On the Galois cohomology of ideal class groups

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Abstract. We use étale cohomology to prove some explicit results on the Galois cohomology of ideal class groups.

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1. Introduction

Let K/k be a finite Galois extension of number fields. In several interesting cases, explicit class-field theoretic computations have been used to describe the cohomology of the ideal class group of the ring of algebraic integers of K as a $\mathrm{Gal}(K/k)$-module. For example, such explicit descriptions play a key role in important recent work of Schoof [6, 7], of Cornacchia and Greither [3] and of Greither [4] and also in work of the second named author [8]. In this note we wish to point out that étale cohomology provides an alternative approach to the derivation of all such explicit descriptions and also allows one to make some new observations that are in much the same spirit. We do this by showing that the étale cohomology of the multiplicative group gives rise to canonical exact four term sequences in which the two outer terms are related to unit groups and ideal class groups and the two inner terms are cohomologically-trivial.

2. Canonical exact sequences

We set $G := \mathrm{Gal}(K/k)$ and let S denote the set of places of k consisting of all archimedean places and all which ramify in K/k. For any intermediate field E of K/k we write $S(E)$ for the set of places of E lying above those in S, $\mathcal{O}_E$ for the ring of algebraic integers in E, $\mathcal{O}_{E,S}$ for the ring of $S(E)$-integers in E, $Y_{E,S}$ for the (left) $\mathrm{Gal}(E/k)$-module $\bigoplus_w \mathbb{Z}$ where w runs over all non-archimedean places in $S(E)$ and $X_{E,S}$ for the kernel of the natural projection homomorphism $Y_{E,S} \to \mathbb{Z}$.

All G-modules are to be regarded as left modules. We set $\mathbb{Z}' := \mathbb{Z}[\frac{1}{2}]$ and for any G-module M we write M' in place of $M \otimes_{\mathbb{Z}} \mathbb{Z}'$. For each such M we also

write $M[0]$ for the complex $C^{\cdot}$ which has $C^0 = M$ and $C^i = 0$ for all $i \neq 0$. We abbreviate 'cohomologically-trivial' to 'c.t.'.
The following observation is key to our approach.

PROPOSITION 2.1. *There exists a canonical complex of G-modules $I^{\cdot}$ with the following properties: each module I^i is c.t. and for each subgroup J of G and each integer a there is a canonical identification*

$$H^a(H^0(J, I^{\cdot})) \cong \begin{cases} (\mathcal{O}_{K^J,S}^{\times})', & \text{if } a = 0 \\ \mathrm{Pic}(\mathcal{O}_{K^J,S})', & \text{if } a = 1 \\ X'_{K^J,S} \otimes_{\mathbb{Z}'} \mathbb{Q}/\mathbb{Z}', & \text{if } a = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For any intermediate field E of K/k we set $U_E := \mathrm{Spec}(\mathcal{O}_{E,S})$. We also set $\mathbb{G}'_m := \mathbb{G}_m \otimes_{\mathbb{Z}} \mathbb{Z}'$, considered as an object of the category $\mathcal{S}(U_k)$ of sheaves on $(U_k)_{\text{ét}}$, and write $\mathbb{G}'_m \to C^{\cdot}(\mathbb{G}'_m)$ for the Godement resolution of $\mathbb{G}'_m$ (cf. [5, Chap. III, Rem. 1.20(c)]). We regard $C^{\cdot}(\mathbb{G}'_m)$ as a complex of objects of $\mathcal{S}(U_k)$, with $C^0(\mathbb{G}'_m)$ placed in degree 0, and so we obtain a complex of G-modules by setting $I^{\cdot} := \Gamma(U_K, C^{\cdot}(\mathbb{G}'_m))$. Then, for each intermediate field E, with $J = \mathrm{Gal}(K/E)$, and each integer a, there is a canonical identification between $H^a_{\text{ét}}(U_E, \mathbb{G}'_m)$ and the cohomology in degree a of the complex $\Gamma(U_E, C^{\cdot}(\mathbb{G}'_m)) = H^0(J, I^{\cdot})$. The stated description of each module $H^a(H^0(J, I^{\cdot}))$ thus follows by means of a standard computation in étale cohomology (cf. [5, Chap. III, Exam. 2.22]).
To show that each module I^i is c.t. we first fix an injective resolution $C^i(\mathbb{G}'_m) \to D^{i\cdot}$ in $\mathcal{S}(U_k)$. Then, since $C^i(\mathbb{G}'_m)$ is flabby, there is a quasi-isomorphism between $\Gamma(U_E, C^i(\mathbb{G}'_m))[0]$ and $\Gamma(U_E, D^{i\cdot})$.
We next note that for each integer $j \geq 0$ the G-module $\Gamma(U_K, D^{ij})$ is injective. To prove this we fix an algebraic closure k^c of k which contains K and use the fact that each G-module M can be regarded as a $\mathrm{Gal}(k^c/k)$-module and hence gives rise to a sheaf M^c on $\mathrm{Spec}(k)_{\text{ét}}$. Further, if $j : \mathrm{Spec}(k) \to U_k$ is the natural morphism, then $M \mapsto j_*(M^c)$ is an exact fully faithful functor from the category of G-modules to $\mathcal{S}(U_k)$ and there is a canonical isomorphism $\mathrm{Hom}_G(M, \Gamma(U_K, D^{ij})) \cong \mathrm{Hom}_{\mathcal{S}(U_k)}(j_*(M^c), D^{ij})$. Hence, the G-module $\Gamma(U_K, D^{ij})$ is injective because the functor $\mathcal{F} \mapsto \mathrm{Hom}_{\mathcal{S}(U_k)}(\mathcal{F}, D^{ij})$ is exact.
The complex $\Gamma(U_K, D^{i\cdot})$ thus gives an injective resolution of the module $I^i = \Gamma(U_K, C^i(\mathbb{G}'_m))$ and so in each degree $a > 0$ one has $H^a(J, I^i) = H^a(H^0(J, \Gamma(U_K, D^{i\cdot}))) = H^a(\Gamma(U_E, D^{i\cdot})) = H^a(\Gamma(U_E, C^i(\mathbb{G}'_m))[0]) = 0$. As this is true for every intermediate field E and every integer $a > 0$, [1, Th. 9] shows that the G-module I^i is indeed c.t. □

REMARK 2.2. With further effort one can prove a version of Proposition 2.1 (and hence of subsequent results in this manuscript) which does not involve inverting 2. To do this one must replace the role of $I^{\cdot} \cong R\Gamma_{\text{ét}}(U_K, \mathbb{G}'_m)$ in the above argument by $\mathrm{Hom}(R\Gamma_{c,\text{ét}}(U_K, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ where $R\Gamma_{c,\text{ét}}(U_K, \mathbb{Z})$ is the cohomology 'with compact support' used in [2, Prop. 3.1]. However, we prefer to avoid such technicalities in the present manuscript.

For any G-module M, any subgroup J of G and any integer a we write $\hat{H}^a(J, M)$ for the corresponding Tate cohomology group.

COROLLARY 2.3. *Let e be any central idempotent of $\mathbb{Z}'[G]$ which annihilates $X'_{K,S}$ and $\wp$ any prime ideal of $\mathcal{O}_K$ which ramifies in K/k. Then $e\,\mathrm{Pic}(\mathcal{O}_{K,S})'$ is equal to the quotient $e\,\mathrm{Pic}(\mathcal{O}_{K,(\wp)})'$ of $e\,\mathrm{Pic}(\mathcal{O}_K)'$ by the subgroup generated by e-times the ideal class of $\wp$ and there exists a canonical exact sequence of G-modules*

$$0 \to e(\mathcal{O}_{K,S}^\times)' \to A \to B \to e\,\mathrm{Pic}(\mathcal{O}_{K,(\wp)})' \to 0 \tag{1}$$

in which A and B are both c.t.. In particular, if J is any subgroup of G and a is any integer, then there exists a canonical isomorphism

$$\hat{H}^{a+2}(J, e(\mathcal{O}_{K,S}^\times)') \cong \hat{H}^a(J, e\,\mathrm{Pic}(\mathcal{O}_{K,(\wp)})'). \tag{2}$$

Proof. Proposition 2.1 implies that there exists a tautological exact sequence

$$0 \to e(\mathcal{O}_{K,S}^\times)' \to eI^0 \xrightarrow{d^0} e\ker(d^1) \to e\,\mathrm{Pic}(\mathcal{O}_{K,S})' \to 0$$

where $d^\cdot$ denotes the differential of $I^\cdot$. Further, the natural exact sequence $eY'_{K,S} \xrightarrow{c} e\,\mathrm{Pic}(\mathcal{O}_K)' \to e\,\mathrm{Pic}(\mathcal{O}_{K,S})' \to 0$ combines with the equality $eX'_{K,S} = 0$ to imply $\mathrm{im}(c)$ is equal to the subgroup of $e\,\mathrm{Pic}(\mathcal{O}_K)'$ that is generated by e-times the ideal class of $\wp$ and so $e\,\mathrm{Pic}(\mathcal{O}_{K,S})' = e\,\mathrm{Pic}(\mathcal{O}_{K,(\wp)})'$.
Now the G-module $A := eI^0$ is a direct summand of I^0 and so is obviously c.t.. To prove that the G-module $B := e\ker(d^1)$ is also c.t. we shall use the brutal truncation $\sigma_{\geq 1}(eI^\cdot)$ (thus, this complex is 0 in degrees less than 1 and agrees with $eI^\cdot$ in all strictly positive degrees). Indeed, it is clear that $B = H^1(\sigma_{\geq 1}(eI^\cdot))$ and that Proposition 2.1 (with J the identity subgroup of G) combines with the equality $eX'_{K,S} = 0$ to imply $H^i(\sigma_{\geq 1}(eI^\cdot)) = 0$ for all $i > 1$. The (-1-shift of the) complex $\sigma_{\geq 1}(eI^\cdot)$ thus constitutes an acyclic resolution of the G-module B. Hence, for each integer $a > 2$ and each subgroup J of $\mathrm{Gal}(K/k)$, one has $H^{a-1}(J, B) = H^a(H^0(J, \sigma_{\geq 1}(eI^\cdot))) = e(H^a(H^0(J, I^\cdot))) = 0$, where the last equality is by Proposition 2.1. When taken in conjunction with [1, Th. 9] this implies that B is c.t., as required.
At this stage we have proved the existence of an exact sequence (1). It thus only remains to note that the isomorphism (2) follows directly by splitting (1) into two short exact sequences and then considering the associated long exact sequences of Tate cohomology for the subgroup J. □

3. SPECIAL CASES

In this section we consider some special cases of Corollary 2.3.

3.1. CM FIELDS. In this subsection we assume that K is a CM field and that k is totally real. We write K^+ for the maximal real subfield of K (so $k \subseteq K^+$) and τ for the (unique) non-trivial element of $\mathrm{Gal}(K/K^+)$, and we assume that every (non-archimedean) place which ramifies in K/k does not split in K/K^+. Then in this case the idempotent $e^- := \frac{1}{2}(1-\tau)$ is central in $\mathbb{Z}'[G]$ (because K^+/k is Galois) and annihilates $Y'_{K,S}$ and hence also $X'_{K,S}$. In addition, it is clear that $e^-\,\mathrm{Pic}(\mathcal{O}_{K,\{\wp\}})' = e^-\,\mathrm{Pic}(\mathcal{O}_K)'$ and that [9, Th. 4.12] implies $e^-(\mathcal{O}_{K,S}^\times)' = \mu'_K$ where μ_K is the torsion subgroup of $K^\times$. From (2) we thus obtain canonical isomorphisms $\hat{H}^{a+2}(J,\mu'_K) \cong \hat{H}^a(J, e^-\,\mathrm{Pic}(\mathcal{O}_K)')$.
This implies, in particular, that if μ'_K ic c.t., then so also is $e^-\,\mathrm{Pic}(\mathcal{O}_K)'$. The latter fact was first proved (in the case G is abelian) by Greither [4, Th. 2.1] as a slight extension of an earlier result of Schoof [6, Th. 2.1(i)] which was itself proved by means of explicit computations in class field theory.

3.2. $X_{K,S} = 0$. In this subsection we assume that there exists a unique non-archimedean place v in S, that the decomposition subgroup of v is equal to G and that the (unique) prime ideal $\wp$ of $\mathcal{O}_K$ that lies above v is principal.
Then $X_{K,S} = 0$ so one can take $e = 1$ in Corollary 2.3, $e\,\mathrm{Pic}(\mathcal{O}_{K,(\wp)})' = \mathrm{Pic}(\mathcal{O}_K)'$ and there is a natural exact sequence $0 \to \mathcal{O}_K^\times \to \mathcal{O}_{K,S}^\times \to \mathbb{Z} \to 0$. Hence, if J is any subgroup of G, then by combining the exact sequence of Tate cohomology (for J) of the latter sequence together with the isomorphisms (2) (with $e = 1$) and the fact that $\hat{H}^i(J,\mathbb{Z})$ vanishes for both $i = -1$ and $i = 1$, one deduces the existence of a canonical exact sequence

$$\begin{aligned} 0 \to \hat{H}^0(J,\mathcal{O}_K^\times)' \to \hat{H}^{-2}(J,\mathrm{Pic}(\mathcal{O}_K))' \to \mathbb{Z}'/|J| \\ \to \hat{H}^1(J,\mathcal{O}_K^\times)' \to \hat{H}^{-1}(J,\mathrm{Pic}(\mathcal{O}_K))' \to 0. \end{aligned}$$

3.3. CYCLOTOMIC FIELDS. In certain cases one can obtain more precise information on the Galois cohomology of $\mathrm{Pic}(\mathcal{O}_K)$ in the setting of §3.2. For example, let p be a fixed odd prime, n a natural number, K_n the maximal real subfield of $\mathbb{Q}(e^{2\pi i/p^n})$ and set $\mathcal{O}_n := \mathcal{O}_{K_n}$, $G_n := \mathrm{Gal}(K_n/\mathbb{Q})$ and $k = \mathbb{Q}$. Then the unique prime ideal of $\mathcal{O}_n$ lying above p is principal with generator $\zeta_n := e^{2\pi i/p^n} + e^{-2\pi i/p^n}$ and so the conditions of §3.2 are satisfied (with $v = p$). In this case one also has a natural isomorphism $\mathcal{O}_{n,S}^\times/C_n \cong \mathcal{O}_n^\times/\mathrm{Cyc}_n$ where C_n is the G_n-submodule of $\mathcal{O}_{n,S}^\times$ generated by ζ_n and Cyc_n is the submodule of $\mathcal{O}_n^\times$ generated by $\{g(\zeta_n)/\zeta_n : g \in G_n\}$. Now G_n is cyclic and so its Tate cohomology has period 2. In addition, the G_n-module C_n is isomorphic to $\mathbb{Z}[G_n]$ (via $\zeta_n \mapsto 1$) and hence is c.t. and so Corollary 2.3 (with $e = 1$) implies the existence, for each subgroup J of G_n and each integer a, of an isomorphism of G_n/J-modules

$$\hat{H}^a(J,\mathcal{O}_n^\times/\mathrm{Cyc}_n)' \cong \hat{H}^a(J,\mathcal{O}_{n,S}^\times/C_n)' \cong \hat{H}^a(J,\mathcal{O}_{n,S}^\times)' \cong \hat{H}^a(J,\mathrm{Pic}(\mathcal{O}_n))'.$$

We thereby obtain an alternative proof of the results of Schoof [7, Prop. 5.1(ii)] and of Cornacchia and Greither [3, Prop. 11].

Further, if $J \subseteq G_1$, and Cl_∞ and $(E/\mathrm{Cyc})_\infty$ denote the inverse limits over n of $\mathrm{Pic}(\mathcal{O}_n)$ and $\mathcal{O}_n^\times/\mathrm{Cyc}_n$ respectively (each with respect to the natural field theoretic norm maps), then by taking the inverse limit of the above displayed isomorphisms one obtains an isomorphism of $\varprojlim_n \mathbb{Z}[\mathrm{Gal}(K_n/K_1)]$-modules

$$H^a(J, (E/\mathrm{Cyc})_\infty)' \cong H^a(J, \mathrm{Cl}_\infty)'.$$

This isomorphism recovers the main result (Corollary 1.4) of the second named author in [8]. We remark that the latter result was proved in loc. cit. by means of rather lengthy computations in class field theory.

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