interpretation of generalized Stark elements in terms of the theory of zeta elements and then use this interpretation to prove Conjecture 3.5 in some important cases. Finally, in §5, we relate special cases of Conjecture 3.12 to well-known results in the literature and thereby deduce some supporting evidence for it.

1.2. For the reader's convenience we end the introduction by collecting together details concerning notation and conventions that are used in the sequel.

1.2.1. *Algebra.* Let E be a field of characteristic 0. For any abelian group A, we denote $E \otimes_{\mathbb{Z}} A$ by EA. If A is a $\mathbb{Q}$-vector space, we sometimes denote $E \otimes_{\mathbb{Q}} A$ also by EA. Similarly, if E is an extension of $\mathbb{Q}_p$ (p is a prime number) and A is a $\mathbb{Z}_p$-module, we denote $E \otimes_{\mathbb{Z}_p} A$ and $E \otimes_{\mathbb{Q}_p} A$ also by EA. For any integer m, we denote A/mA simply by A/m.

For a commutative ring R and an R-module M we set $M^* := \mathrm{Hom}_R(M, R)$. If M is a free R-module with basis $\{b_1, \dots, b_r\}$, then for each i with $1 \le i \le r$ we write b_i^* for the homomorphism $M \to R$ that sends b_j to 1 if $i = j$ and to 0 if $i \neq j$.

For any field E, the absolute Galois groups is denoted by G_E. Let $c \in G_{\mathbb{R}}$ denote the complex conjugation. For a $\mathbb{Z}[G_{\mathbb{R}}]$-module M, let $M^{\pm}$ be the submodule $\{a \in M \mid c \cdot a = \pm a\}$ of M. We also use the idempotent $e^{\pm} := \frac{1 \pm c}{2}$ of $\mathbb{Z}[\frac{1}{2}][G_{\mathbb{R}}]$ and the decomposition $M = M^+ \oplus M^-$ with $M^{\pm} = e^{\pm} M$ for any $\mathbb{Z}[\frac{1}{2}][G_{\mathbb{R}}]$-module M.

1.2.2. *Arithmetic.* Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For any non-negative integer m, we denote by μ_m the subgroup of all m-th roots of unity in $\overline{\mathbb{Q}}^{\times}$. As usual, we denote μ_{p^n} (p is a prime number) by $\mathbb{Z}/p^n(1)$, and $\varprojlim_n \mu_{p^n}$ by $\mathbb{Z}_p(1)$. For any integer j, $\mathbb{Z}_p(j)$ and $\mathbb{Q}_p(j)$ are defined in the usual way.

For a number field K, i.e. a finite extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$, we write $S_\infty(K)$, $S_{\mathbb{C}}(K)$ and $S_p(K)$ for the set of archimedean, complex and p-adic places of K respectively. We write S_∞ for $S_\infty(K)$ if there is no danger of confusion. The ring of integers of K is denoted by $\mathcal{O}_K$. For a finite set S of places of K, the ring of S-integers of K is denoted by $\mathcal{O}_{K,S}$. If L is a finite extension of K, then the set of places of K which ramify in L is denoted by $S_{\mathrm{ram}}(L/K)$ and the set of places of L lying above any given set of places S of K is denoted by S_L. The ring of S_L-integers of L is denoted by $\mathcal{O}_{L,S}$ instead of $\mathcal{O}_{L,S_L}$.

Let L/K be a finite abelian extension with Galois group G. Let S and T be finite disjoint sets of places of K such that $S_\infty(K) \cup S_{\mathrm{ram}}(L/K) \subset S$. Then, for a character

$\chi \in \widehat{G} := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^\times)$, the S-truncated T-modified L-function is defined by

$$L_{K,S,T}(\chi, s) := \prod_{v \in T}(1 - \chi(\mathrm{Fr}_v)\mathrm{N}v^{1-s}) \prod_{v \notin S}(1 - \chi(\mathrm{Fr}_v)\mathrm{N}v^{-s})^{-1} \ (\mathrm{Re}(s) > 1)$$

where $\mathrm{Fr}_v \in G$ is the Frobenius automorphism at a place of L above v, and $\mathrm{N}v$ is the cardinality of the residue field $\kappa(v)$ of v. The function $L_{K,S,T}(\chi, s)$ continues meromorphically to the whole complex plane and its leading term at an integer j is denoted by $L^*_{K,S,T}(\chi, j)$. The S-truncated T-modified L-function for L/K is defined by setting

$$\theta_{L/K,S,T}(s) := \sum_{\chi \in \widehat{G}} L_{K,S,T}(\chi^{-1}, s) e_\chi \quad \text{with} \quad e_\chi := \frac{1}{\# G} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}$$

and has leading term at $s = j$ equal to $\theta^*_{L/K,S,T}(j) := \sum_{\chi \in \widehat{G}} L^*_{K,S,T}(\chi^{-1}, j) e_\chi \in \mathbb{C}[G]^\times$. When $T = \emptyset$, we omit it from notations (so we denote $L_{K,S,\emptyset}(\chi, s)$ by $L_{K,S}(\chi, s)$, for example). Note that

$$\theta_{L/K,S,T}(s) = \delta_{L/K,T}(s) \cdot \theta_{L/K,S}(s)$$

with $\delta_{L/K,T}(s) := \prod_{v \in T}(1 - \mathrm{N}v^{1-s}\mathrm{Fr}_v^{-1})$.

2. Generalized Stark elements

2.1. **The general set up.** Let L/K be a finite abelian extension of number fields. Set $G := \mathrm{Gal}(L/K)$. Fix an *odd* prime number p. For each place w of L, we fix an algebraic closure $\overline{L}_w$ of L and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{L}_w$. From this, we regard G_{L_w} as a subgroup of G_L, and the localization map of Galois cohomology $H^i(L, \cdot) \to H^i(L_w, \cdot)$ is defined by the restriction map. Also, for each place w in $S_\infty(L)$, we identify $\overline{L}_w$ with $\mathbb{C}$. For each integer j we set

$$S^j_\infty(L) := \begin{cases} S_\infty(L) & \text{if } j \text{ is even,} \\ S_{\mathbb{C}}(L) & \text{if } j \text{ is odd,} \end{cases}$$

and note that

$$Y_L(j) := \bigoplus_{w \in S_\infty(L)} H^0(L_w, \mathbb{Z}_p(j)) = \bigoplus_{w \in S^j_\infty(L)} \mathbb{Z}_p(j).$$

In particular, setting $\xi := (e^{2\pi\sqrt{-1}/p^n})_n \in \mathbb{Z}_p(1)$ one obtains a $\mathbb{Z}_p$-basis $\{w(j)\}_{w \in S^j_\infty(L)}$ of $Y_L(j)$, which is defined by $w(j) = (w(j)_{w'})_{w'}$ where

$$w(j)_{w'} := \begin{cases} \xi^{\otimes j} & \text{if } w' = w, \\ 0 & \text{if } w' \neq w. \end{cases}$$

Next we note that the complex conjugation c in $G_{\mathbb{R}}$ acts on the Betti cohomology

$$H_L(j) := H^0_B(\operatorname{Spec} L(\mathbb{C}), \mathbb{Q}(j)) = \bigoplus_{\iota : L \hookrightarrow \mathbb{C}} (2\pi\sqrt{-1})^j \mathbb{Q}$$

by $c \cdot (a_\iota)_\iota := (c \cdot a_\iota)_{c\circ\iota}$ for each a_ι in $(2\pi\sqrt{-1})^j\mathbb{Q}$ and we set

$$H_L(j)^+ := e^+ H_L(j) \quad \text{with} \quad e^+ := \frac{1+c}{2}.$$

Note that the natural decomposition $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}(-1)$ induces an isomorphism

$$\mathbb{R} \otimes_{\mathbb{Q}} L \simeq \mathbb{R}H_L(j)^+ \oplus \mathbb{R}H_L(j-1)^+. \tag{1}$$

For each embedding $\iota' : L \hookrightarrow \mathbb{C}$ we define $\iota'_j = (\iota'_{j,\iota})_\iota$ in $H_L(j)$ by setting

$$\iota'_{j,\iota} := \begin{cases} (2\pi\sqrt{-1})^j & \text{if } \iota = \iota', \\ 0 & \text{if } \iota \neq \iota'. \end{cases}$$

Then, if for each place w in $S_\infty(L)$ we write $\iota_w : L \to \mathbb{C}$ for the embedding induced by the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{L}_w = \mathbb{C}$, we obtain an isomorphism of $\mathbb{Q}_p[G]$-modules

$$Y_L(j) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} H_L(j)^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p \tag{2}$$

that sends each element $w(j)$ to $e^+\iota_{w,j}$.

For each place v in $S_\infty(K)$ we now fix a place w_v in $S_\infty(L)$ that lies above v and set $S_\infty(L)/G := \{w_v \mid v \in S_\infty(K)\}$. For any idempotent ε in $\mathbb{Z}_p[G]$ we set

$$W^\varepsilon_j := \{w \in S^j_\infty(L) \cap (S_\infty(L)/G) \mid \varepsilon \cdot w(-j) \neq 0\}$$

and then define $r^\varepsilon_j := \# W^\varepsilon_j$.

Lemma 2.1. *If ε is a primitive idempotent of $\mathbb{Z}_p[G]$, then $\varepsilon Y_L(-j)$ is a free $\mathbb{Z}_p[G]\varepsilon$-module of rank r^ε_j with basis $\{\varepsilon \cdot w(-j) \mid w \in W^\varepsilon_j\}$.*

Proof. For each $w \in S^j_\infty(L)$, the $\mathbb{Z}_p[G]$-submodule of $Y_L(-j)$ generated by $w(-j)$ is projective, since p is odd. Thus, if ε is a primitive idempotent (so that the ring $\mathbb{Z}_p[G]\varepsilon$ is local), then $\mathbb{Z}_p[G]\varepsilon \cdot w(-j)$ is either zero or a free $\mathbb{Z}_p[G]\varepsilon$-module of rank one and so the decomposition

$$\varepsilon Y_L(-j) = \bigoplus_{w \in S^j_\infty(L) \cap S_\infty(L)/G} \mathbb{Z}_p[G]\varepsilon \cdot w(-j)$$

implies that $\varepsilon Y_L(-j)$ is free with basis $\{\varepsilon \cdot w(-j) \mid w \in W^\varepsilon_j\}$. $\square$

The algebra $\mathbb{Z}_p[G]$ is semilocal and so every idempotent of $\mathbb{Z}_p[G]$ is a sum of primitive idempotents. By Lemma 2.1, we may consider, without any loss of generality, an idempotent satisfying the following condition.

Hypothesis 2.2. ε is an idempotent of $\mathbb{Z}_p[G]$ such that the $\mathbb{Z}_p[G]\varepsilon$-module $\varepsilon Y_L(-j)$ is free of rank r^ε_j and has as basis the set $\{\varepsilon \cdot w(-j) \mid w \in W^\varepsilon_j\}$.

Example 2.3. Suppose K is totally real and L is CM and write c for the complex conjugation in G. For each integer j we obtain idempotents of $\mathbb{Z}_p[G]$ by setting $e_j^{\pm} := (1 \pm (-1)^j c)/2$ and we abbreviate $W_j^{e_j^{\pm}}$ to $W_j^{\pm}$. Then Hypothesis 2.2 is satisfied in each of the following cases.

(i) If $\varepsilon = e_j^+$, then $\varepsilon \cdot w(-j) = w(-j)$ for each w in $S_\infty^j(L) \cap S_\infty(L)/G = S_\infty(L)/G$ and so we have $W_j^+ = S_\infty(L)/G$ and $r_j^\varepsilon = \# S_\infty(L)/G = \# S_\infty(K) = [K : \mathbb{Q}]$.

(ii) If $\varepsilon = e_j^-$, then $\varepsilon \cdot w(-j) = 0$ for each w in $S_\infty^j(L)$ so W_j^- is empty and $r_j^\varepsilon = 0$.

2.2. **The period-regulator isomorphisms.** In this section we assume the idempotent ε satisfies Hypothesis 2.2 with respect to j.

In the sequel we fix a finite set S of places of K which contains $S_\infty(K) \cup S_p(K) \cup S_{\mathrm{ram}}(L/K)$. We also fix (and do not explicitly mention) an isomorphism of fields $\mathbb{C} \simeq \mathbb{C}_p$.

We write $\widehat{G}^\varepsilon$ for the subset of $\widehat{G}$ comprising characters χ for which $\varepsilon \cdot e_\chi \neq 0$. We define a subset of $\widehat{G}^\varepsilon$ by setting

$$\widehat{G}_j^\varepsilon := \{\chi \in \widehat{G}^\varepsilon \mid \dim_{\mathbb{C}_p}(e_\chi \mathbb{C}_p H^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))) = r_j^\varepsilon\}$$

and then obtain an idempotent of $\mathbb{Q}_p[G]\varepsilon$ by setting

$$\varepsilon_j := \sum_{\chi \in \widehat{G}_j^\varepsilon} e_\chi.$$

Remark 2.4. Lemma 4.1(ii) below implies that for each $\chi \in \widehat{G}^\varepsilon$ one has

$$\chi \in \widehat{G}_j^\varepsilon \Longleftrightarrow \begin{cases} e_\chi \mathbb{C}_p H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j)) \text{ vanishes}, & \text{if } j \neq 1, \\ e_\chi(\mathbb{C}_p \oplus \mathbb{C}_p H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p)) \text{ vanishes}, & \text{if } j = 1. \end{cases}$$

By using this description one can deduce the following facts.

(i) If $j < 0$, then $\widehat{G}_j^\varepsilon = \widehat{G}^\varepsilon$ (by Soulé [19, Th. 10.3.27]) and so $\varepsilon_j = \varepsilon$.

(ii) If $j = 0$, then

$$\begin{aligned} \widehat{G}_0^\varepsilon &= \{\chi \in \widehat{G}^\varepsilon \mid e_\chi \mathbb{C}_p X_{L, S \setminus S_\infty} = 0\} \\ &= \{\chi \in \widehat{G}^\varepsilon \mid \mathrm{ord}_{s=0} L_{K,S}(\chi, s) = r_0^\varepsilon\}. \end{aligned}$$

Here (and in the sequel), for any finite set Σ of places of K we write $X_{L,\Sigma}$ for the kernel of the homomorphism $\bigoplus_{w \in \Sigma_L} \mathbb{Z}_p w \to \mathbb{Z}_p$ sending each w to 1. The first displayed equality then follows by noting that, by class field theory, there is a canonical isomorphism $H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p X_{L, S \setminus S_\infty}$, and the second equality follows directly from [27, Chap. I, Prop. 3.4].

(iii) If $j = 1$, then Leopoldt's Conjecture for L is equivalent to the vanishing of $H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p)$ and hence implies that

$$\widehat{G}_1^\varepsilon = \{\chi \in \widehat{G}^\varepsilon \mid \chi \neq \mathbf{1}\},$$

where we write $\mathbf{1}$ for the trivial character of G, and so $\varepsilon_1 = \varepsilon(1 - e_{\mathbf{1}})$.

(iv) If $j > 1$, then Schneider's Conjecture [24] for L is equivalent to the vanishing of $H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))$ and hence implies $\widehat{G}_j^\varepsilon = \widehat{G}^\varepsilon$ and so $\varepsilon_j = \varepsilon$.

In the remainder of this section we define for each integer j a canonical isomorphism of $\mathbb{C}_p[G]$-modules

$$\lambda_j : \varepsilon_j \mathbb{C}_p {\bigwedge}_{\mathbb{Z}_p[G]}^{r_j^\varepsilon} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \xrightarrow{\sim} \varepsilon_j \mathbb{C}_p {\bigwedge}_{\mathbb{Z}_p[G]}^{r_j^\varepsilon} Y_L(-j).$$

The explicit definition that we give is motivated by the Tamagawa number conjecture of Bloch and Kato (see, in particular, the proof of Corollary 4.4 below).

2.2.1. *The case* $j < 0$. In this case the known validity of the Quillen-Lichtenbaum Conjecture (which follows from the recent proof by Rost and Voevodsky of the Bloch-Kato Conjecture) gives a canonical Chern character isomorphism

$$\mathrm{ch}_j : K_{1-2j}(\mathcal{O}_L) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$$

where we write $K_*(-)$ for Quillen's higher algebraic K-theory functor.

One also has $\varepsilon_j = \varepsilon$ (by Remark 2.4(i)) and we define λ_j to be the r_j^ε-th exterior power of the composite isomorphism of $\mathbb{C}_p[G]$-modules

$$\varepsilon\mathbb{C}_p H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \xrightarrow{\sim} \varepsilon\mathbb{C}_p K_{1-2j}(\mathcal{O}_L) \xrightarrow{\sim} \varepsilon\mathbb{C}_p H_L(-j)^+ \xrightarrow{\sim} \varepsilon\mathbb{C}_p Y_L(-j),$$

where the first map is induced by the inverse of the isomorphism ch_j^{-1}, the second by (-1)-times the Borel regulator map

$$b_j : \mathbb{R}K_{1-2j}(\mathcal{O}_L) \xrightarrow{\sim} \mathbb{R}H_L(-j)^+$$

and the third by the isomorphism in (2).

2.2.2. *The case* $j = 0$. We note that $H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ is identified with $\mathbb{Z}_p\mathcal{O}_{L,S}^\times$ via Kummer theory and we define λ_0 to be the r_0^ε-th exterior power of the composite isomorphism of $\mathbb{C}_p[G]$-modules

$$\varepsilon_0\mathbb{C}_p H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)) = \varepsilon_0\mathbb{C}_p\mathcal{O}_{L,S}^\times \xrightarrow{\sim} \varepsilon_0\mathbb{C}_p X_{L,S} \xrightarrow{\sim} \varepsilon_0\mathbb{C}_p Y_L(0) \tag{3}$$

where the first map is the restriction of the Dirichlet regulator (sending each a in $\mathcal{O}_{L,S,T}^\times$ to $-\sum_{w\in S_L} \log|a|_w w$) and the second isomorphism follows from the vanishing of $\varepsilon_0\mathbb{C}_p X_{L,S\setminus S_\infty}$ (see Remark 2.4(ii)).

2.2.3. *The case* $j = 1$. We write $\Gamma_{L,S}$ for the Galois group of the maximal abelian pro-p extension of L unramified outside S. Then the module $H^1(\mathcal{O}_{L,S}, \mathbb{Q}_p)$ identifies with $\mathrm{Hom}_{\mathrm{cont}}(\Gamma_{L,S},\mathbb{Q}_p)$ and so, by combining the global class field theory with Remark 2.4(iii), one obtains a canonical short exact sequence of $\mathbb{C}_p[G]$-modules

$$\begin{tikzcd}
\varepsilon_1\mathbb{C}_pH^1(\mathcal{O}_{L,S},\mathbb{Q}_p) \arrow[r, hook] & \varepsilon_1\bigoplus_{w\in S_p(L)}(\mathbb{C}_p\mathcal{O}_{L_w}^\times)^* \arrow[r, two heads] \arrow[d, "\exp_p^*"', "\simeq"] & \varepsilon_1(\mathbb{C}_p\mathcal{O}_L^\times)^* \\
 & \varepsilon_1(\mathbb{C}_p\otimes_{\mathbb{Q}}L)^* & \varepsilon_1(\mathbb{C}_pH_L(0)^+)^*. \arrow[u, "\simeq"']
\end{tikzcd} \tag{4}$$

Here the first vertical isomorphism is induced by the linear dual of the p-adic exponential map homomorphisms $L_w\to\mathbb{Q}_p\mathcal{O}_{L_w}^\times$ for w in $S_p(L)$ and the second by the linear dual of the isomorphism $\mathbb{C}_p\mathcal{O}_L^\times\simeq\mathbb{C}_pX_{L,S_\infty}$ induced by the Dirichlet regulator map and the fact that $\varepsilon_1\mathbb{Q}_pX_{L,S_\infty}$ is equal to $\varepsilon_1\mathbb{Q}_pY_L(0)$ and hence isomorphic to $\varepsilon_1\mathbb{Q}_pH_L(0)^+$ by (2).

Abbreviating $\det_{\mathbb{C}_p[G]}(-)$ to $\mathrm{D}(-)$, we then define λ_1 to be the composite isomorphism of $\mathbb{C}_p[G]$-modules

$$\begin{aligned}
\varepsilon_1\mathbb{C}_p{\bigwedge}_{\mathbb{Z}_p[G]}^{r_1^\varepsilon}H^1(\mathcal{O}_{L,S},\mathbb{Z}_p) &= \varepsilon_1\mathrm{D}(\mathbb{C}_pH^1(\mathcal{O}_{L,S},\mathbb{Z}_p))\\
&\simeq \varepsilon_1\left(\mathrm{D}((\mathbb{C}_p\otimes_{\mathbb{Q}}L)^*)\otimes_{\mathbb{C}_p[G]}\mathrm{D}^{-1}((\mathbb{C}_pH_L(0)^+)^*)\right)\\
&\simeq \varepsilon_1\mathrm{D}((\mathbb{C}_pH_L(1)^+)^*)\simeq\varepsilon_1\mathrm{D}(\mathbb{C}_pY_L(1)^*)\simeq\varepsilon_1\mathbb{C}_p{\bigwedge}_{\mathbb{Z}_p[G]}^{r_1^\varepsilon}Y_L(-1).
\end{aligned}$$

Here the first isomorphism is the canonical isomorphism induced by (4), the second is induced by the linear dual of (1) (with $j=1$), the third by (2) and the last by the canonical identification $Y_L(1)^*\simeq Y_L(-1)$.

2.2.4. *The case $j>1$.* In this case the vanishing of $\varepsilon_jH^2(\mathcal{O}_{L,S},\mathbb{Q}_p(1-j))$ (see Remark 2.4) combines with the local and global duality theorems to give a canonical short exact sequence of $\mathbb{C}_p[G]$-modules

$$\begin{tikzcd}
\varepsilon_j\mathbb{C}_pH^1(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j)) \arrow[r, hook] & \varepsilon_j\bigoplus_{w\in S_p(L)}\mathbb{C}_pH^1(L_w,\mathbb{Z}_p(j))^* \arrow[r, two heads] \arrow[d, "\mathrm{syn}_p"', "\simeq"] & \varepsilon_j\mathbb{C}_pH^1(\mathcal{O}_{L,S},\mathbb{Z}_p(j))^* \\
 & \varepsilon_j(\mathbb{C}_p\otimes_{\mathbb{Q}}L)^* & \varepsilon_j\mathbb{C}_pH_L(j-1)^{+,*} \arrow[u, "\simeq"']
\end{tikzcd}$$

in which the second vertical homomorphism is induced by the dual of $-b_{1-j}\circ\mathrm{ch}_{1-j}^{-1}$ and the first is induced by the linear duals for each w in $S_p(L)$ of the canonical composite homomorphisms $L_w\to H^1_{\mathrm{syn}}(\mathcal{O}_{L_w},j)\to H^1(L_w,\mathbb{Q}_p(j))$ involving syntomic cohomology that are discussed by Besser in [1, (5.3) and Cor. 9.10].

We then define λ_j to be the isomorphism of $\mathbb{C}_p[G]$-modules obtained from the above diagram in just the same way that λ_1 is obtained from (4).

Remark 2.5. In [1, Prop. 9.11] Besser proves that for w in $S_p(L)$ the composite homomorphism $L_w\to H^1_{\mathrm{syn}}(\mathcal{O}_{L_w},j)\to H^1(L_w,\mathbb{Q}_p(j))$ used above coincides with the exponential map of Bloch and Kato for $\mathbb{Q}_p(j)$ over L_w. In this way the definition of λ_j for $j > 1$ is naturally analogous to the definition of λ_1.

Remark 2.6. A closer analysis of the discussions used to define λ_j for $j > 0$ shows that, in this case, if $\varepsilon Y_L(1-j)$ vanishes, then $\varepsilon_j = \varepsilon$.

2.3. **The definition of generalized Stark elements.**

Definition 2.7. Fix an integer j, an idempotent ε of $\mathbb{Z}_p[G]$ that satisfies Hypothesis 2.2 (with respect to j). Fix also finite sets S and T of places of K satisfying

- $S_\infty(K) \cup S_p(K) \cup S_{\mathrm{ram}}(L/K) \subset S$;
- $S \cap T = \emptyset$;
- $T = \emptyset$ if $j = 1$.

Then the 'Stark element of rank r_j^ε and weight $-2j$' for $(L/K, S, T, \varepsilon)$ is the unique element $\eta^\varepsilon_{L/K,S,T}(j)$ of $\varepsilon_j \mathbb{C}_p {\bigwedge}^{r_j^\varepsilon}_{\mathbb{Z}_p[G]} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ that satisfies

$$\lambda_j(\eta^\varepsilon_{L/K,S,T}(j)) = \varepsilon_j \theta^*_{L/K,S,T}(j) \cdot {\bigwedge}_{w \in W_j^\varepsilon} w(-j) \quad \text{in} \quad \varepsilon_j \mathbb{C}_p {\bigwedge}^{r_j^\varepsilon}_{\mathbb{Z}_p[G]} Y_L(-j).$$

Remark 2.8. It is natural to regard $\eta^\varepsilon_{L/K,S,T}(j)$ to be of weight $-2j$ since it is associated to the motive $h^0(\mathrm{Spec}\, L)(j)$.

Example 2.9. Definition 2.7 generalizes the classical notion of Rubin-Stark element introduced by Rubin in [22]. In fact, we have

$$\varepsilon_0 \theta^*_{L/K,S,T}(0) = \varepsilon \cdot \lim_{s \to 0} s^{-r_0^\varepsilon} \theta_{L/K,S,T}(s)$$

by Remark 2.4(ii) and [27, Chap. I, Prop. 3.4] and so $\eta^\varepsilon_{L/K,S,T}(0)$ coincides with (the 'ε-component' of) the Rubin-Stark element for the data $(L/K, S, T, W_0^\varepsilon)$.

The following proposition is a natural analogue of [22, Prop. 6.1].

Proposition 2.10. *Suppose that $(L'/K, S', T', \varepsilon')$ is another collection of data as in Definition 2.7 (with respect to j) for which all of the following properties are satisfied: $L \subset L'$, $S \subset S'$, $T \subset T'$, with $G' := \mathrm{Gal}(L'/K)$ the natural surjection $\mathbb{Z}_p[G'] \to \mathbb{Z}_p[G]$ sends ε' to ε and W_j^ε is the set of places of L obtained by restricting places in $W_j^{\varepsilon'}$.*

Then $r_j^{\varepsilon'} = r_j^\varepsilon =: r$ and the map

$$\varepsilon'_j \mathbb{C}_p {\bigwedge}^r_{\mathbb{Z}_p[G']} H^1(\mathcal{O}_{L',S'}, \mathbb{Z}_p(1-j)) \to \varepsilon_j \mathbb{C}_p {\bigwedge}^r_{\mathbb{Z}_p[G]} H^1(\mathcal{O}_{L,S'}, \mathbb{Z}_p(1-j))$$

induced by the corestriction map $\mathrm{Cor}_{L'/L} : H^1(\mathcal{O}_{L',S'}, \mathbb{Z}_p(1-j)) \to H^1(\mathcal{O}_{L,S'}, \mathbb{Z}_p(1-j))$ sends $\eta^{\varepsilon'}_{L'/K,S',T'}(j)$ to

$$\delta_{L/K,T' \setminus T}(j) \cdot \left(\prod_{v \in S' \setminus S} (1 - \mathrm{N}v^{-j} \mathrm{Fr}_v^{-1}) \right) \cdot \eta^\varepsilon_{L/K,S,T}(j).$$

Proof. This follows easily from the fact that the natural surjection $\mathbb{C}[G'] \to \mathbb{C}[G]$ sends $\varepsilon'_j\theta^*_{L'/K,S',T'}(j)$ to

$$\varepsilon_j\theta^*_{L/K,S',T'}(j) = \varepsilon_j\delta_{L/K,T'\setminus T}(j)\cdot\left(\prod_{v\in S'\setminus S}(1-\mathrm{N}v^{-j}\mathrm{Fr}_v^{-1})\right)\cdot\theta^*_{L/K,S,T}(j).$$

□

3. Statement of the conjectures

3.1. A Rubin-Stark Conjecture in arbitrary weight.

3.1.1. *Exterior power biduals and pairings.* Fix a commutative ring R and a finitely generated R-module M. In the following, we abbreviate $\bigwedge_R^r(M^*)$ to $\bigwedge_R^r M^*$.

For non-negative integers r and s with $r \le s$ there is a canonical pairing

$$\bigwedge\nolimits_R^s M \times \bigwedge\nolimits_R^r M^* \to \bigwedge\nolimits_R^{s-r} M$$

defined by

$$(a_1\wedge\cdots\wedge a_s, \varphi_1\wedge\cdots\wedge\varphi_r) \mapsto \sum_{\sigma\in\mathfrak{S}_{s,r}}\mathrm{sgn}(\sigma)\det(\varphi_i(a_{\sigma(j)}))_{1\le i,j\le r}a_{\sigma(r+1)}\wedge\cdots\wedge a_{\sigma(s)},$$

with $\mathfrak{S}_{s,r} := \{\sigma\in\mathfrak{S}_s \mid \sigma(1)<\cdots<\sigma(r) \text{ and } \sigma(r+1)<\cdots<\sigma(s)\}$. We denote the image of (a,Φ) under the above pairing by $\Phi(a)$.

We also use the following construction (compare [22, §1.2]).

Definition 3.1. For each non-negative integer r the r-th exterior bidual of M is the module

$$\bigcap\nolimits_R^r M := \left(\bigwedge\nolimits_R^r M^*\right)^*.$$

Remark 3.2. If $R = \mathbb{Z}_p[G]$ with a finite abelian group G, then the map $a \mapsto (\Phi \mapsto \Phi(a))$ induces an identification

$$\left\{a\in\mathbb{Q}_p\bigwedge\nolimits_{\mathbb{Z}_p[G]}^r M \;\middle|\; \Phi(a)\in\mathbb{Z}_p[G] \text{ for every } \Phi\in\bigwedge\nolimits_{\mathbb{Z}_p[G]}^r M^*\right\} \simeq \bigcap\nolimits_{\mathbb{Z}_p[G]}^r M$$

and so we may regard $\bigcap_{\mathbb{Z}_p[G]}^r M$ as a subset of $\mathbb{Q}_p\bigwedge_{\mathbb{Z}_p[G]}^r M$.

Lemma 3.3. *Suppose that $R = \mathbb{Z}_p[G]$ with a finite abelian group G and that M is $\mathbb{Z}_p$-free. Let H be a subgroup of G and denote the natural surjection $\mathbb{Q}_p[G] \to \mathbb{Q}_p[G/H]$ by π_H. Then, for any $a \in \mathbb{Q}_p\bigwedge_{\mathbb{Z}_p[G]}^r M$, we have*

$$\pi_H\left(\left\{\Phi(a) \;\middle|\; \Phi\in\bigwedge\nolimits_{\mathbb{Z}_p[G]}^r M^*\right\}\right) = \left\{\Psi(\mathrm{N}_H^r(a)) \;\middle|\; \Psi\in\bigwedge\nolimits_{\mathbb{Z}_p[G/H]}^r (M^H)^*\right\},$$

where

$$\mathrm{N}_H^r : \mathbb{Q}_p\bigwedge\nolimits_{\mathbb{Z}_p[G]}^r M \to \mathbb{Q}_p\bigwedge\nolimits_{\mathbb{Z}_p[G/H]}^r M^H$$

is the map induced by the norm map

$$M \to M^H;\ m \mapsto \sum_{\sigma\in H} \sigma \cdot m.$$

Proof. This follows from [23, Rem. 2.9 and Lem. 2.10]. □

3.2. *T***-modified cohomology.** Let j be an integer, and S and T sets of places of K as in Definition 2.7.

Let now R denote any of the rings $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{Z}/p^n$ for some natural number n. Then, as T is disjoint from S, for each w in T_L there is a natural momorphism of étale cohomology complexes $R\Gamma(\mathcal{O}_{L,S}, R(1-j)) \to R\Gamma(\kappa(w), R(1-j))$.

We define $R\Gamma_T(\mathcal{O}_{L,S}, R(1-j))$ to be a complex that lies in an exact triangle in the derived category $D(R[G])$ of complexes of $R[G]$-modules of the form

$$R\Gamma_T(\mathcal{O}_{L,S}, R(1-j)) \to R\Gamma(\mathcal{O}_{L,S}, R(1-j)) \to \bigoplus_{w\in T_L} R\Gamma(\kappa(w), R(1-j)) \to, \tag{5}$$

where the second arrow is the diagonal map induced by the morphisms described above. In each degree i we then set

$$H^i_T(\mathcal{O}_{L,S}, R(1-j)) := H^i(R\Gamma_T(\mathcal{O}_{L,S}, R(1-j)))$$

and we note that $H^i_T(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j)) = H^i(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))$ (this follows from the fact that $R\Gamma(\kappa(w), \mathbb{Q}_p(1-j))$ is acyclic if $j \neq 1$ and the assumption that $T = \emptyset$ if $j = 1$). In particular, we can regard $\bigcap^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ as a lattice of $\mathbb{Q}_p \bigwedge^r_{\mathbb{Z}_p[G]} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$.

Example 3.4. Kummer theory identifies $H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ with the p-completion of the (S,T)-unit group $\mathcal{O}^\times_{L,S,T} := \ker(\mathcal{O}^\times_{L,S} \to \bigoplus_{w\in T_L} \kappa(w)^\times)$ of L.

3.2.1. *Statement of the conjecture.* In the sequel for each non-negative integer i we write $\mathrm{Fitt}^i_{\mathbb{Z}_p[G]}(M)$ for the i-th Fitting ideal of a $\mathbb{Z}_p[G]$-module M. We also write I_G for the augmentation ideal of $\mathbb{Z}_p[G]$.

Conjecture 3.5. *Fix an integer j, an idempotent ε of $\mathbb{Z}_p[G]$ satisfying Hypothesis 2.2 (with respect to j) and sets of places S and T as in Definition 2.7. Assume $\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free and in addition that $\varepsilon \in I_G$ if $j = 1$.*

Then, with $\eta = \eta^\varepsilon_{L/K,S,T}(j)$, one has

$$\left\{\Phi(\eta) \,\middle|\, \Phi \in \bigwedge\nolimits^{r^\varepsilon_j}_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))^*\right\} = \varepsilon \cdot \mathrm{Fitt}^0_{\mathbb{Z}_p[G]}(H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))) \tag{6}$$

and hence also

$$\eta \in \bigcap\nolimits^{r^\varepsilon_j}_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)). \tag{7}$$

Remark 3.6. One sees that $H^1(\mathcal{O}_{L,S},\mathbb{Z}_p)$ is always $\mathbb{Z}_p$-free. When $j\neq 1$, we see that $\varepsilon H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free if and only if the composite map

$$\varepsilon H^0(L,\mathbb{Q}_p/\mathbb{Z}_p(1-j))\to\varepsilon H^1(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))\to\varepsilon\bigoplus_{w\in T_L}H^1(\kappa(w),\mathbb{Z}_p(1-j))$$

is injective, where the first map is the natural boundary homomorphism.

Remark 3.7. With the assumption that $\varepsilon\in I_G$ if $j=1$, we can use the complex $R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))[1]\oplus Y_L(-j)[-1]$ to construct an exact sequence of $\mathbb{Z}_p[G]\varepsilon$-modules of the form

$$0\to\varepsilon H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))\to F\to F\to\varepsilon(H^2_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))\oplus Y_L(-j))\to 0$$

where F is both finitely generated and free (see §4). This sequence is a natural analogue of classical 'Tate sequences' (as discussed, for example, in [5, §2.3]) and plays a key role in our analysis. (For $j=1$ and the trivial character, a Tate sequence of similar kind is studied in [12] by Greither and the second author.)

Example 3.8. If we identify $H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ with the p-completion of the (S,T)-unit group $\mathcal{O}^\times_{L,S,T}$ of L (see Example 3.4) then in the case of $j=0$ the equality (6) recovers the 'ε-component' of the p-completion of [5, Conj. 7.3] and thus constitutes a refinement of a range of well-known conjectures in the literature. For the same reason, in the setting of Example 2.9, the $j=0$ case of the containment (7) recovers the 'ε-component' of the p-completion of the Rubin-Stark Conjecture [22, Conj. B$'$] for the data $(L/K,S,T,W_0^\varepsilon)$.

Example 3.9. Assume K totally real, L CM, $j\le 0$ and take ε to be the idempotent e_j^- in Example 2.3(ii).
(i) In this case the inclusion (7) is unconditionally valid. To see this, note $r_j^\varepsilon=0$ so

$$\eta^\varepsilon_{L/K,S,T}(j)=e_j^-\theta^*_{L/K,S,T}(j)=\theta_{L/K,S,T}(j)=\delta_{L/K,T}(j)\theta_{L/K,S}(j).$$

In addition, there is a natural exact sequence

$$0\to\bigoplus_{v\in T}\mathbb{Z}_p[G]\xrightarrow{(1-\mathrm{N}v^{1-j}\mathrm{Fr}_v^{-1})_v}\bigoplus_{v\in T}\mathbb{Z}_p[G]\to\bigoplus_{w\in T_L}H^1(\kappa(w),\mathbb{Z}_p(1-j))\to 0$$

which implies

$$\delta_{L/K,T}(j)\cdot\mathbb{Z}_p[G]\varepsilon=\varepsilon\mathrm{Fitt}^0_{\mathbb{Z}_p[G]}\left(\bigoplus_{w\in T_L}H^1(\kappa(w),\mathbb{Z}_p(1-j))\right)\tag{8}$$

and the assumed injectivity of the displayed map in Remark 3.6 implies this ideal is contained in $\varepsilon\mathrm{Ann}_{\mathbb{Z}_p[G]}(H^0(L,\mathbb{Q}_p/\mathbb{Z}_p(1-j)))$. The claimed inclusion (7) thus follows from the fact that Deligne and Ribet [9] have shown that $a\cdot\theta_{L/K,S}(j)$ belongs to $\mathbb{Z}_p[G]$ for any a in $\mathrm{Ann}_{\mathbb{Z}_p[G]}(H^0(L,\mathbb{Q}_p/\mathbb{Z}_p(1-j)))$.

(ii) If $j < 0$, then the conjectural equality (6) implies

$$\begin{aligned}\mathbb{Z}_p[G]\cdot\theta_{L/K,S,T}(j) &= \varepsilon\mathrm{Fitt}^0_{\mathbb{Z}_p[G]}(H^2_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j)))\\ &\subset \varepsilon\mathrm{Fitt}^0_{\mathbb{Z}_p[G]}(H^2(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))) = \varepsilon\mathbb{Z}_p\mathrm{Fitt}^0_{\mathbb{Z}[G]}(K_{-2j}(\mathcal{O}_{L,S})),\end{aligned}$$

where the inclusion is true as $H^2(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))$ is a quotient of $H^2_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))$ and the final equality because the validity of the Quillen-Lichtenbaum Conjecture gives a canonical isomorphism $H^2(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j)) \simeq \mathbb{Z}_pK_{-2j}(\mathcal{O}_{L,S})$. Noting that $K_{-2j}(\mathcal{O}_L) \subset K_{-2j}(\mathcal{O}_{L,S})$ (see [7, Prop. 5.7]), this displayed inclusion shows (6) refines the classical Coates-Sinnott Conjecture, which predicts $\mathbb{Z}_p[G]\cdot\theta_{L/K,S,T}(j) \subset \mathbb{Z}_p\mathrm{Ann}_{\mathbb{Z}[G]}(K_{-2j}(\mathcal{O}_L))$.

In §4 we interpret generalized Stark elements in terms of the theory of arithmetic zeta elements and use this connection to obtain the following evidence in support of Conjecture 3.5.

Theorem 3.10. *Conjecture 3.5 is valid in both of the following cases.*

(i) *L is an abelian extension of $\mathbb{Q}$.*

(ii) *K is totally real, L is CM, $j \le 0$, ε is the idempotent e_j^- in Example 2.3(ii), and the Iwasawa μ-invariant vanishes for the cyclotomic $\mathbb{Z}_p$-extension L_∞/L.*

We end this section by stating some functorial properties of Conjecture 3.5.

Proposition 3.11. *Let $(L'/K,S',T',\varepsilon')$ be as in Proposition 2.10.*

(i) *Suppose $S' = S$ and $T' = T$. Then (6) (resp. (7)) for $(L'/K,S,T,\varepsilon',j)$ implies (6) (resp. (7)) for $(L/K,S,T,\varepsilon,j)$.*

(ii) *Suppose that $L' = L$, $T' = T$ and $\varepsilon' = \varepsilon$. Then (7) for $(L/K,S,T,\varepsilon,j)$ implies that for $(L/K,S',T,\varepsilon,j)$.*

(iii) *Suppose that $L' = L$, $S' = S$ and $\varepsilon' = \varepsilon$. Then (7) for $(L/K,S,T,\varepsilon,j)$ implies that for $(L/K,S,T',\varepsilon,j)$.*

Proof. We know that $\varepsilon' R\Gamma_T(\mathcal{O}_{L',S},\mathbb{Z}_p(1-j))$ is a perfect complex of $\mathbb{Z}_p[G']$-modules and acyclic outside degrees one and two (see Lemma 4.1 below), and that

$$R\Gamma_T(\mathcal{O}_{L',S},\mathbb{Z}_p(1-j))\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G']}\mathbb{Z}_p[G] \simeq R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))$$

(see [11, Prop. 1.6.5], for example). From this we see that

$$\varepsilon' H^1_T(\mathcal{O}_{L',S},\mathbb{Z}_p(1-j))^{\mathrm{Gal}(L'/L)} \simeq \varepsilon H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))$$

and that

$$\varepsilon' H^2_T(\mathcal{O}_{L',S},\mathbb{Z}_p(1-j))\otimes_{\mathbb{Z}_p[G']}\mathbb{Z}_p[G] \simeq \varepsilon H^2_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j)).$$

Noting this, claim (i) follows from Proposition 2.10 and Lemma 3.3.

Next, we show claim (ii). We have an exact sequence

$$0 \to H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j)) \to H^1_T(\mathcal{O}_{L,S'},\mathbb{Z}_p(1-j)) \to \bigoplus_{w\in(S'\setminus S)_L} H^1_{/f}(L_w,\mathbb{Z}_p(1-j)).$$

Since the last term is $\mathbb{Z}_p$-free, we see that the restriction map

$$H^1_T(\mathcal{O}_{L,S'}, \mathbb{Z}_p(1-j))^* \to H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))^*$$

is surjective. From this, we see that

$$\bigcap\nolimits^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \subset \bigcap\nolimits^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S'}, \mathbb{Z}_p(1-j)).$$

Now the assertion in claim (ii) is clear by Proposition 2.10.

One can prove claim (iii) in the same way as [20, Prop. 5.3.1] by using the exact triangle (5) and the equality (8). □

3.3. **Congruences between Stark elements of differing weights.**

The 'refined class number formula for $\mathbb{G}_m$', as independently conjectured by Mazur-Rubin [18] and the third author [23], constitutes a family of congruence relations between Stark elements of weight zero and differing ranks. In this section we formulate precise families of conjectural congruences between Stark elements of fixed rank and different weights.

At the outset we fix an integer j, an idempotent ε of $\mathbb{Z}_p[G]$ satisfying Hypothesis 2.2 (with respect to j) and sets of places S and T as in §3.2.

We set $r := r_j^\varepsilon$ and $W := W_j^\varepsilon$. We also fix a labeling $W = \{w_1, \ldots, w_r\}$ and use this to define the wedge product $\bigwedge_{w\in W}$.

We fix a positive integer n such that $\mu_{p^n} \subset L^\times$ and use the cyclotomic character

$$\chi_{\rm cyc} : G \to \mathrm{Aut}(\mu_{p^n}) \simeq (\mathbb{Z}/p^n)^\times.$$

For each integer a we also write tw_a for the ring automorphism of $\mathbb{Z}/p^n[G]$ that sends each element σ of G to $\chi_{\rm cyc}(\sigma)^a\sigma$. We then fix an integer k and define δ to be the unique idempotent of $\mathbb{Z}_p[G]$ which projects to $\mathrm{tw}_{k-j}(\overline{\varepsilon})$ in $\mathbb{Z}/p^n[G]$, where $\overline{\varepsilon}$ denotes the image in $\mathbb{Z}/p^n[G]$ of ε.

Then, by an explicit computation, one checks that $W_k^\delta = W$ (and hence $r_k^\delta = r$) and that $\delta Y_L(-k)$ is a free $\mathbb{Z}_p[G]\delta$-module of rank r with basis $\{\delta \cdot w(-k) \mid w \in W\}$.

We next define a tw_{k-j}-semilinear homomorphism

$$\mathrm{tw}_{j,k} : \varepsilon\bigcap\nolimits^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \to \delta\bigcap\nolimits^r_{\mathbb{Z}/p^n[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}/p^n(1-k))$$

that will play a key role in our conjectural congruences.

For simplicity, for each integer a we set

$$H(\mathbb{Z}_p(a)) := H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(a)) \quad \text{and} \quad H(\mathbb{Z}/p^n(a)) := H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}/p^n(a)).$$

Note that the natural map $H(\mathbb{Z}_p(1-j)) \to H(\mathbb{Z}/p^n(1-j))$ induces a homomorphism

$$\bigcap\nolimits^r_{\mathbb{Z}_p[G]} H(\mathbb{Z}_p(1-j)) \to \bigcap\nolimits^r_{\mathbb{Z}/p^n[G]} H(\mathbb{Z}/p^n(1-j)). \tag{9}$$

Recall that each $w \in S_\infty(L)$ determines the embedding $\iota_w : L \hookrightarrow \mathbb{C}$ (see §2.1) and for each integer i with $1 \le i \le r$, we set

$$\xi_i := \iota_{w_i}^{-1}(e^{2\pi\sqrt{-1}/p^n}) \in H^0(L, \mathbb{Z}/p^n(1)). \tag{10}$$

We write

$$c_i : H(\mathbb{Z}/p^n(1-k))^* \to H(\mathbb{Z}/p^n(1-j))^*$$

for the map induced by cup product with $\xi_i^{\otimes(j-k)}$ and

$$\bigwedge\nolimits_{\mathbb{Z}/p^n[G]}^r H(\mathbb{Z}/p^n(1-k))^* \to \bigwedge\nolimits_{\mathbb{Z}/p^n[G]}^r H(\mathbb{Z}/p^n(1-j))^*$$

for the map sending each element $a_1 \wedge \cdots \wedge a_r$ to $c_1(a_1) \wedge \cdots \wedge c_r(a_r)$. Taking the $\mathbb{Z}/p^n$-dual of the last map we obtain a homomorphism

$$\bigcap\nolimits_{\mathbb{Z}/p^n[G]}^r H(\mathbb{Z}/p^n(1-j)) \to \bigcap\nolimits_{\mathbb{Z}/p^n[G]}^r H(\mathbb{Z}/p^n(1-k))$$

and we define $\mathrm{tw}_{j,k}$ to be the composite of this homomorphism with (9).

Conjecture 3.12. *Fix an integer j, an idempotent ε of $\mathbb{Z}_p[G]$ satisfying Hypothesis 2.2 (with respect to j) and sets of places S and T as in §3.2. Assume that the integer k, and associated idempotent δ defined above, are such that*

- *$\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ and $\delta H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-k))$ are both $\mathbb{Z}_p$-free;*
- *$T = \emptyset$ if either $j = 1$ or $k = 1$;*
- *$\varepsilon \in I_G$ if $j = 1$;*
- *$\delta \in I_G$ if $k = 1$.*

Then, if the containment (7) is valid for both pairs (ε, j) and (δ, k), one has

$$\mathrm{tw}_{j,k}(\eta^{\varepsilon}_{L/K,S,T}(j)) = \eta^{\delta}_{L/K,S,T}(k)$$

in the finite module $\delta\bigcap^r_{\mathbb{Z}/p^n[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}/p^n(1-k))$.

We discuss evidence for this conjecture in §5 and, in particular, prove the following result. This result (and its proof) shows that the conjecture incorporates a wide selection of results ranging from classical explicit reciprocity law due to Artin-Hasse and Iwasawa to the classical congruences of Kummer. In this result we use the notation of Example 2.3.

Theorem 3.13. *Assume K is totally real and L is CM.*

(i) *If $K = \mathbb{Q}$, then for all integers j and k Conjecture 3.12 is valid with $\varepsilon = e_j^+$.*
(ii) *For all non-positive integers j and k Conjecture 3.12 is valid with $\varepsilon = e_j^-$.*
(iii) *Conjecture 3.12 for the data $T = \emptyset$, $j = 0$, $\varepsilon = e^+$ and $k = 1$ is a refinement of the 'Congruence Conjecture'* [25, CC$(L/K, S, p, n-1)$] *of Solomon.*

Remark 3.14. The proof of claim (i) relies both (if $j < 0$) on results of Beilinson and Hüber-Wildeshaus on the cyclotomic elements of Deligne-Soulé and (if $j > 0$) on Kato's generalized explicit reciprocity law, whilst claim (ii) relies on results of Deligne and Ribet. Claim (iii) is of interest both because Solomon's Conjecture (recalled in §5.3 below) is formulated as an explicit reciprocity law for Rubin-Stark elements (extending that of Artin-Hasse and Iwasawa [15]) and also because it allows us to interpret the extensive numerical evidence in support of Solomon's conjecture in [21] as evidence for Conjecture 3.12.

Remark 3.15. In a subsequent article we intend to explore connections between Conjecture 3.12 and the very general (conjectural) formalism discussed by Fukaya and Kato in [11].

Proposition 3.16. *Let $(L'/K, S', T', \varepsilon')$ be as in Proposition 2.10.*

(i) *Suppose $S' = S$ and $T' = T$. Then Conjecture 3.12 for $(L'/K, S, T, \varepsilon', j, k)$ implies that for $(L/K, S, T, \varepsilon, j, k)$.*
(ii) *Suppose that $L' = L$ and $\varepsilon' = \varepsilon$. Then Conjecture 3.12 for $(L/K, S, T, \varepsilon, j, k)$ implies that for $(L/K, S', T', \varepsilon, j, k)$.*

Proof. Note that, since $\mu_{p^n} \subset L$, $\mathrm{tw}_{j,k}$ is a $\mathbb{Z}_p[\mathrm{Gal}(L'/L)]$-homomorphism. Note also that $\mathrm{N}^r_{L'/L}(\eta^{\varepsilon'}_{L'/K,S,T}(j)) = \eta^{\varepsilon}_{L/K,S,T}(j)$ and $\mathrm{N}^r_{L'/L}(\eta^{\delta'}_{L'/K,S,T}(j)) = \eta^{\delta}_{L/K,S,T}(j)$ by Proposition 2.10, where $\mathrm{N}^r_{L'/L} = \mathrm{N}^r_{\mathrm{Gal}(L'/L)}$. Assuming Conjecture 3.12 for $(L'/K, S, T, \varepsilon', j, k)$, we have

$$\begin{aligned}\mathrm{tw}_{j,k}(\eta^{\varepsilon}_{L/K,S,T}(j)) &= \mathrm{tw}_{j,k}(\mathrm{N}^r_{L'/L}(\eta^{\varepsilon'}_{L'/K,S,T}(j))) \\ &= \mathrm{N}^r_{L'/L}(\mathrm{tw}_{j,k}(\eta^{\varepsilon'}_{L'/K,S,T}(j))) \equiv \mathrm{N}^r_{L'/L}(\eta^{\delta'}_{L'/K,S,T}(j)) = \eta^{\delta}_{L/K,S,T}(j) \pmod{p^n}.\end{aligned}$$

Hence we have proved claim (i).

Since we have

$$\mathrm{tw}_{k-j}(\delta_{L/K,T'\setminus T}(j)) \equiv \delta_{L/K,T'\setminus T}(k) \pmod{p^n}$$

and

$$\mathrm{tw}_{k-j}\left(\prod_{v\in S'\setminus S}(1-\mathrm{N}v^{-j}\mathrm{Fr}_v^{-1})\right) \equiv \prod_{v\in S'\setminus S}(1-\mathrm{N}v^{-k}\mathrm{Fr}_v^{-1}) \pmod{p^n},$$

claim (ii) follows from the fact that $\mathrm{tw}_{j,k}$ is tw_{k-j}-semilinear, and Propositions 2.10 and 3.11. □

Proposition 3.17. *Suppose that $v \notin S \cup T$ splits completely in L, and assume (7) for both $(L/K, S, T, \varepsilon, j)$ and $(L/K, S, T, \delta, k)$. Then Conjecture 3.12 is valid for $(L/K, S \cup \{v\}, T, \varepsilon, j, k)$.*

Proof. If v is any such place, then p^n divides $\#\kappa(v)^\times = \mathrm{N}v - 1$ (since $\mu_{p^n} \subset L$) and hence also both $(1-\mathrm{N}v^{-j})$ and $(1-\mathrm{N}v^{-k})$. The stated assumptions and Proposition 2.10 therefore imply that

$$\eta^{\varepsilon}_{L/K,S\cup\{v\},T}(j) = (1-\mathrm{N}v^{-j})\eta^{\varepsilon}_{L/K,S,T}(j) \in p^n \cdot {\bigcap}^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))$$

and that

$$\eta^{\delta}_{L/K,S\cup\{v\},T}(k) = (1-\mathrm{N}v^{-k})\eta^{\delta}_{L/K,S,T}(k) \in p^n \cdot {\bigcap}^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-k))$$

and so both sides of the displayed equality in Conjecture 3.12 for $(L/K, S\cup\{v\}, T, \varepsilon, j, k)$ vanish. $\square$

4. Zeta elements and the proof of Theorem 3.10

In this subsection, we interpret generalized Stark elements in terms of the theory of arithmetic zeta elements and use this connection to prove Theorem 3.10.

4.1. **Perfect complexes.** Let $\varepsilon \in \mathbb{Z}_p[G]$ be any idempotent (we do not need to assume Hypothesis 2.2 in this subsection). With Z denoting either $\mathbb{Z}_p$ or $\mathbb{Z}/p^n$ for some natural number n we define an object of $D(Z[G]\varepsilon)$ by setting

$$C^{\varepsilon}_{L,S,T}(j)_Z := Z[G]\varepsilon \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} (R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))[1] \oplus Y_L(-j)[-1]).$$

The properties of these complexes that we use are recorded in the following result.

We write $D^{\mathrm{perf}}(Z[G]\varepsilon)$ for the full triangulated subcategory of $D(Z[G]\varepsilon)$ comprising complexes that are 'perfect' (that is, isomorphic to a bounded complex of finitely generated projective $Z[G]\varepsilon$-modules).

Lemma 4.1. *The following claims are valid for all integers j.*

(i) $C^{\varepsilon}_{L,S,T}(j)_Z$ *belongs to* $D^{\mathrm{perf}}(Z[G]\varepsilon)$ *and is acyclic outside degrees* $-1, 0$ *and* 1.

(ii) *Assume that* $\varepsilon H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))$ *is* $\mathbb{Z}_p$*-free if* $Z = \mathbb{Z}/p^n$ *and that* $\varepsilon \in I_G$ *if* $j = 1$. *Then we have*

$$H^i(C^{\varepsilon}_{L,S,T}(j)_Z) = \begin{cases} 0 & \text{if } i = -1,\\ \varepsilon H^1_T(\mathcal{O}_{L,S}, Z(1-j)) & \text{if } i = 0,\\ \varepsilon H^2_T(\mathcal{O}_{L,S}, Z(1-j)) \oplus \varepsilon(Y_L(-j)\otimes_{\mathbb{Z}_p} Z) & \text{if } i = 1.\end{cases}$$

Furthermore, we have a (non-canonical) isomorphism of $\mathbb{Q}_p[G]$*-modules*

$$\mathbb{Q}_p H^0(C^{\varepsilon}_{L,S,T}(j)_{\mathbb{Z}_p}) \simeq \mathbb{Q}_p H^1(C^{\varepsilon}_{L,S,T}(j)_{\mathbb{Z}_p}).$$

Proof. Since p is odd, it is well-known that $R\Gamma(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))$ belongs to $D^{\mathrm{perf}}(\mathbb{Z}_p[G])$ and is acyclic outside degrees zero, one and two (see, for example, [11, Prop. 1.6.5]). Claim (i) follows from this and the fact that the complex $\bigoplus_{w\in T_L} R\Gamma(\kappa(w), Z(1-j))$ in the triangle (5) belongs to $D^{\mathrm{perf}}(Z[G])$ and is acyclic outside degrees zero and one.

To prove the first assertion of claim (ii) it suffices to show $\varepsilon H^0_T(\mathcal{O}_{L,S}, Z(1-j))$ vanishes under the stated assumptions.

If $j \neq 1$, then $H^0(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$, and hence also, $H^0_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ vanishes. If $\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free, then the exact triangle

$$R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \xrightarrow{p^n} R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \to R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}/p^n(1-j)) \to$$

implies $\varepsilon H^0_T(\mathcal{O}_{L,S}, \mathbb{Z}/p^n(1-j))$ vanishes.

Next, we consider the case when $j = 1$. Recall that we set $T = \emptyset$ in this case (see §3.2). Since $\varepsilon \in I_G$ by assumption, we have

$$\varepsilon H^0_T(\mathcal{O}_{L,S}, Z) = \varepsilon H^0(\mathcal{O}_{L,S}, Z) = \varepsilon \cdot Z = 0.$$

To prove the remaining assertion of claim (ii) we write $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(j))$ for the compactly supported cohomology complex of $\mathbb{Z}_p(j)$ and note that Artin-Verdier duality (as expressed, for example, in [3, (6)]) combines with the triangle (5) to give a canonical exact triangle in $D^{\rm perf}(\mathbb{Z}_p[G])$ of the form

$$\bigoplus_{w \in T_L} R\Gamma(\kappa(w), \mathbb{Z}_p(1-j)) \to C^1_{L,S,T}(j)_{\mathbb{Z}_p} \to R\,\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(j)), \mathbb{Z}_p)[-2] \to .$$

Then, since $C^\varepsilon_{L,S,T}(j)_{\mathbb{Z}_p}$ is acyclic outside degrees zero and one, the final assertion of claim (ii) follows from this triangle and the fact that the $\mathbb{Q}_p[G]$-equivariant Euler characteristics of both $\varepsilon \bigoplus_{w \in T_L} R\Gamma(\kappa(w), \mathbb{Q}_p(1-j))$ and $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Q}_p(j))$ vanish. $\square$

4.2. **Zeta elements.** We quickly review the definition of zeta elements in the context of Conjecture 3.5. To do this we fix notation $L/K, G, p, S, T, j, \varepsilon$ and ε_j as in §2. We often abbreviate $C^\varepsilon_{L,S,T}(j)_{\mathbb{Z}_p}$ to $C^\varepsilon_{L,S,T}(j)$. When $\varepsilon = 1$, we omit it from notations (so we denote $C^1_{L,S,T}(j)$ by $C_{L,S,T}(j)$, for example).

The definition of ε_j combines with Lemma 4.1(ii) to imply $\mathbb{Q}_p[G]\varepsilon_j \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C_{L,S,T}(j)$ is acyclic outside degrees zero and one and that there are canonical isomorphisms

$$\varepsilon_j \mathbb{Q}_p H^i(C_{L,S,T}(j))) \simeq \begin{cases} \varepsilon_j H^1_T(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j)) & \text{if } i = 0, \\ \varepsilon_j \mathbb{Q}_p Y_L(-j) & \text{if } i = 1. \end{cases}$$

Since these $\mathbb{Q}_p[G]\varepsilon_j$-modules are both free of rank r^ε_j there is a canonical 'passage to cohomology' isomorphism of $\mathbb{Q}_p[G]\varepsilon_j$-modules

$$(11) \quad \pi_j : \varepsilon_j \mathbb{Q}_p \mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}(j)) \xrightarrow{\sim} \varepsilon_j \mathbb{Q}_p \left(\bigwedge\nolimits^{r^\varepsilon_j}_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \otimes_{\mathbb{Z}_p[G]} \bigwedge\nolimits^{r^\varepsilon_j}_{\mathbb{Z}_p[G]} Y_L(j) \right).$$

Here we identify $Y_L(j)$ with $Y_L(-j)^*$.

Definition 4.2. The zeta element associated to the data $(L/K, S, T, \varepsilon, j)$ is the unique element $z^\varepsilon_{L/K,S,T}(j)$ of $\varepsilon_j \mathbb{C}_p \mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}(j))$ that satisfies

$$\pi_j(z^\varepsilon_{L/K,S,T}(j)) = \eta^\varepsilon_{L/K,S,T}(j) \otimes \bigwedge\nolimits_{w\in W^\varepsilon_j} w(j),$$

or equivalently,

$$(\mathrm{ev}_L \circ (\lambda_j \otimes \mathrm{id}) \circ \pi_j)(z^\varepsilon_{L/K,S,T}(j)) = \varepsilon_j \theta^*_{L/K,S,T}(j),$$

where ev_L denotes the standard 'evaluation' isomorphism

$$\bigwedge\nolimits^{r^\varepsilon_j}_{\mathbb{C}_p[G]} \mathbb{C}_p Y_L(-j) \otimes_{\mathbb{C}_p[G]} \bigwedge\nolimits^{r^\varepsilon_j}_{\mathbb{C}_p[G]} \mathbb{C}_p Y_L(-j)^* \simeq \mathbb{C}_p[G].$$

4.3. **The proof of Theorem 3.10.** In this section we prove the following results.

Theorem 4.3. *If there exists a $\mathbb{Z}_p[G]\varepsilon$-basis z of $\varepsilon\det_{\mathbb{Z}_p[G]}(C_{L,S,T}(j))$ with $\varepsilon_j z = z^\varepsilon_{L/K,S,T}(j)$, then Conjecture 3.5 is valid.*

Corollary 4.4. *Theorem 3.10 is valid.*

Proof. The first point to note is that the maps that are used in the explicit definition of the isomorphism λ_j given in §2.2 coincide with the maps that occur in the statement of the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}\, L)(j), \mathbb{Z}_p[G]\varepsilon)$ (see [2, Conj. 4(iv)]). This fact is clear if $j \leq 1$ and follows in the case $j > 1$ from the result of Besser recalled in Remark 2.5.

Given this, and our definition of the element $z^\varepsilon_{L/K,S,T}(j)$, the latter conjecture implies the existence of a $\mathbb{Z}_p[G]\varepsilon$-basis of $\varepsilon \cdot \det_{\mathbb{Z}_p[G]}(C_{L,S,T}(j))$ with the property stated in Theorem 4.3. We note that this conjecture is usually formulated without using the set T, but as noted in [5, Prop. 3.4] one can formulate a natural T-modified version of this conjecture, whose validity is independent of the choice of T.

The result of Theorem 3.10(i) now follows directly from Theorem 4.3 and the fact that if L is abelian over $\mathbb{Q}$, then the equivariant Tamagawa number conjecture for $(h^0(\mathrm{Spec}\, L)(j), \mathbb{Z}_p[G])$ is known to be true (by work of the first author and Greither [4], and of Flach [10]).

Theorem 3.10(ii) can be proved by the same method as in [6, Cor. 3.18] by using the Iwasawa main conjecture proved by Wiles. □

The proof of Theorem 4.3 occupies the rest of this section (and is motivated by the argument used to prove [5, Th. 7.5]). We assume that $\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free and that $\varepsilon \in I_G$ if $j = 1$.

We set $\mathcal{A} := \mathbb{Z}_p[G]\varepsilon$, $A := \mathbb{Q}_p[G]\varepsilon$, $W := W^\varepsilon_j$ and $r := r^\varepsilon_j$. We also label (and thereby order) the elements of W as $\{w_i\}_{1\leq i\leq r}$.

Then Lemma 4.1(ii) implies that $C^\varepsilon_{L,S,T}(j)$ is acyclic outside degrees zero and one and we can therefore choose a representative of $C^\varepsilon_{L,S,T}(j)$ of the form $F \xrightarrow{\psi} F$ with F a free $\mathcal{A}$-module with basis $\{b_1, \ldots, b_d\}$ for some sufficiently large integer d so that the natural surjection

$$F \to \mathrm{coker}(\psi) = H^1(C^\varepsilon_{L,S,T}(j)) = \varepsilon H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \oplus \varepsilon Y_L(-j)$$

sends b_i with $1 \le i \le r$ to $\varepsilon \cdot w_i(-j)$ and $\{b_{r+1}, \dots, b_d\}$ to a set of generators of $\varepsilon H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$. See [5, §5.4] for the detail of this construction. Note that the representative chosen in loc. cit. is of the form $P \to F$ with P projective and F free, but in the present case we can identify P with F by Swan's theorem (see [8, (32.1)]). Also, note that the assumption that $\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free is needed here.

We may therefore identify $\det_{\mathcal{A}}(C^{\varepsilon}_{L,S,T}(j))$ with $\bigwedge^d_{\mathcal{A}} F \otimes_{\mathcal{A}} \bigwedge^d_{\mathcal{A}} F^*$. With respect to this identification, any $\mathcal{A}$-basis of $\det_{\mathcal{A}}(C^{\varepsilon}_{L,S,T}(j))$ has the form

$$z_x := x \cdot b_1 \wedge \cdots \wedge b_d \otimes b_1^* \wedge \cdots \wedge b_d^*$$

with $x \in \mathcal{A}^\times$, where we write b_i^* for the $\mathcal{A}$-linear dual of b_i.

Next we write

$$\pi'_j : \mathbb{Q}_p\det_{\mathcal{A}}(C^{\varepsilon}_{L,S,T}(j)) \to \varepsilon_j \mathbb{Q}_p\det_{\mathcal{A}}(C^{\varepsilon}_{L,S,T}(j)) \xrightarrow{\sim} \varepsilon_j {\bigwedge}^r_{\mathbb{Q}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))$$

for the composite homomorphism of A-modules in which the first map is 'multiplication by ε_j' and the second is the composite of the isomorphism π_j in (11) and the isomorphism of A-modules $\varepsilon_j \mathbb{Q}_p {\bigwedge}^r_{\mathbb{Z}_p[G]} Y_L(j) \xrightarrow{\sim} A\varepsilon_j$ that sends the element $\varepsilon_j \cdot w_1(j) \wedge \cdots \wedge w_r(j)$ to ε_j.

Then, with this notation, the argument of [5, Lem. 4.3] implies that

$$\begin{aligned} \pi'_j(z_x) &= (-1)^{r(d-r)} x \left({\bigwedge}_{r<i\le d} \psi_i \right) (b_1 \wedge \cdots \wedge b_d) \\ &= (-1)^{r(d-r)} x \sum_{\sigma \in \mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma) \det(\psi_i(b_{\sigma(k)}))_{r<i,k\le d} b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(r)} \end{aligned} \tag{12}$$

with $\psi_i := b_i^* \circ \psi \in F^*$ for each index i.

In particular, note that the element $({\bigwedge}_{r<i\le d}\psi_i)(b_1 \wedge \cdots \wedge b_d)$ of $\bigwedge^r_{\mathcal{A}} F$ lies in $\varepsilon_j {\bigwedge}^r_{\mathbb{Q}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))$, which is regarded as a submodule of $\mathbb{Q}_p \bigwedge^r_{\mathcal{A}} F$ via the inclusion

$$\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) = H^0(C^{\varepsilon}_{L,S,T}(j)) = \ker \psi \hookrightarrow F. \tag{13}$$

Next we note that the matrix of the endomorphism ψ with respect to the basis $\{b_1, \dots, b_d\}$ of F is $(\psi_i(b_k))_{1\le i,k\le d}$ and that $\psi_i = 0$ for each i with $1 \le i \le r$ since the elements $\{\varepsilon \cdot w_i(-j)\}_{1\le i\le r}$ are an $\mathcal{A}$-basis of $\varepsilon Y_L(-j)$. The matrices $\{\det(\psi_i(b_{\sigma(k)}))_{r<i,k\le d}\}_{\sigma \in \mathfrak{S}_{d,r}}$ are therefore a set of generators of the $\mathcal{A}$-module

$$\mathrm{Fitt}^r_{\mathcal{A}}(H^1(C^{\varepsilon}_{L,S,T}(j))) = \mathrm{Fitt}^0_{\mathcal{A}}(\varepsilon H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))) = \varepsilon \mathrm{Fitt}^0_{\mathbb{Z}_p[G]}(H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))),$$

where the first equality is valid since the $\mathcal{A}$-module $H^1(C^{\varepsilon}_{L,S,T}(j))$ is the direct sum of $\varepsilon H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ and a free module $\varepsilon Y_L(-j)$ of rank r.

Note that the restriction map $F^* \to \varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))^*$ is surjective since the cokernel of (13) is $\mathbb{Z}_p$-free. This fact combines with the equality (12) to imply that

$$(14)\quad \left\{ \Phi(\pi'_j(z_x)) \;\middle|\; \Phi \in \varepsilon \bigwedge\nolimits^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))^* \right\} \\ = \varepsilon \mathrm{Fitt}^0_{\mathbb{Z}_p[G]}(H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))).$$

Now suppose that $\varepsilon_j \cdot z_x = z^{\varepsilon}_{L/K,S,T}(j)$. Then the definition of $z^{\varepsilon}_{L/K,S,T}(j)$ implies $\pi'_j(z_x) = \eta^{\varepsilon}_{L/K,S,T}(j)$ and, given this, the result of Theorem 4.3 follows directly from the equality (14).

5. Some evidence for Conjecture 3.12

In this section, we give a proof of Theorem 3.13. Throughout this section we assume K totally real and L CM.

5.1. **Deligne-Soulé elements, explicit reciprocity and Theorem 3.13(i).** In this subsection we assume $K = \mathbb{Q}$ and $\varepsilon = e^+_j$. In this case we have $r := r^{\varepsilon}_j = \# S_\infty(L)/G = \# S_\infty(\mathbb{Q}) = 1$ (see Example 2.3(i)).

By Proposition 3.16, we may assume $L = \mathbb{Q}(\mu_f)$ with $f \in \mathbb{Z}_{>0}$ such that $f \not\equiv 2 \pmod 4$. Also, we may assume $p^n \mid f$ and that S is equal to the minimal set $\{\infty\} \cup \{\ell \mid f\}$, with ∞ the archimedean place of $\mathbb{Q}$. Finally, we may assume $T = \emptyset$ since p is odd. We note that $\varepsilon \in I_G$ is satisfied when $j = 1$, and that $\varepsilon_j = \varepsilon$ holds when $j \neq 0$ (see Remarks 2.4(i) and 2.6). In the following, we often omit T. (For example, we denote $\eta^{\varepsilon}_{L/\mathbb{Q},S,\emptyset}(j)$ by $\eta^{\varepsilon}_{L/\mathbb{Q},S}(j)$.) For simplicity, we denote $\eta^{e^+_j}_{L/\mathbb{Q},S}(j)$ by $\eta^+_{L/\mathbb{Q},S}(j)$.

Recall that $w \in S_\infty(L)/G$ determines the embedding $\iota_w : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ (see §2.1). We set $\zeta_m := \iota_w^{-1}(e^{2\pi\sqrt{-1}/m})$ for any integer m.

Recall the definition of 'cyclotomic elements' of Deligne-Soulé. First, for a positive integer m, define

$$c_{1-j}(\zeta_f)_m := \mathrm{Cor}_{\mathbb{Q}(\mu_{p^m f})/L}((1-\zeta_{p^m f}) \otimes \zeta_{p^m}^{\otimes(-j)}) \in H^1(\mathcal{O}_{L,S}, \mathbb{Z}/p^m(1-j)).$$

Here we regard $(1-\zeta_{p^m f}) \otimes \zeta_{p^m}^{\otimes(-j)}$ as an element of $H^1(\mathbb{Z}[\mu_{p^m f}, 1/p], \mathbb{Z}/p^m(1-j))$ via the Kummer map

$$\begin{aligned} \mathbb{Z}\left[\mu_{p^m f}, \frac{1}{p}\right]^\times \otimes \mathbb{Z}/p^m(-j) &\to H^1\left(\mathbb{Z}\left[\mu_{p^m f}, \frac{1}{p}\right], \mathbb{Z}/p^m(1)\right) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m(-j) \\ &\simeq H^1\left(\mathbb{Z}\left[\mu_{p^m f}, \frac{1}{p}\right], \mathbb{Z}/p^m(1-j)\right). \end{aligned}$$

The cyclotomic element is defined by the inverse limit

$$c_{1-j}(\zeta_f) := \varprojlim_m c_{1-j}(\zeta_f)_m \in \varprojlim_m H^1(\mathcal{O}_{L,S}, \mathbb{Z}/p^m(1-j)) \simeq H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)).$$

Noting that

$$\eta^+_{L/\mathbb{Q},S}(0) = 2^{-1} \cdot (1-\zeta_f)(1-\zeta_f^{-1}) \in \mathbb{Z}_p\mathcal{O}^\times_{L^+,S} = e^+H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$$

(see Example 2.9 and [27, p.79]), we see by definition that $\mathrm{tw}_{0,j}(\eta^+_{L/\mathbb{Q},S}(0)) = e^+_j c_{1-j}(\zeta_f)_n$. From this, we have

$$\mathrm{tw}_{j,k}(e^+_j c_{1-j}(\zeta_f)) = e^+_k c_{1-k}(\zeta_f)_n$$

for arbitrary integers j and k.

Hence, it is sufficient to show that

$$e^+_j c_{1-j}(\zeta_f) = \eta^+_{L/\mathbb{Q},S}(j).$$

We may assume $j \neq 0$. Suppose first that $j < 0$. In this case, by the definition of $\eta^+_{L/\mathbb{Q},S}(j)$, it is sufficient to show that the image of $e^+_j c_{1-j}(\zeta_f)$ under the isomorphism

$$\lambda_j : e^+_j\mathbb{C}_pH^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \simeq e^+_j\mathbb{C}_pK_{1-2j}(\mathcal{O}_L) \simeq e^+_j\mathbb{C}_pY_L(-j),$$

is $e^+_j\theta^*_{L/\mathbb{Q},S}(j)\cdot w(-j)$. This is a direct consequence of the results of Beilinson and Hüber-Wildeshaus [14, Cor. 9.7] (see also [13, Th. 5.2.1 and 5.2.2]).

Next, suppose $j > 0$. Again, it is sufficient to show that the image of $e^+_j c_{1-j}(\zeta_f)$ under the isomorphism λ_j is $e^+_j\theta^*_{L/\mathbb{Q},S}(j)\cdot w(-j)$. Recalling Remark 2.5, we note that in this case λ_j coincides with the map

$$e^+_j\mathbb{C}_pH^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \overset{\exp^*_p}{\simeq} e^+_j(\mathbb{C}_p\otimes_\mathbb{Q} L)^* \overset{\alpha^*_j}{\simeq} e^+_j\mathbb{C}_pH_L(j)^{+,*} \simeq e^+_j\mathbb{C}_pY_L(-j),$$

where $\exp^*_p$ is the dual exponential map, α^*_j is induced by (1) and the last isomorphism is induced by (2) with the identification $Y_L(j)^* = Y_L(-j)$.

By using the explicit reciprocity law due to Kato [16, Th. 5.12] (see also [17, Chap. II, Th. 2.1.7] and [13, Th. 3.2.6]), we have

$$\exp^*_p(c_{1-j}(\zeta_f)) = \left(x \mapsto -\frac{1}{f^j}\mathrm{Tr}_{L/\mathbb{Q}}(xd_j(\zeta_f))\right) \in L^*,$$

where d_j is the polylogarithmic function

$$d_j(t) := \frac{(-1)^j}{(j-1)!}\mathrm{Li}_{1-j}(t) = \frac{(-1)^j}{(j-1)!}\left(\frac{1}{t}\frac{d}{dt}\right)^{j-1}\left(\frac{t}{1-t}\right).$$

From this and the classical formula

$$e^+_j\theta^*_{L/\mathbb{Q},S}(j) = \frac{1}{4}\left(\frac{2\pi\sqrt{-1}}{f}\right)^j \sum_{1\leq a\leq f,\ (a,f)=1} (d_j(e^{2\pi\sqrt{-1}a/f}) + (-1)^j d_j(e^{-2\pi\sqrt{-1}a/f}))\sigma_a^{-1},$$

where $\sigma_a \in \mathrm{Gal}(L/\mathbb{Q})$ is the automorphism sending ζ_f to ζ_f^a, we see by computation that $\alpha^*_j \circ \exp^*_p(e^+_j c_{1-j}(\zeta_f)) = e^+_j\theta^*_{L/\mathbb{Q},S}(j)\cdot w(-j)$ and this completes the proof.

5.2. **Generalized Kummer congruences and Theorem 3.13(ii).** In the setting of Theorem 3.13(ii) one has $\eta^{e_j^-}_{L/K,S,T}(j) = \theta_{L/K,S,T}(j)$ for any non-positive integer j (see Example 3.9) and $r_j^{e_j^-} = 0$ and the map $\mathrm{tw}_{j,k}$ coincides with the composite homomorphism

$$e_j^- \mathbb{Z}_p[G] \to e_j^- \mathbb{Z}/p^n[G] \overset{\mathrm{tw}_{k-j}}{\to} e_k^- \mathbb{Z}/p^n[G]$$

which sends each $\sigma \in G$ to $\chi_{\mathrm{cyc}}(\sigma)^{k-j}\sigma$. Hence, it is sufficient to show that

$$\mathrm{tw}_j(\theta_{L/K,S,T}(0)) \equiv \theta_{L/K,S,T}(j) \pmod{p^n}$$

for any non-positive integer j.

But, since $\mathrm{tw}_j(\delta_{L/K,T}(0)) \equiv \delta_{L/K,T}(j) \pmod{p^n}$, the above congruence follows directly from the well-known result due to Deligne-Ribet [9] that for any element a of $\mathrm{Ann}_{\mathbb{Z}_p[G]}(H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1)))$ one has $\mathrm{tw}_j(a \cdot \theta_{L/K,S}(0)) \equiv \mathrm{tw}_j(a)\theta_{L/K,S}(j) \pmod{p^n}$.

Remark 5.1. The above argument shows that Conjecture 3.12 constitutes a wide-ranging extension of the classical Kummer congruences. To see this take $K = \mathbb{Q}$, $L = \mathbb{Q}(\mu_{p^n})$, $S = \{\infty, p\}$ and $T = \emptyset$. Then write Δ for the subgroup of G of order $p-1$ and set $e_\Delta := \frac{1}{p-1}\sum_{\sigma\in\Delta}\sigma \in \mathbb{Z}_p[G]$.

Let j, k be odd negative integers such that $j \equiv k \pmod{p^{n-1}(p-1)}$ and $1 - j \not\equiv 0 \pmod{p-1}$. The first condition implies that tw_{j-k} is the identity map, and the second that $e_\Delta H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-j))$ vanishes and hence that $e_\Delta H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free. The same holds for $e_\Delta H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-k))$ since $j \equiv k \pmod{p-1}$.

By Theorem 3.13(ii), we deduce $e_\Delta\theta^*_{L/\mathbb{Q},S}(j) \equiv e_\Delta\theta^*_{L/\mathbb{Q},S}(k) \pmod{p^n}$ and hence $(1-p^{-j})\zeta(j) \equiv (1-p^{-k})\zeta(k) \pmod{p^n}$, where $\zeta(s)$ denotes the Riemann zeta function. This is exactly the formulation of Kummer's congruence.

5.3. **Solomon's Congruence Conjecture and Theorem 3.13(iii).** In this subsection we first review the explicit statement of Solomon's Conjecture and then prove Theorem 3.13(iii).

5.3.1. To review Solomon's conjecture we set $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \prod_{w\in S_p(L)} L_w$ and $U^1_{L_p} := (\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \prod_{w\in S_p(L)} U^1_{L_w}$, where $U^1_{L_w}$ denotes the group of principal units of L_w. As in §2.2.3, we denote by $\Gamma_{L,S}$ the Galois group of the maximal abelian pro-p extension of L unramified outside S. We denote $\mathrm{Gal}(L^+/K)$ by G^+. We write $S_\infty(L)/G = \{w_1, \ldots, w_r\}$. Note that $r = [K : \mathbb{Q}]$.

Note that L/K corresponds to K/k in [25]. In [25, §2.2], a representative $\{\tau_1, \ldots, \tau_r\}$ of the coset space $G_{\mathbb{Q}}/G_K$ is fixed. To fit this choice into our setting, we set

$$\tau_i := \iota_{w_1}^{-1} \circ \iota_{w_i},$$

where each ι_{w_i} is regarded as an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{L}_{w_i} = \mathbb{C}$ (see §2.1). In [25], the algebraic closure of $\mathbb{Q}$ is considered to be in $\mathbb{C}$. This means that an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ is fixed. We take ι_{w_1} for this fixed choice. Also, an embedding $j : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ is chosen in [25, §2.4]. Since we fixed an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$, we take j to be the composition $\overline{\mathbb{Q}} \overset{\iota_{w_1}}{\hookrightarrow} \mathbb{C} \simeq \mathbb{C}_p$.

In [25, Def. 2.14], Solomon defined a map

$$\mathfrak{s}_{L/K,S} \in \mathrm{Hom}_{\mathbb{Z}_p[G]^-}\left(\bigwedge\nolimits^r_{\mathbb{Z}_p[G]^-} U^{1,-}_{L_p}, \mathbb{Q}_p[G]^-\right)$$

by using the zeta value $e^-\theta^*_{L/K,S}(1)$ (we will give the definition in the proof of Proposition 5.2 below). Solomon's Integrality Conjecture [25, IC$(L/K,S,p)$] is equivalent to the containment

$$\mathfrak{s}_{L/K,S} \in \mathrm{Hom}_{\mathbb{Z}_p[G]^-}\left(\bigwedge\nolimits^r_{\mathbb{Z}_p[G]^-} U^{1,-}_{L_p}, \mathbb{Z}_p[G]^-\right).$$

Next, we explain the formulation of Solomon's Congruence Conjecture. Assume that $\mu_{p^n} \subset L$. In [25, §2.3], Solomon constructed a pairing

$$H_{L/K,S,n-1} : \bigcap\nolimits^r_{\mathbb{Z}_p[G^+]} \mathbb{Z}_p\mathcal{O}^\times_{L^+,S} \times \bigwedge\nolimits^r_{\mathbb{Z}_p[G]^-} U^{1,-}_{L_p} \to \mathbb{Z}/p^n[G]^-$$

by using the Hilbert symbol $L^\times_w \times L^\times_w \to \mu_{p^n}$ for p-adic places $w \in S_p(L)$ (see the proof of Proposition 5.2 below).

Assuming the validity of (7) for the data $(L/K, S, \emptyset, e^+, 0)$ (or equivalently, the p-part of the Rubin-Stark Conjecture for the data $(L^+/K, S, \emptyset, S_\infty(K))$, see Example 3.8), Solomon's Congruence Conjecture [25, CC$(L/K,S,p,n-1)$] asserts that for every $u \in \bigwedge^r_{\mathbb{Z}_p[G]^-} U^{1,-}_{L_p}$ one has

$$\mathfrak{s}_{L/K,S}(u) \equiv (-1)^r \chi_{\mathrm{cyc}}(\tau_1 \cdots \tau_r) H_{L/K,S,n-1}(\eta^{e^+}_{L/K,S}(0), u) \pmod{p^n}.$$

(The sign $(-1)^r$ appears here since we use (-1)-times the usual logarithm (see (3))).

5.3.2. We now prove Theorem 3.13(iii). To do this we note that $H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p)^- = \mathrm{Hom}_{\mathrm{cont}}(\Gamma^-_{L,S}, \mathbb{Z}_p)$. The dual of the map $\mathrm{rec}_p : U^1_{L_p} \to \Gamma_{L,S}$ that sends u to $\prod_{w \in S_p(L)} \mathrm{rec}_w(u)$, where rec_w denotes the local reciprocity map at w, therefore induces a homomorphism

$$\begin{aligned} \mathrm{rec}^*_p : \bigcap\nolimits^r_{\mathbb{Z}_p[G]^-} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p)^- &\to \bigcap\nolimits^r_{\mathbb{Z}_p[G]^-} \mathrm{Hom}_{\mathbb{Z}_p}(U^{1,-}_{L_p}, \mathbb{Z}_p) \\ &\simeq \mathrm{Hom}_{\mathbb{Z}_p[G]^-}\left(\bigwedge\nolimits^r_{\mathbb{Z}_p[G]^-} U^{1,-}_{L_p}, \mathbb{Z}_p[G]^-\right)^{\#}, \end{aligned}$$

where for a $\mathbb{Z}_p[G]$-module M we denote by $M^\#$ the module M on which G acts via the involution $\sigma \mapsto \sigma^{-1}$, and the last isomorphism follows from

$$\mathrm{Hom}_{\mathbb{Z}_p}(U^{1,-}_{L_p}, \mathbb{Z}_p) \simeq \mathrm{Hom}_{\mathbb{Z}_p[G]^-}(U^{1,-}_{L_p}, \mathbb{Z}_p[G]^-)^\#;\ f \mapsto \sum_{\sigma \in G} f(\sigma(\cdot))\sigma^{-1}$$

and the definition of r-th exterior bidual. The proof of Theorem 3.13(iii) is thus reduced to the following result.

Proposition 5.2. *One has* $\mathrm{rec}_p^*(\eta^{e^-}_{L/K,S}(1)) = (-1)^r \mathfrak{s}_{L/K,S}$ *and*

$$\mathrm{rec}_p^*(\mathrm{tw}_{0,1}(a)) = \chi_{\mathrm{cyc}}(\tau_1 \cdots \tau_r) H_{L/K,S,n-1}(a, \cdot) \qquad \text{in } \mathrm{Hom}_{\mathbb{Z}_p[G]^-}\Big(\bigwedge\nolimits^r_{\mathbb{Z}_p[G]^-} U^{1,-}_{L_p}, \mathbb{Z}/p^n[G]^-\Big)^{\#}$$

for every $a \in \bigcap^r_{\mathbb{Z}_p[G]^+} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))^+ = \bigcap^r_{\mathbb{Z}_p[G^+]} \mathbb{Z}_p \mathcal{O}^\times_{L^+,S}$.

Proof. We review the definition of $\mathfrak{s}_{L/K,S}$. For an integer i with $1 \leq i \leq r$, we define

$$\log_p^{(i)} : U^{1,-}_{L_p} \to \overline{\mathbb{Q}}_p$$

by $\log_p^{(i)}(u) := \log_p(j(\tau_i u)) = \log_p(\iota_{w_i}(u))$, where $\log_p$ denotes the p-adic logarithm defined on $\{a \in \overline{\mathbb{Q}}_p \mid |a-1|_p < 1\}$. We define

$$\mathrm{Log}_p^{(i)} := \sum_{\sigma \in G} \log_p^{(i)}(\sigma(\cdot))\sigma^{-1} \in \mathrm{Hom}_{\mathbb{Z}_p[G]^-}(U^{1,-}_{L_p}, \overline{\mathbb{Q}}_p[G]^-).$$

Put

$$a^-_{L/K,S} := \iota^{-1}_{w_1}\left(\left(\frac{\sqrt{-1}}{\pi}\right)^r e^- \theta^*_{L/K,S}(1)\right) \in \overline{\mathbb{Q}}[G]^-.$$

We define $\mathfrak{s}_{L/K,S}$ by

$$\mathfrak{s}_{L/K,S} := j(a^-_{L/K,S}) \cdot \mathrm{Log}_p^{(1)} \wedge \cdots \wedge \mathrm{Log}_p^{(r)} \in \mathrm{Hom}_{\mathbb{Z}_p[G]^-}\Big(\bigwedge\nolimits^r_{\mathbb{Z}_p[G]^-} U^{1,-}_{L_p}, \overline{\mathbb{Q}}_p[G]^-\Big)^{\#}.$$

Solomon proved that the image of $\mathfrak{s}_{L/K,S}$ lies in $\mathbb{Q}_p[G]^-$, and that $\mathfrak{s}_{L/K,S}$ is independent of the choice of j (see [25, Prop. 2.16]).

Now we prove the first assertion of the proposition. By the definition of $\eta^{e^-}_{L/K,S}(1)$, it is sufficient to prove that the composition map

$$\bigwedge\nolimits^r_{\mathbb{C}_p[G]^-} \mathbb{C}_p Y_L(1) \xrightarrow{\sim} \bigwedge\nolimits^r_{\mathbb{C}_p[G]^-} (\mathbb{C}_p \otimes_{\mathbb{Q}} L)^- \overset{\exp_p}{\xrightarrow{\sim}} \mathbb{C}_p \bigwedge\nolimits^r_{\mathbb{Z}_p[G]^-} U^{1,-}_{L_p} \xrightarrow{(-1)^r \mathfrak{s}_{L/K,S}} \mathbb{C}_p[G]^-$$

coincides with $e^-\theta^*_{L/K,S}(1) \cdot w_1(1)^* \wedge \cdots \wedge w_r(1)^*$ in $\mathrm{Hom}_{\mathbb{C}_p[G]^-}(\bigwedge^r_{\mathbb{C}_p[G]^-} \mathbb{C}_p Y_L(1), \mathbb{C}_p[G]^-)^{\#}$, where the first map is induced by

$$\mathbb{C}_p Y_L(1) \overset{(2)}{\simeq} \mathbb{C}_p H_L(1)^+ \overset{(1)}{\simeq} (\mathbb{C}_p \otimes_{\mathbb{Q}} L)^-,$$

and the second by the p-adic exponential map. We denote by β and α the maps induced by (2) and (1) respectively.

Noting that the equality $(-1)^r j(a^-_{L/K,S}) = (\pi\sqrt{-1})^{-r} e^- \theta^*_{L/K,S}(1)$ holds in $\mathbb{C}_p[G]^-$ (via the isomorphism $\mathbb{C} \simeq \mathbb{C}_p$), we see that

$$(-1)^r \mathfrak{s}_{L/K,S} = (\pi\sqrt{-1})^{-r} e^- \theta^*_{L/K,S}(1) \cdot \mathrm{Log}_p^{(1)} \wedge \cdots \wedge \mathrm{Log}_p^{(r)}.$$

Hence, it is sufficient to prove that $\iota_{w_i} \circ \alpha \circ \beta(w_i(1)) = \pi\sqrt{-1}$, but this is straightforward to check by definition. Thus we have proved the first assertion of the proposition.

To prove the second assertion we review the definition of $H_{L/K,S,n-1}$. In [25, §2.3], Solomon defined a map $f_u \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{L^+,S}^{\times}, \mathbb{Z}/p^n)$ for any $u \in U_{L_p}^1$ as follows. For each $w \in S_p(L)$, we denote the Hilbert symbol

$$L_w^{\times} \times L_w^{\times} \to \mu_{p^n};\ (x,y) \mapsto \frac{\mathrm{rec}_w(y)x^{1/p^n}}{x^{1/p^n}}$$

by $(x,y)_{w,n}$. The map f_u is then defined by setting $f_u(a) := \sum_{w \in S_p(L)} \xi_1^*((a,u_w)_{w,n})$, where u_w is the w-component of $u \in U_{L_p}^1$ and ξ_1^* is the isomorphism $\mu_{p^n} \xrightarrow{\sim} \mathbb{Z}/p^n$ sending ξ_1 to 1. (For the definition of ξ_i, see (10).)

We define the ring isomorphism

$$\chi_{\mathrm{cyc}}^{\#} : \mathbb{Z}/p^n[G^+] \xrightarrow{\sim} \mathbb{Z}/p^n[G]^-;\ \sigma \mapsto 2^{-1}\sum_{\widetilde{\sigma}} \chi_{\mathrm{cyc}}(\widetilde{\sigma})\widetilde{\sigma}^{-1},$$

where $\widetilde{\sigma} \in G$ runs over the lifts of $\sigma \in G^+$. The pairing $H_{L/K,S,n-1}$ is defined by

$$H_{L/K,S,n-1}(a, u_1 \wedge \cdots \wedge u_r) := 2^r \chi_{\mathrm{cyc}}^{\#}((\widetilde{F}_{u_1} \wedge \cdots \wedge \widetilde{F}_{u_r})(a)),$$

where $\widetilde{F}_{u_i} := \sum_{\sigma \in G^+} \widetilde{f}_{u_i}(\sigma(\cdot))\sigma^{-1}$, and $\widetilde{f}_{u_i} \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{L^+,S}^{\times}, \mathbb{Z}_p)$ is a lift of f_{u_i}.

We compare Solomon's pairing $H_{L/K,S,n-1}$ with the twisting map

$$\begin{aligned}\mathrm{tw}_{0,1} : {\bigcap}^r_{\mathbb{Z}_p[G^+]} \mathbb{Z}_p\mathcal{O}_{L^+,S}^{\times} &= {\bigcap}^r_{\mathbb{Z}_p[G]^+} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))^+ \\ &\to {\bigcap}^r_{\mathbb{Z}/p^n[G]^-} H^1(\mathcal{O}_{L,S}, \mathbb{Z}/p^n)^- = \mathrm{Hom}_{\mathbb{Z}_p[G]^-}\left({\bigwedge}^r_{\mathbb{Z}_p[G]^-} \Gamma_{L,S}^-, \mathbb{Z}/p^n[G]^-\right)^{\#}.\end{aligned}$$

For each i with $1 \leq i \leq r$, we define $h_i : \Gamma_{L,S} \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{L,S}^{\times}, \mathbb{Z}/p^n)$ by

$$h_i(\gamma)(a) := \xi_i^*\left(\frac{\gamma a^{1/p^n}}{a^{1/p^n}}\right),$$

where $\xi_i^* : \mu_{p^n} \xrightarrow{\sim} \mathbb{Z}/p^n$ is defined by $\xi_i \mapsto 1$. Put $H_i := \sum_{\sigma \in G^+} h_i(\sigma(\cdot))\sigma^{-1}$. One checks that

$$\mathrm{tw}_{0,1}(a)(\gamma_1 \wedge \cdots \wedge \gamma_r) = 2^r \chi_{\mathrm{cyc}}^{\#}((\widetilde{H}_1 \wedge \cdots \wedge \widetilde{H}_r)(a)),$$

where $\widetilde{H}_i \in \mathrm{Hom}_{\mathbb{Z}[G^+]}(\mathcal{O}_{L^+,S}^\times, \mathbb{Z}[G^+])$ is a lift of H_i. Note also that for all $a \in \mathcal{O}_{L^+,S}^\times$ and $u \in U_{L_p}^{1,-}$ one has

$$\begin{aligned} h_i(\mathrm{rec}_p(u))(a) &= \xi_i^*\left(\frac{\mathrm{rec}_p(u)a^{1/p^n}}{a^{1/p^n}}\right) = \xi_i^*\left(\prod_{w\in S_p(L)} \frac{\mathrm{rec}_w(u_w)a^{1/p^n}}{a^{1/p^n}}\right) \\ &= \sum_{w\in S_p(L)} \xi_i^*((a,u_w)_{w,n}) = \sum_{w\in S_p(L)} \xi_1^*(\tau_i(a,u_w)_{w,n}) = \chi_{\mathrm{cyc}}(\tau_i)f_u(a). \end{aligned}$$

Hence we have

$$\mathrm{rec}_p^*(\mathrm{tw}_{0,1}(a)) = \chi_{\mathrm{cyc}}(\tau_1\cdots\tau_r)H_{L/K,S,n-1}(a,\cdot)$$

in $\mathrm{Hom}_{\mathbb{Z}_p[G]^-}(\bigwedge^r_{\mathbb{Z}_p[G]^-} U_{L_p}^{1,-}, \mathbb{Z}/p^n[G]^-)^\#$, as required. $\square$

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