ON NON-ABELIAN ZETA ELEMENTS FOR $\mathbb{G}_m$

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ABSTRACT. We introduce explicit non-commutative generalizations of several natural constructions in commutative algebra including the notions of determinant modules of perfect complexes and of higher Fitting invariants of finitely generated modules.

We then use these constructions to define and study a natural notion of 'non-abelian zeta element' for the multiplicative group $\mathbb{G}_m$ over finite Galois extensions of global fields, thereby extending the theory for abelian extensions that we developed in earlier joint work with Kurihara.

In particular, we formulate a precise and explicit conjectural link between these non-abelian zeta elements and the non-commutative determinant modules of certain natural 'Weil-étale cohomology' complexes. We prove that this conjecture is equivalent to a special case of the equivariant Tamagawa number conjecture, and also show that it is a consequence, in the relevant cases, of either the p-adic Stark conjecture of Serre and Tate or the p-adic Gross-Stark conjecture. We show that these connections, and our earlier work with Kurihara, lead to unconditional proofs of our conjecture for several important classes of number fields, including absolutely abelian fields and some classes of non-abelian Galois extensions of degree divisible by a prime p at which the relevant p-adic L-series possess trivial zeroes. In addition, we give an unconditional proof of the analogous statement for global function fields and a proof, in general modulo a standard vanishing conjecture on μ-invariants and in some interesting cases unconditionally, of a natural p-adic analogue of our conjecture.

In another direction, we show that our central conjecture and its p-adic analogue entail very detailed information about the arithmetic properties of generalized (non-abelian) Stark and p-adic Stark elements. These properties include explicit families of integral congruence relations between canonical 'non-abelian Rubin-Stark elements' and between canonical 'non-abelian p-adic Gross-Rubin-Stark elements' that we define (which both refine and extend recent conjectures in the abelian case of Mazur and Rubin and of the second author) and in addition explicit formulas in terms of these elements for the non-commutative higher Fitting invariants of the integral Selmer groups of $\mathbb{G}_m$.

In this way we obtain both a clear and very general approach to, and a notable refinement of, many aspects of the existing theories of refined Stark and p-adic Stark conjectures.

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1. Introduction

In this article we shall formulate and discuss strongly refined versions of Stark's seminal conjectures on the algebraic properties of the values at zero of Artin L-series.

The general approach that we use is a natural extension of that introduced in the context of abelian L-series in our earlier joint work with Kurihara [11, 12].

In particular, an essential aspect of this approach is the (unconditional) definition of natural notions of 'non-abelian zeta element', of 'Selmer group' and of 'Weil-étale cohomology complex' associated to the multiplicative group $\mathbb{G}_m$ over finite Galois extensions of global fields.

A further key feature of the theory developed here is the introduction of explicit generalizations of several classical notions of commutative algebra to a natural non-commutative setting.

These generalizations may themselves be of some independent interest and include, perhaps most notably, an elementary and seemingly natural theory of non-abelian determinant modules of perfect complexes (which avoids any use of either relative algebraic K-theory, of Deligne's theory of virtual objects or of the theory of localized Whitehead groups of Fukaya and Kato) and a natural theory of the higher non-commutative Fitting invariants of finitely generated modules.

Having introduced these notions, our central conjecture is stated as Conjecture 9.2 and simply asserts that, for any finite Galois extension L/K of global fields, the canonical non-abelian zeta element of $\mathbb{G}_m$ with respect to L/K is a 'locally-primitive basis' of the non-abelian determinant module of the Weil-étale cohomology complex of $\mathbb{G}_m$ with respect to L/K.

Despite the straightforward nature of this prediction we are able to show that it is equivalent to the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[G])$, a conjecture which we note has hitherto only ever been discussed using the rather involved formalism of relative algebraic K-theory and virtual objects. (We also remark in passing that the algebraic techniques introduced here can be used to give a similarly explicit reinterpretation of both main conjectures in non-commutative Iwasawa theory and the general case of the equivariant Tamagawa number conjecture, and thereby to the derivation of a wide range of explicit arithmetic results and predictions in much greater generality than is discussed here, and that these aspects of the theory will be discussed elsewhere.)

To describe direct links between Conjecture 9.2 and previously formulated refinements of Stark's conjecture we find it convenient to introduce a natural notion of 'higher non-abelian Stark element' which, in turn, we can show specializes to give a natural notion of 'non-abelian Rubin-Stark element'.

In particular, by these means we can show that Conjecture 9.2 extends all of the conjectures that were formulated for abelian L-series in our earlier work with Kurihara [11] and therefore simultaneously refines, extends and provides a seemingly definitive version of, the conjectures that are formulated by the first author in [7].

We also mention that our approach leads to concrete improvements of several earlier results in this context. For example, it allows us to remove an important technical hypothesis (concerning the cohomological-triviality of roots of unity) from the main result of [7] and also to greatly simplify the proof of the latter result.

Using the connections discussed above we are able to deduce that, upon appropriate specialization, Conjecture 9.2 incorporates natural non-abelian generalizations of, amongst other things, the Rubin-Stark Conjecture (from [44]), the congruences for derivatives of L-series that were formulated (independently) by Mazur and Rubin in [39] and by the second author in [45] and the annihilation results that are proved by Rubin in [43].

We can also show that, at the same time, Conjecture 9.2 predicts explicit formulas for the higher (non-commutative) Fitting invariants of the Selmer groups of $\mathbb{G}_m$ over arbitrary finite Galois extensions L/K of global fields and incorporates strongly refined versions of the conjectures studied both by Stark in [47, 48] and by Chinburg in [14].

Having established its equivalence to a special case of the equivariant Tamagawa number conjecture, previous work of several authors leads directly to the verification of Conjecture 9.2 for several important classes of fields, including both all absolutely abelian fields (as first proved in [11]) and all global fields of positive characteristic.

In addition, we are able to provide important 'new' evidence in support of Conjecture 9.2 by combining results of the first author in [9] with the approach developed in the joint work with Kurihara [12] and results of Darmon, Dasgupta and Pollack [17], and of Ventullo [51], to prove the conjecture for several classes of fields in the technically difficult case of non-abelian CM Galois extensions of totally real fields of degree divisible by a prime p at which the associated p-adic L-series possess trivial zeroes.

As a further application of our general approach, in the final part of the article we restrict to the setting of finite CM Galois extensions of totally real fields and define a natural generalization of the 'p-adic Gross-Rubin-Stark elements' that were introduced (in the setting of abelian extensions) by the first author in [9] and a natural p-adic analogue of the notion of non-abelian zeta elements for $\mathbb{G}_m$.

We use these elements to develop a natural analogue of the above theory in which the roles of Dirichlet regulators and Artin L-series are respectively replaced by Gross's p-adic regulators and the Deligne-Ribet p-adic Artin L-series of totally even p-adic characters (as discussed by Greenberg in [25]). We also prove that the central conjecture of this p-adic theory is valid modulo Iwasawa's conjecture on the vanishing of cyclotomic μ-invariants and even, for some interesting families of extensions, unconditionally.

In a little more detail, the main contents of this article is as follows. In Part I (comprising §2-§6) we introduce natural non-commutative generalizations of relevant constructions in commutative algebra. This entails, amongst other things, defining canonical integral structures on the reduced exterior powers of finitely generated modules over semisimple algebras and introducing natural notions of 'non-abelian Rubin lattice', of 'locally-free pre-envelope' (in the general sense of Enochs [20]), of non-commutative 'Fitting lattices' and higher non-commutative Fitting invariants and a natural, and explicit, notion of non-abelian determinant modules of perfect complexes. In Part II (comprising §7 and §8) we establish the general arithmetic setting in which we shall apply these generalized constructions and, in particular, define both canonical Selmer groups for $\mathbb{G}_m$ and the notion of higher non-abelian Stark element and use class field theoretic techniques to prove that the unit groups of global fields possess canonical families of locally-free pre-envelopes, to define the relevant Weil-étale cohomology complexes and to introduce a natural notion of 'non-abelian reciprocity map' (which extends the reciprocity maps that have been independently introduced in the abelian case by Mazur and Rubin in [39] and by the second author in [45]). In Part III (comprising §9-§13) we introduce the key notion of 'non-abelian zeta element', state our central conjecture (Conjecture 9.2) regarding these elements, derive a wide range of concrete consequences of this conjecture, state the main supporting evidence that we can offer in support of our central conjecture and then in the remainder of the article prove all of these results. Finally, in Part IV (comprising §14-§16) we develop a precise p-adic analogue of the above theory in which the roles of Artin L-series and Dirichlet regulators are played by p-adic Artin L-series and the p-adic regulators of Gross respectively.

This article constitutes a natural continuation of earlier joint work with Masato Kurihara and both authors are extremely grateful to him for his generous encouragement and for many insightful discussions. It is also a great pleasure for the first author to thank Dick Gross, John Tate and Cornelius Greither for discussions concerning this project. In addition, the authors are grateful to Alice Livingstone Boomla and Andreas Nickel for their comments on an earlier version of this article.

Part I: Non-commutative Algebra

2. Exterior powers

In this section we discuss the basic properties of a natural construction of non-commutative exterior powers.

2.1. **Commutative exterior powers.** In this section we quickly review the basic theory of exterior powers over commutative rings.

Let R be a commutative ring, and M be an R-module. Then for every positive integer r, an element $f \in \mathrm{Hom}_R(M, R)$ induces the homomorphism

$$\bigwedge\nolimits_R^r M \to \bigwedge\nolimits_R^{r-1} M$$

which is defined by

$$m_1 \wedge \cdots \wedge m_r \mapsto \sum_{i=1}^{r} (-1)^{i+1} f(m_i) m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_r.$$

This homomorphism is also denoted by f. Using this construction, we define the following pairing:

$$\bigwedge\nolimits_R^r M \times \bigwedge\nolimits_R^s \mathrm{Hom}_R(M, R) \to \bigwedge\nolimits_R^{r-s} M; \quad (m, \wedge_{i=1}^{i=s} f_i) \mapsto f_s \circ \cdots \circ f_1(m),$$

where r and s are non-negative integers with $r \geq s$. We then set

$$(\wedge_{i=1}^{i=s} f_i)(m) := (f_s \circ \cdots \circ f_1)(m).$$

We shall also use the following convenient notation: for any natural numbers r and s with $s \leq r$ we write $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]$ for the subset of S_r comprising permutations σ which satisfy both

$$\sigma(1) < \cdots < \sigma(s) \quad \text{and} \quad \sigma(s+1) < \cdots < \sigma(r).$$

We can now record two results which will play an important role in the sequel.

Lemma 2.1. *For all $m_1, m_2, \cdots, m_r$ in M and $f_1, f_2, \cdots, f_s$ in $\mathrm{Hom}_R(M, R)$ one has*

$$(\wedge_{i=1}^{i=s} f_i)(\wedge_{j=1}^{j=r} m_j) = \sum_{\sigma \in \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]} \mathrm{sgn}(\sigma) \det(f_i(m_{\sigma(j)}))_{1 \leq i,j \leq s} m_{\sigma(s+1)} \wedge \cdots \wedge m_{\sigma(r)}.$$

In particular, if $r = s$, then we have

$$(\wedge_{i=1}^{i=r} f_i)(\wedge_{j=1}^{j=r} m_j) = \det(f_i(m_j))_{1 \leq i,j \leq r}.$$

Proof. This is verified by means of an easy and explicit computation. □

Lemma 2.2. *Let E be a field, and A be an n-dimensional E-vector space. Consider the E-linear map*

$$\Phi : A \longrightarrow E^{\oplus m},$$

where $\Phi = \bigoplus_{i=1}^m \varphi_i$ with $\varphi_1, \ldots, \varphi_m \in \mathrm{Hom}_E(A, E)$ $(m \leq n)$, then we have

$$\mathrm{im}(\bigwedge\nolimits_{1 \leq i \leq m} \varphi_i : \bigwedge\nolimits_E^n A \longrightarrow \bigwedge\nolimits_E^{n-m} A) = \begin{cases} \bigwedge_E^{n-m} \ker(\Phi), & \text{if } \Phi \text{ is surjective}, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Suppose first that Φ is surjective. Then there exists a subspace $B \subset A$ such that $A = \ker \Phi \oplus B$ and Φ maps B isomorphically onto $E^{\oplus m}$. We see that $\bigwedge_{1 \leq i \leq m} \varphi_i$ induces an isomorphism

$$\bigwedge\nolimits_E^m B \xrightarrow{\sim} E.$$

Hence we have an isomorphism

$$\bigwedge_{1\leq i\leq m}\varphi_i : \bigwedge_E^n A = \bigwedge_E^{n-m} \ker\Phi \otimes_E \bigwedge_E^m B \xrightarrow{\sim} \bigwedge_E^{n-m} \ker\Phi.$$

In particular, we have

$$\mathrm{im}(\bigwedge_{1\leq i\leq m}\varphi_i : \bigwedge_E^n A \longrightarrow \bigwedge_E^{n-m} A) = \bigwedge_E^{n-m} \ker\Phi.$$

Next, suppose Φ is not surjective. Then $\varphi_1, \dots, \varphi_m \in \mathrm{Hom}_E(A, E)$ are linearly dependent. Hence we have $\bigwedge_{1\leq i\leq m}\varphi_i = 0$. $\square$

2.2. **Morita theory.** In this subsection, as a preliminary to subsequent subsections, we review some facts from Morita theory ([16]), restricting in an important special case.

Let E be a field, and fix a d-dimensional E-vector space V. Set $A := \mathrm{End}_E(V)$, then V has a natural structure of left A-module. Define the dual of V by $V^* := \mathrm{Hom}_E(V, E)$, then V^* has a structure of right A-module, given by

$$(v^* \cdot a)(v) := v^*(a \cdot v),$$

where $a \in A$, $v^* \in V$, and $v \in V$. We define the pairings

$$(\cdot,\cdot)_E : V^* \times V \to E,$$
$$(\cdot,\cdot)_A : V \times V^* \to A,$$

by

$$(v^*, v)_E := v^*(v),$$
$$(v, v^*)_A(v') := v^*(v') \cdot v,$$

where $v, v' \in V$ and $v^* \in V^*$. The pairing $(\cdot,\cdot)_E$ (resp. $(\cdot,\cdot)_A$) induces an isomorphism of E-vector spaces (resp. two-sided A-modules):

$$V^* \otimes_A V \xrightarrow{\sim} E$$
$$(\text{resp. } V \otimes_E V^* \xrightarrow{\sim} A).$$

The functor $V^* \otimes_A \cdot$ from the category of left A-modules to that of E-vector spaces gives an equivalence of categories. We call this "Morita functor".

2.3. **Semisimple rings.** We now review some basic facts about semisimple rings. In the sequel we write $\zeta(A)$ for the centre of a ring A.

For a ring A, a nonzero left A-module M is called simple if M has no nonzero proper submodules. M is called semisimple if it is a direct sum of simple modules. A is called semisimple if every nonzero left A-module is semisimple. A is called simple if it has no nonzero proper two-sided ideals. A is artinian if the left ideals of A satisfy the descending chain condition. It is known that every simple artinian ring is semisimple. It is also known that every semisimple ring A is decomposed as a finite direct sum of simple artinian rings $A_1, \dots, A_k$:

$$A \simeq \bigoplus_{i=1}^{i=k} A_i.$$

This decomposition is unique up to isomorphism. The converse also holds: a ring which is isomorphic to a finite direct sum of simple artinian rings is semisimple.

Let A be a simple artinian ring. Fix a simple left A-module M, and put $D := \mathrm{End}_A(M)$. Then D is a division ring. Since every simple left A-module is isomorphic to M, D does not depend on M up to isomorphism. We have a canonical ring isomorphism

$$A \xrightarrow{\sim} \mathrm{End}_D(M); \quad a \mapsto (m \mapsto am).$$

(This can be proved by a more general version of Morita theory which we described in §2.2.) Set $F := \zeta(D)$. Then F is a field, and canonically isomorphic to $\zeta(A)$. An extension field E of F is called splitting field of A if $D \otimes_F E$ is isomorphic to a matrix ring $\mathrm{M}_m(E)$ for some m. If E is such a field, then we say "E splits A". If F splits A, or equivalently, if $D = F$, then we say 'A is split'. There always exists a splitting field E of A, and one can take the extension E/F finite and separable. The integer m does not depend on the choice of E, and is called Schur index of A. If E is a splitting field of A, then we have an isomorphism

$$A \otimes_F E \simeq \mathrm{End}_D(M) \otimes_F E \simeq \mathrm{M}_n(D^{\mathrm{op}}) \otimes_F E \simeq \mathrm{M}_n(\mathrm{M}_m(E)) = \mathrm{M}_{nm}(E),$$

where n is the dimension of the left D-vector space M. By this isomorphism, we can embed A into $\mathrm{M}_{nm}(E)$. The reduced norm

$$\mathrm{Nrd}_A : A \to F$$

is defined by $\mathrm{Nrd}_A(a) := \det(a)$, where $a \in A$ is regarded as an element of $\mathrm{M}_{nm}(E)$. One checks that $\det(a)$ is in F, and does not depend on the choice of the splitting field E, so Nrd_A is well-defined.

Another description of the reduced norm is as follows. Take a splitting field E of A, and a simple left $A \otimes_F E$-module V. One sees that left multiplication gives an isomorphism

$$A \otimes_F E \xrightarrow{\sim} \mathrm{End}_E(V).$$

Embedding A into $\mathrm{End}_E(V)$ by this isomorphism, and taking determinant on $\mathrm{End}_E(V)$, we get the reduced norm, which coincides with the definition above.

We extend the reduced norm to $\mathrm{End}_A(M)$ for arbitrary finitely generated left A-module M. We define

$$\mathrm{Nrd}_{\mathrm{End}_A(M)} : \mathrm{End}_A(M) \to F$$

by

$$\mathrm{End}_A(M) \to \mathrm{End}_{A\otimes_F E}(M \otimes_F E) \xrightarrow{\sim} \mathrm{End}_E(V^* \otimes_{(A\otimes_F E)} (M \otimes_F E)) \xrightarrow{\det} E,$$

where the second arrow is induced by the Morita functor. One checks that the image of this map is in F. If $M = A$ and identify A^{op} with $\mathrm{End}_A(A)$, then $\mathrm{Nrd}_{\mathrm{End}_A(A)}$ coincides with the reduced norm $\mathrm{Nrd}_{A^{\mathrm{op}}}(= \mathrm{Nrd}_A)$ defined before.

Since every semisimple ring is decomposed as a finite direct sum of simple artinian rings, the above construction of the reduced norm for simple artinian rings is extended to semisimple rings.

We note that the above construction of the reduced norm for semisimple rings A induces a reduced norm on $K_1(A)$ (which we also denote by Nrd_A):

$$\mathrm{Nrd}_A : K_1(A) \to F^\times$$

(cf. [16, §45A]).

2.4. **Non-commutative exterior powers: definitions.** In this subsection, we construct exterior powers over non-commutative rings.

The general idea is as follows. For a non-commutative ring A, suppose that there exists a functor Φ from the category of A-modules to that of R-modules for some commutative ring R, which leads the equivalence of the categories. Then, for an A-module M, define the exterior power of M over A by $\bigwedge_R^* \Phi(M)$.

If A is a split simple artinian ring, then the Morita functor induces an equivalence between the categories of finitely generated left A-modules and of finite dimensional vector spaces over $\zeta(A)$. This is the key observation of our construction of non-commutative exterior powers.

2.4.1. We start with the basic definition in the case of simple Artinian rings.

Definition 2.3. Let A be a simple Artinian ring. Take a splitting field E of A, and a left simple A_E-module V, where $A_E := A \otimes_{\zeta(A)} E$. For a left A-module M and a non-negative integer r, we define the r-th reduced exterior power of M over A by

$$\bigwedge\nolimits_A^r M := \bigwedge\nolimits_E^{rd} (V^* \otimes_{A_E} M_E),$$

where $d := \dim_E(V)$, $M_E := M \otimes_{\zeta(A)} E$, and $V^* := \mathrm{Hom}_E(V, E)$. We note that this depends on E, but is independent of V up to isomorphism.

Let A, V, M and E be as in the definition above and fix an E-basis $\{v_1, \ldots, v_d\}$ of V. Then for any subset $\{m_i\}_{1 \le i \le r}$ of M we set

$$\wedge_{i=1}^{i=r} m_i := \bigwedge\nolimits_{1 \le i \le r} \left(\bigwedge\nolimits_{1 \le j \le d} v_j^* \otimes m_i \right) \in \bigwedge\nolimits_E^{rd} (V^* \otimes_{A_E} M_E) = \bigwedge\nolimits_A^r M. \tag{1}$$

Here we regard m_i as an element of M_E by identifying m_i with $m_i \otimes 1$ and write $v_1^*, \ldots, v_d^*$ for the basis of V^* that is dual to $v_1, \ldots, v_d$.

We then define the subspace of 'primitive elements' $(\bigwedge_A^r M)^{\mathrm{prim}}$ of $\bigwedge_A^r M$ to be the E-linear span of all elements of the form $\wedge_{i=1}^{i=r} m_i$ with each m_i in M.

2.4.2. To make the analogous constructions for linear duals we write A^{op} for the opposite ring of A.

We note $\mathrm{Hom}_A(M, A)$ has a natural structure as left A^{op}-module and we can consider the exterior power over A^{op}. We also note that V^* is a simple left A_E^{op}-module, and that its dual V^{**} is canonically isomorphic to V. In this case, the definition above therefore gives

$$\bigwedge\nolimits_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M, A) = \bigwedge\nolimits_E^{rd} (V \otimes_{A_E^{\mathrm{op}}} \mathrm{Hom}_{A_E}(M_E, A_E)).$$

For any subset $\{\varphi_i\}_{1 \le i \le r}$ of $\mathrm{Hom}_A(M, A)$ we set

$$\wedge_{i=1}^{i=r} \varphi_i := \bigwedge\nolimits_{1 \le i \le r} \left(\bigwedge\nolimits_{1 \le j \le d} v_j \otimes \varphi_i \right) \in \bigwedge\nolimits_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M, A), \tag{2}$$

where each φ_i is regarded as an element of $\mathrm{Hom}_{A_E}(M_E, A_E)$, and, just as above, we define the primitive subspace $(\bigwedge_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M, A))^{\mathrm{prim}}$ of $\bigwedge_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M, A)$ to be the E-linear span of all elements of the form $\wedge_{i=1}^{i=r} \varphi_i$.

One has a natural isomorphism

$$V \otimes_{A_E^{\mathrm{op}}} \mathrm{Hom}_{A_E}(M_E, A_E) \xrightarrow{\sim} \mathrm{Hom}_E(V^* \otimes_{A_E} M_E, E); \quad v \otimes f \mapsto (v^* \otimes m \mapsto v^*(f(m)v))$$

and hence an induced identification

$$\bigwedge\nolimits_{A^{\mathrm{op}}}^{r} \mathrm{Hom}_A(M,A) = \bigwedge\nolimits_{E}^{rd} \mathrm{Hom}_E(V^* \otimes_{A_E} M_E, E).$$

Using this identification, we obtain a pairing

$$\bigwedge\nolimits_{A}^{r} M \times \bigwedge\nolimits_{A^{\mathrm{op}}}^{s} \mathrm{Hom}_A(M,A) \to \bigwedge\nolimits_{A}^{r-s} M \tag{3}$$

by applying the construction in §2.1 for the E-vector space $V^* \otimes_{A_E} M_E$. We denote the image of (m, φ) under this pairing by $\varphi(m)$.

2.4.3. The above constructions extend to general semisimple rings in the obvious way.

Definition 2.4. Let A be a semisimple ring, with corresponding decomposition as a direct sum of simple artinian rings

$$A = \bigoplus_{i=1}^{i=k} A_i. \tag{4}$$

Then for any left A-module M and any non-negative integer r, we define

$$\bigwedge\nolimits_{A}^{r} M := \bigoplus_{i=1}^{i=k} \bigwedge\nolimits_{A_i}^{r} (A_i \otimes_A M),$$

with each exterior power in the direct sum being defined with respect to a fixed choice of splitting field E of A. We note that $\bigwedge_A^r M$ has then a natural structure of $\zeta(A_E)$-module.

After fixing a decomposition (4) one defines componentwise the elements $\wedge_{i=1}^{i=r} m_i$ and $\wedge_{i=1}^{i=r} \varphi_i$ and a corresponding duality pairing (3).

In particular, if $r = s$ then the pairing (3) is non-degenerate and takes values in $\zeta(A_E)$. In this setting the orthogonal complement of $(\bigwedge_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M,A))^{\mathrm{prim}}$ in $\bigwedge_A^r M$ is the subspace

$$(\bigwedge\nolimits_{A}^{r} M)^0 := \{x \in \bigwedge\nolimits_{A}^{r} M : \theta(x) = 0 \quad \text{for all } \theta \in (\bigwedge\nolimits_{A^{\mathrm{op}}}^{r} \mathrm{Hom}_A(M,A))^{\mathrm{prim}}\}.$$

Lemma 2.5.

(i) *If A is commutative, then $(\bigwedge_A^r M)^0$ vanishes.*

(ii) *If A is non-commutative and M is a free A-module of rank t, then $(\bigwedge_A^r M)^0$ vanishes if and only if $r \geq t$.*

Proof. Since the pairing (3) with $s = r$ is non-degenerate the space $(\bigwedge_A^r M)^0$ vanishes if and only if the spaces $\bigwedge_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M,A)$ and $(\bigwedge_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M,A))^{\mathrm{prim}}$ coincide.

If A is commutative, then it is clear that $\bigwedge_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M,A) = (\bigwedge_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M,A))^{\mathrm{prim}}$ and so claim (i) is verified.

To prove claim (ii) we note that if M is a free A-module of rank t, then $\mathrm{Hom}_A(M,A)$ is a free A^{op}-module of rank t and we fix a basis $\{\varphi_i\}_{1 \leq i \leq r}$ of $\mathrm{Hom}_A(M,A)$. We also fix a simple A_E-module V with E-basis $\{v_i\}_{1 \leq i \leq d}$ and note that $V \otimes_{A_E^{\mathrm{op}}} \mathrm{Hom}_{A_E}(M_E, A_E)$ has as an E-basis the set $\{v_i \otimes \varphi_j\}_{1 \leq i \leq d, 1 \leq j \leq t}$.

In particular, if $r > t$, then each space $\bigwedge_E^{rd}(V \otimes_{A_E^{\mathrm{op}}} \mathrm{Hom}_{A_E}(M_E, A_E))$ vanishes and hence also both $\bigwedge_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M,A)$ and $(\bigwedge_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M,A))^{\mathrm{prim}}$ vanish.

If $r = t$, then the space $\bigwedge_{A^{\rm op}}^r \mathrm{Hom}_A(M, A)$ is generated over $\zeta(A_E)$ by the single element $\wedge_{i=1}^{i=r}\varphi_i$ and so $\bigwedge_{A^{\rm op}}^r \mathrm{Hom}_A(M, A) = (\bigwedge_{A^{\rm op}}^r \mathrm{Hom}_A(M, A))^{\rm prim}$.

To prove claim (ii) it thus suffices to show that if A is non-commutative and $r < t$, then $\bigwedge_{A^{\rm op}}^r \mathrm{Hom}_A(M, A)$ is strictly bigger than $(\bigwedge_{A^{\rm op}}^r \mathrm{Hom}_A(M, A))^{\rm prim}$.

To do this we note that for any finite set of elements $\{\vartheta_i\}_{1\le i\le r}$ in $\mathrm{Hom}_A(M, A)$ the element $\wedge_{1\le i\le r}\vartheta_i$ is an E-linear combination of elements of the form $\wedge_{1\le a\le rd}(x_a \otimes y_a)$ where each basis element v_i occurs precisely r times in the set $\{x_a\}_{1\le a\le rd}$ and each y_a is an element of $\{\varphi_i\}_{1\le i\le t}$. This property is therefore also satisfied by any element that lies in the E-linear span Λ of such elements.

The claimed result is thus true since if $d > 1$ (as we can always assume if A is non-commutative) and $r < t$, then Λ does not contain the element

$$(\wedge_{1\le b\le d}(x_b \otimes y_b)) \wedge (\wedge_{1\le i\le d} \wedge_{1\le j\le r-1} (v_i \otimes \varphi_j))$$

where we set

$$x_b := \begin{cases} v_1, & \text{if } b \in \{1, 2\}, \\ v_b, & \text{if } 3 \le b \le d, \end{cases} \quad \text{and} \quad y_b = \begin{cases} \varphi_{r+1}, & \text{if } b = 1, \\ \varphi_r, & \text{if } 2 \le b \le d. \end{cases}$$

□

2.5. **Non-commutative exterior powers: basic properties.** In this section we record several basic properties of the exteriors powers constructed above that will be used in the sequel.

Proposition 2.6. *Let A be a semisimple ring and W be a left A-module. Then for all subsets $\{w_i\}_{1\le i\le r}$ of W and $\{\varphi_j\}_{1\le j\le r}$ of $\mathrm{Hom}_A(W, A)$ one has*

$$(\wedge_{i=1}^{i=r}\varphi_i)(\wedge_{j=1}^{j=r}w_j) = \mathrm{Nrd}_{\mathrm{M}_r(A^{\rm op})}((\varphi_i(w_j))_{1\le i,j\le r}).$$

Proof. We may assume that A is simple. Note first that we have a canonical isomorphism

$$A_E \simeq \mathrm{End}_E(V).$$

Since we fixed the E-basis of V, we identify A_E with the matrix ring $\mathrm{M}_d(E)$. By definition, we have

$$(\wedge_{i=1}^{i=r}\varphi_i)(\wedge_{j=1}^{j=r}w_j) = (\bigwedge\nolimits_{1\le i\le r}(\bigwedge\nolimits_{1\le j\le d} v_j \otimes \varphi_i))(\bigwedge\nolimits_{1\le i\le r}(\bigwedge\nolimits_{1\le j\le d} v_j^* \otimes w_i)).$$

We see that $(v_{i'} \otimes \varphi_i)(v_{j'}^* \otimes w_j) = v_{j'}^*(\varphi_i(w_j)v_{i'}) \in E$ is the (j', i')-component of the matrix $\varphi_i(w_j) \in A \subset \mathrm{M}_d(E)$. Hence, regarding $({}^t\varphi_i(w_j))_{1\le i,j\le r} \in \mathrm{M}_{rd}(E)$, where ${}^t\varphi_i(w_j)$ is the transpose of $\varphi_i(w_j) \in \mathrm{M}_d(E)$, we have

$$(\wedge_{i=1}^{i=r}\varphi_i)(\wedge_{j=1}^{j=r}w_j) = \det({}^t\varphi_i(w_j))_{1\le i,j\le r},$$

and the right hand side is equal to $\mathrm{Nrd}_{\mathrm{M}_r(A^{\rm op})}((\varphi_i(w_j))_{1\le i,j\le r})$ by the definition of the reduced norm. □

Remark 2.7. Proposition 2.6 implies both that the value $(\wedge_{i=1}^{i=r}\varphi_i)(\wedge_{j=1}^{j=r}w_j)$ belongs to $\zeta(A)$ and depends only on the elements $w_1, \ldots, w_r$ and homomorphisms $\varphi_1, \ldots, \varphi_r$. This fact is important to the formulation of our conjectures since the definitions (1) and (2) clearly depend on the choice of basis $\{v_j\}_{1\le j\le d}$.

Corollary 2.8. *Let A be a semisimple ring and W a free A-module of rank r. Then there is a canonical isomorphism of $\zeta(A)$-modules*

$$\iota_W : {\bigwedge}^r_{A^{\mathrm{op}}} \mathrm{Hom}_A(W, A) \cong \mathrm{Hom}_{\zeta(A)}({\bigwedge}^r_A W, \zeta(A))$$

with the following property: for any A-basis $\{b_i\}_{1\le i\le r}$ of W one has $\iota_W(\wedge_{i=1}^{i=r} b_i^*)(\wedge_{j=1}^{j=r} b_j) = 1$ where for each index i we write b_i^* for the element of $\mathrm{Hom}_A(W, A)$ that is dual to b_i.

Proof. In this case the pairing (3) with $s = r$ induces a homomorphism of free rank one $\zeta(A)$-modules $\iota_W : \bigwedge^r_{A^{\mathrm{op}}} \mathrm{Hom}_A(W, A) \cong \mathrm{Hom}_{\zeta(A)}(\bigwedge^r_A W, \zeta(A))$.

Both the bijectivity of this pairing and the equality $\iota_W(\wedge_{i=1}^{i=r} b_i^*)(\wedge_{j=1}^{j=r} b_j) = 1$ follow directly from Proposition 2.6. □

Corollary 2.9. *Let A be a semisimple ring and W a free A-module of rank r. Fix an A-basis $\{b_i\}_{1\le i\le r}$ of W. Then for each φ in $\mathrm{End}_A(W)$ one has*

$$\wedge_{i=1}^{i=r}\varphi(b_i) = \mathrm{Nrd}_{\mathrm{End}_A(W)}(\varphi)\cdot(\wedge_{i=1}^{i=r} b_i) \in {\bigwedge}^r_A W.$$

Proof. This formula follows immediately upon comparing the results of Proposition 2.6 and Corollary 2.8. □

Finally we record a natural non-commutative generalization of Lemma 2.2.

Proposition 2.10. *Let A be a semisimple ring and W a free A-module of rank r. For a natural number s with $s \le r$ and a subset $\{\varphi_i\}_{1\le i\le s}$ of $\mathrm{Hom}_A(W, A)$ we consider the map*

$$\Phi := \bigoplus_{i=1}^{i=s} \varphi_i : W \to A^{\oplus s}.$$

If Φ is surjective, then the image of the map

$${\bigwedge}^r_A W \to {\bigwedge}^{r-s}_A W; \quad b \mapsto ({\bigwedge}_{1\le i\le s}\varphi_i)(b)$$

is $\bigwedge^{r-s}_A \ker(\Phi)$.

If Φ is not surjective, then the image of this map vanishes.

Proof. This follows easily by combining Lemma 2.2 with our definition of reduced exterior powers. □

2.6. **Non-commutative exterior powers: descent theory.** The definition of reduced exterior powers involves the choice of splitting fields and so (to obtain a natural theory) one must check that it behaves functorially with respect to field extensions.

To do this we fix a field K and a finite dimensional semisimple K-algebra A (that is, A is both a finite dimensional K-algebra and a semisimple ring). We also fix an embedding of fields $K \to K'$.

We show first that if $\zeta(A)$ is étale over K, then the scalar extension $A' := K' \otimes_K A$ is a semisimple K'-algebra with a Wedderburn decomposition that is induced by that of A.

Proposition 2.11. *Let the decomposition of A by simple artinian rings be given by*

$$A = \bigoplus_{i=1}^{i=k} A_i.$$

Assume that $\zeta(A)$ is étale over K (that is, for each index i the field $F_i := \zeta(A_i)$ is a finite separable extension of K). Consider the (finite) set

$$\Sigma(F_i/K, K') := \{K\text{-embeddings } F_i \hookrightarrow \overline{K'}\}/\sim,$$

where the equivalence relation $\sim$ is defined by

$$\sigma \sim \sigma' \Leftrightarrow \text{there exists } \tau \in \mathrm{Aut}_{K'}(\overline{K'}) \text{ such that } \sigma = \tau \circ \sigma'.$$

Then, for each $1 \leq i \leq k$ and $\sigma \in \Sigma(F_i/K, K')$, $A_i \otimes_{F_i} \sigma(F_i)K'$ is a simple artinian ring with center $\sigma(F_i)K'$, where $\sigma(F_i)K'$ is the composite field of $\sigma(F_i)$ and K' in $\overline{K'}$ (this is independent of a representative of σ), and we have a decomposition of A':

$$A' \simeq \bigoplus_{i=1}^{i=k} \bigoplus_{\sigma \in \Sigma(F_i/K,K')} (A_i \otimes_{F_i} \sigma(F_i)K').$$

In particular, A' is semisimple.

Proof. Since each F_i is separable over K, we have an isomorphism

$$F_i \otimes_K K' \simeq \bigoplus_{\sigma \in \Sigma(F_i/K,K')} \sigma(F_i)K'.$$

Hence we have

$$A_i \otimes_K K' \simeq A_i \otimes_{F_i} (F_i \otimes_K K') \simeq \bigoplus_{\sigma \in \Sigma(F_i/K,K')} (A_i \otimes_{F_i} \sigma(F_i)K').$$

Since A_i is a central simple algebra over F_i, $A_i \otimes_{F_i} \sigma(F_i)K'$ is also a central simple algebra over $\sigma(F_i)K'$. $\square$

Example 2.12. Let G be a finite group. Then $\mathbb{Q}[G]$ is a finite dimensional semisimple $\mathbb{Q}$-algebra. Note that this algebra is étale over $\mathbb{Q}$ since $\mathbb{Q}$ is of characteristic 0. Let K be a local field of characteristic 0. Let $\widehat{G}$ denote the set of $\mathbb{C}$-valued irreducible characters of G. For each $\chi \in \widehat{G}$, define the primitive central idempotent

$$e_\chi := \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma \in \mathbb{C}[G].$$

Define the equivalence relation $\sim_{\mathbb{Q}}$ (resp. $\sim_K$) on $\widehat{G}$ by

$$\chi \sim_{\mathbb{Q}} \text{ (resp. } \sim_K) \ \chi' \Leftrightarrow \text{there exists } \tau \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{C}) \text{ (resp. } \mathrm{Aut}_K(\mathbb{C})) \text{ such that } \chi = \tau \circ \chi,$$

where we fix an isomorphism between the completion of $\overline{K}$ and $\mathbb{C}$, and hence regard K as a subset of $\mathbb{C}$. By standard representation theory, there are decompositions of $\mathbb{Q}[G]$ and $K[G]$ into simple artinian rings:

$$\mathbb{Q}[G] = \bigoplus_{[\chi] \in \widehat{G}/\sim_{\mathbb{Q}}} \mathbb{Q}[G]e_{[\chi]} \quad \text{and} \quad K[G] = \bigoplus_{[\chi] \in \widehat{G}/\sim_K} K[G]e_{[\chi]},$$

where for $[\chi]$ in $\widehat{G}/\sim_{\mathbb{Q}}$, respectively in $\widehat{G}/\sim_K$), we define an element of $\mathbb{Q}[G]$, respectively $K[G]$, by setting $e_{[\chi]} := \sum_{\chi' \in [\chi]} e_{\chi'}$.

Let $[\chi]_{\mathbb{Q}}$ (resp. $[\chi]_K$) denote the equivalence class of χ via the relation $\sim_{\mathbb{Q}}$ (resp. $\sim_K$). The relation $\sim_K$ defines an equivalence relation on $[\chi]_{\mathbb{Q}}$. We see that the map

$$[\chi]_{\mathbb{Q}}/\sim_K \to \Sigma(F_{[\chi]_{\mathbb{Q}}}/\mathbb{Q}, K); \quad [\chi']_K \mapsto (a \mapsto \chi'(a))$$

is bijective, where $F_{[\chi]_{\mathbb{Q}}} := \zeta(\mathbb{Q}[G]e_{[\chi]_{\mathbb{Q}}})$. Therefore, by Proposition 2.11, for every $\chi \in \widehat{G}$ there is a natural isomorphism of algebras $K[G]e_{[\chi]_K} \simeq \mathbb{Q}[G]e_{[\chi]_{\mathbb{Q}}} \otimes_{F_{[\chi]_{\mathbb{Q}}}} F_{[\chi]_{\mathbb{Q}}}K$.

For a left A-module M we set $M' := K' \otimes_K M$. Then, assuming $\zeta(A)$ to be étale over K, we now construct a natural embedding (or 'scalar extension')

$$\bigwedge\nolimits_A^r M \hookrightarrow \bigwedge\nolimits_{A'}^r M'$$

as follows.

By passing to components we may assume that A is simple. We then set $F := \zeta(A)$ and fix both an algebraic extension E of F which splits A and a simple left A_E-module V. Then, by definition, one has

$$\bigwedge\nolimits_A^r M = \bigwedge\nolimits_E^{rd} (V^* \otimes_{A_E} M_E),$$

where $d = \dim_E(V)$. For each $\sigma \in \Sigma(F/K, K')$, where $\Sigma(F/K, K')$ is as in Proposition 2.11, fix a K-embedding $\widetilde{\sigma}$ of E into $\overline{K'}$ which extends σ (such a $\widetilde{\sigma}$ exists since E/F is algebraic). Then $\widetilde{\sigma}(E)K'$ splits the simple ring $A \otimes_F \sigma(F)K'$. For simplicity, set $E_\sigma := \widetilde{\sigma}(E)K'$ and $A_{E_\sigma} := A \otimes_F E_\sigma$. We see that $V_\sigma := V \otimes_E E_\sigma$ is a simple left A_{E_σ}-module, so by Proposition 2.11 and the definition of reduced exterior powers, we have

$$\bigwedge\nolimits_{A'}^r M' = \bigoplus_{\sigma \in \Sigma(F/K, K')} \bigwedge\nolimits_{E_\sigma}^{rd} (V_\sigma^* \otimes_{A_{E_\sigma}} M_{E_\sigma}),$$

where $M_{E_\sigma} := M \otimes_F E_\sigma$. For each $\sigma \in \Sigma(F/K, K')$, there is a canonical embedding

$$V^* \otimes_{A_E} M_E \hookrightarrow V_\sigma^* \otimes_{A_{E_\sigma}} M_{E_\sigma}.$$

This induces an embedding

$$f_\sigma : \bigwedge\nolimits_E^{rd} (V^* \otimes_{A_E} M_E) \hookrightarrow \bigwedge\nolimits_{E_\sigma}^{rd} (V_\sigma^* \otimes_{A_{E_\sigma}} M_{E_\sigma}).$$

We finally define the required scalar extension

$$\bigwedge\nolimits_A^r M = \bigwedge\nolimits_E^{rd} (V^* \otimes_{A_E} M_E) \hookrightarrow \bigoplus_{\sigma \in \Sigma(F/K, K')} \bigwedge\nolimits_{E_\sigma}^{rd} (V_\sigma^* \otimes_{A_{E_\sigma}} M_{E_\sigma}) = \bigwedge\nolimits_{A'}^r M'$$

to be the tuple $\bigoplus_\sigma f_\sigma$.

3. Integral structures

In this section we fix a Dedekind domain R with field of fractions F. We also fix a finite-dimensional semisimple F-algebra A and an R-order $\mathcal{A}$ in A.

For any $\mathcal{A}$-module M we abbreviate the A-module $F \otimes_R M$ to M_F.

We shall extend an idea used (in the commutative case) by Rubin in [44] to introduce, for each finitely generated $\mathcal{A}$-module M and each non-negative integer r, a canonical integral structure on the reduced exterior power $\bigwedge_A^r M_F$.

3.1. **The canonical central order.**

We first introduce a canonical R-submodule of $\zeta(A)$.

Definition 3.1. We write $\xi(\mathcal{A})$ for the R-submodule of $\zeta(A)$ that is generated by the elements $\mathrm{Nrd}_A(M)$ as M runs over all matrices in $\bigcup_{n\geq 0} \mathrm{M}_n(\mathcal{A})$.

The basic properties of this module are described in the following result.

Lemma 3.2.

(i) *$\xi(\mathcal{A})$ is an R-order in $\zeta(A)$.*

(ii) *If $\mathcal{A}$ is commutative, then $\xi(\mathcal{A}) = \zeta(\mathcal{A}) = \mathcal{A}$.*

(iii) *If $\mathcal{A}$ is maximal, then $\xi(\mathcal{A}) \subseteq \zeta(\mathcal{A})$ and $\xi(\mathcal{A})^\times$ has finite 2-power index in $\zeta(\mathcal{A})^\times$.*

Proof. Since the module $\xi(\mathcal{A})$ is clearly a subring of $\zeta(A)$ it is an R-order if and only if it is finitely generated over R. This is true since for any $n > 0$ and any matrix M in $\mathrm{M}_n(\mathcal{A})$ the element $\mathrm{Nrd}_A(M)$ is integral over R. This proves claim (i).

If $\mathcal{A}$ is commutative, then for each matrix M in $\mathrm{M}_n(\mathcal{A})$ one has $\mathrm{Nrd}_A(M) = \det(M) \in \mathcal{A}$. In this case it is therefore clear that $\xi(\mathcal{A})$ is equal to $\mathcal{A} = \zeta(\mathcal{A})$. This proves claim (ii).

Claim (iii) is true because if $\mathcal{A}$ is maximal, then $\zeta(\mathcal{A})$ is equal to the maximal R-order Λ of $\zeta(A)$ and the subgroup of $\Lambda^\times$ that is generated by the set $\{\mathrm{Nrd}_A(M) : M \in \mathrm{GL}_n(\mathcal{A}), n \geq 1\}$ has finite 2-power index (as a consequence of [16, Th. (45.7)]). □

3.2. **Integral structures on non-commutative exterior powers.**

We now define a canonical integral structure on the reduced exterior powers of A-modules. This structure will play a key role in the sequel.

Definition 3.3. Let M be a finitely generated R-torsion-free left $\mathcal{A}$-module. For every non-negative integer r, we define the r-th Rubin lattice of M by setting

$$\bigcap\nolimits_{\mathcal{A}}^r M := \{a \in \bigwedge\nolimits_A^r M_F : (\wedge_{i=1}^{i=r} \varphi_i)(a) \in \xi(\mathcal{A}) \text{ for all } \varphi_1, \ldots, \varphi_r \in \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A})\}.$$

Remark 3.4. If $\mathcal{A}$ is equal to $\mathbb{Z}[G]$ for some abelian group G, then $\xi(\mathbb{Z}[G]) = \mathbb{Z}[G]$ (by Lemma 3.2(ii)) and $\bigcap_{\mathbb{Z}[G]}^r M$ coincides with the lattice defined by Rubin in [44].

The basic properties of such lattices in the general case are recorded in the next result.

Proposition 3.5. *Let M and r be as in Definition 3.3.*

(i) *$\bigcap_{\mathcal{A}}^r M$ contains $(\bigwedge_A^r M_F)^0$, is stable under multiplication by $\xi(\mathcal{A})$ and spans $\bigwedge_A^r M_F$.*

(ii) *The quotient $\bigcap_{\mathcal{A}}^r M/(\bigwedge_A^r M_F)^0$ is finitely generated over R.*

(iii) *If M is a free $\mathcal{A}$-module of rank d with $d \geq r$, then for any choice of basis $b = \{b_1, \ldots, b_d\}$ of M there is a natural surjective homomorphism of $\xi(\mathcal{A})$-modules*

$$\theta_b : {\bigcap}_{\mathcal{A}}^{r} M \to \bigoplus_{\sigma \in \left[{d \atop r}\right]} \xi(\mathcal{A}).$$

This homomorphism splits and is, in addition, bijective if either $\mathcal{A}$ is commutative or $r = d$.

Proof. Claim (i) is clear.

To prove claim (ii) we write $\mathcal{O}$ for the integral closure of R in a splitting field E of A and note that the pairing (3) identifies $({\bigwedge}_{A^{\mathrm{op}}}^{r} \mathrm{Hom}_A(M_F, A))^{\mathrm{prim}}$ with the $\zeta(A_E)$-linear dual of the quotient ${\bigwedge}_{A}^{r} M_F/({\bigwedge}_{A}^{r} M_F)^0$. Since $\xi(\mathcal{A})$ is an R-order in $\zeta(A)$, the finite generation of the R-module ${\bigcap}_{\mathcal{A}}^{r} M/({\bigwedge}_{A}^{r} M_F)^0$ then follows directly from the definition of ${\bigcap}_{\mathcal{A}}^{r} M$ and the fact that the $\xi(\mathcal{A})$-linear span of $\{\wedge_{i=1}^{i=r} \varphi_i : \varphi_i \in \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A})\}$ generates a full $\mathcal{O}$-lattice in $({\bigwedge}_{A^{\mathrm{op}}}^{r} \mathrm{Hom}_A(M_F, A))^{\mathrm{prim}}$.

To prove claim (iii) we define θ_b to be the homomorphism of $\xi(\mathcal{A})$-modules which satisfies

$$\theta_b(x) = ((\wedge_{i=1}^{i=r} b_{\sigma(i)}^*)(x))_{\sigma \in \left[{d \atop r}\right]}$$

for all x in ${\bigcap}_{\mathcal{A}}^{r} M$.

We also write θ_b' for the homomorphism of $\xi(\mathcal{A})$-modules $\bigoplus_{\sigma \in \left[{d \atop r}\right]} \xi(\mathcal{A}) \to {\bigcap}_{\mathcal{A}}^{r} M$ which satisfies

$$\theta_b'((c_\sigma)_\sigma) = \sum_{\sigma \in \left[{d \atop r}\right]} c_\sigma \cdot \wedge_{i=1}^{i=r} b_{\sigma(i)}$$

for all $(c_\sigma)_\sigma$ in $\bigoplus_{\sigma \in \left[{d \atop r}\right]} \xi(\mathcal{A})$.

Then Proposition 2.6 implies that $(b_{\sigma(1)}^* \wedge \cdots \wedge b_{\sigma(r)}^*)(b_{\tau(1)} \wedge \cdots \wedge b_{\tau(r)}) = \delta_{\sigma\tau}$ for all σ and τ in $\left[{d \atop r}\right]$ and so the composite $\theta_b \circ \theta_b'$ is the identity on $\bigoplus_{\sigma \in \left[{d \atop r}\right]} \xi(\mathcal{A})$. This shows that θ_b' is a section to θ_b, as required.

Next we note that if $\mathcal{A}$ is commutative, then ${\bigcap}_{\mathcal{A}}^{r} M = {\bigwedge}_{\mathcal{A}}^{r} M$ (as M is free) and $\xi(\mathcal{A}) = \mathcal{A}$ and using these equalities it is easily seen that θ_b is an isomorphism.

Finally we assume $r = d$ and fix a decomposition $A = \prod_{i \in I} A_i$ and splitting field E as in Definition 2.4. Then in this case each element x of ${\bigwedge}_{A}^{d} M_F$ can be written uniquely as

$$x = (c_i(x) \cdot \wedge_{1 \leq j \leq d}(\wedge_{1 \leq k \leq d_i}(v_{ik})^* \otimes e_i b_j))_{i \in I} = \sum_{i \in I} c_i(x) \cdot \wedge_{j=1}^{j=d} b_j,$$

with each element $c_i(x)$ in $E \subseteq A_{i,E}$, and so $\theta_b(x) = \sum_{i \in I} c_i(x)$.

Thus, if x belongs to $\ker(\theta_b)$, then $\sum_{i \in I} c_i(x) = 0$ and hence also

$$x = \sum_{i \in I} c_i(x) \cdot \wedge_{j=1}^{j=d} b_j = 0,$$

as required to complete the proof of claim (iii). □

Remark 3.6. The lattice ${\bigcap}_{\mathcal{A}}^{r} M$ depends on the choice of E-bases (of the simple A_E-modules V) that occur in the definition (2) of exterior powers. However, Remark 2.7 ensures that this dependence is natural in the following sense. Let $\{\underline{b}_V\}_V$ be any other choice of

E-bases of the modules V and write τ for the automorphism of the $\zeta(A_E)$-module $\bigwedge_A^r M_F$ sending each element $\wedge_{j=1}^{j=r} m_j$ to $\tilde{\wedge}_{j=1}^{j=r} m_j$, where each m_j belongs to M and $\tilde{\wedge}$ indicates that the exterior power is defined with respect to the bases $\{\underline{b}_V\}_V$. Then, writing $\tilde{\bigcap}_{\mathcal{A}}^r M$ for the Rubin lattice that is defined relative to exterior powers with respect to $\{\underline{b}_V\}_V$, Remark 2.7 implies that $\tau(\bigcap_{\mathcal{A}}^r M) = \tilde{\bigcap}_{\mathcal{A}}^r M$.

Remark 3.7. Proposition 3.5(i) combines with Lemma 2.5(ii) to imply that the $\xi(\mathcal{A})$-module $\bigcap_{\mathcal{A}}^r M$ is not always finitely generated. However, under certain natural conditions on $\mathcal{A}$ and M (which are always satisfied in the relevant arithmetic settings) we shall later define a collection of canonical finitely generated $\xi(\mathcal{A})$-submodules $\mathrm{F}_{0_{\mathcal{P}}}(\bigcap_{\mathcal{A}}^r M)$ of $\bigcap_{\mathcal{A}}^r M$ which each coincide, when $\mathcal{A}$ is commutative, with $\bigcap_{\mathcal{A}}^r M$. For more details see §5.2 (and in particular Proposition 5.3(i), (ii) and (vi)).

4. Locally-free Pre-envelopes

In this section we introduce the notion of a 'strict family of locally-free pre-envelopes'. This notion is motivated by the theory developed by Enochs in [20] and will play a key role in the sequel. In particular, in §7 we see that such families occur naturally in arithmetic.

We continue to use the general notation of §3.

4.1. **Locally-free modules.** For each prime ideal $\mathfrak{p}$ of R we write $R_{(\mathfrak{p})}$ and $R_{\mathfrak{p}}$ for the localization and completion of R at $\mathfrak{p}$. For any finite set of prime ideals $\mathcal{P}$ of R we set $R_{\langle\mathcal{P}\rangle} := \bigcap_{\mathfrak{p}} R_{(\mathfrak{p})}$ where the intersection is over all prime ideals of R that do not belong to $\mathcal{P}$.

For each R-module M, each prime ideal $\mathfrak{p}$ of R and each finite set of prime ideals $\mathcal{P}$ of R we set $M_{(\mathfrak{p})} := R_{(\mathfrak{p})} \otimes_R M$, $M_{\mathfrak{p}} := R_{\mathfrak{p}} \otimes_R M$ and $M_{\langle\mathcal{P}\rangle} := R_{\langle\mathcal{P}\rangle} \otimes_R M$. We regard these modules as endowed with natural actions of the algebras $\mathcal{A}_{(\mathfrak{p})}$, $\mathcal{A}_{\mathfrak{p}}$ and $\mathcal{A}_{\langle\mathcal{P}\rangle}$ respectively.

A finitely generated module M over an R-order $\mathcal{A}$ will be said to be 'locally-free' if $M_{(\mathfrak{p})}$ is a free $\mathcal{A}_{(\mathfrak{p})}$-module, or equivalently (as an easy consequence of Maranda's Theorem - see [16, Th. (30.14)]) if $M_{\mathfrak{p}}$ is a free $\mathcal{A}_{\mathfrak{p}}$-module, for all prime ideals $\mathfrak{p}$. For any such module M the rank of the $\mathcal{A}_{(\mathfrak{p})}$-module $M_{(\mathfrak{p})}$ is independent of $\mathfrak{p}$ and will be referred to as the 'rank' $\mathrm{rk}_{\mathcal{A}}(M)$ of M. A locally-free $\mathcal{A}$-module of rank one will often be referred to as an 'invertible' $\mathcal{A}$-module.

Since localization at $\mathfrak{p}$ is an exact functor a locally-free $\mathcal{A}$-module is automatically projective. We record two important examples (that will be much used in the sequel and) for which the converse is also true.

Example 4.1.
(i) If $\mathcal{A}$ is a Dedekind domain, with quotient field E, then every finitely generated torsion-free $\mathcal{A}$-module M is locally-free, with $\mathrm{rk}_{\mathcal{A}}(M)$ equal to the dimension of the E-space spanned by M.

(ii) If G is a finite group for which no prime divisor of $|G|$ is invertible in R and $\mathcal{A} = R[G]$ then, by a theorem of Swan (see, for example, [16, (32.1)]), a finitely generated projective $\mathcal{A}$-module is locally-free. For any such module M the product $\mathrm{rk}_{R[G]}(M) \cdot |G|$ is equal to the dimension of the F-space spanned by M.

4.2. **Families of locally-free pre-envelopes.** Let M be a finitely generated $\mathcal{A}$-module.

4.2.1. By a 'family of locally-free pre-envelopes of M' we shall mean a collection $\mathcal{P}$ of injective homomorphisms of $\mathcal{A}$-modules $\iota : M \to P$ where P is finitely generated locally-free and the following property is satisfied: for any other homomorphism $\iota' : M \to P'$ in $\mathcal{P}$ there exists a commutative diagram of $\mathcal{A}$-modules of the form

(5)
$$\begin{tikzcd} & M \arrow[dl, "\iota"'] \arrow[dr, "\iota'"] & \\ P \arrow[rr, "\kappa_{\iota,\iota'}"] & & P'. \end{tikzcd}$$

We shall say that such a family $\mathcal{P}$ is 'strict' if it has the following two properties:

($\mathcal{P}_1$) for each ι and ι' in $\mathcal{P}$ there exists a diagram as above in which the map $\kappa_{\iota,\iota'}$ is bijective;

($\mathcal{P}_2$) for any (and therefore every) ι in $\mathcal{P}$ the map $\mathrm{Hom}_{\mathcal{A}}(P, \mathcal{A}) \to \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A})$ induced by restriction through ι is surjective.

We write $\mathrm{slfp}_{\mathcal{A}}(M)$ for the set of strict families of locally-free pre-envelopes for the $\mathcal{A}$-module M.

We define the 'rank' $\mathrm{rk}_{\mathcal{A}}(\mathcal{P})$ of a family $\mathcal{P}$ in $\mathrm{slfp}_{\mathcal{A}}(M)$ to be equal to $\mathrm{rk}_{\mathcal{A}}(P)$ for any, and therefore every, locally-free $\mathcal{A}$-module P that occurs in a diagram of the form (5).

By a family of 'strict free envelopes' we mean an object of $\mathrm{slfp}_{\mathcal{A}}(M)$ with the property that any (and therefore every) $\mathcal{A}$-module P that occurs in a diagram of the form (5) is free. We write $\mathrm{sfe}_{\mathcal{A}}(M)$ for the set of families of strict free envelopes.

4.2.2. For the classes of order that will be of most interest to us in the sequel the above notion of 'pre-envelope' coincides with that used by Enochs in [20].

To explain this point, and also to prepare for the construction of canonical strict families of locally-free pre-envelopes in an arithmetic setting, we consider orders $\mathcal{A}$ that satisfy the following two conditions.

($\mathcal{A}_1$) there exists an R-linear anti-involution $\iota_{\mathcal{A}}$ of $\mathcal{A}$ such that for every prime ideal $\mathfrak{p}$ of R the linear dual $\mathrm{Hom}_{R_{(\mathfrak{p})}}(\mathcal{A}_{(\mathfrak{p})}, R_{(\mathfrak{p})})$ is a free rank one $\mathcal{A}_{(\mathfrak{p})}$-module when endowed with the left action $(a\theta)(a') := \theta(\iota_{\mathcal{A}}(a)a')$.

($\mathcal{A}_2$) The functor $M \mapsto \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A})$ is exact on the category of finitely generated R-torsion-free $\mathcal{A}$-modules.

There are two standard examples of such orders that we will use in subsequent sections.

Example 4.2. The conditions ($\mathcal{A}_1$) and ($\mathcal{A}_2$) are satisfied in both of the following cases.

(i) $\mathcal{A}$ is both commutative and locally-Gorenstein relative to R and $\iota_{\mathcal{A}}$ is the identity anti-involution.

(ii) $\mathcal{A} = R[G]$ for a finite group G and $\iota_{R[G]}$ is the R-linear anti-involution which inverts elements of G. In this case ($\mathcal{A}_2$) is satisfied because the functor $M \mapsto \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A})$ is naturally equivalent to $M \mapsto \mathrm{Hom}_R(M, R)$. When both R and G are clear from context we often simply denote the anti-involution $\iota_{R[G]}$ by $\iota_{\#}$.

Remark 4.3. For later use we record some straightforward consequences of condition ($\mathcal{A}_2$).

(i) $\mathrm{Ext}^i_{\mathcal{A}}(M, \mathcal{A}) = 0$ for all integers i with $i \geq 2$.

(ii) There is a natural identification $\mathrm{Ext}^1_{\mathcal{A}}(M,\mathcal{A}) \cong \mathrm{Hom}_{\mathcal{A}}(M_{\mathrm{tor}}, A/\mathcal{A})$ with M_{tor} the R-torsion submodule of M.
(iii) The functor $M \mapsto \mathrm{Hom}_{\mathcal{A}}(M, A/\mathcal{A})$ is exact on the category of finite $\mathcal{A}$-modules.

The following result will be useful in the sequel.

Lemma 4.4. *Let $\mathcal{A}$ be an R-order and $\iota : M \to P$ an injective homomorphism of finitely generated $\mathcal{A}$-modules in which P is locally-free and* $\mathrm{cok}(\iota)$ *is R-torsion-free.*

(i) *If $\mathcal{A}$ satisfies $(\mathcal{A}_1)$, then ι is a locally-free pre-envelope of M in the sense of* [20].
(ii) *If $\mathcal{A}$ satisfies $(\mathcal{A}_2)$, then the map* $\mathrm{Hom}_{\mathcal{A}}(P,\mathcal{A}) \to \mathrm{Hom}_{\mathcal{A}}(M,\mathcal{A})$ *induced by restriction through ι is surjective.*

Proof. To prove claim (i) we must show that for *any* injective homomorphism $\iota' : M \to P'$ of $\mathcal{A}$-modules, where P' is finitely generated and locally-free, there exists a commutative diagram of the form (5).

To do this we consider the following diagram of $\mathcal{A}$-modules (where each module is endowed with the left action described in condition $(\mathcal{A}_1)$)

$$\begin{tikzcd} & \mathrm{Hom}_R(M,R) & \\ \mathrm{Hom}_R(P,R) \arrow[ur, two heads, "\mathrm{Hom}_R(\iota,R)"] & & \mathrm{Hom}_R(P',R). \arrow[ul, "\mathrm{Hom}_R(\iota',R)"'] \arrow[ll, dashed, "\kappa"] \end{tikzcd}$$

Here the surjectivity of $\mathrm{Hom}_R(\iota,R)$ follows from the assumption that $\mathrm{cok}(\iota)$ is R-torsion-free and the existence of a homomorphism of $\mathcal{A}$-modules κ which makes the diagram commute then follows from the fact that $(\mathcal{A}_1)$ implies $\mathrm{Hom}_R(P',R)$ is a projective $\mathcal{A}$-module. By applying the functor $N \mapsto \mathrm{Hom}_R(N,R)$ to this diagram, and using the natural identification $\mathrm{Hom}_R(\mathrm{Hom}_R(N,R),R) \cong N$ for each $\mathcal{A}$-lattice N, one obtains a commutative diagram of the required form (5) in which $\kappa_{\iota,\iota'}$ is equal to $\mathrm{Hom}_R(\kappa,R)$.

Since $\mathcal{A}$ is assumed to satisfy condition $(\mathcal{A}_2)$, claim (ii) is proved by applying the functor $M \mapsto \mathrm{Hom}_{\mathcal{A}}(M,\mathcal{A})$ to the tautological short exact sequence $0 \to M \to P \to \mathrm{cok}(\iota) \to 0$. $\square$

Example 4.5. Let G be a finite group and set $\mathcal{A} := R[G]$ and $T_G := \sum_{g\in G} g \in \mathcal{A}$. Then for any $\mathcal{A}$-lattice M the module $M \otimes_R \mathcal{A}$ has a diagonal action of G with respect to which it is a locally-free $\mathcal{A}$-module and lies in an exact sequence of $\mathcal{A}$-modules

$$0 \to M \xrightarrow{\iota_M} M \otimes_R \mathcal{A} \to M \otimes_R (\mathcal{A}/(T_G)) \to 0$$

in which ι_M sends each m to $m \otimes_R T_G$ and the module $M \otimes_R (\mathcal{A}/(T_G))$ is R-torsion-free. This exact sequence combines with Lemma 4.4 to imply both that ι_M is a locally-free pre-envelope of M (in the sense of Enochs) and that the singleton $\mathcal{P}_M := \{\iota_M\}$ is a canonical element of $\mathrm{slfp}_{\mathcal{A}}(M)$. This family is however different from the canonical families that we shall construct in an arithmetic setting in §7.

4.3. **Morphism bundles.** In this section we fix a strict family of pre-envelopes $\mathcal{P}$ in $\mathrm{slfp}_{\mathcal{A}}(M)$ and an $\mathcal{A}$-module N.

4.3.1. We define an '$\mathcal{A}$-module morphism' π from $\mathcal{P}$ to N to be a choice for each $\iota : M \to P$ in $\mathcal{P}$ of a homomorphism of $\mathcal{A}$-modules $\pi_\iota : P \to N$ such that the following property is satisfied: for all other $\iota' : M \to P'$ in $\mathcal{P}$ there exists a commutative diagram of $\mathcal{A}$-modules

$$\begin{tikzcd} P \arrow[rr, "\kappa_{\iota,\iota',\pi}"] \arrow[dr, "\pi_\iota"'] & & P' \arrow[dl, "\pi_{\iota'}"] \\ & N & \end{tikzcd} \tag{6}$$

in which $\kappa_{\iota,\iota',\pi}$ is bijective and occurs in a commutative diagram of the form (5).

We shall say that such a morphism $\pi : \mathcal{P} \to N$ is surjective if for any (and therefore every) ι in $\mathcal{P}$ the homomorphism π_ι is surjective.

4.3.2. We now assume to be given for each prime ideal $\mathfrak{p}$ of R a family $\mathcal{P}_\mathfrak{p}$ in $\mathrm{slfp}_{\mathcal{A}_{(\mathfrak{p})}}(M_{(\mathfrak{p})})$ that contains the localization $\iota_{(\mathfrak{p})} := \mathcal{A}_{(\mathfrak{p})} \otimes_\mathcal{A} \iota$ of every ι in $\mathcal{P}$.

In this case we define a 'bundle (relative to the families $\{\mathcal{P}_\mathfrak{p}\}_\mathfrak{p}$) of $\mathcal{A}$-module morphisms' $\pi = \{\pi_\mathfrak{p}\}_\mathfrak{p}$ from $\mathcal{P}$ to N to be a choice for each prime ideal $\mathfrak{p}$ of R of an $\mathcal{A}_{(\mathfrak{p})}$-module morphism $\pi_\mathfrak{p} : \mathcal{P}_\mathfrak{p} \to N_{(\mathfrak{p})}$ such that the following property is satisfied: for every $\iota : M \to P$ and $\iota' : M \to P'$ in $\mathcal{P}$ there exists a commutative diagram of $\mathcal{A}$-modules

$$\begin{tikzcd} P \arrow[rr, "\tilde{\kappa}_{\iota,\iota'}"] \arrow[dr, "\tilde{\pi}_\iota"'] & & P' \arrow[dl, "\tilde{\pi}_{\iota'}"] \\ & N & \end{tikzcd}$$

the $\mathfrak{p}$-localization of which is contained, for almost all prime ideals $\mathfrak{p}$ of R, in the set of diagrams (6) (with P, P' and N replaced by $P_{(\mathfrak{p})}, P'_{(\mathfrak{p})}$ and $N_{(\mathfrak{p})}$) which verifies that $\pi_\mathfrak{p}$ is an $\mathcal{A}_{(\mathfrak{p})}$-module morphism.

We refer to the morphism $\pi_\mathfrak{p}$ as the '$\mathfrak{p}$-component' of a bundle of morphisms π. We shall also then say that a bundle of morphisms is surjective if for every prime ideal $\mathfrak{p}$ of R its $\mathfrak{p}$-component is surjective.

Remark 4.6. Given $\mathcal{P}$ in $\mathrm{slfp}_\mathcal{A}(M)$ one obtains for each prime ideal $\mathfrak{p}$ of R a canonical family $\mathcal{P}_{(\mathfrak{p})}$ in $\mathrm{slfp}_{\mathcal{A}_{(\mathfrak{p})}}(M_{(\mathfrak{p})})$ by setting $\mathcal{P}_{(\mathfrak{p})} := \{\mathcal{A}_{(\mathfrak{p})} \otimes_\mathcal{A} \iota : \iota \in \mathcal{P}\}$.

In particular, each $\mathcal{A}$-module morphism π from $\mathcal{P}$ to N defines a canonical bundle (relative to $\{\mathcal{P}_{(\mathfrak{p})}\}_\mathfrak{p}$) of $\mathcal{A}$-module morphisms $\pi^{\mathrm{bundle}} : \mathcal{P} \to N$ and in the sequel we do not distinguish between π and π^{bundle}.

4.4. **The case of group rings.** Let G be a finite group. Then, in the case $\mathcal{A} = \mathbb{Z}[G]$, we often abbreviate $\mathrm{slfp}_\mathcal{A}(M)$ to $\mathrm{slfp}_G(M)$.

Fix a normal subgroup H of G. Then for each family $\mathcal{P}$ in $\mathrm{slfp}_G(M)$ we write $\mathcal{P}^H$ for the object $\{\mathrm{Hom}_H(\mathbb{Z}, \iota) : \iota \in \mathcal{P}\}$ of $\mathrm{slfp}_{G/H}(M^H)$.

For any locally-free G-module P the homomorphism $\mathrm{Tr}_{P,H} : P_H \cong P^H$ induced by sending each x in P to $\sum_{h \in H} h(x)$ is bijective. In particular, for each bundle of G-module morphisms $\pi : \mathcal{P} \to N$ we can define π_H to be the bundle of G/H-module morphisms $\mathcal{P}^H \to N_H$ that is represented, for any prime ideal $\mathfrak{p}$ of R, by the composite $P_\mathfrak{p}^H \cong P_{\mathfrak{p},H} \to N_{\mathfrak{p},H}$ where

$\iota : M \to P$ is representative of $\mathcal{P}$, the first arrow is the bijective map induced by the inverse of $\mathrm{Tr}_{P,H}$ and the second arrow is induced by (the H-coinvariants of) $\pi_{\mathfrak{p},\iota}$.

5. Fitting lattices and higher Fitting invariants

In this section we introduce a natural non-commutative generalization of the classical notion of 'higher Fitting ideal' together with a related, and very useful, notion of 'non-commutative Fitting lattice'.

To do so we fix data $R, F, \mathcal{A}$ and A as in §3. We also assume to be given a finitely generated $\mathcal{A}$-module M, a family $\mathcal{P}$ in $\mathrm{slfp}_{\mathcal{A}}(M)$ and a bundle (relative to $\{\mathcal{P}_{(\mathfrak{p})}\}_{\mathfrak{p}}$) of $\mathcal{A}$-module morphisms $\pi = \{\pi_{\mathfrak{p}}\}_{\mathfrak{p}}$ with domain $\mathcal{P}$.

We set $d := \mathrm{rk}_{\mathcal{A}}(\mathcal{P})$.

5.1. Module presentations.

We first introduce some convenient classes of module presentation.

5.1.1. By a 'presentation' h of a finitely generated $\mathcal{A}$-module X we shall mean an exact sequence of $\mathcal{A}$-modules of the form

$$\mathcal{A}^{r_{h,1}} \xrightarrow{\theta_h} \mathcal{A}^{r_{h,2}} \xrightarrow{\pi_h} X \to 0 \tag{7}$$

in which (without loss of generality) one has $r_{h,1} \geq r_{h,2}$.

We say that a presentation h' of an $\mathcal{A}$-module X' is 'finer' than h if both $r_{h',1} = r_{h,1}$ and $r_{h',2} = r_{h,2}$ and there exists an automorphism of the $\mathcal{A}$-module $\mathcal{A}^{r_{h,2}}$ which induces a well-defined surjective homomorphism of $\mathcal{A}$-modules $X' \to X$.

We say that $\mathcal{A}$-module presentations h and h' are equivalent if both h is finer than h' and h' is finer than h (and we note that in this case the $\mathcal{A}$-modules X and X' are isomorphic).

5.1.2. We define a 'presentation bundle' h of an $\mathcal{A}$-module X to be a collection $\{h_{\mathfrak{p}}\}_{\mathfrak{p}}$ over all prime ideals $\mathfrak{p}$ of R of presentations $h_{\mathfrak{p}}$ of the $\mathcal{A}_{(\mathfrak{p})}$-modules $X_{(\mathfrak{p})}$ with the following properties. Set $r_{h,1} := r_{h_{(0)},1}$ and $r_{h,2} := r_{h_{(0)},2}$: then for all prime ideals $\mathfrak{p}$ one has $r_{h_{\mathfrak{p}},1} = r_{h,1}$ and $r_{h_{\mathfrak{p}},2} = r_{h,2}$ and the induced homomorphism $A \otimes_{\mathcal{A}_{(\mathfrak{p})}} h_{\mathfrak{p}} : A^{r_{h,1}} \to A^{r_{h,2}}$ coincides with $h_{(0)}$.

We shall say that an $\mathcal{A}$-module presentation bundle h' is finer than h if for each prime ideal $\mathfrak{p}$ the $\mathcal{A}_{(\mathfrak{p})}$-module presentation $h'_{\mathfrak{p}}$ is finer than $h_{\mathfrak{p}}$.

We say that $\mathcal{A}$-module presentation bundles h and h' are equivalent if both h is finer than h' and h' is finer than h.

Remark 5.1. Each presentation h of an $\mathcal{A}$-module X gives rise to an associated presentation bundle h^{bundle} of X (in which one has $\theta_{h^{\mathrm{bundle}}_{\mathfrak{p}}} = \mathcal{A}_{\mathfrak{p}} \otimes_{\mathcal{A}} \theta_h$ and $\pi_{h^{\mathrm{bundle}}_{\mathfrak{p}}} = \mathcal{A}_{\mathfrak{p}} \otimes_{\mathcal{A}} \pi_h$ for all primes $\mathfrak{p}$) and we shall usually not distinguish between h and h^{bundle}.

It is also straightforward to check that to each presentation bundle h for X one can associate a canonical exact sequence of $\mathcal{A}$-modules $P' \to P \to X \to 0$ in which P' and P are locally-free of ranks $r_{h,1}$ and $r_{h,2}$ respectively.

5.1.3. In the sequel we refer to a presentation bundle h as 'quadratic' if one has $r_{h,1} = r_{h,2}$.

If $\mathcal{P}$ belongs to $\mathrm{sfe}_{\mathcal{A}}(M)$ and π is a morphism of $\mathcal{A}$-modules with domain $\mathcal{P}$, then we say that a presentation h (of some $\mathcal{A}$-module X) 'factors through' π if $r_{h,2} = \mathrm{rk}_{\mathcal{A}}(\mathcal{P})$ and for any, and therefore every, representative $\iota : M \to P$ of $\mathcal{P}$ there exists an isomorphism of $\mathcal{A}$-modules $\kappa_{h,\iota} : \mathcal{A}^{r_{h,2}} \to P$ for which $\mathrm{im}(\kappa_{h,\iota} \circ \theta_h) \subseteq \ker(\pi_\iota)$.

Such isomorphisms $\kappa_{h,\iota}$ are not unique but the precise choice will not matter in the sequel. (It is also clear that the existence of such an isomorphism implies that there is a surjective homomorphism of $\mathcal{A}$-modules from X to $\mathrm{im}(\pi_\iota)$.)

If $\mathcal{P}$ belongs to $\mathrm{sfp}_{\mathcal{A}}(M)$ and π is a bundle (relative to $\{\mathcal{P}_{(\mathfrak{p})}\}_{\mathfrak{p}}$) of $\mathcal{A}$-module morphisms with domain $\mathcal{P}$, then we say that a presentation bundle h 'factors through' π if for every prime ideal $\mathfrak{p}$ of R the presentation $h_{\mathfrak{p}}$ factors through $\pi_{\mathfrak{p}}$, and hence therefore $r_{h,2} = \mathrm{rk}_{\mathcal{A}}(\mathcal{P})$.

5.2. **Higher Fitting lattices: definitions.** In this section we define a notion of the higher Fitting lattice of the module M relative to the given bundle of $\mathcal{A}$-module morphisms $\pi = \{\pi_{\mathfrak{p}}\}_{\mathfrak{p}}$.

5.2.1. We first assume to be given a quadratic presentation h of an $\mathcal{A}$-module X as in (7). We set $t := r_{h,1} = r_{h,2}$ and write b for the standard (ordered) $\mathcal{A}$-basis $(b_1, \dots, b_t)$ of $\mathcal{A}^t$.

For each integer i with $1 \le i \le t$ we write b_i^* for the dual of b_i in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^t, \mathcal{A})$ and then define an element $\theta_{h,i}$ of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^t, \mathcal{A})$ by setting $\theta_{h,i} := b_i^* \circ \theta_h$.

For each integer r with $1 \le r \le t$ this gives rise to a well-defined 'exterior product' homomorphism of $\xi(\mathcal{A})$-modules

$$\wedge_{j=r+1}^{j=t}\theta_{h,j} : {\bigcap}_{\mathcal{A}}^{t}\mathcal{A}^t \to {\bigcap}_{\mathcal{A}}^{r}\mathcal{A}^t.$$

In addition, if h factors through an $\mathcal{A}$-module morphism π with domain $\mathcal{P}$ in $\mathrm{sfe}_{\mathcal{A}}(M)$, then for each such integer r and any choice of homomorphism $\kappa_{h,\iota}$ as in §5.1.3 the map ${\bigwedge}_A^r(F \otimes_R \kappa_{h,\iota})$ restricts to give a homomorphism of $\xi(\mathcal{A})$-modules

$$\cap_{\mathcal{A}}^r\kappa_{h,\iota} : {\bigcap}_{\mathcal{A}}^{r}\mathcal{A}^t \to {\bigcap}_{\mathcal{A}}^{r}P.$$

We define the 'r-th Fitting lattice of h relative to $\kappa_{h,\iota}$' to be the $\xi(\mathcal{A})$-module

$$\mathrm{FL}^r_{\kappa_{h,\iota}}(h) := \mathrm{im}((\cap_{\mathcal{A}}^r\kappa_{h,\iota}) \circ (\wedge_{j=r+1}^{j=t}\theta_{h,j})) \subseteq {\bigcap}_{\mathcal{A}}^{r}P$$

and then set

$$\mathrm{F}_\pi({\bigcap}_{\mathcal{A}}^{r}P) := \sum_h \mathrm{FL}^r_{\kappa_{h,\iota}}(h)$$

where in the sum h runs over all quadratic presentations which factor through π, and $\kappa_{h,\iota}$ over all choices of isomorphisms as in §5.1.3.

Noting that ι induces an injective homomorphism of F-spaces

$${\bigwedge}_A^r(F \otimes \iota) : {\bigwedge}_A^r M_F \to {\bigwedge}_A^r P_F$$

we then define the 'r-th Fitting lattice of M with respect to π' by setting

$$\mathrm{F}_\pi({\bigcap}_{\mathcal{A}}^{r}M) := \left\{ x \in {\bigwedge}_A^r M_F \ : ({\bigwedge}_A^r(A \otimes_{\mathcal{A}} \iota))(x) \in \mathrm{F}_\pi({\bigcap}_{\mathcal{A}}^{r}P) \right\}.$$

5.2.2. If π is any bundle of $\mathcal{A}$-module morphisms with domain $\mathcal{P}$ in $\mathrm{slfp}_{\mathcal{A}}(M)$, then we define the '$r$-th Fitting lattice of M with respect to π' by setting

$$\mathrm{F}_{\pi}(\bigcap\nolimits_{\mathcal{A}}^{r} M) := \bigcap_{\mathfrak{p}} \mathrm{F}_{\pi_{\mathfrak{p}}}(\bigcap\nolimits_{\mathcal{A}_{(\mathfrak{p})}}^{r} M_{\mathfrak{p}})$$

where the intersection runs over all prime ideals $\mathfrak{p}$ of R.

Remark 5.2. This definition ensures that if π is any $\mathcal{A}$-module morphism with domain $\mathcal{P}$, then one has $\mathrm{F}_{\pi}(\bigcap_{\mathcal{A}}^{r} M) = \mathrm{F}_{\pi^{\mathrm{bundle}}}(\bigcap_{\mathcal{A}}^{r} M)$, where π^{bundle} is the bundle of $\mathcal{A}$-module morphisms defined in Remark 4.6.

5.3. **Higher Fitting lattices: basic properties.** In the following result we record the basic properties of Fitting lattices.

Before stating this result we introduce some notation. We assume to be given a surjective homomorphism of R-orders $\rho : \mathcal{A} \to \mathcal{B}$ and write B for the algebra spanned by $\mathcal{B}$. For any $\mathcal{A}$-module M we write ${}_{\rho}M$ for the image of $\mathcal{B} \otimes_{\mathcal{A}} M$ in $B \otimes_{\mathcal{A}} M$. Since for any finitely generated $\mathcal{A}$-module N the module $\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{B}, N)$ is R-torsion (as B is a projective A-module), any embedding of $\mathcal{A}$-modules $\iota : M \to P$ induces an embedding of $\mathcal{B}$-modules ${}_{\rho}\iota : {}_{\rho}M \to {}_{\rho}P$. In this way, for any element $\mathcal{P}$ of $\mathrm{slfp}_{\mathcal{A}}(M)$ we obtain an element ${}_{\rho}\mathcal{P}$ of $\mathrm{slfp}_{\mathcal{A}}({}_{\rho}M)$ by setting ${}_{\rho}\mathcal{P} := \{{}_{\rho}\iota : \iota \in \mathcal{P}\}$.

We write $\mathrm{id}_{\mathcal{P}}$ and $0_{\mathcal{P}}$ for the identity and zero endomorphisms of each family $\mathcal{P}$ in $\mathrm{slfp}_{\mathcal{A}}(M)$.

Finally, for any commutative ring Λ, finitely generated Λ-module M and non-negative integer a we write $\mathrm{Fit}_{\Lambda}^{a}(M)$ for the a-th Fitting ideal of the Λ-module M.

Proposition 5.3. *Fix a family of pre-envelopes $\mathcal{P}$ in $\mathrm{slfp}_{\mathcal{A}}(M)$ and an $\mathcal{A}$-module morphism π with domain $\mathcal{P}$.*

(i) $\mathrm{F}_{\pi}(\bigcap_{\mathcal{A}}^{r} M)$ *is a finitely generated $\xi(\mathcal{A})$-submodule of $\bigcap_{\mathcal{A}}^{r} M$ that is independent of the choice of representative ι in $\mathcal{P}$.*

(ii) *If π' is any homomorphism of $\mathcal{A}$-modules with domain $\mathcal{P}$ and the property that* $\ker(\pi_{\iota}) \subseteq \ker(\pi'_{\iota})$ *for any, and therefore every, ι in $\mathcal{P}$, then* $\mathrm{F}_{\pi}(\bigcap_{\mathcal{A}}^{r} M) \subseteq \mathrm{F}_{\pi'}(\bigcap_{\mathcal{A}}^{r} M)$. *In particular,* $\mathrm{F}_{0_{\mathcal{P}}}(\bigcap_{\mathcal{A}}^{r} M)$ *is the (unique) maximal Fitting lattice in $\bigcap_{\mathcal{A}}^{r} M$.*

(iii) $\mathrm{F}_{\mathrm{id}_{\mathcal{P}}}(\bigcap_{\mathcal{A}}^{r} M) = 0$.

(iv) *Let $\rho : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism of R-orders and write ${}_{\rho}\pi$ for the $\mathcal{B}$-module morphism with domain ${}_{\rho}\mathcal{P}$ that has $({}_{\rho}\pi)_{{}_{\rho}\iota} = {}_{\rho}(\pi_{\iota})$ for each ι in $\mathcal{P}$. Then the lattice* $\mathrm{F}_{{}_{\rho}\pi}(\bigcap_{\mathcal{B}}^{r}({}_{\rho}M))$ *is equal to the image of* $\mathrm{F}_{\pi}(\bigcap_{\mathcal{A}}^{r} M)$ *under the natural map* $\bigwedge_{A}^{r} M_F \to \zeta(B) \otimes_{\zeta(A)} \bigwedge_{A}^{r} M_F \cong \bigwedge_{B}^{r}(B \otimes_A M_F)$.

(v) *If M is a free $\mathcal{A}$-module and $\mathcal{P}$ is represented by the identity map on M, then for any choice of $\mathcal{A}$-basis b of M one has* $\mathrm{F}_{0_{\mathcal{P}}}(\bigcap_{\mathcal{A}}^{r} M) + \ker(\theta_b) = \bigcap_{\mathcal{A}}^{r} M$, *where θ_b is the homomorphism of $\xi(\mathcal{A})$-modules that occurs in Proposition 3.5. In particular, if either $\mathcal{A}$ is commutative or* $\mathrm{rk}_{\mathcal{A}}(M) = r$, *then* $\mathrm{F}_{0_{\mathcal{P}}}(\bigcap_{\mathcal{A}}^{r} M) = \bigcap_{\mathcal{A}}^{r} M$.

(vi) *If $\mathcal{A}$ is commutative and satisfies condition ($\mathcal{A}_2$), then* $\mathrm{F}_{0_{\mathcal{P}}}(\bigcap_{\mathcal{A}}^{r} M) = \bigcap_{\mathcal{A}}^{r} M$.

(vii) *If $\mathcal{A}$ is commutative, M is a locally-free $\mathcal{A}$-module and $\mathcal{P}$ is represented by the identity map on M, then* $\mathrm{F}_{\pi}(\bigcap_{\mathcal{A}}^{r} M) \subseteq \mathrm{Fit}_{\mathcal{A}}^{r}(\mathrm{im}(\pi)) \cdot \bigwedge_{\mathcal{A}}^{r} M$, *with equality if for any, and therefore every, representative ι of $\mathcal{P}$ the $\mathcal{A}$-module* $\ker(\pi_{\iota})$ *can be generated by d elements.*

Proof. It is clear, by its very construction, that $\mathrm{F}_\pi(\bigcap^r_{\mathcal{A}}M)$ is a $\xi(\mathcal{A})$-module and straightforward to check that it is contained in $\bigcap^r_{\mathcal{A}}M$.

To show that it is independent of ι in $\mathcal{P}$ one need only note that if one takes any diagram of the form (6), then $\bigwedge^r_A(A\otimes_{\mathcal{A}}\kappa_{\iota,\iota',\pi})$ restricts to give an isomorphism of $\xi(\mathcal{A})$-modules $\bigcap^r_{\mathcal{A}}P\cong\bigcap^r_{\mathcal{A}}P'$ which sends the lattice $\mathrm{FL}^r_{\kappa_{h,\iota}}(h)$ to $\mathrm{FL}^r_{\kappa_{\iota,\iota',\pi}\circ\kappa_{h,\iota}}(h)$.

To complete the proof of claim (i) it suffices to show that $\mathrm{F}_\pi(\bigcap^r_{\mathcal{A}}M)$ is a finitely generated R-module and to do this it is enough to consider the case that the ring R is local and the $\mathcal{A}$-module M is free, of rank t say. In this case if h is any quadratic presentation of $\mathcal{A}$-modules which factors through π, then for any isomorphism of $\mathcal{A}$-modules $\kappa:\mathcal{A}^t\to M$, Proposition 3.5(iii) combines with the commutative diagram of $\mathcal{A}$-modules

$$\begin{array}{ccc}\mathcal{A}^t & \xrightarrow{\theta_h} & \mathcal{A}^t\\ \kappa\big\downarrow & & \big\downarrow\kappa\\ M & \xrightarrow{\kappa\circ\theta_h\circ\kappa^{-1}} & M\end{array}$$

to imply that

$$\mathrm{FL}^r_\kappa(h)=\xi(\mathcal{A})\cdot\wedge_{j=r+1}^{j=t}(\kappa(b_j)^*\circ(\kappa\circ\theta_h\circ\kappa^{-1}))(\wedge_{j=1}^{j=t}\kappa(b_j))\subseteq\bigcap\nolimits^r_{\mathcal{A}}M \tag{8}$$

where we write $\{b_i\}_{1\le i\le t}$ for the standard basis of $\mathcal{A}^t$.

Now, as h and κ vary, all of the homomorphisms $\kappa(b_j)^*\circ(\kappa\circ\theta_h\circ\kappa^{-1})$ belong to $\mathrm{Hom}_{\mathcal{A}}(M,\mathcal{A})$. The finite generation of $\mathrm{F}_\pi(\bigcap^r_{\mathcal{A}}M)$ as an R-module is therefore a consequence of the fact that the $\xi(\mathcal{A})$-linear span of $\{\wedge_{i=1}^{i=t}\varphi_i:\varphi_i\in\mathrm{Hom}_{\mathcal{A}}(M,\mathcal{A})\}$ in $\bigwedge^r_{A^{\mathrm{op}}}\mathrm{Hom}_A(M_F,A)$ is finitely generated.

The first assertion of claim (ii) is true because the stated assumptions imply that for every prime ideal $\mathfrak{p}$ of R any presentation $h_\mathfrak{p}$ of $\mathcal{A}_{(\mathfrak{p})}$-modules which factors through $\pi_\mathfrak{p}$ also factors through $\pi'_\mathfrak{p}$. The second assertion of claim (ii) then follows immediately from the first assertion.

Claim (iii) is true because if π is the identity endomorphism of $M=P$, then $\ker(\pi_\iota)$ vanishes. This combines with the injectivity of $\kappa_{h,\iota}$ to imply that θ_h, and hence also each projection $\theta_{h,j}$, is the zero map and hence that $\mathrm{FL}^r_{\kappa_{h,\iota}}(h)$ vanishes.

To prove claim (iv) we can again assume that R is local and that M is a free $\mathcal{A}$-module, of rank t say. We note that ρ induces a surjective homomorphism of rings $\xi(\mathcal{A})\to\xi(\mathcal{B})$ and we write $\hat\rho$ for the natural map $\bigwedge^r_AM_F\to\zeta(B)\otimes_{\zeta(A)}\bigwedge^r_AM_F\cong\bigwedge^r_BB\otimes_AM_F$.

By using the formula (8) one deduces that $\mathrm{F}_\pi(\bigcap^r_{\mathcal{A}}M)$ is equal to the $\xi(\mathcal{A})$-module generated by all elements of the form $\wedge_{j=r+1}^{j=t}(c_j^*\circ\theta)(\wedge_{j=1}^{j=t}c_j)$ where $\{c_j\}_{1\le j\le t}$ is any choice of $\mathcal{A}$-basis of M and θ any homomorphism of $\mathcal{A}$-modules $M\to M$ for which $\mathrm{im}(\theta)$ is contained in $\ker(\pi_\iota)$ for some choice of ι in $\mathcal{P}$. In the same way one obtains a similar description of $\mathrm{F}_{\rho\pi}(\bigcap^r_{\mathcal{B}}({}_\rho M))$.

In particular, since for any such θ one has $\mathrm{im}({}_\rho\theta)\subseteq\ker({}_\rho\pi_{\rho\iota})$, the inclusion $\hat\rho(\mathrm{F}_\pi(\bigcap^r_{\mathcal{A}}M))\subseteq\mathrm{F}_{\rho\pi}(\bigcap^r_{\mathcal{B}}({}_\rho M))$ follows directly from the fact that

$$\hat\rho(\wedge_{j=r+1}^{j=t}(c_j^*\circ\theta)(\wedge_{j=1}^{j=t}c_j))=\wedge_{j=r+1}^{j=t}(d_j^*\circ{}_\rho\theta)(\wedge_{j=1}^{j=t}d_j)$$

where we write d_j for the image of c_j under the natural surjection $M\to{}_\rho M$.

To prove the reverse inclusion, and hence complete the proof of claim (iv), we note first that for any homomorphism of $\mathcal{B}$-modules $\theta' : {}_\rho M \to {}_\rho M$ for which $\mathrm{im}(\theta') \subseteq \ker({}_\rho\pi_{\rho\iota})$ there exists a homomorphism of $\mathcal{A}$-modules $M \to M$ for which both $\mathrm{im}(\theta) \subseteq \ker(\pi_\iota)$ and ${}_\rho\theta = \theta'$. It then suffices to note that, since $\mathcal{A}$ is semi-local, Bass's Theorem implies that any $\mathcal{B}$-basis of ${}_\rho M$ can be lifted to an $\mathcal{A}$-basis of M.

To prove claim (v) we fix an $\mathcal{A}$-basis $\{m_i\}_{1\le i\le t}$ of M. Then the proof of Proposition 3.5(iii) shows claim (v) is true provided that $\mathrm{F}_\pi(\bigcap^r_{\mathcal{A}} M)$ contains the element $\wedge_{k=1}^{k=r} m_{\sigma(i)}$ for each choice of σ in $\left[{t \atop r}\right]$. To show this we fix such a permutation σ and, writing $\{b_i\}_{1\le i\le t}$ for the standard basis of $\mathcal{A}^t$, we define θ_σ to be the natural projection map from $\mathcal{A}^t$ to the $\mathcal{A}$-module direct summand that is generated by the set $\{b_{\sigma(i)}\}_{r<i\le t}$. Then the formula of Lemma 2.1 implies $(\wedge_{j=r+1}^{j=t}(b_j^* \circ \theta_\sigma))(\wedge_{k=1}^{k=t} b_k) = \pm \wedge_{k=1}^{k=r} b_{\sigma(i)}$.

It then suffices to note that, since π is the zero endomorphism of M, there exists a presentation h of the form (7) with $r_{h,2} = t$ and $\theta_h = \theta_\sigma$ and which factors through π by means of the isomorphism of $\mathcal{A}$-modules $\mathcal{A}^t \cong M$ which sends each element b_i to m_i.

To prove claim (vi) we assume $\mathcal{A}$ is commutative, fix $\iota : M \to P$ in $\mathcal{P}$, write 0_P for the zero endomorphism of P and use the map $\bigwedge^r_A (A \otimes_{\mathcal{A}} \iota)$ to regard $\bigwedge^r_A M_F$ as a submodule of $\bigwedge^r_A P_F$. Then, since claim (v) implies $\mathrm{F}_{0_P}(\bigcap^r_{\mathcal{A}} P) = \bigcap^r_{\mathcal{A}} P$, it suffices to show that in $\bigwedge^r_A P_F$ one has $\bigcap^r_{\mathcal{A}} M = (\bigwedge^r_A M_F) \cap \bigcap^r_{\mathcal{A}} P$. But it is clear that $\bigcap^r_{\mathcal{A}} M \subseteq (\bigwedge^r_A M_F) \cap \bigcap^r_{\mathcal{A}} P$ and if one assumes that $\mathcal{A}$ satisfies $(\mathcal{A}_2)$, then the reverse inclusion is also very easy to check.

Turning to claim (vii) we continue to assume $\mathcal{A}$ is commutative. We first note that, in this case, if a presentation h of an $\mathcal{A}$-module X (as in (7)) is both quadratic and factors through π, then there exists a surjective homomorphism of $\mathcal{A}$-modules $X \to \mathrm{im}(\pi)$ and hence, by standard properties of higher Fitting ideals, an inclusion $\mathrm{Fit}^r_{\mathcal{A}}(X) \subseteq \mathrm{Fit}^r_{\mathcal{A}}(\mathrm{im}(\pi))$.

We further note that if $\ker(\pi_\iota)$ can be generated by d elements, then there is an exact sequence of $\mathcal{A}$-modules $\mathcal{A}^d \to P \xrightarrow{\pi_\iota} M \to 0$. By applying Roiter's Lemma to the locally-free module P one deduces the existence for any natural number n of a quadratic presentation h of an $\mathcal{A}$-module X' which factors through π and is such that $X'_{(\mathfrak{p})} = M_{(\mathfrak{p})}$ for all prime ideals that do not divide n.

Given these observations, claim (vii) will follow if we can show that for any $\mathcal{A}$-module X that has a quadratic presentation h of the form (7), there is an equality

$$\sum_{h'} \mathrm{im}(f^r_{h'}) = \mathrm{Fit}^r_{\mathcal{A}}(X) \cdot \bigwedge\nolimits^r_{\mathcal{A}} \mathcal{A}^{r_{h,2}}$$

where in the sum h' ranges over all quadratic presentations of $\mathcal{A}$-modules that are finer than h and for each such h' we write $f^r_{h'}$ for the map $\wedge_{j=r+1}^{j=r_{h',2}} \theta_{h',j}$ which occurs in §5.2.1.

To do this we set $t = r_{h,2}$ and we recall Proposition 3.5(iii) implies that $\bigcap^t_{\mathcal{A}} \mathcal{A}^t$ is equal to $\bigwedge^t_{\mathcal{A}} \mathcal{A}^t$ and so is free of rank one with basis $b_1 \wedge \cdots \wedge b_t$, where $b = \{b_1, \ldots, b_t\}$ is the standard basis of $\mathcal{A}^t$. By Lemma 2.1, we also know that $f^r_{h'}(b_1 \wedge \cdots \wedge b_t)$ is a sum of the elements of the form

$$\pm \det((\theta_{h',j}(b_{\sigma(i)}))_{r<i,j\le t}) \cdot \wedge_{k=1}^{k=r} b_{\sigma(k)} \in \det((\theta_{h',j}(b_{\sigma(i)}))_{r<i,j\le t}) \cdot \bigwedge\nolimits^r_{\mathcal{A}} \mathcal{A}^d,$$

where σ runs over $\left[{t \atop r}\right]$.

We next make two observations which follow immediately from the definition of higher Fitting ideal.

Firstly, since each matrix $(\theta_{h',j}(b_{\sigma(i)}))_{r<i,j\le t}$ is a $(t-r)\times(t-r)$ minor of the matrix of $\theta_{h'}$ with respect to the basis b, its determinant belongs to $\mathrm{Fit}^r_{\mathcal{A}}(X') \subseteq \mathrm{Fit}^r_{\mathcal{A}}(X)$, thus proving the inclusion of claim (vii).

Secondly, the $\mathcal{A}$-module $\mathrm{Fit}^r_{\mathcal{A}}(X)\cdot \bigwedge^r_{\mathcal{A}}\mathcal{A}^d$ is generated by all elements of the form

$$\det((\theta_{h,\sigma(i)}(b_{\tau(j)}))_{r<i,j\le t})\cdot \wedge_{k=1}^{k=r} b_{\mu(k)},$$

as σ, τ and μ range over $\begin{bmatrix} t \\ r\end{bmatrix}$ and so it suffices to show that each such element is contained in $\mathrm{im}(f^r_{h'})$ for a suitable choice of presentation h'.

Fixing a choice of σ, τ and μ we write θ' for the unique endomorphism of the $\mathcal{A}$-module $\mathcal{A}^t$ which at each integer i with $1\le i\le t$ satisfies

$$\theta'(b_i) := \begin{cases} \iota_{\sigma\mu^{-1}}(\theta_h(b_{\tau\mu^{-1}(i)})), & \text{if } i\notin\{\mu(1),\dots,\mu(r)\},\\ 0, & \text{if } i\in\{\mu(1),\dots,\mu(r)\},\end{cases}$$

where $\iota_{\sigma\mu^{-1}}$ is the automorphism of the $\mathcal{A}$-module $\mathcal{A}^t$ which satisfies $\iota_{\sigma\mu^{-1}}(b_{\mu(i)}) = b_{\sigma(i)}$ for all i with $1\le i\le t$. Then, since $\mathrm{im}((\iota_{\sigma\mu^{-1}})^{-1}\circ\theta')\subseteq \mathrm{im}(\theta_h)$ there exists a quadratic $\mathcal{A}$-module presentation h' that is finer than h and such that $\theta' = \theta_{h'}$. In addition, Lemma 2.1 implies that

$$\begin{aligned} f^r_{h'}(\wedge_{k=1}^{k=d} b_k) &= \pm\det(((b^*_{\mu(i)}\circ\theta')(b_{\mu(j)}))_{r<i,j\le t})\wedge_{k=1}^{k=r} b_{\mu(k)}\\ &= \pm\det(((b^*_{\mu(i)}\circ\iota_{\sigma\mu^{-1}})(\theta_h(b_{\tau(j)}))_{r<i,j\le t}))\cdot\wedge_{k=1}^{k=r} b_{\mu(k)}\\ &= \pm\det((b^*_{\sigma(i)}(\theta_h(b_{\tau(j)}))_{r<i,j\le t}))\cdot\wedge_{k=1}^{k=r} b_{\mu(k)}\\ &= \pm\det((\theta_{h,\sigma(i)}(b_{\tau(j)}))_{r<i,j\le t})\cdot\wedge_{k=1}^{k=r} b_{\mu(k)}.\end{aligned}$$

This shows that $\det((\theta_{h,\sigma(i)}(b_{\tau(j)}))_{r<i,j\le t})\cdot\wedge_{k=1}^{k=r} b_{\mu(k)}$ belongs to $\mathrm{im}(f^r_{h'})$ and hence completes the proof of claim (vii). □

Remark 5.4. Fix $\mathcal{P}$ in $\mathrm{slfp}_{\mathcal{A}}(M)$ and a non-negative integer r. Then the proof of Proposition 5.3(i) also allows us (by a slight abuse of notation) to define $\mathrm{F}_\pi(\bigcap^r_{\mathcal{A}}\mathcal{P})$ to be equal to $\mathrm{F}_\pi(\bigcap^r_{\mathcal{A}}P)$ for any choice of embedding $\iota: M\to P$ in $\mathcal{P}$ and to regard $\bigwedge^r_A(A\otimes_{\mathcal{A}}\iota)$ as inducing a canonical map $\mathrm{F}_\pi(\bigcap^r_{\mathcal{A}}M)\to\mathrm{F}_\pi(\bigcap^r_{\mathcal{A}}\mathcal{P})$.

In a similar way, if $\mathcal{A}$ is commutative, then we occasionally use $\bigwedge^r_{\mathcal{A}}\mathcal{P}$ to denote the lattice $\bigwedge^r_{\mathcal{A}}P$ for any choice of P as above. We note that, in terms of this notation, Proposition 3.5(iii) implies there is a natural identification $\bigwedge^r_{\mathcal{A}}\mathcal{P} = \bigcap^r_{\mathcal{A}}\mathcal{P}$.

5.4. **Non-commutative higher Fitting invariants: definitions.** We next introduce a natural notion of 'higher Fitting invariant' in the non-commutative setting.

For any strictly positive integer n we write $\{b_i\}_{1\le i\le n}$ for the standard basis of the free $\mathcal{A}$-module $\mathcal{A}^n$.

For any non-negative integer t we write $[n]_t$ for the set of subsets of $\{1,2,\dots,n\}$ that are of cardinality $\min\{t,n\}$.

5.4.1. We fix a presentation h of $\mathcal{A}$-modules of the form (7). We write $M(h)$ for the matrix of θ_h with respect to the bases $\{b_i\}_{1\le i\le r_{h,1}}$ and $\{b_i\}_{1\le i\le r_{h,2}}$.

Then for any non-negative integer t and any $\varphi = (\varphi_i)_{1\le i\le t}$ in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{r_{h,1}}, \mathcal{A})^t$ we write $\mathrm{Min}_{\varphi}^{r_{h,2}}(\theta_h)$ for the set of all $r_{h,2}\times r_{h,2}$ minors of the matrices $M(h,J,\varphi)$ that are obtained from $M(h)$ by choosing any J in $[r_{h,2}]_t$, with $J = \{i_1, i_2, \cdots, i_t\}$ and $i_1 < i_2 < \cdots < i_t$, and setting

$$M(h,J,\varphi)_{ij} := \begin{cases} \varphi_a(b_i), & \text{if } j = i_a \text{ with } 1 \le a \le t \\ M(h)_{ij}, & \text{otherwise.} \end{cases} \tag{9}$$

For any non-negative integer r we define the 'r-th (non-commutative) Fitting invariant of the presentation h' to be the ideal of $\xi(\mathcal{A})$ obtained by setting

$$\mathrm{FI}_{\mathcal{A}}^{r}(h) := \xi(\mathcal{A})\cdot\{\mathrm{Nrd}_{A}(N) : N \in \mathrm{Min}_{\varphi}^{r_{h,2}}(\theta_h),\ \varphi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{r_{h,1}}, \mathcal{A})^t,\ t \le r\}.$$

We then define the 'total r-th (non-commutative) Fitting invariant of h' by setting

$$\mathrm{FI}_{\mathcal{A}}^{r,\mathrm{tot}}(h) := \sum_{h'} \mathrm{FI}_{\mathcal{A}}^{r}(h')$$

where in the sum h' runs over all $\mathcal{A}$-module presentations that are finer than h.

5.4.2. For each $\mathcal{A}$-module presentation bundle h and each non-negative integer r we define ideals of $\xi(\mathcal{A})$ by setting

$$\mathrm{FI}_{\mathcal{A}}^{r}(h) := \bigcap_{\mathfrak{p}} \mathrm{FI}_{\mathcal{A}_{(\mathfrak{p})}}^{r}(h_{\mathfrak{p}})$$

and

$$\mathrm{FI}_{\mathcal{A}}^{r,\mathrm{tot}}(h) := \bigcap_{\mathfrak{p}} \mathrm{FI}_{\mathcal{A}_{(\mathfrak{p})}}^{r,\mathrm{tot}}(h_{\mathfrak{p}}),$$

where in both cases in the intersection $\mathfrak{p}$ runs over all prime ideals of R.

We refer to $\mathrm{FI}_{\mathcal{A}}^{r}(h)$ and $\mathrm{FI}_{\mathcal{A}}^{r,\mathrm{tot}}(h)$ as the r-th, respectively the total r-th, (non-commutative) Fitting invariant of the presentation bundle h.

5.5. **Non-commutative higher Fitting invariants: basic properties.** In this section we record some basic properties of the higher Fitting invariants defined above and also describe their connection to the Fitting lattices defined earlier.

5.5.1. Our first result shows that higher Fitting invariants of presentations share some of the same properties as do higher Fitting ideals of modules in the commutative setting.

Proposition 5.5. *Let h be a presentation bundle of a finitely generated $\mathcal{A}$-module X. Then for each non-negative integer r the following claims are valid.*

(i) $\mathrm{FI}_{\mathcal{A}}^{r}(h) \subseteq \mathrm{FI}_{\mathcal{A}}^{r+1}(h)$ *and* $\mathrm{FI}_{\mathcal{A}}^{r,\mathrm{tot}}(h) \subseteq \mathrm{FI}_{\mathcal{A}}^{r+1,\mathrm{tot}}(h)$

(ii) $\mathrm{FI}_{\mathcal{A}}^{r}(h) = \mathrm{FI}_{\mathcal{A}}^{r,\mathrm{tot}}(h) = \xi(\mathcal{A})$ *if* $r \ge r_{h,2}$.

(iii) *If the $\mathcal{A}$-module presentation bundle h' is finer than h, then* $\mathrm{FI}_{\mathcal{A}}^{r,\mathrm{tot}}(h') \subseteq \mathrm{FI}_{\mathcal{A}}^{r,\mathrm{tot}}(h)$. *In particular, if h and h' are equivalent, then* $\mathrm{FI}_{\mathcal{A}}^{r,\mathrm{tot}}(h) = \mathrm{FI}_{\mathcal{A}}^{r,\mathrm{tot}}(h')$.

(iv) *If $\mathcal{A}$ is commutative, then* $\mathrm{FI}_{\mathcal{A}}^{r}(h) = \mathrm{FI}_{\mathcal{A}}^{r,\mathrm{tot}}(h)$ *and this ideal coincides with the r-th Fitting ideal* $\mathrm{Fit}_{\mathcal{A}}^{r}(X)$ *of X as an $\mathcal{A}$-module.*

Proof. Claims (i) and (ii) follow directly from the definitions of (total) higher Fitting invariant.

The first assertion of claim (iii) follows directly from the fact that if h' is finer than h, then for each prime ideal $\mathfrak{p}$ of R any $\mathcal{A}_{(\mathfrak{p})}$-module presentation that is finer than $h'_{\mathfrak{p}}$ is automatically finer than $h_{\mathfrak{p}}$ and hence $\mathrm{FI}^{r,\mathrm{tot}}_{\mathcal{A}_{(\mathfrak{p})}}(h'_{\mathfrak{p}}) \subseteq \mathrm{FI}^{r,\mathrm{tot}}_{\mathcal{A}_{(\mathfrak{p})}}(h_{\mathfrak{p}})$.

The second assertion of claim (iii) then follows immediately upon combining the first assertion with the definition of equivalence of presentations.

It suffices to prove claim (v) after localizing at a prime ideal $\mathfrak{p}$. Then, setting $\Lambda := \mathcal{A}_{(\mathfrak{p})}$, claim (iii) implies it suffices to show that $\mathrm{FI}^r_\Lambda(h_{\mathfrak{p}}) = \mathrm{Fit}^r_\Lambda(X_{(\mathfrak{p})})$.

Next we note that, if Λ is abelian, then $\mathrm{FI}^r_\Lambda(h_{\mathfrak{p}})$ is generated over Λ by elements of the form $\det(N)$ where N is an $r_{h,2}\times r_{h,2}$ matrix, at least $r_{h,2}-r$ columns of which coincide with the columns of an $r_{h,2}\times r_{h,2}$ minor of $M(h_{\mathfrak{p}})$. The Laplace expansion of $\det(N)$ therefore shows that it is contained in the ideal of Λ that is generated by the set of $(r_{h,2}-r)\times(r_{h,2}-r)$ minors of $M(h_{\mathfrak{p}})$. Thus, since the latter ideal is, by definition, equal to $\mathrm{Fit}^r_\Lambda(X_{(\mathfrak{p})})$ one has $\mathrm{FI}^r_\Lambda(h_{\mathfrak{p}}) \subseteq \mathrm{Fit}^r_\Lambda(X_{(\mathfrak{p})})$.

To prove the reverse inclusion it suffices to show that for each $(r_{h,2}-r)\times(r_{h,2}-r)$ minor N of $M(h_{\mathfrak{p}})$ the term $\det(N)$ belongs to $\mathrm{FI}^r_\Lambda(h_{\mathfrak{p}})$. To show this we assume that N is obtained by first deleting from $M(h_{\mathfrak{p}})$ the columns corresponding to a subset $J=\{i_1,i_2,\cdots,i_r\}$ of $[r_{h,2}]_r$ with $i_1<i_2<\cdots<i_r$, and then taking the rows corresponding to an element J_1 of $[r_{h,1}]_{r_{h,2}-r}$. We choose an element J'_1 of $[r_{h,1}]_{r_{h,2}}$ which contains J_1, label the elements of $J'_1\setminus J_1$ as $k_1<k_2<\cdots<k_r$ and then define an element $(\varphi_a)_{1\le a\le r}$ of $\mathrm{Hom}_\Lambda(\Lambda^{r_{h,1}},\Lambda)^r$ by setting $\varphi_a(b_i)=\delta_{ik_a}$ for each i with $1\le i\le r_{h,1}$.

Then an explicit computation shows that, with these choices, the determinant of the matrix $M(h_{\mathfrak{p}},J,\varphi)$ defined in (9) is equal to $\pm\det(N)$ and hence implies that $\det(N)$ belongs to $\mathrm{FI}^r_\Lambda(h_{\mathfrak{p}})$, as required. □

5.5.2. In this subsection we focus on the zero-th Fitting invariant and, in particular, relate it to an earlier construction of Nickel in [40].

To do this we note first that for every natural number m and every matrix M in $\mathrm{M}_m(\mathcal{A})$ there is a unique matrix M^* in $\mathrm{M}_m(A)$ with $MM^* = M^*M = \mathrm{Nrd}_A(M)\cdot I_m$ and such that for every primitive central idempotent e of A the matrix M^*e is non-zero if and only if $\mathrm{Nrd}_A(M)e$ is non-zero. Motivated by the result of [40, Th. 4.2], we then use this construction to define a subset of $\zeta(A)$ by setting

$$\mathfrak{A}(\mathcal{A}) := \{x\in\zeta(A) : \forall d\ge 1,\ \forall M\in \mathrm{M}_d(\mathcal{A}) \text{ one has } xM^*\in\mathrm{M}_d(\mathcal{A})\}.$$

The basic properties of this set are described in the following result.

Lemma 5.6.

(i) *$\mathfrak{A}(\mathcal{A})$ is an ideal of $\zeta(\mathcal{A})$.*

(ii) *An element x of $\zeta(A)$ belongs to $\mathfrak{A}(\mathcal{A})$ if and only if there exists a non-negative integer m_x such that for all $a\ge m_x$ and all $M\in\mathrm{M}_a(\mathcal{A})$ one has $xM^*\in\mathrm{M}_a(\mathcal{A})$.*

(iii) *$\mathfrak{A}(\mathcal{A})\cdot\xi(\mathcal{A}) = \mathfrak{A}(\mathcal{A})$.*

Proof. The set $\mathfrak{A}(\mathcal{A})$ is clearly an additive subgroup of $\zeta(A)$ and stable under multiplication by $\zeta(\mathcal{A})$. One also has $\mathfrak{A}(\mathcal{A})\subseteq\zeta(\mathcal{A})$ since if M is the 1×1 identity matrix, then $x = xM = xM^*$ and so $x = xM^*\in\mathrm{M}_1(\mathcal{A})$ implies $x\in\mathcal{A}\cap\zeta(A)=\zeta(\mathcal{A})$. This proves claim (i).

To prove claim (ii) it obviously suffices to show that the stated condition is sufficient to imply x belongs to $\mathfrak{A}(\mathcal{A})$. To do this we fix a natural number d and a matrix M in $\mathrm{M}_d(\mathcal{A})$ and note that in $\mathrm{M}_{d+m_x}(A)$ one has

$$x\begin{pmatrix} M & 0 \\ 0 & \mathrm{I}_{m_x}\end{pmatrix}^* = x\begin{pmatrix} M^* & 0 \\ 0 & \mathrm{Nrd}_A(M)\cdot \mathrm{I}_{m_x}\end{pmatrix} = \begin{pmatrix} xM^* & 0 \\ 0 & x\mathrm{Nrd}_A(M)\cdot \mathrm{I}_{m_x}\end{pmatrix}.$$

In particular, since $d+m_x > m_x$, the stated condition on x (with $a = d+m_x$ and M replaced by $\begin{pmatrix} M & 0 \\ 0 & \mathrm{I}_{m_x}\end{pmatrix}$) implies that xM^* belongs to $\mathrm{M}_d(\mathcal{A})$, as required.

Since 1 belongs to $\xi(\mathcal{A})$, to prove claim (iii) it suffices to show that for any x in $\mathfrak{A}(\mathcal{A})$, any natural number n, and any matrix N in $\mathrm{M}_n(\mathcal{A})$, the element $x' := x\cdot \mathrm{Nrd}_A(N)$ belongs to $\mathfrak{A}(\mathcal{A})$. We do this by showing that x' satisfies the condition described in claim (ii) with $m_{x'}$ taken to be n.

We thus fix an integer d with $d \geq n$ and choose N' in $\mathrm{M}_d(\mathcal{A})$ with $\mathrm{Nrd}_A(N') = \mathrm{Nrd}_A(N)$. Then, for any M in $\mathrm{M}_d(\mathcal{A})$ one has $M^*(N')^* = (N'M)^*$ and hence

$$x'M^* = x\cdot \mathrm{Nrd}_A(N)M^* = x\cdot \mathrm{Nrd}_A(N')M^* = x\cdot M^*((N')^*N') = (x\cdot (N'M)^*)N' \in \mathrm{M}_d(\mathcal{A})$$

where the containment is valid since, by assumption, the product $x\cdot (N'M)^*$ belongs to $\mathrm{M}_d(\mathcal{A})$. $\square$

Remark 5.7. The ideal $\mathfrak{A}(\mathcal{A})$ differs slightly from an ideal $\mathcal{H}(\mathcal{A})$ defined by Johnston and Nickel in [30] (the reason being that the above definition of M^* differs slightly from the 'generalized adjoint matrices' M^* defined in loc. cit.) Nevertheless, the extensive computations of $\mathcal{H}(\mathcal{A})$ made in loc. cit can be used to give concrete information about the ideal $\mathfrak{A}(\mathcal{A})$.

We can now state the main result of this subsection.

Proposition 5.8. *Let h be a presentation bundle of a finitely generated $\mathcal{A}$-module X. Then all of the following claims are valid.*

(i) $\zeta(\mathcal{A})\cdot \mathrm{FI}^0_{\mathcal{A}}(h) = \xi(\mathcal{A})\cdot \mathrm{Fit}_{\mathcal{A}}(h)$, *where* $\mathrm{Fit}_{\mathcal{A}}(h)$ *is the noncommutative Fitting invariant of h, as defined by Nickel.*

(ii) $\mathfrak{A}(\mathcal{A})\cdot \mathrm{FI}^0_{\mathcal{A}}(h) \subseteq \mathfrak{A}(\mathcal{A})\cdot \mathrm{FI}^{0,\mathrm{tot}}_{\mathcal{A}}(h) \subseteq \mathrm{Ann}_{\mathcal{A}}(X)$.

(iii) *If h is quadratic, then* $\mathrm{FI}^{0,\mathrm{tot}}_{\mathcal{A}}(h) = \mathrm{FI}^0_{\mathcal{A}}(h) = \xi(\mathcal{A})\cdot \mathrm{Nrd}_A(M(h))$ *and this ideal is independent of h. In particular, in any such case we may denote both* $\mathrm{FI}^{0,\mathrm{tot}}_{\mathcal{A}}(h)$ *and* $\mathrm{FI}^0_{\mathcal{A}}(h)$ *by* $\mathrm{FI}^0_{\mathcal{A}}(X)$.

(iv) *Let $0 \to X_1 \to X_2 \to X_3 \to 0$ be a short exact sequence of $\mathcal{A}$-modules. If X_1 and X_3 have quadratic presentation bundles, then so does X_2 and there is an equality* $\mathrm{FI}^0_{\mathcal{A}}(X_2) = \mathrm{FI}^0_{\mathcal{A}}(X_1)\mathrm{FI}^0_{\mathcal{A}}(X_3)$.

Proof. After localising (which we do not explicitly indicate) we can assume that h is a presentation of the form (7).

We write $\xi'(\mathcal{A})$ for the R-order in $\zeta(A)$ that is generated over $\zeta(\mathcal{A})$ by the elements $\mathrm{Nrd}_A(M)$ as M runs over matrices in $\bigcup_{n\geq 1}\mathrm{GL}_n(\mathcal{A})$. Then the invariant $\mathrm{Fit}_{\mathcal{A}}(h)$ is defined in [30, (3.3)] to be the $\xi'(\mathcal{A})$-submodule of $\zeta(A)$ that is generated by the elements $\mathrm{Nrd}_A(N)$ as N runs over all $r_{h,2}\times r_{h,2}$ minors of the matrix $M(h)$. Thus, since $\mathrm{FI}^0_{\mathcal{A}}(h)$ is defined to be the ideal of $\xi(\mathcal{A})$ that is generated by the same elements $\mathrm{Nrd}_A(N)$, the equality $\zeta(\mathcal{A})\cdot\mathrm{FI}^0_{\mathcal{A}}(h)=\xi(\mathcal{A})\cdot\mathrm{Fit}_{\mathcal{A}}(h)$ of claim (i) follows directly from the fact that $\zeta(\mathcal{A})\cdot\xi(\mathcal{A})=\xi(\mathcal{A})\cdot\xi'(\mathcal{A})$.

The first inclusion of claim (ii) is obvious. In addition, if h' is any presentation bundle that is finer than h and $\mathfrak{p}$ is a prime ideal of R, then $h'_{\mathfrak{p}}$ is the presentation bundle of an $\mathcal{A}_{(\mathfrak{p})}$-module Y for which there exists a surjective homomorphism of $\mathcal{A}_{(\mathfrak{p})}$-modules $Y\to X_{(\mathfrak{p})}$ and hence an inclusion $\mathrm{Ann}_{\mathcal{A}_{(\mathfrak{p})}}(Y)\subseteq\mathrm{Ann}_{\mathcal{A}}(X)_{(\mathfrak{p})}$. This observation shows that the second assertion of claim (ii) is reduced to proving $\mathfrak{A}(\mathcal{A})\cdot\mathrm{FI}^0_{\mathcal{A}}(h)\subseteq\mathrm{Ann}_{\mathcal{A}}(X)$.

To do this we fix x in $\mathfrak{A}(\mathcal{A})$ and y in $\xi(\mathcal{A})$. Then Lemma 5.6(iii) implies that the product $x':=xy$ belongs to $\mathfrak{A}(\mathcal{A})$. Next we let N be any $r_{h,2}\times r_{h,2}$ minor of the matrix $M(h)$. Then the argument of [40, Th. 4.2] combines with the assumption $r_{h,2}\geq m_{x'}$ to imply that $x'\cdot\mathrm{Nrd}_A(N)=x(y\cdot\mathrm{Nrd}_A(N))$ belongs to $\mathrm{Ann}_{\mathcal{A}}(X)$.

This implies the inclusion of claim (ii) since, as y varies over $\xi(\mathcal{A})$ and N over the $r_{h,2}\times r_{h,2}$ minors of $M(h)$, the elements $y\cdot\mathrm{Nrd}_A(N)$ run over a set of generators of the R-module $\mathrm{FI}^0_{\mathcal{A}}(h)$.

If h is quadratic, then it is clear that $\mathrm{FI}^0_{\mathcal{A}}(h)=\xi(\mathcal{A})\cdot\mathrm{Nrd}_A(M(h))$ and so the first assertion of claim (iii) is true if one can show that $\mathrm{FI}^{0,\mathrm{tot}}_{\mathcal{A}}(h)=\mathrm{FI}^0_{\mathcal{A}}(h)$. To show this it is enough to show that if h' is any $\mathcal{A}$-module presentation that is finer than the quadratic presentation h, then $\mathrm{FI}^0_{\mathcal{A}}(h')\subseteq\mathrm{FI}^0_{\mathcal{A}}(h)$.

We set $t:=r_{h,1}=r_{h,2}$ and note that, since h' is finer than h, one has $r_{h',1}=r_{h',2}=t$ and there exists a matrix U in $\mathrm{GL}_t(\mathcal{A})$ and a matrix V in $\mathrm{M}_t(\mathcal{A})$ with $M(h')U=VM(h)$. It follows that $\mathrm{Nrd}_A(M(h'))=\mathrm{Nrd}_A(U^{-1})\mathrm{Nrd}_A(V)\mathrm{Nrd}_A(M(h))$ and this implies the required inclusion since $\mathrm{FI}^0_{\mathcal{A}}(h')$ and $\mathrm{FI}^0_{\mathcal{A}}(h)$ are respectively generated over $\xi(\mathcal{A})$ by $\mathrm{Nrd}_A(M(h'))$ and $\mathrm{Nrd}_A(M(h))$ and the product $\mathrm{Nrd}_A(U^{-1})\mathrm{Nrd}_A(V)$ belongs to $\xi(\mathcal{A})$.

Finally we note that the second assertion of claim (iii), and the whole of claim (iv), are proved by a simple adaptation of the proofs of Nickel [40, Th. 3.2ii)] and [40, Prop. 3.5iii)] respectively. $\square$

5.5.3. We end this section by recording the connection between higher Fitting invariants and the Fitting lattices that were defined earlier.

Lemma 5.9. *Let h be a quadratic presentation of an $\mathcal{A}$-module X of the form (7) and set $t:=r_{h,2}$. Write $\mathcal{P}$ for the family in $\mathrm{slfp}_{\mathcal{A}}(\mathcal{A}^t)$ represented by the identity automorphism ι of $\mathcal{A}^t$ and π for the homomorphism of $\mathcal{A}$-modules $\mathcal{P}\to X$ induced by π_h.*

Then for each non-negative integer r one has

$$\{(\bigwedge_{i=1}^{i=r}\varphi_i)(\mathrm{F}_\pi({\textstyle\bigcap}^r_{\mathcal{A}}\mathcal{A}^t)):(\varphi_i)_{1\leq i\leq r}\in\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^t,\mathcal{A})^r\}=\mathrm{FI}^{r,\mathrm{tot}}_{\mathcal{A}}(h). \tag{10}$$

Proof. Note first that, with these definitions, a quadratic presentation of $\mathcal{A}$-modules h' factors through π if and only if it is finer than h.

Further, if we fix any such presentation h', set $\kappa:=\kappa_{h',\iota}$ and write $\{b_k\}_{1\leq k\leq t}$ for the standard basis of $\mathcal{A}^t$, then Proposition 2.6 implies that for any $(\varphi_i)_{1\leq i\leq r}$ in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^t,\mathcal{A})^r$

one has

$$
\begin{aligned}
(\bigwedge_{i=1}^{i=r}\varphi_i)(((\cap^r_{\mathcal{A}}\kappa_{h,\iota})\circ(\wedge_{j=r+1}^{j=t}\theta_{h',j}))(\wedge_{k=1}^{k=t}b_k)) &= (\bigwedge_{i=1}^{i=r}(\varphi_i\circ\kappa_{h,\iota})((\wedge_{j=r+1}^{j=t}\theta_{h',j})(\wedge_{k=1}^{k=t}b_k))\\
&= (\wedge_{i=1}^{i=r}(\varphi_i\circ\kappa_{h,\iota})\wedge(\wedge_{j=r+1}^{j=t}\theta_{h',j}))(\wedge_{k=1}^{k=t}b_k)\\
&= \mathrm{Nrd}_{\mathrm{M}_r(A^{\mathrm{op}})}(M),
\end{aligned}
$$

with $M=(M_{ij})$ the element of $\mathrm{M}_t(\mathcal{A})$ defined by

$$
M_{ij}:=\begin{cases}(\varphi_i\circ\kappa_{h,\iota})(b_j), & \text{if } 1\le i\le r \text{ and } 1\le j\le t,\\ \theta_{h',j}(b_j), & \text{if } r+1\le i\le t \text{ and } 1\le j\le t.\end{cases}
$$

Now, in terms of the notation (9), one has $M=M(h',J,\varphi')$ with $J=\{1,2,\cdots,r\}\in[t]_r$ and $\varphi'=(\varphi_i\circ\kappa_{h,\iota})_{1\le i\le r}\in\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^t,\mathcal{A})^r$. The above displayed expression therefore belongs to $\mathrm{FI}^r_{\mathcal{A}}(h')$. Since $\bigcap^t_{\mathcal{A}}\mathcal{A}^t=\xi(\mathcal{A})\cdot\wedge_{k=1}^{k=t}b_i$ (by Proposition 3.5(iii)) this in turn shows that the left hand side of (10) is contained in $\mathrm{FI}^{r,\mathrm{tot}}_{\mathcal{A}}(h)$.

To prove the reverse inclusion it suffices to fix a refinement h' of h and show that for any $\varphi=(\varphi_i)_{1\le i\le r}$ in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^t,\mathcal{A})^r$ and any J in $[t]_r$ there exists a refinement h'' of h and an element $\varphi'=(\varphi'_i)_{1\le i\le r}$ in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^t,\mathcal{A})^r$ such that $\mathrm{Nrd}_{A^{\mathrm{op}}}(M(h',J,\varphi))=(\bigwedge_{i=1}^{i=r}\varphi'_i)(x)$ for a suitable element x of $\mathrm{FL}^r_{\kappa_{h'',\iota}}(h'')$.

To do this we write $J=\{i_1,i_2,\cdots,i_r\}$ with $i_1<i_2<\cdots<i_r$. We then choose a permutation σ of $\{1,2,\cdots,t\}$ which satisfies $\sigma(j)=i_j$ for each j with $1\le j\le r$ and define κ_J to be the $\mathcal{A}$-module automorphism of $\mathcal{A}^t$ with $\kappa_J(b_j)=b_{\sigma(j)}$ for all j with $1\le j\le t$. We then define h'' to be the unique refinement of h for which $\theta_{h''}=\kappa_J^{-1}\circ\theta_{h'}$ and $\pi_{h''}=\pi_{h'}\circ\kappa_J$ and note that, with this definition, one can take $\kappa_{h'',\iota}=\kappa_{h',\iota}\circ\kappa_J$. We define the element φ' by setting $\varphi'_i=\varphi_i\circ\kappa^{-1}_{h'',\iota}$ for each i with $1\le i\le r$.

The proof is then completed by noting that $x:=((\cap^r_{\mathcal{A}}\kappa_{h'',\iota})\circ(\wedge_{j=r+1}^{j=t}\theta_{h'',j}))(\wedge_{k=1}^{k=t}b_k)$ belongs to $\mathrm{FL}^r_{\kappa_{h'',\iota}}(h'')$ and that

$$
\begin{aligned}
(\bigwedge_{i=1}^{i=r}\varphi'_i)(x) &= (\bigwedge_{i=1}^{i=r}(\varphi_i\circ\kappa^{-1}_{h'',\iota}))(((\cap^r_{\mathcal{A}}\kappa_{h'',\iota})\circ(\wedge_{j=r+1}^{j=t}\theta_{h'',j}))(\wedge_{k=1}^{k=t}b_k))\\
&= (\bigwedge_{i=1}^{i=r}\varphi_i)((\wedge_{j=r+1}^{j=t}\theta_{h',\sigma(j)})(\wedge_{k=1}^{k=t}b_k))\\
&= ((\wedge_{i=1}^{i=r}\varphi_i)\wedge(\wedge_{j=r+1}^{j=t}\theta_{h',\sigma(j)}))(\wedge_{k=1}^{k=t}b_k)\\
&= \pm\mathrm{Nrd}_{A^{\mathrm{op}}}(M(h',J,\varphi)),
\end{aligned}
$$

where the last equality follows from Corollary 2.9 and the definition (9) of $M(h',J,\varphi)$. □

Remark 5.10. If $\mathcal{A}$ is commutative, then Propositions 5.3(vii) and 5.5(iv) respectively imply that the left and right hand sides of the equality (10) are equal to $\mathrm{Fit}^r_{\mathcal{A}}(X)$.

In general, for any family $\mathcal{P}$ in $\mathrm{slfp}_{\mathcal{A}}(M)$, any homomorphism of $\mathcal{A}$-modules π with domain $\mathcal{P}$ and any integer r with $0\le r\le\mathrm{rk}_{\mathcal{A}}(\mathcal{P})$ the result of Lemma 5.9 leads to an expression for the $\xi(\mathcal{A})$-ideal

$$\{(\bigwedge_{i=1}^{i=r}\varphi_i)(\mathrm{F}_\pi({\bigcap}^r_{\mathcal{A}}P)) : (\varphi_i)_{1\le i\le r}\in \mathrm{Hom}_{\mathcal{A}}(P,\mathcal{A})^r\}$$

in terms of the higher Fitting invariants of quadratic presentation bundles which factor through π. For brevity, we leave the derivation of a precise such result to the reader.

Remark 5.11. Let M be a finitely generated free $\mathcal{A}$-module of rank d and fix a basis b. Let r be any non-negative integer with $0 \le r \le d$.

Then Proposition 3.5(iii) shows that there is an isomorphism of $\xi(\mathcal{A})$-modules

$$\kappa_b : {\bigcap}^r_{\mathcal{A}} M \cong (\bigoplus_{\sigma\in\left[{d \atop r}\right]} \xi(\mathcal{A}))\oplus \ker(\theta_b).$$

In this context, the result of Lemma 5.9 suggests that for any quadratic $\mathcal{A}$-module presentation h with $r_{h,2}=d$ the lattices $\kappa_b(\mathrm{F}_{\pi_h}({\bigcap}^r_{\mathcal{A}}M))$ and $(\bigoplus_{\sigma\in\left[{d \atop r}\right]}\mathrm{FI}^{r,\mathrm{tot}}_{\mathcal{A}}(h))\oplus\ker(\theta_b)$ should be closely related.

6. Non-commutative determinant modules

In this section we continue to fix data $R, F, \mathcal{A}$ and A as in §3.

We write $D(\mathcal{A})$ for the derived category of (left) $\mathcal{A}$-modules. We also write $C^{\mathrm{lf}}(\mathcal{A})$ for the category of bounded complexes of finitely generated locally-free $\mathcal{A}$-modules and $D^{\mathrm{lf}}(\mathcal{A})$ for the full triangulated subcategory of $D(\mathcal{A})$ comprising complexes that are isomorphic to a complex in $C^{\mathrm{lf}}(\mathcal{A})$.

We write $K_0^{\mathrm{lf}}(\mathcal{A})$ for the Grothendieck group of the category of finitely generated locally-free $\mathcal{A}$-modules. We observe that each object C of $D^{\mathrm{lf}}(\mathcal{A})$ gives rise to a canonical 'Euler characteristic' in $K_0^{\mathrm{lf}}(\mathcal{A})$ and we write this element as $\chi_{\mathcal{A}}(C)$.

We recall that the 'reduced locally-free classgroup' $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$ of $\mathcal{A}$ is defined to be the kernel of the homomorphism $K_0^{\mathrm{lf}}(\mathcal{A})\to\mathbb{Z}$ that is induced by sending each locally-free module M to $\mathrm{rk}_{\mathcal{A}}(M)$.

We write $C^{\mathrm{lf},0}(\mathcal{A})$ for the subcategory of $C^{\mathrm{lf}}(\mathcal{A})$ comprising complexes $P^\bullet$ for which $\chi_{\mathcal{A}}(P^\bullet)$ belongs to $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$ and $D^{\mathrm{lf},0}(\mathcal{A})$ for the full triangulated subcategory of $D^{\mathrm{lf}}(\mathcal{A})$ comprising complexes C for which $\chi_{\mathcal{A}}(C)$ belongs to $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$. (The latter condition is equivalent to requiring that C be isomorphic in $D(\mathcal{A})$ to an object of $C^{\mathrm{lf},0}(\mathcal{A})$).

In this section we shall associate a canonical invertible $\xi(\mathcal{A})$-module to each object of the category $D^{\mathrm{lf},0}(\mathcal{A})$. We show that such modules constitute a natural theory of 'non-commutative determinants' and hence provide a more explicit alternative (in our setting) to both the category of virtual objects constructed by Deligne in [18] and to the theory of noncommutative determinants and 'localized K_1-groups' constructed by Fukaya and Kato in [24, §1.2 and §1.3].

6.1. **Definitions and basic properties.** In this first section we associate a canonical determinant module to each complex in $C^{\mathrm{lf},0}(\mathcal{A})$ and investigate its basic properties.

6.1.1. At the outset we assume to be a given a locally-free $\mathcal{A}$-module P and set $r := \mathrm{rk}_{\mathcal{A}}(P)$ and $P_F := F \otimes_R P$.

For each prime ideal $\mathfrak{p}$ of R we fix an $\mathcal{A}_{(\mathfrak{p})}$-basis $\underline{b}_{\mathfrak{p}} = \{b_{\mathfrak{p},j}\}_{1\le j\le r}$ of the localization $P_{(\mathfrak{p})}$ and define a free rank one $\xi(\mathcal{A}_{(\mathfrak{p})})$-submodule of $\bigwedge_A^r P_F$ by setting

$$\bigwedge\nolimits^r_{\mathcal{A}_{(\mathfrak{p})}} P_{(\mathfrak{p})} := \xi(\mathcal{A}_{(\mathfrak{p})}) \cdot \wedge_{j=1}^{j=r} b_{\mathfrak{p},j}.$$

We then obtain a $\xi(\mathcal{A})$-submodule of $\bigwedge_A^r P_F$ by setting

$$\bigwedge\nolimits^r_{\mathcal{A}} P := \bigcap_{\mathfrak{p}} \bigwedge\nolimits^r_{\mathcal{A}_{(\mathfrak{p})}} P_{(\mathfrak{p})}$$

where the intersection is taken over all prime ideals $\mathfrak{p}$ of R.

The basic properties of this construction are recorded in the following result.

Lemma 6.1. *For each module P as above the following claims are valid.*

(i) *$\bigwedge_{\mathcal{A}}^r P$ is independent of the choice of bases $\{\underline{b}_{\mathfrak{p}}\}_{\mathfrak{p}}$.*

(ii) *If P is a free $\mathcal{A}$-module, with basis $\{b_j\}_{1\le j\le r}$, then $\bigwedge_{\mathcal{A}}^r P$ is a free rank one $\xi(\mathcal{A})$-module with basis $\wedge_{j=1}^{j=r} b_j$.*

(iii) *$\bigwedge_{\mathcal{A}}^r P$ is an invertible $\xi(\mathcal{A})$-module, with $(\bigwedge_{\mathcal{A}}^r P)_{(\mathfrak{p})} = \bigwedge^r_{\mathcal{A}_{(\mathfrak{p})}} P_{(\mathfrak{p})}$ for all prime ideals $\mathfrak{p}$ of R.*

(iv) *Let $\varrho : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism of R-orders. Write B for the F-algebra spanned by $\mathcal{B}$ and $\varrho_1 : A \to B$, $\varrho_2 : \zeta(A) \to \zeta(B)$ and $\varrho_3 : \xi(\mathcal{A}) \to \xi(\mathcal{B})$ for the surjective ring homomorphisms induced by ϱ. Then $\mathcal{B} \otimes_{\mathcal{A},\varrho} P$ is a locally-free $\mathcal{B}$-module and the natural isomorphism of $\zeta(B)$-modules $\zeta(B) \otimes_{\zeta(A),\varrho_2} \bigwedge_A^r P_F \cong \bigwedge_B^r (B \otimes_{A,\varrho_1} P_F)$ restricts to give an isomorphism of invertible $\xi(\mathcal{B})$-modules $\xi(\mathcal{B}) \otimes_{\xi(\mathcal{A}),\varrho_3} \bigwedge_{\mathcal{A}}^r P \cong \bigwedge_{\mathcal{B}}^r (\mathcal{B} \otimes_{\mathcal{A},\varrho} P)$, where the exterior powers in the latter module are defined with respect to the same E-bases of those simple A_E-modules which factor through B.*

(v) *If $0 \to P_1 \xrightarrow{\theta} P_2 \xrightarrow{\phi} P_3 \to 0$ is a short exact sequence of locally-free $\mathcal{A}$-modules, then there is a natural isomorphism of $\xi(\mathcal{A})$-modules $\bigwedge_{\mathcal{A}}^{r_2} P_2 \cong \bigwedge_{\mathcal{A}}^{r_1} P_1 \otimes_{\xi(\mathcal{A})} \bigwedge_{\mathcal{A}}^{r_3} P_3$ where we set $r_i := \mathrm{rk}_{\mathcal{A}}(P_i)$ for $i = 1, 2, 3$.*

Proof. To prove claim (i) it suffices to fix a prime ideal $\mathfrak{p}$ of R and to show that $\bigwedge^r_{\mathcal{A}_{(\mathfrak{p})}} P_{(\mathfrak{p})}$ is independent of the choice of bases $\underline{b}_{\mathfrak{p}}$.

To show this we note that if $\{b'_{\mathfrak{p},j}\}_{1\le j\le r}$ is any other choice of $\mathcal{A}_{(\mathfrak{p})}$-basis of $P_{(\mathfrak{p})}$, then Corollary 2.9 implies $\wedge_{j=1}^{j=r} b'_{\mathfrak{p},j} = \mathrm{Nrd}_A(U_{\mathfrak{p}}) \cdot \wedge_{j=1}^{j=r} b_{\mathfrak{p},j}$ for a matrix $U_{\mathfrak{p}}$ in $\mathrm{GL}_r(\mathcal{A}_{(\mathfrak{p})})$. This implies the required result since $\mathrm{Nrd}_A(U_{\mathfrak{p}})$ is a unit of $\xi(\mathcal{A}_{(\mathfrak{p})})$.

Claim (ii) is true since the stated conditions imply that one can take $b_{\mathfrak{p},i} = b_i$ for all i with $1 \le i \le r$.

To prove claim (iii) it suffices to show $\bigwedge_{\mathcal{A}}^r P$ is a full R-submodule of $\bigwedge_A^r P_F$. To show this we note that Roiter's Lemma implies the existence of a free rank r submodule P' of P and we choose an $\mathcal{A}$-basis $\{b'_i\}_{1\le i\le r}$ of P'. Then for each prime ideal $\mathfrak{p}$ of R Corollary 2.9 implies $\wedge_{j=1}^{j=r} b'_j = \mathrm{Nrd}_A(M_{\mathfrak{p}}) \cdot \wedge_{j=1}^{j=r} b_{\mathfrak{p},j}$ for a matrix $M_{\mathfrak{p}}$ in $\mathrm{M}_r(\mathcal{A}_{(\mathfrak{p})})$. Hence, since $\mathrm{Nrd}_A(M_{\mathfrak{p}})$

belongs to $\xi(\mathcal{A}_{(\mathfrak{p})})$, claim (ii) implies that

$$\bigwedge\nolimits_{\mathcal{A}}^{r} P' = \xi(\mathcal{A}) \cdot \wedge_{j=1}^{j=r} b'_j \subseteq \bigcap_{\mathfrak{p}} \bigwedge\nolimits_{\mathcal{A}_{(\mathfrak{p})}}^{r} P_{(\mathfrak{p})} = \bigwedge\nolimits_{\mathcal{A}}^{r} P.$$

This inclusion implies the required result since $\bigwedge_{\mathcal{A}}^{r} P'$ is a free $\xi(\mathcal{A})$-module.

Claim (iv) is verified by a straightforward exercise that we leave to the reader.

Turning to claim (v) we fix an $\mathcal{A}$-module section σ to ϕ. We note that $r_2 = r_1 + r_3$ and that for any given A-bases $\underline{b}_j := \{b_{j,a}\}_{1\le a\le r_j}$ of $P_{j,F}$ for $j = 1,3$ we obtain an A-basis $\underline{b}^{\sigma}_{1,3} := \{b^{\sigma}_{1,3,a}\}_{1\le a\le r_2}$ of $P_{2,F}$ by setting $b^{\sigma}_{1,3,a} = b_{1,a}$ if $1 \le a \le r_1$ and $b^{\sigma}_{1,3,a} = \sigma_i(b_{3,a-r_1})$ if $r_1 < a \le r_3$. We then write

$$\Delta : \bigwedge\nolimits_{A}^{r_2} P_{2,F} \cong \bigwedge\nolimits_{A}^{r_1} P_{1,F} \otimes_{\zeta(A)} \bigwedge\nolimits_{A}^{r_3} P_{3,F}$$

for the unique isomorphism of $\zeta(A)$-modules for which

$$\Delta(\wedge_{j=1}^{j=r_2} b^{\sigma}_{1,3,j}) = (\wedge_{s=1}^{s=r_1} b_{1,s}) \otimes_{\zeta(A)} (\wedge_{t=1}^{t=r_3} b_{3,t}). \tag{11}$$

By using Corollary 2.9 one shows easily that this isomorphism is independent of both the choices of bases $\underline{b}_1$ and $\underline{b}_3$ and the choice of section σ and we prove claim (iv) by showing that $\Delta(\bigwedge_{\mathcal{A}}^{r_2} P_2) = \bigwedge_{\mathcal{A}}^{r_1} P_1 \otimes_{\xi(\mathcal{A})} \bigwedge_{\mathcal{A}}^{r_3} P_3$.

It is enough to show this after localizing at each prime ideal $\mathfrak{p}$ of R and in this case the equality follows from claim (i) and the fact that we can choose the elements $\{b_{1,s}\}_{1\le s\le r_1}$ and $\{b_{3,t}\}_{1\le t\le r_3}$ to be $\mathcal{A}_{(\mathfrak{p})}$-bases of $P_{1,(\mathfrak{p})}$ and $P_{3,(\mathfrak{p})}$ and then our choice of section σ implies the set $\{b^{\sigma}_{1,3,j}\}_{1\le j\le r_2}$ defined above is an $\mathcal{A}_{(\mathfrak{p})}$-bases of $P_{2,(\mathfrak{p})}$. □

Remark 6.2. Let P be a free $\mathcal{A}$-module of rank one. If $\mathcal{A}$ is commutative, then there is clearly a natural identification $\bigwedge_{\mathcal{A}}^{1} P \cong P$. However, if $\mathcal{A}$ is not commutative, and P is non-zero, then the $\xi(\mathcal{A})$-modules $\bigwedge_{\mathcal{A}}^{1} P$ and P are not isomorphic.

6.1.2. In the sequel we use the following convenient notation. For any free rank one $\zeta(A)$-module W we set $W^1 := W$ and $W^{-1} := \mathrm{Hom}_{\zeta(A)}(W, \zeta(A))$, regarded as a (free rank one) $\zeta(A)$-module via the natural composition action. For each basis element w of W we set $w^1 := w$ and write w^{-1} for the (unique) basis element of W^{-1} which sends w to 1. For any invertible $\xi(\mathcal{A})$-module $\mathcal{L}$ we similarly define invertible $\xi(\mathcal{A})$-modules by setting $\mathcal{L}^1 := \mathcal{L}$ and $\mathcal{L}^{-1} := \mathrm{Hom}_{\xi(\mathcal{A})}(\mathcal{L}, \xi(\mathcal{A}))$.

We now assume to be given a complex $P^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$ of the form

$$\cdots \to P^i \xrightarrow{d^i} P^{i+1} \to \cdots \tag{12}$$

We set $P^\bullet_F := A \otimes_{\mathcal{A}} P^\bullet$ and for each integer i also $r_i := \mathrm{rk}_{\mathcal{A}}(P^i)$.

We define a free rank one $\zeta(A)$-module by setting

$$\det_A(P^\bullet_F) := \bigotimes_{i\in\mathbb{Z}} (\bigwedge\nolimits_{A}^{r_i} P^i_F)^{(-1)^i},$$

where the tensor product is over $\zeta(A)$. We also note that, following Corollary 2.8, if i is odd, then the $\zeta(A)$-module $(\bigwedge_{A}^{r_i} P^i_F)^{(-1)^i}$ can be identified with $\bigwedge_{A^{\mathrm{op}}}^{r_i} \mathrm{Hom}_A(P^i_F, A)$.

Following Lemma 6.1 we then obtain an invertible $\xi(\mathcal{A})$-submodule of $\det_A(P_F^\bullet)$ by setting

$$\det_{\mathcal{A}}(P^\bullet) := \bigotimes_{i\in\mathbb{Z}}(\bigwedge\nolimits_{\mathcal{A}}^{r_i}P^i)^{(-1)^i},$$

where the tensor product is over $\xi(\mathcal{A})$.

6.1.3. The basic properties of this construction are recorded in the following result.

Lemma 6.3. *Let $P^\bullet$ be a complex as in (12).*

(i) *For each prime ideal $\mathfrak{p}$ of R and each integer i fix an $\mathcal{A}_{(\mathfrak{p})}$-basis $\underline{b}_{\mathfrak{p},i} = \{b_{\mathfrak{p},i,j}\}_{1\le j\le r_i}$ of $P^i_{(\mathfrak{p})}$. Then $\bigotimes_{i\in\mathbb{Z}}(\wedge_{j=1}^{j=r_i}b_{\mathfrak{p},i,j})^{(-1)^i}$ is a $\xi(\mathcal{A}_{(\mathfrak{p})})$-basis of $\det_{\mathcal{A}}(P^\bullet)_{(\mathfrak{p})} = \det_{\mathcal{A}_{(\mathfrak{p})}}(P^\bullet_{(\mathfrak{p})})$.*

(ii) *If each $\mathcal{A}$-module P^i is free, with basis $\{b_{i,j}\}_{1\le j\le r_i}$, then $\bigotimes_{i\in\mathbb{Z}}(\wedge_{j=1}^{j=r_i}b_{i,j})^{(-1)^i}$ is a $\xi(\mathcal{A})$-basis of $\det_{\mathcal{A}}(P^\bullet)$.*

(iii) *If $\chi_{\mathcal{A}}(P^\bullet)$ belongs to $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$, then $\det_{\mathcal{A}}(P^\bullet)$ is independent of the choice of bases of the simple A_E-modules used to define exterior products (via (1) and (2)).*

(iv) *Let $\varrho : \mathcal{A}\to\mathcal{B}$ be a surjective homomorphism of R-orders and write $\varrho' : \xi(\mathcal{A})\to\xi(\mathcal{B})$ for the surjective ring homomorphism induced by ϱ and B for the F-algebra spanned by $\mathcal{B}$. Then $\mathcal{B}\otimes_{\mathcal{A},\varrho}P^\bullet$ is an object of $C^{\mathrm{lf}}(\mathcal{B})$ and there is a natural isomorphism of invertible $\xi(\mathcal{B})$-modules $\xi(\mathcal{B})\otimes_{\xi(\mathcal{A}),\varrho'}\det_{\mathcal{A}}(P^\bullet)\cong\det_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{A},\varrho}P^\bullet)$, where the exterior powers in the latter module are defined with respect to the same E-bases of those simple A_E-modules which factor through B.*

(v) *If $0\to P_1^\bullet\to P_2^\bullet\to P_3^\bullet\to 0$ is a short exact sequence in $C^{\mathrm{lf}}(\mathcal{A})$, then there is a natural isomorphism of $\xi(\mathcal{A})$-modules $\det_{\mathcal{A}}(P_2^\bullet)\cong\det_{\mathcal{A}}(P_1^\bullet)\otimes_{\xi(\mathcal{A})}\det_{\mathcal{A}}(P_3^\bullet)$.*

(vi) *If $P^\bullet$ is acyclic, then $\det_{\mathcal{A}}(P^\bullet)$ is naturally isomorphic to $\xi(\mathcal{A})$.*

(vii) *The invertible $\xi(\mathcal{A})$-modules $\det_{\mathcal{A}}(P^\bullet[1])$ and $\det_{\mathcal{A}}(P^\bullet)^{-1}$ are naturally isomorphic.*

(viii) *Each quasi-isomorphism $\lambda : P_1^\bullet\to P_2^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$ induces a canonical isomorphism of $\xi(\mathcal{A})$-modules $\det_{\mathcal{A}}(\lambda) : \det_{\mathcal{A}}(P_1^\bullet)\cong\det_{\mathcal{A}}(P_2^\bullet)$.*

Proof. Claim (i) follows easily from the fact that in each degree i the element $(\wedge_{j=1}^{j=r_i}b_{\mathfrak{p},i,j})^{(-1)^i}$ is a basis of the $\xi(\mathcal{A}_{(\mathfrak{p})})$-module $(\bigwedge_A^{r_i}P_F^i)^{(-1)^i})_{(\mathfrak{p})}$.

Claim (ii) follows directly from the definition of $\det_{\mathcal{A}}(P^\bullet)$ and the result of Lemma 6.1(ii).

To prove claim (iii) we fix E-bases $\{v_j\}_{1\le j\le d}$ and $\{w_j\}_{1\le j\le d}$ of a choice of simple A_E-module V. We write $M = (M_{st})$ for the matrix in $\mathrm{GL}_d(E)$ which satisfies $w_s = \sum_{t=1}^{t=d}M_{st}v_t$ for each integer s and note that this implies $w_s^* = \sum_{t=1}^{t=d}N_{st}v_t^*$ for each integer s where we set $N := (M^{\mathrm{tr}})^{-1}$. Using these equalities one computes that in each even degree i there is an equality

$$\bigwedge\nolimits_{1\le j\le r_i}(\bigwedge\nolimits_{1\le s\le d}w_s^*\otimes b_{\mathfrak{p},i,j}) = \det(N)^{r_i}\cdot\bigwedge\nolimits_{1\le j\le r_i}(\bigwedge\nolimits_{1\le s\le d}v_s^*\otimes b_{\mathfrak{p},i,j})$$

and in each odd degree i an equality

$$\bigwedge\nolimits_{1\le j\le r_i}(\bigwedge\nolimits_{1\le s\le d}w_s\otimes b_{\mathfrak{p},i,j}^{-1}) = \det(M)^{r_i}\cdot\bigwedge\nolimits_{1\le j\le r_i}(\bigwedge\nolimits_{1\le s\le d}v_s\otimes b_{\mathfrak{p},i,j}^{-1}).$$

Since $\det(M) = \det(N)^{-1}$ this implies that the tensor product over all integers i of these respective terms differ by a factor of $\det(N)^{\sum_{i\in\mathbb{Z}}(-1)^ir_i}$. This then implies the stated result since if the image of $\chi_{\mathcal{A}}(P^\bullet)$ in $K_0(A)$ vanishes, then one has $\sum_{i\in\mathbb{Z}}(-1)^ir_i = 0$.

The isomorphism in claim (iv) is obtained by applying the result of Lemma 6.1(iv) to each of the modules P^i

Turning to claim (v) we note that the given exact sequence induces in each degree i a short exact sequence of locally-free $\mathcal{A}$-modules $0 \to P_1^i \to P_2^i \to P_3^i \to 0$ and hence, via Lemma 6.1(v), a canonical isomorphism of $\xi(\mathcal{A})$-modules

$$\kappa_i : \bigwedge\nolimits_{\mathcal{A}}^{r_{2i}} P_2^i \cong \bigwedge\nolimits_{\mathcal{A}}^{r_{1i}} P_1^i \otimes_{\xi(\mathcal{A})} \bigwedge\nolimits_{\mathcal{A}}^{r_{3i}} P_3^i$$

where for each $j = 1, 2, 3$ we set $r_{ji} := \mathrm{rk}_{\mathcal{A}}(P_j^i)$. We note also that this map induces a composite isomorphism of $\xi(\mathcal{A})$-modules

$$\kappa_i^* : (\bigwedge\nolimits_{\mathcal{A}}^{r_{2i}} P_2^i)^{-1} \to (\bigwedge\nolimits_{\mathcal{A}}^{r_{1i}} P_1^i \otimes_{\xi(\mathcal{A})} \bigwedge\nolimits_{\mathcal{A}}^{r_{3i}} P_3^i)^{-1} \cong (\bigwedge\nolimits_{\mathcal{A}}^{r_{1i}} P_1^i)^{-1} \otimes_{\xi(\mathcal{A})} (\bigwedge\nolimits_{\mathcal{A}}^{r_{3i}} P_3^i)^{-1}$$

where the first map is $\mathrm{Hom}_{\xi(\mathcal{A})}(\kappa^i, \xi(\mathcal{A}))^{-1}$ and the second is obtained by restriction of the unique isomorphism of $\zeta(A)$-modules

$$(\bigwedge\nolimits_{A}^{r_{1i}} P_{1,F}^i \otimes_{\zeta(A)} \bigwedge\nolimits_{A}^{r_{3i}} P_{3,F}^i)^{-1} \cong (\bigwedge\nolimits_{A}^{r_{1i}} P_{1,F}^i)^{-1} \otimes_{\zeta(A)} (\bigwedge\nolimits_{A}^{r_{3i}} P_{3,F}^i)^{-1}$$

which, for any (and therefore every) choice of $\zeta(A)$-bases w_{1i} and w_{3i} of $\bigwedge_A^{r_{1i}} P_{1,F}^i$ and $\bigwedge_A^{r_{3i}} P_{3,F}^i$, sends the element $(w_{1i} \otimes_{\zeta(A)} w_{3i})^{-1}$ to $w_{1i}^{-1} \otimes_{\zeta(A)} w_{3i}^{-1}$.

We then define the stated isomorphism $\det_{\mathcal{A}}(P_2^\bullet) \cong \det_{\mathcal{A}}(P_1^\bullet) \otimes_{\xi(\mathcal{A})} \det_{\mathcal{A}}(P_3^\bullet)$ to be that which is induced by the map $\bigotimes_{i\in\mathbb{Z}} \kappa_i'$ with $\kappa_i' = \kappa_i$ if i is even and $\kappa_i' = \kappa_i^*$ if i is odd.

To prove claim (vi) we assume $P^\bullet$ is acyclic. This implies, by an easy downward induction on i, firstly that each $\mathcal{A}$-module $\mathrm{im}(d^i) = \ker(d^{i+1})$ is projective and hence that there is an isomorphism of $\mathcal{A}$-modules $P^i \cong \ker(d^i) \oplus \mathrm{im}(d^i)$, and secondly that each module $P^i/\ker(d^i) \cong \mathrm{im}(d^i)$ is locally-free.

We now write a for the lowest degree in which the module P^i is non-zero, $P_1^\bullet$ for the complex $P^a \xrightarrow{d^a} \mathrm{im}(d^a)$ where the first term is placed in degree a, ι for the natural inclusion of complexes $P_1^\bullet \to P^\bullet$ and we note the complex $\mathrm{cok}(\iota)$ is acyclic and belongs to $C^{\mathrm{lf}}(\mathcal{A})$. In particular, by applying claim (iii) to the tautological exact sequence of complexes in $C^{\mathrm{lf}}(\mathcal{A})$

$$0 \to P_1^\bullet \xrightarrow{\iota} P^\bullet \to \mathrm{cok}(\iota) \to 0, \tag{13}$$

one can use an induction on the number of non-zero modules P^i to reduce claim (iv) to the case that $P^\bullet$ is concentrated in degrees a and $a+1$, for some integer a.

In this case, if we choose an A-basis $\underline{b} = \{b_i\}_{1\le i\le r_a}$ of P_F^a, then

$$\det_{\mathcal{A}}(P^\bullet)_F = \zeta(A) \cdot (\wedge_{j=1}^{j=r_a} b_j)^{(-1)^a} \otimes_{\zeta(A)} (\wedge_{j=1}^{j=r_a} d^a(b_j))^{(-1)^{a+1}}$$

and we write $\Delta' : \det_{\mathcal{A}}(P^\bullet)_F \to \zeta(A)$ for the $\zeta(A)$-module isomorphism which sends the element $(\wedge_{j=1}^{j=r_a} b_j)^{(-1)^a} \otimes_{\zeta(A)} (\wedge_{j=1}^{j=r_a} d^a(b_j))^{(-1)^{a+1}}$ to 1.

We claim that Δ' is independent of the choice of basis $\underline{b}$. To show this let $\{b_i'\}_{1\le i\le r_a}$ be another A-basis of P_F^a and define a matrix $M = (M_{ij})$ in $\mathrm{GL}_{r_a}(A)$ by the equalities $b_i' = \sum_{j=1}^{j=r_a} M_{ij} \cdot b_j$, or equivalently $d^a(b_i') = \sum_{j=1}^{j=r_a} M_{ij} \cdot d^a(b_j)$, for all i with $1 \le i \le r_a$.

By applying Corollary 2.9 in this context we can therefore deduce that

$$\begin{aligned}
&(\wedge_{j=1}^{j=r_a} b'_j)^{(-1)^a} \otimes_{\zeta(A)} (\wedge_{j=1}^{j=r_a} d^a(b'_j))^{(-1)^{a+1}}\\
=&(\mathrm{Nrd}_A(M)\cdot \wedge_{j=1}^{j=r_a} b_j)^{(-1)^a} \otimes_{\zeta(A)} (\mathrm{Nrd}_A(M)\cdot \wedge_{j=1}^{j=r_a} d^a(b_j))^{(-1)^{a+1}}\\
=&\mathrm{Nrd}_A(M)^{(-1)^a}\mathrm{Nrd}_A(M)^{(-1)^{a+1}}\cdot (\wedge_{j=1}^{j=r_a} b_j)^{(-1)^a} \otimes_{\zeta(A)} (\wedge_{j=1}^{j=r_a} d^a(b_j))^{(-1)^{a+1}}\\
=&(\wedge_{j=1}^{j=r_a} b_j)^{(-1)^a} \otimes_{\zeta(A)} (\wedge_{j=1}^{j=r_a} d^a(b_j))^{(-1)^{a+1}},
\end{aligned}$$

as required.

Now, since Δ' is independent of the choice of $\underline{b}$, for any given prime ideal $\mathfrak{p}$ one can choose $\underline{b}$ to be an $\mathcal{A}_{(\mathfrak{p})}$-basis of $P^a_{(\mathfrak{p})}$ so that

$$\det_{\mathcal{A}}(P^\bullet)_{(\mathfrak{p})} = \xi(\mathcal{A})_{(\mathfrak{p})}\cdot (\wedge_{j=1}^{j=r_a} b_j^{(-1)^a}) \otimes_{\zeta(A)} (\wedge_{j=1}^{j=r_a} d^a(b_j)^{(-1)^{a+1}}).$$

In particular, with this choice it is clear that $\Delta'(\det_{\mathcal{A}}(P^\bullet)_{(\mathfrak{p})}) = \xi(\mathcal{A})_{(\mathfrak{p})}$ and, since this equality is true for all prime ideals $\mathfrak{p}$, one thus has $\Delta'(\det_{\mathcal{A}}(P^\bullet)) = \xi(\mathcal{A})$. In this case therefore the isomorphism in claim (iv) is obtained by restricting Δ' to $\det_{\mathcal{A}}(P^\bullet)$.

To prove claim (vii) we write $\mathrm{Cone}_{P^\bullet}$ for the mapping cone of the identity endomorphism of $P^\bullet$ and use the composite isomorphism

$$\kappa : \det_{\mathcal{A}}(P^\bullet) \otimes_{\xi(\mathcal{A})} \det_{\mathcal{A}}(P^\bullet[1]) \cong \det_{\mathcal{A}}(\mathrm{Cone}_{P^\bullet}) \cong \xi(\mathcal{A}).$$

Here the first isomorphism results from applying claim (iii) to the natural short exact sequence $0 \to P^\bullet \to \mathrm{Cone}_{P^\bullet} \to P^\bullet[1] \to 0$ in $C^{\mathrm{lf}}(\mathcal{A})$ and the second is the isomorphism of claim (iv) for the acyclic complex $\mathrm{Cone}_{P^\bullet}$. One then obtains a canonical isomorphism of invertible $\xi(\mathcal{A})$-modules $\det_{\mathcal{A}}(P^\bullet[1]) \to \mathrm{Hom}_{\xi(\mathcal{A})}(\det_{\mathcal{A}}(P^\bullet), \xi(\mathcal{A}))$ by sending each element x to the map $y \mapsto \kappa(y \otimes x)$.

Finally, to prove claim (viii) we adapt an argument of Knudsen and Mumford [33, proof of Th. 1]. To do this we denote by $Z^\bullet_\lambda$ the complex with $Z^i_\lambda = P^i_1 \oplus P^i_2 \oplus P^{i+1}_1$ and the differential in degree i is represented by the matrix

$$\begin{pmatrix} d^i_1 & 0 & -1\\ 0 & d^i_2 & \lambda^{i+1}\\ 0 & 0 & -d^{i+1}_1 \end{pmatrix}$$

where we write d^i_j for the differential of $P^\bullet_j$ in degree i. With this notation there are quasi-isomorphisms $\lambda_1 : P^\bullet_1 \to Z^\bullet_\lambda$, $\lambda_2 : P^\bullet_2 \to Z^\bullet_\lambda$ and $\lambda'_2 : Z^\bullet_\lambda \to P^\bullet_2$ in $C^{\mathrm{lf}}(\mathcal{A})$ with

$$\lambda_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\quad \lambda_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}\quad \text{and}\quad \lambda'_2 = \begin{pmatrix}\lambda\\1\\0\end{pmatrix}.$$

One checks that $\lambda'_2 \circ \lambda_1 = \lambda$ and $\lambda'_2 \circ \lambda_2 = \mathrm{id}_{P^\bullet_2}$. In addition, the complexes $\mathrm{cok}(\lambda_1)$ and $\mathrm{cok}(\lambda_2)$ are acyclic objects of $C^{\mathrm{lf}}(\mathcal{A})$ and so there are natural composite isomorphisms

$$\det_{\mathcal{A}}(\lambda_i)' : \det_{\mathcal{A}}(P^\bullet_i) \cong \det_{\mathcal{A}}(P^\bullet_i) \otimes_{\xi(\mathcal{A})} \det_{\mathcal{A}}(\mathrm{cok}(\lambda_i)) \cong \det_{\mathcal{A}}(Z^\bullet_\lambda) \tag{14}$$

for $i = 1$ and $i = 2$, where the first map is obtained by applying claim (v) to the complex $\mathrm{cok}(\lambda_i)$ and the second by applying claim (iii) to the tautological short exact sequence $0 \to P_i^\bullet \xrightarrow{\lambda_i} Z_\lambda^\bullet \to \mathrm{cok}(\lambda_1) \to 0$.

We then obtain an isomorphism of the required sort by setting

$$\det_{\mathcal{A}}(\lambda) := (\det_{\mathcal{A}}(\lambda_2)')^{-1} \circ \det_{\mathcal{A}}(\lambda_1)'.$$

□

6.1.4. The result of Lemma 6.3(i) motivates us to make the following definition.

Definition 6.4. We say that a complex $P^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$ is 'free' if each $\mathcal{A}$-module P^i is free.

For any such complex $P^\bullet$ we shall say that a basis element b of the (free rank one) $\mathcal{A}$-module $\det_{\mathcal{A}}(P^\bullet)$ is 'primitive' if it is equal to $\bigotimes_{i\in\mathbb{Z}}(\wedge_{j=1}^{j=r_i} b_{i,j}^{(-1)^i})$ for some choice of bases $\{b_{i,j}\}_{1\le j\le r_i}$ of the modules P^i.

The key properties of such bases that we shall use in the sequel are recorded in the next two results.

In the sequel we write $\mathrm{sr}(\mathcal{A})$ for the stable range of $\mathcal{A}$. We recall that Bass has shown that $\mathrm{sr}(\mathcal{A}) = 1$ if R is local, and hence $\mathcal{A}$ is semi-local, and that $\mathrm{sr}(\mathcal{A}) = 2$ in all other cases. (For more details see [16, Th. (40.31)] and [16, Th. (40.41)] respectively).

Lemma 6.5. *Let $P^\bullet$ be a free complex in $C^{\mathrm{lf}}(\mathcal{A})$ of the form (12) for which there exists an integer i with $r_i \ge \mathrm{sr}(\mathcal{A})$.*

Let b be a primitive basis of $\det_{\mathcal{A}}(P^\bullet)$. Then any other element b' of $\det_{\mathcal{A}}(P^\bullet)_F$ is a primitive basis of $\det_{\mathcal{A}}(P^\bullet)$ if and only if $b' = u \cdot b$ with u in $\mathrm{Nrd}_A(K_1(\mathcal{A}))$.

Proof. Necessity of the given conditions follows from Corollary 2.9 (in just the same way as did Lemma 6.1(i)).

To prove sufficiency we first apply [16, Th. (40.42)] to deduce the existence of a matrix $u_i = (u_{i,ab})$ in $\mathrm{GL}_{r_i}(\mathcal{A})$ with $\mathrm{Nrd}_A(u_i)^{(-1)^i} = u$.

We then fix bases $\{b_{s,t}\}_{1\le t\le r_s}$ of the $\mathcal{A}$-modules P^s such that $b = \bigotimes_{s\in\mathbb{Z}}(\wedge_{t=1}^{t=r_s} b_{s,t})^{(-1)^s}$ and write $\{b'_{s,t}\}_{1\le t\le r_s}$ for the basis of each module P^s obtained by setting $b'_{s,t} = b_{s,t}$ if $s \neq i$ and $b'_{i,t} = \sum_{w=1}^{w=r_i} u_{i,tw} b_{i,w}$. Then Corollary 2.9 implies that

$$u \cdot b = \mathrm{Nrd}_A(u_i)^{(-1)^i} \cdot \bigotimes_{s\in\mathbb{Z}}(\wedge_{t=1}^{t=r_s} b_{s,t})^{(-1)^s} = \bigotimes_{s\in\mathbb{Z}}(\wedge_{t=1}^{t=r_s} b'_{s,t})^{(-1)^s},$$

as required. □

In the next result we use the result of Lemma 6.3(viii).

Lemma 6.6. *Let $\lambda : P_1^\bullet \to P_2^\bullet$ be a quasi-isomorphism in $C^{\mathrm{lf}}(\mathcal{A})$ between free complexes, each of which has at least one term of rank at least $\mathrm{sr}(\mathcal{A})$.*

Then an element b of $\det_{\mathcal{A}}(P_1^\bullet)_F$ is a primitive basis of $\det_{\mathcal{A}}(P_1^\bullet)$ if and only if the image of b under $A \otimes_{\mathcal{A}} \det_{\mathcal{A}}(\lambda)$ is a primitive basis of $\det_{\mathcal{A}}(P_2^\bullet)$.

Proof. For $j = 1$ and $j = 2$ we choose in each degree i an $\mathcal{A}$-basis $\{b_{j,ik}\}_{1\le k\le r_{ji}}$ of P_j^i, where we set $r_{ji} := \mathrm{rk}_{\mathcal{A}}(P_j^i)$ and then write b_j for the corresponding primitive basis $\bigotimes_{i\in\mathbb{Z}}(\wedge_{k=1}^{k=r_{ji}} b_{j,ik})^{(-1)^i}$ of $\det_{\mathcal{A}}(P_j^\bullet)$.

Then Lemma 6.5 implies that the stated claim is true if and only if there exists an element u of $\mathrm{Nrd}_A(K_1(\mathcal{A}))$ with $\det_{\mathcal{A}}(\lambda)(b_1) = u \cdot b_2$. To show this we adopt the notation of the proof of Lemma 6.3(viii).

We observe first that in each degree i the set of elements $(b_{1,ia_1}, b_{2,ia_2}, b_{1,(i+1)a_3})$, with $1 \le a_1 \le r_{1i}, 1 \le a_2 \le r_{2i}$ and $1 \le a_3 \le r_{1(i+1)}$ constitutes an $\mathcal{A}$-basis of Z^i_λ and we write b_λ for the corresponding primitive basis of $\det_{\mathcal{A}}(Z^\bullet_\lambda)$. In the same way the elements $(0, b_{2,ia_2}, b_{1,(i+1)a_3})$ and $(b_{1,ia_1}, 0, b_{1,(i+1)a_3})$ give rise to primitive bases b'_1 and b'_2 of $\det_{\mathcal{A}}(\mathrm{cok}(\lambda_1))$ and $\det_{\mathcal{A}}(\mathrm{cok}(\lambda_2))$ respectively.

An explicit computation shows that, for both $j = 1$ and $j = 2$, the isomorphism $\det(\lambda_j)'$ in (14) sends b_j to $\mu_j(b'_j)^{-1} \cdot b_\lambda$ with μ_j the isomorphism $\det_{\mathcal{A}}(\mathrm{cok}(\lambda_j)) \cong \xi(\mathcal{A})$ induced by applying Lemma 6.3(vi) to the acyclic complex $\mathrm{cok}(\lambda_j)$, and hence that

$$\det_{\mathcal{A}}(\lambda)(b_1) = (\det_{\mathcal{A}}(\lambda_2)')^{-1}(\det_{\mathcal{A}}(\lambda_1)'(b_1)) = \mu_2(b'_2)\mu_1(b'_1)^{-1} \cdot b_2.$$

Given this, the required equality follows directly from the result of Lemma 6.7 below. □

In the following result we use the result of Lemma 6.3(vi).

Lemma 6.7. *Let $P^\bullet$ be an acyclic free complex in $C^{\mathrm{lf}}(\mathcal{A})$. Then the canonical isomorphism $\det_{\mathcal{A}}(P^\bullet) \cong \xi(\mathcal{A})$ sends each primitive basis of $\det_{\mathcal{A}}(P^\bullet)$ to an element of $\mathrm{Nrd}_A(K_1(\mathcal{A}))$.*

Proof. Lemma 6.5 reduces us to proving that in each degree i one can choose an $\mathcal{A}$-basis $\{b_{i,j}\}_{1\le j\le r_i}$ of P^i such that the isomorphism constructed in Lemma 6.3(vi) sends $\bigotimes_{i\in\mathbb{Z}}(\wedge_{j=1}^{j=r_i} b_{i,j})^{(-1)^i}$ to 1.

To show this we adopt the notation of the proof of Lemma 6.3(vi) and so argue by induction on the number of non-zero modules P^i.

We choose $\mathcal{A}$-bases $\{b_{i,j}\}_{1\le j\le r_i}$ of P^i for each $i \in \mathbb{Z}\setminus\{a+1\}$ and an $\mathcal{A}$-basis $\{b'_j\}_{1\le j\le t}$ of $P^{a+1}/\mathrm{im}(d^a)$. We also choose an $\mathcal{A}$-invariant splitting σ of the tautological exact sequence $0 \to \mathrm{im}(d^a) \to P^{a+1} \to P^{a+1}/\mathrm{im}(d^a) \to 0$ and then use it to define a basis $\{b_{i,j}\}_{1\le j\le r_{a+1}}$ of P^{a+1} by setting $b_{a+1,j} = d^a(b_{a,j})$ if $1 \le j \le r_a$ and $b_{a+1,j} = \sigma(b'_{j-r_a})$ if $r_a < j \le r_{a+1}$.

Now the isomorphism $\Delta : \det_{\mathcal{A}}(P^\bullet) \cong \det_{\mathcal{A}}(P^\bullet) \otimes_{\xi(\mathcal{A})} \det_{\mathcal{A}}(\mathrm{cok}(\iota))$ obtained by applying Lemma 6.3(v) to the exact sequence (13) sends the primitive basis $\bigotimes_{i\in\mathbb{Z}}(\wedge_{j=1}^{j=r_i} b_{i,j})^{(-1)^i}$ of $\det_{\mathcal{A}}(P^\bullet)$ to $x \otimes y$ for the primitive bases $x = (\wedge_{j=1}^{j=r_a} b_{a,j})^{(-1)^a} \otimes_{\xi(\mathcal{A})} (\wedge_{j=1}^{j=r_{a+1}} d^a(b_{a,j}))^{(-1)^{a+1}}$ of $\det_{\mathcal{A}}(P_1^\bullet)$ and $y = \bigotimes_{i\ge a+1}(\wedge_{j=1}^{j=r_i} b_{i,j})^{(-1)^i}$ of $\det_{\mathcal{A}}(\mathrm{cok}(\iota))$.

Arguing by induction, it is therefore enough to note that our definition of the isomorphism $\det_{\mathcal{A}}(P_1^\bullet) \cong \xi(\mathcal{A})$ ensures that x is sent to 1. □

6.2. **Extension to the derived category.** In this section we extend relevant aspects of the above construction of determinant modules to objects of the category $D^{\mathrm{lf},0}(\mathcal{A})$.

To do this we choose for any C in $D^{\mathrm{lf},0}(\mathcal{A})$ a representative complex $P^\bullet$ in $C^{\mathrm{lf},0}(\mathcal{A})$ and then set

$$\det_{\mathcal{A}}(C) := \det_{\mathcal{A}}(P^\bullet).$$

6.2.1. The basic properties of this definition are recorded in the following result.

Proposition 6.8.

(i) *For any object C of $D^{\mathrm{lf},0}(\mathcal{A})$ the $\xi(\mathcal{A})$-module $\det_{\mathcal{A}}(C)$ defined above depends, up to canonical isomorphism, only on C.*
(ii) *If $C_1 \to C_2 \to C_3 \to C_1[1]$ is any exact triangle in $D^{\mathrm{lf},0}(\mathcal{A})$, then there is a canonical isomorphism of $\xi(\mathcal{A})$-modules $\det_{\mathcal{A}}(C_2) \cong \det_{\mathcal{A}}(C_1) \otimes_{\xi(\mathcal{A})} \det_{\mathcal{A}}(C_3)$.*
(iii) *Let $\varrho : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism of R-orders and write $\varrho' : \xi(\mathcal{A}) \to \xi(\mathcal{B})$ for the ring homomorphism induced by ϱ. Then for any object C of $D^{\mathrm{lf},0}(\mathcal{A})$ the derived tensor product $\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A},\varrho} C$ belongs to $D^{\mathrm{lf},0}(\mathcal{B})$ and there is a canonical isomorphism of invertible $\xi(\mathcal{B})$-modules $\xi(\mathcal{B}) \otimes_{\xi(\mathcal{A}),\varrho'} \det_{\mathcal{A}}(C) \cong \det_{\mathcal{A}}(\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A},\varrho} C)$.*

Proof. To prove claim (i) we note first that, by a standard argument, if $P_1^\bullet$ and $P_2^\bullet$ are any objects of $C^{\mathrm{lf}}(\mathcal{A})$ for which there are isomorphisms $\theta_1 : P_1^\bullet \cong C^\bullet$ and $\theta_2 : P_2^\bullet \cong C^\bullet$ in $D^{\mathrm{lf}}(\mathcal{A})$, then there exists a quasi-isomorphism $\lambda : P_1^\bullet \cong P_2^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$ that is unique up to homotopy and such that $\theta_1 = \theta_2 \circ \lambda$ in $D^{\mathrm{lf}}(\mathcal{A})$.

Recalling the results of Lemma 6.3(iii) and (v), claim (i) will therefore follow if for any quasi-isomorphisms $P_1^\bullet \xrightarrow{\lambda} P_2^\bullet$ and $P_1^\bullet \xrightarrow{\mu} P_2^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$ which differ by a homotopy one has $\det_{\mathcal{A}}(\lambda) = \det_{\mathcal{A}}(\mu)$.

To prove this we follow the proof of [33, Prop. 2]. Thus, we note that if H is any choice of homotopy with $\lambda - \mu = d_2 \circ H + H \circ d_1$, where d_1 and d_2 denotes the differentials of $P_1^\bullet$ and $P_2^\bullet$, then, in terms of the notation in the proof of Lemma 6.3(vi), there is an isomorphism $\theta : Z_\lambda^\bullet \to Z_\mu^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$ that is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & H \\ 0 & 0 & 1 \end{pmatrix}$$

and is such that the following diagram in $C^{\mathrm{lf}}(\mathcal{A})$ commutes

$$\begin{array}{ccccc} P_1^\bullet & \xrightarrow{\lambda_1} & Z_\lambda^\bullet & \xleftarrow{\lambda_2} & P_2^\bullet \\ \| & & \downarrow{\scriptstyle\theta} & & \| \\ P_1^\bullet & \xrightarrow{\mu_1} & Z_\mu^\bullet & \xleftarrow{\mu_2} & P_2^\bullet . \end{array} \tag{15}$$

The first and second squares in this diagram can be completed to give an isomorphism of short exact sequences in $C^{\mathrm{lf}}(\mathcal{A})$ of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_i^\bullet & \xrightarrow{\lambda_i} & Z_\lambda^\bullet & \longrightarrow & \mathrm{cok}(\lambda_i) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\kappa_i} & & \\ 0 & \longrightarrow & P_i^\bullet & \xrightarrow{\mu_i} & Z_\lambda^\bullet & \longrightarrow & \mathrm{cok}(\mu_i) & \longrightarrow & 0 \end{array}$$

with $i = 1$ and $i = 2$ respectively, in which in each degree j, the map

$$\kappa_i^j : P_2^j \oplus P_1^{j+1} = \mathrm{cok}(\lambda_i)^j \to \mathrm{cok}(\mu_i)^j = P_2^j \oplus P_1^{j+1}$$

is induced by the matrix $\begin{pmatrix} 1 & H \\ 0 & 1 \end{pmatrix}$ if $i = 1$ and is the identity map if $i = 2$.

In particular, since for both $i = 1$ and $i = 2$ one has $\mathrm{Nrd}_A(\kappa_i^j) = 1$ in each degree j, there is a commutative diagram of isomorphisms of $\xi(\mathcal{A})$-modules

$$\begin{array}{ccccc}
\det_{\mathcal{A}}(P_1^\bullet) & \xrightarrow{\det(\lambda_1)'} & \det_{\mathcal{A}}(Z_\lambda^\bullet) & \xleftarrow{\det(\lambda_2)'} & \det_{\mathcal{A}}(P_2^\bullet) \\
\| & & \downarrow & & \| \\
\det_{\mathcal{A}}(P_1^\bullet) & \xrightarrow{\det(\mu_1)'} & \det_{\mathcal{A}}(Z_\mu^\bullet) & \xleftarrow{\det(\mu_2)'} & \det_{\mathcal{A}}(P_2^\bullet)
\end{array}$$

in which the unlabeled map is induced by the isomorphisms $\{\theta^i\}_{i\in\mathbb{Z}}$. This commutative diagram implies immediately that $\det_{\mathcal{A}}(\lambda) = \det_{\mathcal{A}}(\mu)$, as claimed.

Turning to claim (ii) we first choose complexes $P_1^\bullet$ and $P_3^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$ for which there exist isomorphisms $\alpha_j : P_j^\bullet \to C_j$ in $D^{\mathrm{lf}}(\mathcal{A})$ for $i = 1, 3$. We then choose a morphism $\mu : P_3^\bullet[-1] \to P_1^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$ such that $\alpha_1 \circ \mu = -w[-1] \circ \alpha_3[-1]$ in $D(\mathcal{A})$ and write $P_2^\bullet$ for the mapping cone of μ. Then, by the axioms of a triangulated category, there is an isomorphism $\alpha_2 : P_2^\bullet \to C_2$ in $D(\mathcal{A})$ making the following diagram commute (in $D(\mathcal{A})$)

$$\begin{array}{ccccccc}
P_3^\bullet[-1] & \xrightarrow{\mu} & P_1^\bullet & \xrightarrow{\theta} & P_2^\bullet & \xrightarrow{\phi} & P_3^\bullet \\
\downarrow{\scriptstyle \alpha_3[-1]} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} \\
C_3[-1] & \xrightarrow{-w[-1]} & C_1 & \xrightarrow{u} & C_2 & \xrightarrow{v} & C_3
\end{array}$$

where θ and ϕ are the natural morphisms (coming from the definition of $P_2^\bullet$ as the mapping cone of μ) and the lower row is the exact triangle that is induced by shifting the given triangle.

Given this construction, the proof of claim (i) reduces the proof of claim (ii) to showing that for any morphism of short exact sequences in $C^{\mathrm{lf}}(\mathcal{A})$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_{11}^\bullet & \xrightarrow{\theta_1} & P_{12}^\bullet & \xrightarrow{\phi_1} & P_{13}^\bullet & \longrightarrow & 0 \\
 & & {\scriptstyle \lambda_1}\downarrow & & {\scriptstyle \lambda_2}\downarrow & & {\scriptstyle \lambda_3}\downarrow & & \\
0 & \longrightarrow & P_{21}^\bullet & \xrightarrow{\theta_2} & P_{22}^\bullet & \xrightarrow{\phi_2} & P_{23}^\bullet & \longrightarrow & 0
\end{array} \tag{16}$$

in which the vertical maps are quasi-isomorphisms there is a commutative diagram of isomorphisms of $\xi(\mathcal{A})$-modules

$$\begin{array}{ccc}
\det_{\mathcal{A}}(P_{12}^\bullet) & \longrightarrow & \det_{\mathcal{A}}(P_{11}^\bullet) \otimes_{\xi(\mathcal{A})} \det_{\mathcal{A}}(P_{13}^\bullet) \\
{\scriptstyle \det_{\mathcal{A}}(\lambda_2)}\downarrow & & \downarrow{\scriptstyle \det_{\mathcal{A}}(\lambda_1)\otimes_{\xi(\mathcal{A})}\det_{\mathcal{A}}(\lambda_3)} \\
\det_{\mathcal{A}}(P_{22}^\bullet) & \longrightarrow & \det_{\mathcal{A}}(P_{21}^\bullet) \otimes_{\xi(\mathcal{A})} \det_{\mathcal{A}}(P_{23}^\bullet)
\end{array} \tag{17}$$

where the horizontal isomorphisms are obtained by applying Lemma 6.3(v) to the upper and lower rows of (16).

To show this we use the fact that, in terms of the notation used in the proof of Lemma 6.3(vi), the diagram (16) gives rise for both $i = 1$ and $i = 2$ to a commutative diagram in

$C^{\mathrm{lf}}(\mathcal{A})$ of the form

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P^\bullet_{i1} \arrow[r, "\theta_i"] \arrow[d, "\lambda_{1i}"'] & P^\bullet_{i2} \arrow[r, "\phi_i"] \arrow[d, "\lambda_{2i}"'] & P^\bullet_{i3} \arrow[r] \arrow[d, "\lambda_{3i}"'] & 0 \\
0 \arrow[r] & Z^\bullet_{\lambda_1} \arrow[r, "\theta'"] \arrow[d] & Z^\bullet_{\lambda_2} \arrow[r, "\phi'"] \arrow[d] & Z^\bullet_{\lambda_3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{cok}(\lambda_{1i}) \arrow[r, "\theta'_i"] \arrow[d] & \mathrm{cok}(\lambda_{2i}) \arrow[r, "\phi'_i"] \arrow[d] & \mathrm{cok}(\lambda_{3i}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{18}
$$

in which each row and column is a short exact sequence. Here θ' and ϕ' are respectively induced by the matrices $\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_1\end{pmatrix}$ and $\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_1\end{pmatrix}$, the unlabeled vertical arrows are the tautological projections and θ'_i and ϕ'_i are the maps that are induced by the (obvious) commutativity of the upper squares.

We use (18) to construct the following diagram of isomorphisms of $\xi(\mathcal{A})$-modules.

$$
\begin{tikzcd}
\mathrm{D}(P^\bullet_{12}) \arrow[r, "\rho_{11}"] & \mathrm{D}(P^\bullet_{11}) \otimes \mathrm{D}(P^\bullet_{13}) \\
\mathrm{D}(P^\bullet_{12}) \otimes \mathrm{D}(\mathrm{cok}(\lambda_{12})) \arrow[u, "\mathrm{id}\otimes\epsilon_{12}"] \arrow[r, "\rho_{11}\otimes\rho_{13}"] \arrow[d, "\kappa_{12}"'] & (\mathrm{D}(P^\bullet_{11}) \otimes \mathrm{D}(\mathrm{cok}(\lambda_{11}))) \otimes (\mathrm{D}(P^\bullet_{13}) \otimes \mathrm{D}(\mathrm{cok}(\lambda_{13}))) \arrow[u, "(\mathrm{id}\otimes\epsilon_{11})\otimes(\mathrm{id}\otimes\epsilon_{13})"'] \arrow[d, "\kappa_{11}\otimes\kappa_{13}"] \\
\mathrm{D}(Z^\bullet_{\lambda_2}) \arrow[r, "\rho_2"] & \mathrm{D}(Z^\bullet_{\lambda_1}) \otimes \mathrm{D}(Z^\bullet_{\lambda_3}) \\
\mathrm{D}(P^\bullet_{22}) \otimes \mathrm{D}(\mathrm{cok}(\lambda_{22})) \arrow[u, "\kappa_{22}"] \arrow[r, "\rho_{21}\otimes\rho_{23}"] \arrow[d, "\mathrm{id}\otimes\epsilon_{22}"'] & (\mathrm{D}(P^\bullet_{21}) \otimes \mathrm{D}(\mathrm{cok}(\lambda_{21}))) \otimes (\mathrm{D}(P^\bullet_{23}) \otimes \mathrm{D}(\mathrm{cok}(\lambda_{23}))) \arrow[u, "\kappa_{21}\otimes\kappa_{23}"'] \arrow[d, "(\mathrm{id}\otimes\epsilon_{21})\otimes(\mathrm{id}\otimes\epsilon_{23})"] \\
\mathrm{D}(P^\bullet_{22}) \arrow[r, "\rho_{21}"] & \mathrm{D}(P^\bullet_{21}) \otimes \mathrm{D}(P^\bullet_{23}).
\end{tikzcd}
$$

In this diagram we abbreviate $\det_{\mathcal{A}}(-)$ to $\mathrm{D}(-)$, write ϵ_{ij} for the isomorphism of Lemma 6.3(vi) for the acyclic complex $\mathrm{cok}(\lambda_{ij})$, κ_{ij} and ρ_{ij} for the isomorphisms obtained by applying Lemma 6.3(v) to the j-th column, resp. j-th row, of the diagram (18) and set $\rho_2 = \rho_{12} = \rho_{22}$. The commutativity of the upper and lower squares is straightforward to check and the commutativity of the two remaining squares follows readily after ensuring that the sections that are chosen in the construction (in the proof of Lemma 6.3(v)) of the isomorphisms κ_{ij} and ρ_{ij} are compatible in each degree with the diagram (18), as described in Lemma 6.10 below (with $\Lambda = \mathcal{A}$).

Finally, to complete the proof of claim (ii) we note that the left and right hand side composite isomorphisms in the above diagram are (by definition) respectively equal to $\det_{\mathcal{A}}(\lambda_2)$ and $\det_{\mathcal{A}}(\lambda_1) \otimes_{\xi(\mathcal{A})} \det_{\mathcal{A}}(\lambda_3)$, and so the commutativity of this diagram is equivalent to that of the required diagram (17).

Turning to claim (iii), it is first clear that $\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A},\varrho} C$ belongs to $D^{\mathrm{lf},0}(\mathcal{B})$. In addition, the required isomorphism $\xi(\mathcal{B}) \otimes_{\xi(\mathcal{A}),\varrho'} \det_{\mathcal{A}}(C) \cong \det_{\mathcal{A}}(\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A},\varrho} C)$ is then simply obtained by combining claim (i) with the result of Lemma 6.3(iv) (and noting that, in this case, the choice of bases of the simple A_E-modules used to define the exterior products (1) and (2) is irrelevant by virtue of Lemma 6.3(iii)). □

Remark 6.9. Proposition 6.8(i) implies, in particular, that for any acyclic object C of $D^{\mathrm{lf}}(\Lambda)$ there exists a canonical isomorphism of $\xi(\Lambda)$-modules $\det_\Lambda(C) \cong \xi(\Lambda)$.

Lemma 6.10. *Let Λ be a noetherian ring. We assume to be given a commutative diagram of short exact sequences of finitely generated projective Λ-modules of the form*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \xrightarrow{d_1'} & N_1 & \xrightarrow{d_1} & P_1 & \longrightarrow & 0 \\
 & & {\scriptstyle\epsilon_1}\downarrow & & {\scriptstyle\phi_1}\downarrow & & {\scriptstyle\kappa_1}\downarrow & & \\
0 & \longrightarrow & M_2 & \xrightarrow{d_2'} & N_2 & \xrightarrow{d_2} & P_2 & \longrightarrow & 0 \\
 & & {\scriptstyle\epsilon_2}\downarrow & & {\scriptstyle\phi_2}\downarrow & & {\scriptstyle\kappa_2}\downarrow & & \\
0 & \longrightarrow & M_3 & \xrightarrow{d_3'} & N_3 & \xrightarrow{d_3} & P_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Then there exist Λ-equivariant sections $\sigma_i : P_i \to N_i$ to d_i for $i = 1, 2$ and 3 such that there are commutative diagrams of Λ-modules

$$\begin{array}{ccc}
N_1 & \xleftarrow{\sigma_1} & P_1 \\
{\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle\kappa_1} \\
N_2 & \xleftarrow[\sigma_2]{} & P_2
\end{array}
\quad \textit{and} \quad
\begin{array}{ccc}
N_2 & \xleftarrow{\sigma_2} & P_2 \\
{\scriptstyle\phi_2}\downarrow & & \downarrow{\scriptstyle\kappa_2} \\
N_3 & \xleftarrow[\sigma_3]{} & P_3.
\end{array} \tag{19}$$

Proof. First choose any Λ-equivariant section σ to d_2 and write θ for the composite homomorphism $\phi_2 \circ \sigma \circ \kappa_1 : P_1 \to N_3$.

The commutativity of the given diagram implies that there exists a unique homomorphism θ_1 in $\mathrm{Hom}_\Lambda(P_1, M_3)$ such that $\theta = d_3' \circ \theta_1$. Since P_1 is a projective Λ-module we can then choose a homomorphism θ_2 in $\mathrm{Hom}_\Lambda(P_1, M_2)$ with $\theta_1 = \epsilon_2 \circ \theta_2$.

Next we note that, since P_3 is a projective Λ-module, the group $\mathrm{Ext}^1_\Lambda(P_3, M_2)$ vanishes and so there exists a homomorphism θ_3 in $\mathrm{Hom}_\Lambda(P_2, M_2)$ with $\theta_2 = \theta_3 \circ \kappa_1$.

We now set $\sigma_2 := \sigma - d_2' \circ \theta_3 \in \mathrm{Hom}_\Lambda(P_2, N_2)$. Then σ_2 is a section to d_2 since $d_2 \circ \sigma_2 = d_2 \circ \sigma - (d_2 \circ d_2') \circ \theta_3 = d_2 \circ \sigma$. In addition, for x in P_1 one has

$$\begin{aligned}
\phi_2(\sigma_2(\kappa_1(x))) &= \phi_2(\sigma(\kappa_1(x))) - \phi_2(d_2' \circ \theta_3(\kappa_1(x))) \\
&= \theta(x) - d_3'(\epsilon_2(\theta_3 \circ \kappa_1)(x)) \\
&= \theta(x) - d_3'((\epsilon_2 \circ \theta_2)(x)) \\
&= \theta(x) - (d_3' \circ \theta_1)(x) \\
&= \theta(w') - \theta(w') = 0.
\end{aligned}$$

Since P_1 is a projective Λ-module this implies there exists a unique homomorphism σ_1 in $\mathrm{Hom}_\Lambda(P_1, N_1)$ which makes the first diagram in (19) commute (with respect to our fixed map σ_2) and hence that $\kappa_1(d_1 \circ \sigma_1) = (d_2 \circ \sigma_2) \circ \kappa_1 = \kappa_1$ so that σ_1 is a section to d_1.

Finally we note that the commutativity of the first diagram in (19) implies there exists a (unique) homomorphism σ_3 in $\mathrm{Hom}_\Lambda(P_3, N_3)$ which makes the second diagram in (19) commute and one checks easily that this homomorphism is a section to d_3, as required. $\square$

6.2.2. In this section we prove a consequence of Proposition 6.8(ii) in a special case that will play an important role in the sequel.

To do so we assume to be given an exact triangle in $D^{\mathrm{lf},0}(\mathcal{A})$

$$C_1 \xrightarrow{\theta_1} C_2 \xrightarrow{\theta_2} C_3 \xrightarrow{\theta_3} C_1[1], \tag{20}$$

in which each cohomology group $H^a(C_j)$ is finite for all $a \notin \{0,1\}$, an extension field E of F which splits A and a commutative diagram of A_E-modules

$$\begin{tikzcd}
0 \arrow[r] & H^0(C_1)_E \arrow[r, "H^0(\theta_1)_E"] \arrow[d, "\tau_1"'] & H^0(C_2)_E \arrow[r, "H^0(\theta_2)_E"] \arrow[d, "\tau_2"'] & H^0(C_3)_E \arrow[d, "\tau_3"'] & \\
& H^1(C_1)_E \arrow[r, "H^1(\theta_1)_E"] & H^1(C_2)_E \arrow[r, "H^1(\theta_2)_E"] & H^1(C_3)_E \arrow[r] & 0
\end{tikzcd} \tag{21}$$

in which the vertical maps are bijective (and the horizontal rows exact).

We write $A = \prod_{i \in I} A_i$ for the decomposition of A into simple components and for each index i we fix a simple left A_i-module V_i. Then for each index $j = 1, 2, 3$ we define a composite isomorphism of $\zeta(A_E)$-modules

$$\vartheta_{\tau_j} : \mathrm{det}_{\mathcal{A}}(C_j)_E = \mathrm{det}_{A_E}(C_{j,E}) = \prod_{i \in I} \mathrm{det}_E(\mathrm{Hom}_{A_i}(V_i, C_{j,E})) \xrightarrow{(\vartheta_{j,i})_i} \prod_{i \in I} E = \zeta(A_E). \tag{22}$$

Here each $\vartheta_{j,i}$ is the composite isomorphism of E-spaces

$$\begin{aligned}&\det_E(\mathrm{Hom}_{A_i}(V_i, C_{j,E}))\\ &\qquad\cong {\bigwedge}_E^{n_{j,i}} \mathrm{Hom}_{A_i}(V_i, H^0(C_j)_E) \otimes_E \mathrm{Hom}_E({\bigwedge}_E^{n_{j,i}} \mathrm{Hom}_{A_i}(V_i, H^1(C_j)_E), E) \cong E\end{aligned}$$

with $n_{j,i} = \dim_E \mathrm{Hom}_{A_i}(V_i, H^0(C_j)_E)$, where the first map is the natural 'passage to cohomology' isomorphism and the second is the composite of the map that is induced by the restriction to $\mathrm{Hom}_{A_i}(V_i, H^0(C_j)_E)$ of the isomorphism τ_j and the natural evaluation pairing on $\mathrm{Hom}_{A_i}(V_i, H^1(C_j)_E)$.

Proposition 6.11. *Fix an exact triangle (20) and a commuting diagram (21). Assume that each complex C_j is isomorphic in $D(\mathcal{A})$ to a complex in $C^{\mathrm{lf}}(\mathcal{A})$ that is concentrated in degrees zero and one and, in addition, that the map $H^0(\theta_3)$ has finite image.*

Then there is a commutative diagram of $\zeta(A_E)$-module isomorphisms

$$\begin{array}{ccc} \det_{\mathcal{A}}(C_2)_E & \longrightarrow & \det_{\mathcal{A}}(C_1)_E \otimes_{\zeta(A_E)} \det_{\mathcal{A}}(C_3)_E \\ {\scriptstyle \vartheta_{\tau_2}}\big\downarrow & & \big\downarrow{\scriptstyle \vartheta_{\tau_1}\otimes_{\zeta(A_E)}\vartheta_{\tau_3}} \\ \zeta(A_E) & = & \zeta(A_E) \otimes_{\zeta(A_E)} \zeta(A_E), \end{array} \tag{23}$$

in which the upper horizontal map is the scalar extension of the isomorphism of $\xi(\mathcal{A})$-modules obtained by applying Proposition 6.8(ii) to (20).

Proof. Just as in the proof of Proposition 6.8(ii) we can replace the exact triangle (20) by a short exact sequence of complexes as in Lemma 6.3(v). We can also assume in this case that the complexes $P_j^\bullet$ are concentrated in degrees zero and one.

Then there are commutative diagrams of A_E-modules

$$\begin{array}{ccccccc} P^0_{1,E} & \xrightarrow{d_{1,E}} & B_1 & \qquad & P^1_{1,E} & \xrightarrow{\pi_{1,E}} & H^1(C_1)_E \\ {\scriptstyle \theta^0_{1,E}}\big\downarrow & & \big\downarrow{\scriptstyle \theta^1_{1,E}} & & {\scriptstyle \theta^1_{1,E}}\big\downarrow & & \big\downarrow{\scriptstyle H^1(\theta_1)_E} \\ P^0_{2,E} & \xrightarrow{d_{2,E}} & B_2 & & P^1_{2,E} & \xrightarrow{\pi_{2,E}} & H^1(C_2)_E \\ {\scriptstyle \theta^0_{2,E}}\big\downarrow & & \big\downarrow{\scriptstyle \theta^1_{2,E}} & & {\scriptstyle \theta^1_{2,E}}\big\downarrow & & \big\downarrow{\scriptstyle H^1(\theta_2)_E} \\ P^0_{3,E} & \xrightarrow{d_{3,E}} & B_3 & & P^1_{3,E} & \xrightarrow{\pi_{3,E}} & H^1(C_3)_E, \end{array}$$

where we write d_i for the differential (in degree zero) of $P_i^\bullet$, set $B_i := \mathrm{im}(d_i)_E$ and write π_i for the tautological surjection $P_i^1 \to H^1(P_i^\bullet)$.

Each column in these diagrams is a short exact sequence (by virtue of our assumption that $H^0(\theta_3)_E$ is the zero map) and each horizontal map is surjective and so we can apply Lemma 6.10 (with $\Lambda = A_E$) to choose A_E-module sections σ_i to $d_{i,E}$ and σ'_j to $\pi_{j,E}$ that are compatible with each of these diagrams.

In this way one obtains from the exact commutative diagram (21) a commutative diagram of A_E-modules

$$\begin{tikzcd}
P^0_{1,E} \arrow[r, "\theta^0_{1,E}"] & P^0_{2,E} \arrow[r, "\theta^0_{2,E}"] & P^0_{3,E} \\
B_1 \oplus H^0(C_1)_E \arrow[u, "{(\sigma_1,\subseteq)}"] \arrow[r, "{(\theta^1_{1,E},H^0(\theta_1)_E)}"] \arrow[d, "{(\mathrm{id},\tau_1)}"'] & B_2 \oplus H^0(C_2)_E \arrow[u, "{(\sigma_2,\subseteq)}"] \arrow[r, "{(\theta^1_{2,E},H^0(\theta_2)_E)}"] \arrow[d, "{(\mathrm{id},\tau_2)}"'] & B_3 \oplus H^0(C_3)_E \arrow[u, "{(\sigma_3,\subseteq)}"] \arrow[d, "{(\mathrm{id},\tau_3)}"'] \\
B_1 \oplus H^1(C_1)_E \arrow[r, "{(\theta^1_{1,E},H^1(\theta_1)_E)}"] \arrow[d, "{(\subseteq,\sigma'_1)}"'] & B_2 \oplus H^1(C_2)_E \arrow[r, "{(\theta^1_{2,E},H^1(\theta_2)_E)}"] \arrow[d, "{(\subseteq,\sigma'_2)}"'] & B_3 \oplus H^1(C_3)_E \arrow[d, "{(\subseteq,\sigma'_3)}"'] \\
P^1_{1,E} \arrow[r, "\theta^1_{1,E}"] & P^1_{2,E} \arrow[r, "\theta^1_{2,E}"] & P^1_{3,E}
\end{tikzcd} \tag{24}$$

in which every row is a short exact sequence and every vertical map is bijective.

For each of $j = 1, 2, 3$ and $i = 0, 1$ we set $r_j := \mathrm{rk}_{\mathcal{A}}(P^0_j) = \mathrm{rk}_{\mathcal{A}}(P^1_j)$ and write W^i_j for the free rank one $\zeta(A_E)$-module $\bigwedge^{r_j}_{A_E} P^i_{j,E}$. We also write λ_i for the isomorphism of $\zeta(A_E)$-modules $W^0_j \to W^1_j$ that is induced by the composite vertical isomorphism given by the j-th column of (24).

Then the commutativity of (24) directly implies the commutativity of the following diagram of $\zeta(A_E)$-modules

$$\begin{tikzcd}
W^0_2 \otimes (W^1_2)^{-1} \arrow[r, "\kappa"] \arrow[d, "\lambda_2\otimes\mathrm{id}"'] & (W^0_1 \otimes (W^1_1)^{-1}) \otimes (W^0_3 \otimes (W^1_3)^{-1}) \arrow[d, "(\lambda_1\otimes\mathrm{id})\otimes(\lambda_3\otimes\mathrm{id})"] \\
W^1_2 \otimes (W^1_2)^{-1} \arrow[d] & (W^1_1 \otimes (W^1_1)^{-1}) \otimes (W^1_3 \otimes (W^1_3)^{-1}) \arrow[d] \\
\zeta(A_E) \arrow[r, equal] & \zeta(A_E) \otimes \zeta(A_E),
\end{tikzcd}$$

in which all tensor products are taken over $\zeta(A_E)$, κ is induced by the maps κ_0 and κ_1^* which occur in the proof of Lemma 6.3(v) and the unlabeled left and right hand vertical arrows are induced by the evaluation maps on W^1_2, and on W^1_1 and W^1_3, respectively.

It is straightforward to check that the left and right hand vertical maps in this last commutative diagram are respectively equal to ϑ_{τ_2} and $\vartheta_{\tau_1} \otimes_{\zeta(A_E)} \vartheta_{\tau_3}$. To deduce the claimed commutativity of the diagram (23) we therefore need only note that the upper horizontal arrow in that diagram is, by definition, equal to κ. $\square$

Remark 6.12. The assumption made in Proposition 6.11 that C_j is isomorphic to a complex in $C^{\mathrm{lf}}(\mathcal{A})$ that is concentrated in degrees zero and one is an additional restriction (since it implies, for example, that the module $H^0(C_j)$ is R-torsion-free). Using more sophisticated techniques (as in [4]) it is possible to prove more general versions of Proposition 6.11 in which, for example, both this hypothesis and the hypothesis that the image of $H^0(\theta_3)$ is finite is removed. However, the result of Proposition 6.11 is sufficient for our present purposes and so, for brevity, we do not discuss such generalizations here.

6.3. **Primitive and locally-primitive bases.** In this section we use the results of §6.1.4 to defines notions of 'primitive basis' and 'locally-primitive basis' in the context of the determinant modules defined in §6.2.

6.3.1. The following definitions will play a key role in the formulation of our central arithmetic conjecture (see Conjecture 9.2 and Proposition 9.4).

Definition 6.13. Let C be an object of $D^{\mathrm{lf},0}(\mathcal{A})$.

We say that an element b of $\det_{\mathcal{A}}(C)$ is a 'primitive basis' if C is isomorphic in $D^{\mathrm{lf}}(\mathcal{A})$ to a complex $P^\bullet$ in $C^{\mathrm{lf},0}(\mathcal{A})$ that is both free and such that in some degree i one has $\mathrm{rk}_{\mathcal{A}}(P^i) \geq \mathrm{sr}(\mathcal{A})$ and, with respect to the induced identification $\det_{\mathcal{A}}(C^\bullet) = \det_{\mathcal{A}}(P^\bullet)$, the element b is a primitive basis of $\det_{\mathcal{A}}(P^\bullet)$.

Definition 6.14. Let C be an object of $D^{\mathrm{lf},0}(\mathcal{A})$.

Then we say that an element b of $\det_{\mathcal{A}}(C)$ is a 'locally-primitive basis' if for every prime ideal $\mathfrak{p}$ of R the image of b in $\det_{\mathcal{A}}(C)_{(\mathfrak{p})} = \det_{\mathcal{A}_{(\mathfrak{p})}}(C_{(\mathfrak{p})})$ is a primitive basis of $\det_{\mathcal{A}_{(\mathfrak{p})}}(C_{(\mathfrak{p})})$.

Remark 6.15. It is clear that any primitive basis of $\det_{\mathcal{A}}(C)$ is also a locally-primitive basis of $\det_{\mathcal{A}}(C)$. In addition, Lemma 6.3(ii) implies that each locally-primitive basis of $\det_{\mathcal{A}}(C)$ is a basis of the $\xi(\mathcal{A})$-module $\det_{\mathcal{A}}(C)$.

Finally we note that Lemma 6.6 implies the notion of primitive basis is intrinsic to C. More precisely, it shows that if b is a primitive basis of $\det_{\mathcal{A}}(P_1^\bullet)$ for any free complex $P_1^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$ that is both isomorphic in $D^{\mathrm{lf}}(\mathcal{A})$ to C and such that in some degree i one has $\mathrm{rk}_{\mathcal{A}}(P_1^i) \geq \mathrm{sr}(\mathcal{A})$, then it also corresponds to a primitive basis of $\det_{\mathcal{A}}(P_2^\bullet)$ for any other such complex $P_2^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$.

6.3.2. We show that, for any complex C in $D^{\mathrm{lf},0}(\mathcal{A})$, both the freeness of the $\xi(\mathcal{A})$-module $\det_{\mathcal{A}}(C)$ and the existence of a primitive basis of $\det_{\mathcal{A}}(C)$ are determined by properties of the Euler characteristic $\chi_{\mathcal{A}}(C)$. However, before stating the precise result, we must make some observations concerning classgroups of orders.

We note first that the argument of [16, Rem. (49.11)(iv)] shows that $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$ is naturally isomorphic to the 'locally-free classgroup' $\mathrm{Cl}(\mathcal{A})$ of $\mathcal{A}$, as defined in [16, (49.10)].

We recall $\mathrm{Cl}(\mathcal{A})$ is finite, that it is equal to the set of stable isomorphism classes $[I]$ of invertible $\mathcal{A}$-modules I and that the addition is defined by setting $[I_1]+[I_2] := [I_3]$ whenever there is an isomorphism of $\mathcal{A}$-modules of the form $I_1 \oplus I_2 \cong \mathcal{A} \oplus I_3$.

We recall further that if $\mathcal{A}$ is commutative, then $\mathrm{Cl}(\mathcal{A})$ is naturally isomorphic to the multiplicative group of isomorphism classes of invertible $\mathcal{A}$-submodules of A.

Lemma 6.16. *The association $P \mapsto \bigwedge_{\mathcal{A}}^{\mathrm{rk}_{\mathcal{A}}(P)} P$ for each locally-free $\mathcal{A}$-module P induces a well-defined homomorphism of abelian groups $\det_{\mathcal{A}}^{\mathrm{red}} : \mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A}) \to \mathrm{Cl}(\xi(\mathcal{A}))$.*

Proof. This is equivalent to the following two claims: firstly, if I_1 and I_2 are any invertible $\mathcal{A}$-modules that are stably isomorphic, then the $\xi(\mathcal{A})$-modules $\bigwedge_{\mathcal{A}}^1 I_1$ and $\bigwedge_{\mathcal{A}}^1 I_2$ are isomorphic; secondly, if I_1, I_2 and I_3 are any invertible $\mathcal{A}$-modules for which the $\mathcal{A}$-modules $I_1 \oplus I_2 \cong \mathcal{A} \oplus I_3$, then the $\xi(\mathcal{A})$-modules $\bigwedge_{\mathcal{A}}^1 I_2$ and $(\bigwedge_{\mathcal{A}}^1 I_1) \otimes_{\xi(\mathcal{A})} (\bigwedge_{\mathcal{A}}^1 I_3)$ are isomorphic.

To prove the first claim we note that if I_1 and I_2 are stably-isomorphic, then there are locally-free $\mathcal{A}$-modules N_1, N_2 and N_3 which lie in a short exact sequence of the form

$0 \to N_1 \to I_j \oplus N_2 \to N_3 \to 0$ for both $j = 1$ and $j = 2$. Thus by applying Lemma 6.1(v) to these sequences we obtain isomorphisms of invertible $\xi(\mathcal{A})$-modules

$$(\bigwedge\nolimits_{\mathcal{A}}^{1} I_1) \otimes_{\xi(\mathcal{A})} (\bigwedge\nolimits_{\mathcal{A}}^{n_2} N_2) \cong (\bigwedge\nolimits_{\mathcal{A}}^{n_1} N_1) \otimes_{\xi(\mathcal{A})} (\bigwedge\nolimits_{\mathcal{A}}^{n_3} N_3) \cong (\bigwedge\nolimits_{\mathcal{A}}^{1} I_2) \otimes_{\xi(\mathcal{A})} (\bigwedge\nolimits_{\mathcal{A}}^{n_2} N_2),$$

where we set $n_j := \mathrm{rk}_{\mathcal{A}}(N_j)$, and this in turn induces an isomorphism of $\xi(\mathcal{A})$-modules of the required form $\bigwedge_{\mathcal{A}}^1 I_1 \cong \bigwedge_{\mathcal{A}}^1 I_2$.

To prove the second claim we note Lemma 6.1(v) combines with the given isomorphism $I_1 \oplus I_2 \cong \mathcal{A} \oplus I_3$ to give an isomorphism of $\xi(\mathcal{A})$-modules

$$(\bigwedge\nolimits_{\mathcal{A}}^{1} I_1) \otimes_{\xi(\mathcal{A})} (\bigwedge\nolimits_{\mathcal{A}}^{1} I_2) \cong (\bigwedge\nolimits_{\mathcal{A}}^{1} \mathcal{A}) \otimes_{\xi(\mathcal{A})} (\bigwedge\nolimits_{\mathcal{A}}^{1} I_3).$$

This isomorphism implies the second claim since, by Lemma 6.1(ii), the $\xi(\mathcal{A})$-module $\bigwedge_{\mathcal{A}}^1 \mathcal{A}$ is free of rank one. □

We can now state the main result of this section.

Proposition 6.17. *Let C be an object of $D^{\mathrm{lf},0}(\mathcal{A})$ (so that $\chi_{\mathcal{A}}(C)$ belongs to $\mathrm{SK}_0(\mathcal{A})$). Then the following claims are valid.*

(i) $\det_{\mathcal{A}}(C)$ *has a $\xi(\mathcal{A})$-basis if and only if $\chi_{\mathcal{A}}(C)$ belongs to* $\ker(\det_{\mathcal{A}}^{\mathrm{red}})$.

(ii) $\det_{\mathcal{A}}(C)$ *has a primitive $\xi(\mathcal{A})$-basis if and only if $\chi_{\mathcal{A}}(C)$ vanishes.*

(iii) $\det_{\mathcal{A}}(C)$ *has a locally-primitive $\xi(\mathcal{A})$-basis if and only if for all finite sets of prime ideals $\mathcal{P}$ of R the modules* $\det_{\mathcal{A}}(C)_{\langle \mathcal{P}\rangle}$ *have a common primitive $\xi(\mathcal{A})_{\langle \mathcal{P}\rangle}$-basis.*

Proof. We first fix $P^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$ that is isomorphic in $D(\mathcal{A})$ to C. Then, by a standard construction of homological algebra, there is a quasi-isomorphism of $\mathcal{A}$-module complexes of the form $\theta : Q^\bullet \to P^\bullet$ where $Q^\bullet$ is bounded and has the property that if a is the lowest degree of a non-zero module Q^j, then Q^j is a finitely generated free $\mathcal{A}$-module for all $j > a$. We set $r_i := \mathrm{rk}_{\mathcal{A}}(Q^i)$ in each degree i and note that, if necessary after replacing $Q^\bullet$ by the direct sum of $Q^\bullet$ and the (acyclic) complex $\mathcal{A} \xrightarrow{\mathrm{id}} \mathcal{A}$, where the first term is placed in degree a, we can assume that $r_a \geq 2$, and hence also that $r_a \geq \mathrm{sr}(\mathcal{A})$.

Now the mapping cone $D^\bullet$ of θ is an acyclic complex for which in each degree j one has $D^j = P^j \oplus Q^{j+1}$. In particular, since D^j is a locally-free $\mathcal{A}$-module for all $j \neq a-1$, the acyclicity of $D^\bullet$ combines with the Krull-Schmidt Theorem to imply that Q^a is a locally-free $\mathcal{A}$-module, and hence that $Q^\bullet$ belongs to $C^{\mathrm{lf}}(\mathcal{A})$.

To prove claim (i) we use [16, Prop. (49.3)] to choose an isomorphism of $\mathcal{A}$-modules of the form $Q^a \cong I \oplus M$ where I is invertible and M is free of rank $r_a - 1$. Then, since each of the modules Q^j for $j \neq a$ is free, Lemma 6.1(ii) and (iv) combine to give an isomorphism of $\xi(\mathcal{A})$-modules

$$\begin{aligned}\det_{\mathcal{A}}(C) &= \bigotimes_{i\in\mathbb{Z}} (\bigwedge\nolimits_{\mathcal{A}}^{i} Q^i)^{(-1)^i} \\ &\cong (\bigwedge\nolimits_{\mathcal{A}}^{1} I)^{(-1)^a} \otimes_{\xi(\mathcal{A})} ((\bigwedge\nolimits_{\mathcal{A}}^{r_a-1} M)^{(-1)^a} \otimes_{\xi(\mathcal{A})} \bigotimes_{i\in\mathbb{Z}\setminus\{a\}} (\bigwedge\nolimits_{\mathcal{A}}^{i} Q^i)^{(-1)^i}) \\ &\cong (\bigwedge\nolimits_{\mathcal{A}}^{1} I)^{(-1)^a}.\end{aligned}$$

To deduce claim (i) we need only now note that the natural isomorphism $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A}) \cong \mathrm{Cl}(\mathcal{A})$ sends

$$\chi_{\mathcal{A}}(C) = \sum_{i\in\mathbb{Z}}(-1)^i(Q^i) = (-1)^a(I) + ((-1)^a(r_a-1) + \sum_{i\in\mathbb{Z}\setminus\{a\}}(-1)^i r_i)(\mathcal{A}) \tag{25}$$

to $(-1)^a$ times the stable-isomorphism class of I.

Turning to claim (ii) we note $\det_{\mathcal{A}}(C)$ has a primitive basis if and only if C is isomorphic in $D(\mathcal{A})$ to a complex $K^\bullet$ in $C^{\mathrm{lf},0}(\mathcal{A})$ that is both free and such that in some degree i one has $\mathrm{rk}_{\mathcal{A}}(K^i) \geq \mathrm{sr}(\mathcal{A})$. In particular, if such a complex exists, then it is clear that $\chi_{\mathcal{A}}(C) = \chi_{\mathcal{A}}(K^\bullet)$ vanishes.

To prove the converse, we note that if $\chi_{\mathcal{A}}(C)$ vanishes, then the sum (25) vanishes and so $Q^a \cong I \oplus M$ is a stably-free $\mathcal{A}$-module. Then, since $\mathrm{rk}_{\mathcal{A}}(Q^a) \geq 2$, the Bass Cancelation Theorem (cf. [16, Th. (41.20)]) implies Q^a is a free $\mathcal{A}$-module of rank at least $\mathrm{sr}(\mathcal{A})$ and hence that $\det_{\mathcal{A}}(C)$ has a primitive basis, as required.

The proof of claim (iii) is a straightforward exercise that we leave to the reader. □

The results of Proposition 6.17(ii) and (iii) combine to imply that, in general, the $\xi(\mathcal{A})$-module $\det_{\mathcal{A}}(C)$ need not possess a primitive basis even if it is free.

However, if $\mathcal{A}$ is commutative, then the situation is much more straightforward.

Corollary 6.18. *Let C be an object of $D^{\mathrm{lf},0}(\mathcal{A})$. If $\mathcal{A}$ is commutative, then an element of $\det_{\mathcal{A}}(C)_F$ is a primitive basis of $\det_{\mathcal{A}}(C)$ if and only if it is a basis of $\det_{\mathcal{A}}(C)$ as a $\xi(\mathcal{A})$-module.*

Proof. Necessity of the given condition is clear (cf. Remark 6.15).

To prove sufficiency we note that if $\mathcal{A}$ is commutative, then $\xi(\mathcal{A}) = \mathcal{A}$ and the map $\det_{\mathcal{A}}^{\mathrm{red}}$ identifies with the identity automorphism of $\mathrm{Cl}(\mathcal{A}) = \mathrm{Cl}(\xi(\mathcal{A}))$.

In particular, if in this case the $\xi(\mathcal{A})$-module $\det_{\mathcal{A}}(C)$ is free, then Proposition 6.17(ii) implies $\chi_{\mathcal{A}}(C)$ vanishes and then Proposition 6.17(iii) implies that $\det_{\mathcal{A}}(C)$ has a primitive basis b.

Now any basis b' of the (free rank one) $\xi(\mathcal{A})$-module $\det_{\mathcal{A}}(C)$ must differ from b by multiplication by an element of $\mathcal{A}^\times = \mathrm{Nrd}_A(\mathcal{A}^\times)$ and then the argument of Lemma 6.6 implies that b' is also a primitive basis of $\det_{\mathcal{A}}(C)$. □

6.3.3. In this final section we record two further properties of primitive bases that will be useful in the sequel.

Proposition 6.19. *Let C be an object of $D^{\mathrm{lf},0}(\mathcal{A})$.*

(i) *If C is acyclic, then the canonical isomorphism $\det_{\mathcal{A}}(C) \cong \xi(\mathcal{A})$ (from Remark 6.9) sends each primitive basis of $\det_{\mathcal{A}}(C)$ to an element of $\mathrm{Nrd}_A(K_1(\mathcal{A}))$.*

(ii) *Let b be a primitive basis of $\det_{\mathcal{A}}(C)$. Then an element b' of $\det_{\mathcal{A}}(C)_F$ is a primitive basis of $\det_{\mathcal{A}}(C)$ if and only if $b' = u \cdot b$ with u in $\mathrm{Nrd}_A(K_1(\mathcal{A}))$.*

Proof. Claim (i) follows directly from Lemma 6.7 and claim (ii) from Lemma 6.5. □

Proposition 6.20. *Let $C_1 \to C_2 \to C_3 \to C_1[1]$ be an exact triangle in $D^{\mathrm{lf},0}(\mathcal{A})$ and write Δ for the induced isomorphism of $\xi(\mathcal{A})$-modules $\det_{\mathcal{A}}(C_2) \cong \det_{\mathcal{A}}(C_1) \otimes_{\xi(\mathcal{A})} \det_{\mathcal{A}}(C_3)$ (as in Proposition 6.8(ii)).*

If x_1 and x_3 are (locally-)primitive bases of $\det_{\mathcal{A}}(C_1)$ *and* $\det_{\mathcal{A}}(C_3)$*, then* $\Delta^{-1}(x_1 \otimes_{\xi(\mathcal{A})} x_3)$ *is a (locally-)primitive basis of* $\det_{\mathcal{A}}(C_2)$*.*

Proof. We first choose for $j = 1, 3$ a complex $P_j^\bullet$ in $C^{\mathrm{lf},0}(\mathcal{A})$ that is isomorphic in $D(\mathcal{A})$ to C_j, has $\mathrm{rk}_{\mathcal{A}}(P_j^a) \geq \mathrm{sr}(\mathcal{A})$ in at least one degree a and is free if x_i is a primitive basis of $\det_{\mathcal{A}}(C_j)$.

We then define $P_2^\bullet$ to be the mapping cone of a morphism $\mu : P_3^\bullet[-1] \to P_1^\bullet$ in $C^{\mathrm{lf}}(\mathcal{A})$ chosen as in the proof of Proposition 6.8(ii) and we recall that the isomorphism Δ is obtained by applying in each degree i the construction of Lemma 6.1(v) to the natural short exact sequences $0 \to P_1^i \xrightarrow{\theta} P_2^i \xrightarrow{\phi^i} P_3^i \to 0$.

We assume first that x_1 and x_3 are primitive bases of $\det_{\mathcal{A}}(C_1)$ and $\det_{\mathcal{A}}(C_3)$. In this case Remark 6.15 implies that in each degree i there exists an $\mathcal{A}$-basis $\{b_{j,i,t}\}_{1 \leq j \leq r_{ji}}$ of P_j^i such that $x_j = \bigotimes_{i \in \mathbb{Z}} (\wedge_{t=1}^{t=r_{ji}} b_{j,i,t})^{(-1)^i}$.

This implies that the element $\Delta^{-1}(x_1 \otimes_{\xi(\mathcal{A})} x_3)$ is a primitive basis of $\det_{\mathcal{A}}(P_2^\bullet)$ because, after choosing an $\mathcal{A}$-equivariant section σ_i to ϕ_i, the set $\{b_{1,3,j}^{\sigma_i}\}_{1 \leq j \leq r_{2i}}$ constructed in the proof of Lemma 6.1(v) is an $\mathcal{A}$-basis of P_2^i.

If x_1 and x_3 are only locally-primitive bases of $\det_{\mathcal{A}}(C_1)$ and $\det_{\mathcal{A}}(C_3)$, then for every prime ideal $\mathfrak{p}$ of R one can use the same argument (after replacing each complex $P_j^\bullet$ by $P_{j,(\mathfrak{p})}^\bullet$) to show that $\Delta^{-1}(x_1 \otimes_{\xi(\mathcal{A})} x_3)$ is a primitive basis of $\det_{\mathcal{A}_{(\mathfrak{p})}}(P_{2,(\mathfrak{p})}^\bullet) = \det_{\mathcal{A}}(P_2^\bullet)_{(\mathfrak{p})}$. This implies that $\Delta^{-1}(x_1 \otimes_{\xi(\mathcal{A})} x_3)$ is a locally-primitive bases of $\det_{\mathcal{A}}(C_2)$, as required. □

Part II: The Arithmetic Setting

In the sequel we fix a finite Galois extension L/K of global fields and set $G := \mathrm{Gal}(L/K)$. For any finite non-empty set of places Σ of K and any intermediate field E of L/K we write Σ_E for the set of places of E lying above those in Σ, $Y_{E,\Sigma}$ for the free abelian group on the set Σ_E and $X_{E,\Sigma}$ for the submodule of $Y_{E,\Sigma}$ comprising elements whose coefficients sum to zero. If E/K is Galois, we often abbreviate $\mathrm{Gal}(E/K)$ to $G_{E/K}$.

If Σ contains the set S_K^∞ of archimedean places of K (in the number field case), then we write $\mathcal{O}_{E,\Sigma}$ for the subring of E comprising elements integral at all places outside Σ_E and $\mathcal{O}_{E,\Sigma}^\times$ for the unit group of $\mathcal{O}_{E,\Sigma}$. (If $\Sigma = S_K^\infty$, then we abbreviate $\mathcal{O}_{E,\Sigma}$ to $\mathcal{O}_E$.)

In this case, for any finite non-empty set of places T of K which is disjoint from such a set Σ, we write $\mathcal{O}_{E,\Sigma,T}^\times$ for the (finite index) subgroup of $\mathcal{O}_{E,\Sigma}^\times$ consisting of those elements congruent to 1 modulo all places in T_E. In addition, we write $\mathrm{Cl}_\Sigma^T(E)$ for the ray class group of $\mathcal{O}_{E,\Sigma}$ modulo $\prod_{w \in T_E} w$ (that is, the quotient of the group of fractional $\mathcal{O}_E$-ideals whose supports are coprime to all places in $\Sigma_E \cup T_E$ by the subgroup of principal ideals with a generator congruent to 1 modulo all places in T_E).

We note that if E/K is Galois, then each of $Y_{E,\Sigma}, X_{E,\Sigma}, \mathcal{O}_{E,\Sigma}^\times$, $\mathcal{O}_{E,\Sigma,T}^\times$ and $\mathrm{Cl}_\Sigma^T(E)$ are stable under the natural action of $G_{E/K}$.

7. Canonical pre-envelopes and reciprocity maps for $\mathbb{G}_m$

For any abelian group M we set $M^\vee := \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$. If M is a G-module, then we endow $M^\vee$ with the natural contragredient action of G.

7.1. **Statement of the main results.** For any finite non-empty set of places Σ of K that contains S_K^∞ (in the number field case), and any finite non-empty set of places T that is disjoint from Σ, the '(Σ-relative T-trivialized) integral dual Selmer group for $\mathbb{G}_m$ over L' is defined in [11] by setting

$$\mathcal{S}_{\Sigma,T}(\mathbb{G}_m/L) := \mathrm{cok}(\prod_{w\notin \Sigma_L\cup T_L}\mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(L_T^\times,\mathbb{Z})),$$

where $L_T^\times$ is the group $\{a\in L^\times : \mathrm{ord}_w(a-1)>0 \text{ for all } w\in T_L\}$ and the homomorphism on the right hand side sends $(x_w)_w$ to the map $(a\mapsto \sum_{w\notin\Sigma_L\cup T_L}\mathrm{ord}_w(a)x_w)$.

We recall it is also shown in loc. cit. that this module lies in a canonical exact sequence

$$0\to \mathrm{Cl}_\Sigma^T(L)^\vee \to \mathcal{S}_{\Sigma,T}(\mathbb{G}_m/L)\to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{L,\Sigma,T}^\times,\mathbb{Z})\to 0, \tag{26}$$

and has a canonical transpose $\mathcal{S}_{\Sigma,T}^{\mathrm{tr}}(\mathbb{G}_m/L)$, in the sense of Jannsen's homotopy theory of modules [29], which itself lies in a canonical exact sequence

$$0\longrightarrow \mathrm{Cl}_\Sigma^T(L)\longrightarrow \mathcal{S}_{\Sigma,T}^{\mathrm{tr}}(\mathbb{G}_m/L)\longrightarrow X_{L,\Sigma}\longrightarrow 0. \tag{27}$$

In the sequel we shall fix a finite non-empty set of places S of K that contains both S_K^∞ (in the number field case) and all places which ramify in L/K, and a finite non-empty set of places T of K that is disjoint from S and such that the group $\mathcal{O}_{K,S,T}^\times$ is torsion-free.

We can now state the main results of this section.

Proposition 7.1. *Fix sets of places S and T as above.*

Then there exists a canonical strict family $\mathcal{P}=\mathcal{P}_{L,S,T}$ of locally-free pre-envelopes for $\mathcal{O}_{L,S,T}^\times$ that depends only on L,S and T and for which there exists a surjective bundle of G-module morphisms $\pi:\mathcal{P}\to\mathcal{S}_{S,T}^{\mathrm{tr}}(\mathbb{G}_m/L)$.

Remark 7.2. Our methods will also show that for a natural class of extensions L/K the family $\mathcal{P}_{L,S,T}$ constructed in Proposition 7.1 should be free (in the sense described in §4.2.1) and, in addition, that there should exist a surjective G-module morphism $\tilde\pi:\mathcal{P}\to\mathcal{S}_{S,T}^{\mathrm{tr}}(\mathbb{G}_m/L)$ for which, in terms of the notation of Remark 4.6, one has $\pi=\tilde\pi^{\mathrm{bundle}}$. For more details see Remark 11.5.

With S as above, we now find it convenient to set $n:=|S|-1$, label (and thereby order) the elements of S as $\{v_i : 0\le i\le n\}$ and then set $S_0 := S\setminus\{v_0\}$. (Except for one argument that is made in §16.2 the precise choice of the place v_0 will not matter in the sequel.)

For any normal subgroup H of G, with $E=L^H$, we write V_E for the subset of S_0 comprising places which split completely in E/K and then write

$$\varrho_{L,E,S}:\mathcal{S}_{S,T}^{\mathrm{tr}}(\mathbb{G}_m/L)\to X_{L,S}\to Y_{L,V_E\setminus V_L} \tag{28}$$

for the natural composite (surjective) homomorphism, where the first arrow is as in (27) and the second is the natural projection. We also set $r_E := |V_E|$ and note that $r_E\ge r_L$.

Finally, with $\mathcal{P}$ and π as in Proposition 7.1, we consider the composite surjective bundle of G-module morphisms

$$\pi_E : \mathcal{P} \xrightarrow{\pi} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L) \xrightarrow{\varrho_{L,E,S}} Y_{L,V_E\setminus V_L}.$$

(In particular, note that if $r_E = r_L$, then $V_E = V_L$ and so π_E is equal to $0_{\mathcal{P}}$.)

In §7.3 we will define, in terms of the notation used in Remark 5.4, a natural $\xi(\mathbb{Z}[G])$-module of 'H-coinvariants' $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G\mathcal{P})_H$ of the Fitting lattice $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G\mathcal{P})$.

Proposition 7.3. *We fix $L/K, S, T, \mathcal{P} = \mathcal{P}_{L,S,T}$ and π as in Proposition 7.1. We also fix a normal subgroup H of G and set $E := L^H$ and $\Gamma := G/H \cong G_{E/K}$.*

(i) *If $r_E > r_L$, then the group $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G\mathcal{P})_H$ is finite of order dividing a power of $|G|$.*

(ii) *There exists a canonical 'reciprocity' homomorphism of $\xi(\mathbb{Z}[G])$-modules*

$$\mathrm{Rec}^{\mathcal{P}}_H : \mathrm{F}_{0_{\mathcal{P}^H}}(\bigcap\nolimits^{r_E}_{\Gamma}\mathcal{P}^H) \to \mathrm{F}_{\pi_E}(\bigcap\nolimits^{r_L}_{G}\mathcal{P})_H$$

that depends only on $\mathcal{P}$ and H.

(iii) *If G is abelian, then $\mathrm{F}_{0_{\mathcal{P}^H}}(\bigcap^{r_E}_\Gamma\mathcal{O}^\times_{E,S,T}) = \bigcap^{r_E}_\Gamma\mathcal{O}^\times_{E,S,T}$ and there is a natural commutative diagram of Γ-modules*

(29)
$$\begin{tikzcd}
 & (\bigcap^{r_L}_\Gamma\mathcal{O}^\times_{E,S,T})\otimes_{\mathbb{Z}}(J_{V_E\setminus V_L})_H \arrow[dr] & \\
\bigcap^{r_E}_\Gamma\mathcal{O}^\times_{E,S,T} \arrow[ur, "\mathrm{Rec}_{V_E\setminus V_L}"] \arrow[dr, "\mathrm{Rec}^{\mathcal{P}}_H"'] & & (\bigwedge^{r_L}_{\mathbb{Z}[\Gamma]}\mathcal{P}^H)\otimes_{\mathbb{Z}}(J_{V_E\setminus V_L})_H \\
 & \mathrm{F}_{\pi_E}(\bigcap^{r_L}_G\mathcal{P})_H. \arrow[ur, "\varsigma"'] &
\end{tikzcd}$$

Here we write $\mathrm{Rec}_{V_E\setminus V_L}$ for the reciprocity map defined independently by Mazur and Rubin in [39] *and by the second author in* [45] *(see also* [11, §5.3]*), we regard $\mathrm{F}_{0_{\mathcal{P}^H}}(\bigcap^{r_E}_\Gamma\mathcal{O}^\times_{E,S,T})$ as a submodule of $\mathrm{F}_{0_{\mathcal{P}^H}}(\bigcap^{r_E}_\Gamma\mathcal{P}^H)$ (as per Remark 5.4), the unlabeled arrow is induced by the natural identification $\bigcap^{r_L}_\Gamma\mathcal{P}^H = \bigwedge^{r_L}_{\mathbb{Z}[\Gamma]}\mathcal{P}^H$ and the homomorphism ς is induced by the result of Proposition 5.3(vii) (as discussed in Lemma 7.12(ii) below).*

7.2. **The proof of Proposition 7.1.** In this section we prove Proposition 7.1. We therefore fix data $L/K, S$ and T as in the statement of this result.

7.2.1. We first recall the construction of a canonical complex of G-modules. As the notation suggests, this complex can be naturally interpreted in terms of the Weil-étale cohomology theory that Lichtenbaum has constructed for global function fields in [35] and conjectured to exist for number fields in [36] (see Remark 7.6 below for more details). However, other than perhaps for motivational purposes, such interpretations of our complexes play no role in the sequel.

Lemma 7.4. *There exists a complex*

$$C_{L,S,T} := R\,\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})[-2]$$

in $D^{\mathrm{lf},0}(\mathbb{Z}[G])$ *that is defined up to canonical isomorphism and has the following properties.*

(i) $C_{L,S,T}$ *is acyclic outside degrees zero and one and there are canonical identifications* $H^0(C_{L,S,T}) = \mathcal{O}^\times_{L,S,T}$ *and* $H^1(C_{L,S,T}) = \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$.

(ii) *For any normal subgroup* H *of* G *there is a canonical 'projection formula' isomorphism in* $D^{\mathrm{lf},0}(\mathbb{Z}[G/H])$ *of the form* $\mathbb{Z}[G/H] \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C_{L,S,T} \cong C_{L^H,S,T}$.

Proof. The complex $C_{L,S,T}$ is constructed in [11, §2.2]. More precisely, the descriptions in claim (i) follow directly from [11, Def. 2.6 and Rem. 2.7]. In addition, since S is assumed to contain all places which ramify in L/K, claim (ii) follows from the argument used to prove [11, Lem. 2.8].

Lastly we note that the isomorphism in claim (ii) follows by combining the construction of $C_{L,S,T}$ in [11] with the canonical projection formula isomorphism in étale cohomology $\mathbb{Z}_p[G/H] \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} R\Gamma_c((\mathcal{O}_{L,S})_{\text{ét}}, \mathbb{Z}) \cong R\Gamma_c((\mathcal{O}_{L^H,S})_{\text{ét}}, \mathbb{Z})$. □

Remark 7.5. Since $C_{L,S,T}$ is acyclic in all degrees greater than one (by Lemma 7.4(i)), the isomorphism in Lemma 7.4(ii) implies that for every normal subgroup H of G the transpose Selmer group $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L^H)$ identifies naturally with the module of H-coinvariants of $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$.

Remark 7.6. Assume L is a function field. Write C_L for the corresponding smooth projective curve, j for the open immersion $\mathrm{Spec}(\mathcal{O}_{L,S}) \longrightarrow C_L$ and $(\mathcal{O}_{L,S})_{W\text{ét}}$ and $(C_L)_{W\text{ét}}$ for the Weil-étale sites on $\mathrm{Spec}(\mathcal{O}_{L,S})$ and C_L that are defined by Lichtenbaum in [35, §2]. Then the complex $R\Gamma_{c,T}((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})$ constructed in [11] is canonically isomorphic to a natural 'T-modification' of the complex $R\Gamma((C_L)_{W\text{ét}}, j_!\mathbb{Z})$ that arises naturally in Lichtenbaum's theory (for more details see [11, §2.2]). In particular, in this case, the duality theorem in Weil-étale cohomology for curves over finite fields that is proved in [35] implies that the complex $C_{L,S,T}$ defined above is canonically isomorphic to a natural 'T-modification' of the Weil-étale cohomology complex $R\Gamma((\mathcal{O}_{L,S})_{W\text{ét}}, \mathbb{G}_m)$ of $\mathbb{G}_m$ over $\mathcal{O}_{L,S}$.

7.2.2. We next adapt arguments that were used by Macias Castillo and the first author in [13] when developing the theory of 'organising matrices'.

Our first result describes a convenient resolution of the transpose Selmer group $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$ and uses the natural homomorphism

$$\varrho_{L,S} : \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L) \to X_{L,S} \to Y_{L,S_0}.$$

For any subgroup H of G we set $T_H := \sum_{h\in H} h \in \mathbb{Z}[G]$ and, in the sequel, we always identify $X_{L^H,S}$ with $T_H(X_{L,S})$ (and also Y_{L^H,S_0} with $T_H(Y_{L,S_0})$) by means of the homomorphism which sends each place v' in S_{L^H} to $T_H(w')$ where w' is any choice of place of L above v'.

In the sequel we shall also use the following convenient notation: for each natural number e we write $[e]$ for the set of integers i with $1 \le i \le e$.

In particular, for each i in $[n]$ we now fix a place w_i of L above v_i and for each intermediate field E of L/K write $w_{i,E}$ for the place of E obtained by restriction of w_i (so $w_{i,L} = w_i$).

Lemma 7.7. *There exists a class* $\mathcal{C}_{S,T}(L/K)$ *of pairs* $(\varpi, \underline{b})$ *where* ϖ *is a surjective homomorphism of* G*-modules of the form* $P \to \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$, *where* P *is free of finite rank, and* $\underline{b}$ *is an ordered* G*-basis of* P, *for which all of the following properties are satisfied.*

(i) *The rank d of P is independent of P and not less than $n+2$.*
(ii) *Write $\underline{b}=\{b_i\}_{i\in[d]}$. Then the following two properties are satisfied.*
  (a) *For each i in $[n]$ one has $\varrho_{L,S}(\varpi(b_i))=w_{i,L}$.*
  (b) *For each i in $[d]\setminus[n]$ one has $\varrho_{L,S}(\varpi(b_i))=0$.*
(iii) *If $(\tilde{\varpi},\tilde{\underline{b}})$ is any other pair in $\mathcal{C}_{S,T}(L/K)$, with $\tilde{P}$ the domain of $\tilde{\varpi}$, then there exists a commutative diagram of G-modules*

$$\begin{tikzcd} P \arrow[rr, "\iota"] \arrow[dr, "\varpi"'] & & \tilde{P} \arrow[dl, "\tilde{\varpi}"] \\ & \mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_m/L) & \end{tikzcd} \tag{30}$$

in which ι is an isomorphism and there is an equality of ordered sets $\tilde{\underline{b}}=\iota(\underline{b})$.
(iv) *Fix a pair $(\varpi,\underline{b})$ in $\mathcal{C}_{S,T}(L/K)$. Then, for any normal subgroup H of G, the class $\mathcal{C}_{S,T}(L^H/K)$ contains the pair $(\varpi^H,T_H(\underline{b}))$, where we set $T_H(\underline{b}):=\{T_H(b_i)\}_{1\le i\le d}$ and write ϖ^H for the composite homomorphism*

$$P^H\to P_H\to \mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_m/L)_H\to \mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_m/L^H).$$

Here the first arrow sends each element $T_H(b_i)$ to the image of b_i in P_H, the second is ϖ_H and the third is the isomorphism described in Remark 7.5.

Proof. The construction of a class $\mathcal{C}_{S,T}(L/K)$ satisfying claims (i), (ii) and (iii) follows directly from the argument used to prove [13, Lem. 3.1]. One can also, of course, make the same construction with L/K replaced by L^H/K to define the class $\mathcal{C}_{S,T}(L^H/K)$ and to prove claim (iv) one must show that this class contains the pair $(\varpi^H,T_H(\underline{b}))$.

We note first that $T_H(\underline{b})$ is an ordered G/H-basis of P^H since the projectivity of P implies that $P^H=T_H(P)$. It is then enough to note that, after identifying Y_{L^H,S_0} with $T_H(Y_{L,S_0})$ in the manner described above, this basis satisfies the properties in claim (ii) with L replaced by L^H. □

The key point in the proof of Proposition 7.1 is now provided by the following result.

Proposition 7.8. *Set $C:=C_{L,S,T}$. Then for each pair $(\varpi,\underline{b})$ in $\mathcal{C}_{S,T}(L/K)$ there exists a class $\mathcal{C}^{\rm lf}_{(\varpi,\underline{b})}(C)$ of complexes $P^\bullet$ in $C^{{\rm lf},0}(\mathbb{Z}[G])$ with each of the following properties.*

(i) *$P^\bullet$ has the form $P^0\xrightarrow{\phi}P$, where P^0 is a finitely generated locally-free G-module placed in degree zero and P is the domain of ϖ.*
(ii) *One has $H^0(P^\bullet)=\ker(\phi)=\mathcal{O}^\times_{L,S,T}$ and $H^1(P^\bullet)={\rm cok}(\phi)\cong\mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_m/L)$ where the isomorphism is induced by ϖ.*
(iii) *There exists an isomorphism $\vartheta:P^\bullet\to C$ in $D(\mathbb{Z}[G])$ such that for both $i=0,1$ the map $H^i(\vartheta)$ is the identity map with respect to the identification $H^i(P^\bullet)=H^i(C)$ that is induced by the descriptions in claim (ii) and in Lemma 7.4(i).*

(iv) *If* $(\tilde{\varpi}, \underline{\tilde{b}})$ *is any other element of* $\mathcal{C}_{S,T}(L/K)$*, with* $\underline{\tilde{b}} = \{\tilde{b}_i\}_{1\le i\le d}$*, and* $\tilde{P}^\bullet$ *belongs to* $\mathcal{C}^{\mathrm{lf}}_{(\tilde{\varpi},\underline{\tilde{b}})}(C)$*, then there exists a commutative diagram of* G*-modules*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^0(P^\bullet) & \xrightarrow{\iota} & P^0 & \xrightarrow{\phi} & P & \longrightarrow & H^1(P^\bullet) & \longrightarrow & 0 \\ & & \Big\| & & {\scriptstyle\kappa^0_{\iota,\tilde{\iota}}}\Big\downarrow & & \Big\downarrow{\scriptstyle\kappa^1_{\iota,\tilde{\iota}}} & & \Big\| & & \\ 0 & \longrightarrow & H^0(\tilde{P}^\bullet) & \xrightarrow{\tilde{\iota}} & \tilde{P}^0 & \xrightarrow{\tilde{\phi}} & \tilde{P} & \longrightarrow & H^1(\tilde{P}^\bullet) & \longrightarrow & 0 \end{array} \tag{31}$$

in which the rows are the tautological exact sequences, both $\kappa^0_{\iota,\tilde{\iota}}$ *and* $\kappa^1_{\iota,\tilde{\iota}}$ *are bijective and* $\kappa^1_{\iota,\tilde{\iota}}(b_i) = \tilde{b}_i$ *for each* i *in* $[n]$*.*

(v) *Fix a complex* $P^\bullet$ *as in claim (i) and for each normal subgroup* H *of* G *set* $P^{\bullet,H} := \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, P^\bullet)$*. Then the complex* $P^{\bullet,H}$ *is naturally isomorphic to* $\mathbb{Z}\otimes_{\mathbb{Z}[H]} P^\bullet$ *and, with respect to the identification* $\mathbb{Z}[G/H]\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C_{L,S,T} \cong C_{L^H,S,T}$ *in Lemma 7.4(ii), belongs to* $\mathcal{C}^{\mathrm{lf}}_{(\varpi^H,T_H(\underline{b}))}(C_{L^H,S,T})$*, where the homomorphism* ϖ^H *is as defined in Lemma 7.7(v).*

Proof. The existence of a class $\mathcal{C}^{\mathrm{lf}}_{(\varpi,\underline{b})}(C)$ satisfying claims (i), (ii), (iii) and (iv) is proved by mimicking the argument used to prove [13, Prop. 3.2], with the role of the set $\mathcal{C}(H^2(C))$ constructed in [13, Lem. 3.1] now being played by the set $\mathcal{C}_{S,T}(L/K)$ constructed in Lemma 7.7 above.

The fact that each complex $P^\bullet$ constructed in this way belongs to $C^{\mathrm{lf},0}(\mathbb{Z}[G])$ follows from the fact that every finitely generated projective G-module is locally-free (see Remark 4.1(ii)) and that $C_{L,S,T}$ belongs to $D^{\mathrm{lf},0}(\mathbb{Z}[G])$.

The same construction with C and $(\varpi,\underline{b})$ replaced by $C_H := C_{L^H,S,T}$ and $(\varpi^H, T_H(\underline{b}))$ defines the class $\mathcal{C}^{\mathrm{lf}}_{(\varpi^H,T_H(\underline{b}))}(C_H)$. The (termwise) isomorphism $\mathbb{Z}\otimes_{\mathbb{Z}[H]}P^\bullet \cong P^{\bullet,H}$ is induced by applying T_H and the rest of claim (iv) is then straightforward to check. □

7.2.3. We now turn to the proof of Proposition 7.1.

Recalling the identification $H^0(C_{L,S,T}) = \mathcal{O}^\times_{L,S,T}$ from Lemma 7.4, we first define the family $\mathcal{P} = \mathcal{P}_{L,S,T}$ to comprise all embeddings $\iota : \mathcal{O}^\times_{L,S,T} = H^0(C_{L,S,T}) = H^0(P^\bullet) \to P^0$ which arise in any diagram constructed as in Proposition 7.8(iv).

Then it is clear from the latter diagram that the cokernel of each ι in $\mathcal{P}$ is torsion-free and so Lemma 4.4(ii) (in the setting of Example 4.2(ii)) implies that ι satisfies the condition $(\mathcal{P}_2)$ in §4.2.1. Given this fact, the result of Proposition 7.8(iv) directly translates into the statement that $\mathcal{P}$ is a strict family of locally-free pre-envelopes for $\mathcal{O}^\times_{L,S,T}$. Closer analysis of the construction also shows that $\mathcal{P}$ depends only on L, S and T, as required.

To define a surjective bundle of G-module morphisms $\pi : \mathcal{P} \to \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$ we now proceed as follows. We first fix an embedding ι as in diagram (31). Then, since P^0 is a locally-free G-module, Roiter's Lemma implies the existence of a finite index G-submodule $\hat{P}^0$ of P^0 which is free and hence, since $P^\bullet$ belongs to $C^{\mathrm{lf},0}(\mathbb{Z}[G])$, isomorphic to the module P in (31). We fix such an isomorphism of G-modules $j : \hat{P}^0 \to P$.

In a similar way, for each prime p the $\mathbb{Z}_{(p)}[G]$-modules $P^0_{(p)}$ and $P_{(p)}$ are isomorphic and so we can fix an isomorphism of $\mathbb{Z}_{(p)}[G]$-modules $j_p : P^0_{(p)} \to P_{(p)}$. We may, and will, assume that for any prime p which does not divide the index $[P^0 : \hat{P}^0]$ the isomorphism j_p is equal to the p-localization of the homomorphism $P^0 \to \hat{P}^0 \to P$ where the first arrow is multiplication by $[P^0 : \hat{P}^0]$ and the second is j.

For each prime p we now define a composite homomorphism of $\mathbb{Z}_{(p)}[G]$-modules

$$\pi_{\iota,p} : P^0_{(p)} \to P_{(p)} \to H^1(P^\bullet)_{(p)} = \mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_m/L)_{(p)}$$

where the first arrow is j_p and the second is the p-localization of the map that occurs in the upper row of (31).

For any $\iota' : \mathcal{O}^\times_{L,S,T} \to P'$ in $\mathcal{P}$ we write $\kappa_{\iota',\iota'}$ for the identity map on P' and $\kappa_{\iota,\iota'}$ for the bijective map $\kappa^0_{\iota,\iota'}$ that occurs in diagram (31) with $\tilde{\iota} = \iota'$. For each prime p we then set $\pi_{\iota',p} := \pi_{\iota,p} \circ (\kappa_{\iota,\iota'})^{-1}_{(p)}$. Finally, for any ι' and ι'' in $\mathcal{P}$ we set $\kappa_{\iota',\iota'',\pi,p} := (\kappa^{-1}_{\iota,\iota'} \circ \kappa_{\iota,\iota''})_{(p)}$.

For every prime p the data $\{\pi_{\iota',p}, \kappa_{\iota',\iota'',\pi,p}\}_{\iota',\iota''}$ constitutes, as ι' and ι'' vary over $\mathcal{P}$, a $\mathbb{Z}_{(p)}[G]$-module morphism $\pi_p : \mathcal{P}_{(p)} \to \mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_m/L)_{(p)}$.

It is then straightforward to check that as p varies over all primes the data $\{\pi_p\}_p$ constitutes a bundle (with respect to the families $\{\mathcal{P}_{(p)}\}_p$) of G-module morphisms $\pi : \mathcal{P} \to \mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_m/L)$, as per the definition given in §4.3.2.

Remark 7.9. If the module P^0 in diagram (31) is free, then in the above construction one can take $\hat{P}^0 = P^0$. In this case one can then take each map $\pi_{\iota,p}$ to be the p-localization of the composite homomorphism $\hat{\pi}_\iota : P^0 \to P \to H^1(P^\bullet)$ where the first arrow is the map j and the second is as in the upper row of (31). Then, as ι' and ι'' vary over $\mathcal{P}$, the data $\{\hat{\pi}_\iota \circ \kappa^{-1}_{\iota,\iota'}, \kappa^{-1}_{\iota,\iota'} \circ \kappa_{\iota,\iota''}\}$ constitutes a G-module morphism $\hat{\pi} : \mathcal{P} \to \mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_m/L)$ with the property that the bundle of morphisms π constructed above is equal to $\hat{\pi}^{\rm bundle}$.

7.3. **Modules of coinvariants.** In this section we discuss the module of 'H-coinvariants' $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G \mathcal{P})_H$ of $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G \mathcal{P})$.

For each i in $[n]$ and we write G_i for the decomposition subgroup of w_i in G. We also write $I_{G,i}$ for the left ideal of $\mathbb{Z}[G]$ that is generated by the set $\{x-1\}_{x\in G_i}$ and note that this is equal to the kernel of the surjective homomorphism of G-modules $\mathbb{Z}[G] \to Y_{L,\{v_i\}}$ which sends 1 to w_i. For each normal subgroup H of G we write I_H for the kernel of the natural projection map $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$.

We also fix a representative $\iota : \mathcal{O}^\times_{L,S,T} \to P^0$ of $\mathcal{P}$ as in the upper row of (31) and, in the sequel, we always use the isomorphism j_p (from §7.2.3) to identify the $\mathbb{Z}_{(p)}[G]$-modules $P^0_{(p)}$ and $P_{(p)}$.

In this way, for example, for each normal subgroup H of G we regard $T_H(\underline{b})$ as a $\mathbb{Z}_{(p)}[G/H]$-basis of $P^{0,H}_{(p)}$ and we then write $\{T_H(b_{i,p})^*\}_{i\in[d]}$ for the corresponding dual basis of $\mathrm{Hom}_{\mathbb{Z}_{(p)}[G/H]}(P^{0,H}_{(p)}, \mathbb{Z}_{(p)}[G/H])$.

7.3.1. For each prime p we write $\pi_{E,p}$ for the p-component of the bundle π_E that occurs in Proposition 7.3.

Recalling (from §5.2.2) the definition of $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G \mathcal{P})$ via its p-localizations, we define its module of H-coinvariants by first specifying $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} \mathcal{P}_{(p)})_H$ for every prime p.

For each intermediate field E of L/K we write Z_E for the subset of $[n]$ comprising the r_E integers i for which v_i belongs to V_E (and so splits completely in E/K).

Lemma 7.10. *Fix a prime p and a normal subgroup H of G and set $E := L^H$.*

(i) *A quadratic presentation h of $\mathbb{Z}_{(p)}[G]$-modules factors through $\pi_{E,p}$ if and only if $r_{h,2} = d$ and there exists an isomorphism of $\mathbb{Z}_p[G]$-modules $\kappa : \mathbb{Z}_{(p)}[G]^d \cong P_{(p)}$ such that $b^*_{i,p}(\mathrm{im}(\kappa \circ \theta_h)) \subseteq I_{G,i}$ for every i in $Z_E \setminus Z_L$.*

(ii) *The lattice $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})$ is equal to the image of the map*

$$\bigoplus_{\Phi \in \Delta_E} \bigcap\nolimits^{d}_{\mathbb{Z}_{(p)}[G]} P_{(p)} \to \bigcap\nolimits^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)}; \quad (a_\Phi)_\Phi \mapsto \sum_{\Phi} \Phi(a_\Phi)$$

where Δ_E denotes the set of homomorphisms of the form $\Phi = \bigwedge_{j \in [d] \setminus Z_L} \varphi_j$ with

$$\varphi_j \in \begin{cases} \mathrm{Hom}_{\mathbb{Z}_{(p)}[G]}(P_{(p)}, I_{G,j}), & \text{if } j \in Z_E \setminus Z_L, \\ \mathrm{Hom}_{\mathbb{Z}_{(p)}[G]}(P_{(p)}, \mathbb{Z}_{(p)}[G]), & \text{if } j \in [d] \setminus Z_E. \end{cases}$$

Proof. Using Lemma 7.7(ii) and (iv) one shows that an element x of $P_{(p)}$ belongs to $\ker(\pi_{E,p})$ if and only if one has $b^*_{i,p}(x) \in I_{G,i}$ for all i in $Z_E \setminus Z_L$.

Given this fact, claim (i) follows directly from the definition (in §5.1.3) of what it means for h to factor through $\pi_{E,p}$.

We note next that for any quadratic h which factors through $\pi_{E,p}$ (as in claim (i)) the argument used in the proof of Proposition 5.3(i) implies that the lattice $\mathrm{FL}^{r_L}_\kappa(h)$ is equal to the image of the homomorphism

$$\wedge_{j \in [d] \setminus Z_L} (b^*_{j,p} \circ \kappa \circ \theta_h \circ \kappa^{-1}) : \bigcap\nolimits^{d}_{\mathbb{Z}_p[G]} P_{(p)} \to \bigcap\nolimits^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)}.$$

Claim (i) now implies claim (ii), with each φ_j equal to $b^*_{j,p} \circ \kappa \circ \theta_h \circ \kappa^{-1}$. □

For each i in $Z_E \setminus Z_L$ we write G^*_i for the normal closure of G_i in G (so $G^*_i \subseteq H$) and define a two-sided ideal of $\mathbb{Z}[G]$ by setting $I^*_{G,i} := I_{G^*_i}$.

Then, following the description of $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})$ that is given in Lemma 7.10(ii), we define $\mathrm{F}^H_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})$ to be the $\xi(\mathbb{Z}_{(p)}[G])$-submodule of $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})$ that is generated by all finite sums of the form $\sum_{j \in J} \Phi_j(a_{\Phi_j})$ with each $\Phi_j = \bigwedge_{a \in [d] \setminus Z_L} \varphi_{j,a}$ in Δ_E, each a_{Φ_j} in $\bigcap^{d}_{\mathbb{Z}_{(p)}[G]} P_{(p)}$ and both of the following relations satisfied:

(R_1) for each j in J and each i in $Z_E \setminus Z_L$ the homomorphisms $\varphi_{j,i}$ have the same image under the natural homomorphism $\mathrm{Hom}_{\mathbb{Z}_{(p)}[G]}(P_{(p)}, I_{G,i}) \to \mathrm{Hom}_{\mathbb{Z}_{(p)}[G]}(P_{(p)}, I^*_{G,i}/I_H I^*_{G,i})$;

(R_2) the image of $\sum_{j \in J} \bigwedge_{a \in [d] \setminus Z_E} \varphi_{j,a}(a_{\Phi_j})$ in $\bigcap^{r_E}_{\mathbb{Z}_{(p)}[G/H]} P_{H,(p)}$ vanishes.

We then define the $\xi(\mathbb{Z}_{(p)}[G])$-module of H-coinvariants of $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})$ by setting

$$\mathrm{F}_{\pi_{E,p}}(\bigcap\nolimits^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})_H := \mathrm{F}_{\pi_{E,p}}(\bigcap\nolimits^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)}) / \mathrm{F}^H_{\pi_{L,p}}(\bigcap\nolimits^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)}).$$

7.3.2. The basic properties of the above definition are recorded in the following result.

Lemma 7.11.

(i) *If* $r_E = r_L$*, then* $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})_H$ *identifies with the image of the composite homomorphism*

$$\mathrm{F}_{\pi_{E,p}}(\bigcap\nolimits^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)}) \to \bigcap\nolimits^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)} \to \bigcap\nolimits^{r_E}_{\mathbb{Z}_{(p)}[G/H]} P_{H,(p)},$$

where the first arrow in the natural inclusion and the second the natural projection.

(ii) *If* $r_E > r_L$*, then* $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_G P_{(p)})_H$ *is finite and vanishes if* p *is coprime to* $|G|$*.*

Proof. If $r_E = r_L$, then $Z_E = Z_L$ and so the relation (R_1) is satisfied vacuously. In addition, in this case (R_2) asserts only that $\sum_{j\in J} \Phi_j(a_{\Phi_j})$ belongs to the kernel of the projection map $\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)} \to \bigcap^{r_E}_{\mathbb{Z}_{(p)}[G/H]} P_{H,(p)}$ and so claim (i) is clear.

To prove claim (ii) we note each $\mathbb{Z}_{(p)}$-module $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})_H$ is finitely generated (as a consequence of Proposition 5.3(i)) and hence that the group $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G \mathcal{P})_H$ is finite provided that $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})_H$ is a torsion group that vanishes for almost all p.

To prove this we note first that $\mathrm{F}^H_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})$ contains the $\mathbb{Z}_{(p)}$-module generated by elements $\Phi(a_\Phi)$ with a_Φ in $\bigcap^{d}_{\mathbb{Z}_{(p)}[G]} P_{(p)}$ and $\Phi = \wedge_{a\in[d]\setminus Z_L} \varphi_a$ with

$$\varphi_a \in \begin{cases} \mathrm{Hom}_{\mathbb{Z}_{(p)}[G]}(P_{(p)}, I_H I^*_{G,a}), & \text{if } a \in Z_E \setminus Z_L, \\ \mathrm{Hom}_{\mathbb{Z}_{(p)}[G]}(P_{(p)}, \mathbb{Z}_{(p)}[G]), & \text{otherwise.} \end{cases}$$

(To show that such an element $\Phi(a_\Phi)$ satisfies the relations (R_1) and (R_2) one need only note that it is equal to $\sum_{j\in J} \Phi_j(a_{\Phi_j})$ with $J = \{1,2\}$ $a_{\Phi_1} = -a_{\Phi_2} = a_\Phi$, $\Phi_1 = \Phi$ and $\Phi_2 = \wedge_{a\in[d]\setminus Z_L} \varphi'_{2,a}$ with $\varphi'_{2,a} = \varphi_a$ for $a \in [d] \setminus Z_E$ and $\varphi'_{2,a} = 0$ for $a \in Z_E \setminus Z_L$.)

Next we recall that for each a in $Z_E \setminus Z_L$ one has $G_a \subseteq H$ and hence $I^*_{G,a} \subseteq I_H$. This implies that $I^*_{G,a}/I_H I^*_{G,a}$ is a quotient of $I^*_{G,a}/(I^*_{G,a})^2$ which is itself easily shown to be a group of order $|(G^*_a)^{\mathrm{ab}}|^{[G:G^*_a]}$.

Taken together these facts imply that $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})_H$ is a torsion group which vanishes whenever p is coprime to the order of G. This proves claim (ii). □

Motivated by the results of Lemma 7.11 we define the $\xi(\mathbb{Z}[G])$-module of H-coinvariants $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G \mathcal{P})_H$ of $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G \mathcal{P})$ to be the image of the natural composite homomorphism

$$\mathrm{F}_{0_{\mathcal{P}}}(\bigcap\nolimits^{r_L}_G P) \to \bigcap\nolimits^{r_L}_G P \to \bigcap\nolimits^{r_E}_{G/H} P_H$$

if $r_E = r_L$ and to be the module

$$\bigoplus_p \mathrm{F}_{\pi_{E,p}}(\bigcap\nolimits^{r_L}_{\mathbb{Z}_{(p)}[G]} P_{(p)})_H$$

if $r_E > r_L$, where in the direct sum p runs over all primes.

Lemma 7.12. *Assume that* $r_E > r_L$*.*

(i) *Then the module* $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G \mathcal{P})_H$ *is finite of order dividing a power of* $|G|$*.*

(ii) *If* G *is abelian, then there is a natural injective homomorphism*

$$\varsigma : \mathrm{F}_{\pi_E}(\bigcap\nolimits^{r_L}_G \mathcal{P})_H \to (\bigwedge\nolimits^{r_L}_G \mathcal{P}) \otimes_{\mathbb{Z}} (\prod_{a\in Z_E\setminus Z_L} I_{G,a})_H.$$

Proof. Claim (i) is an immediate consequence of Lemma 7.11(ii).

The injective homomorphism in claim (ii) is obtained by combining the argument proving Proposition 5.3(vii) together with the fact that if G is abelian then $\prod_{a\in Z_E\setminus Z_L} I_{G,a}$ is equal to $\mathrm{Fit}_{\mathbb{Z}[G]}(Y_{L,V_E\setminus V_L})$. $\square$

7.4. **The proof of Proposition 7.3.** In this section we prove Proposition 7.3.

At the outset we note Proposition 7.3(i) coincides with the statement of Lemma 7.12(i).

7.4.1. To prove Proposition 7.3(ii) we explicitly construct a canonical 'reciprocity' map $\mathrm{Rec}^{\mathcal{P}}_H : \mathrm{F}_{0_{\mathcal{P}^H}}(\bigcap^{r_E}_{\Gamma}\mathcal{P}^H) \to \mathrm{F}_{\pi_E}(\bigcap^{r_L}_{G}\mathcal{P})_H$.

If firstly $r_E = r_L$, then the relevant case of Proposition 5.3(iv) induces an identification of $\mathrm{F}_{0_{\mathcal{P}}}(\bigcap^{r_L}_{G}\mathcal{P})_H$ with $\mathrm{F}_{0_{\mathcal{P}^H}}(\bigcap^{r_E}_{\Gamma}\mathcal{P}^H)$ and we define $\mathrm{Rec}^{\mathcal{P}}_H$ to be the identity map.

To deal with the case $r_E > r_L$ we use the following result.

Proposition 7.13. *For each integer a in $[n]$ there exists a canonical G-module morphism $\mathrm{Rec}^{\mathcal{P}}_a : \mathcal{P} \to I_{G,a}$ with the following property. For any $\iota : \mathcal{O}^\times_{L,S,T} \to P^0$ in $\mathcal{P}$ and any normal subgroup H of G which contains G_a there exists a commutative diagram of G-modules*

$$\begin{tikzcd}
(P^0)^H & P^0 \arrow[l, two heads, "T_H"'] \arrow[r, "(\mathrm{Rec}^{\mathcal{P}}_a)_\iota"] & I_{G,a} \arrow[d, "\varrho"] \\
\mathcal{O}^\times_{E,S,T} \arrow[u, "\iota^H"] \arrow[rr, "\mathrm{Rec}_a"] & & I_{G,a}/I_H I_{G,a}
\end{tikzcd}$$

Here ϱ is the natural projection map, we set $E := L^H$ and we write Rec_a *for the (well-defined) homomorphism of G-modules with*

$$\mathrm{Rec}_a(u) = \sum_{\tau\in G/H} \varrho(\hat{\tau}^{-1}(\mathrm{rec}_a(\hat{\tau}(u)) - 1))$$

for each u in $\mathcal{O}^\times_{E,S,T}$, where rec_a *is the reciprocity map $E^\times_{w_{a,E}} \to G_a$ and $\hat{\tau}$ is any choice of lift of τ to G.*

Proof. For each ι in $\mathcal{P}$ we choose a complex $P^\bullet$ in $\mathcal{C}^{\mathrm{lf}}_{(\varpi,\underline{b})}(C)$ as in the upper row of diagram (31). We then set $(\mathrm{Rec}^{\mathcal{P}}_a)_\iota := b^*_a \circ \phi$ and note that this homomorphism belongs to $\mathrm{Hom}_G(P^0, I_{G,a})$ as a consequence of Lemma 7.10.

The commutativity of diagram (31) then implies both that $\{(\mathrm{Rec}^{\mathcal{P}}_a)_\iota\}_{\iota\in\mathcal{P}}$ constitutes a G-module morphism $\mathcal{P} \to I_{G,a}$ and that it depends only upon $\mathcal{P}$ and the integer a.

To prove the commutativity of the given diagram one can then mimic the argument of [5, §10]. (More precisely, in terms of the notation used in loc. cit., one need only replace ϕ, I_j and G_j by $\phi, I_{G,a}$ and G^*_a respectively.) $\square$

Since we are assuming that $r_E > r_L$ the group $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_{G}\mathcal{P})_H$ is finite (by Lemma 7.12(i)) and so it suffices to describe $\mathrm{Rec}^{\mathcal{P}}_H$ after localization at each prime p. (This step can actually be avoided if the pre-envelope $\mathcal{P}$ is free.)

To do this we fix $\iota : \mathcal{O}^\times_{L,S,T} \to P^0$ in $\mathcal{P}$ as in the upper row of (31) and we note Proposition 3.5(iii) implies that every element of $\mathrm{F}_{0_{\mathcal{P}^H}}(\bigcap^{r_E}_{\mathbb{Z}_{(p)}[G/H]}\mathcal{O}^\times_{E,S,T})$ is the $\mathbb{Z}_p$-linear span of elements of the form

$$\xi := \mathrm{Nrd}_{\mathbb{Q}[G/H]}(U)\cdot(\wedge_{a\in[d]\setminus Z_E}T_H(b_{a,p})^*\circ\varphi)(\wedge_{i\in[d]}T_H(b_{i,p}))$$

with U in $\mathrm{M}_s(\mathbb{Z}_{(p)}[G/H])$ for some s and φ in $\mathrm{Hom}_{\mathbb{Z}_{(p)}[G/H]}(P^{0,H}_{(p)},P^{0,H}_{(p)})$.

We choose a lift U' of U to $\mathrm{M}_s(\mathbb{Z}_{(p)}[G])$ and a lift φ' to $\mathrm{Hom}_{\mathbb{Z}_{(p)}[G]}(P^0_{(p)},P^0_{(p)})$ of the composite homomorphism $\varphi\circ T_H : P^0_{(p)}\to P^{0,H}_{(p)}$ through the surjection $P^0_{(p)}\to P^{0,H}_{(p)}$ induced by T_H. Then Lemma 7.10(ii) implies the element

$$\mathrm{Nrd}_{\mathbb{Q}[G]}(U')\cdot((\wedge_{a\in Z_E\setminus Z_L}(\mathrm{Rec}^{\mathcal{P}}_a)_\iota)\wedge(\wedge_{a\in[d]\setminus Z_E}b^*_{a,p}\circ\varphi'))(\wedge_{i\in[d]}b_{i,p})$$

belongs to $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]}P^0_{(p)})$ and we define $\mathrm{Rec}^{\mathcal{P}}_H(\xi)$ to be equal to the image of this element in $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_L}_{\mathbb{Z}_{(p)}[G]}P^0_{(p)})_H$.

Lemma 7.14. *The association $\xi\mapsto\mathrm{Rec}^{\mathcal{P}}_H(\xi)$ is a well-defined homomorphism of $\xi(\mathbb{Z}_{(p)}[G])$-modules that depends only on the pair $(\mathcal{P},H)$.*

Proof. To show that the map is a well-defined homomorphism of $\xi(\mathbb{Z}_{(p)}[G])$-modules we consider any relation in $\mathrm{F}_{0_{\mathcal{P}^H}}(\bigcap^{r_E}_{\Gamma}\mathcal{O}^\times_{E,S,T})$ of the form

$$\sum_{j\in J}c_j\mathrm{Nrd}_{\mathbb{Q}[G/H]}(U_j)\cdot(\wedge_{a\in[d]\setminus Z_E}T_H(b_{a,p})^*\circ\varphi_j)(\wedge_{i\in[d]}T_H(b_{i,p}))=0$$

with each c_j in $\mathbb{Z}_p$, U_j in $\mathrm{M}_{s_j}(\mathbb{Z}_{(p)}[G/H])$ and each φ_j in $\mathrm{Hom}_{\mathbb{Z}_{(p)}[G/H]}(P^{0,H}_{(p)},P^{0,H}_{(p)})$. It then suffices to note that for any lift U'_j of U_j to $\mathrm{M}_{s_j}(\mathbb{Z}_p[G])$ and any lift φ'_j of $\varphi_j\circ T_H$ to $\mathrm{Hom}_{\mathbb{Z}_{(p)}[G]}(P^0_{(p)},P^0_{(p)})$ the element

$$\sum_{j\in J}c_j\mathrm{Nrd}_{\mathbb{Q}[G]}(U'_j)\cdot((\wedge_{a\in Z_E\setminus Z_L}(\mathrm{Rec}^{\mathcal{P}}_a)_\iota)\wedge(\wedge_{a\in[d]\setminus Z_E}b^*_{a,p}\circ\varphi'_j))(\wedge_{i\in[d]}b_{i,p})$$

satisfies both of the relations (R_1) and (R_2).

It is also clear from the construction that this map depends only on the pair $(\mathcal{P},H)$. $\square$

7.4.2. We now consider Proposition 7.3(iii).

The first assertion of claim (iii) is proved in Proposition 5.3(vi) and so we need only prove commutativity of the diagram (29) with the homomorphisms $\mathrm{Rec}^{\mathcal{P}}_H$ and ς as constructed in §7.4.1 and described in Proposition 7.12(ii) respectively. Moreover, this commutativity is in turn only a restatement of the result of [11, Lem. 5.20] in which the map $\widetilde{\mathrm{Rec}}_W$ corresponds to the restriction of the composite $\varsigma\circ\mathrm{Rec}^{\mathcal{P}}_H$ to the subgroup $\bigcap^{r_E}_{\Gamma}\mathcal{O}^\times_{E,S,T}$ of $\bigwedge^{r_E}_{\mathbb{Z}[\Gamma]}\mathcal{P}^H$.

This completes the proof of Proposition 7.3.

7.5. **Non-abelian reciprocity maps and determinant modules.** In this section we record an alternative description of the reciprocity maps $\mathrm{Rec}^{\mathcal{P}}_H$ which will be useful in the sequel.

To do this we assume that $r_E > r_L$, we fix an embedding $\iota:\mathcal{O}^\times_{L,S,T}\to P^0$ in $\mathcal{P}_{L,S,T}$ and a complex $P^\bullet$ in $\mathcal{C}^{\mathrm{lf}}_{(\varpi,\underline{b})}(C)$ as in the upper row of diagram (31) and, just as in §7.3, for each prime p we use the isomorphism j_p (from §7.2.3) to identify the $\mathbb{Z}_{(p)}[G]$-modules $P^0_{(p)}$ and $P_{(p)}$.

For each normal subgroup H of G, with $E := L^H$, we define $\Delta^{H,p}_{L,S,T}$ to be the unique homomorphism of $\xi(\mathbb{Z}_{(p)}[G])$-modules

$$\det_{\mathbb{Z}[G]}(C_{L,S,T})_{(p)} \to \mathrm{F0}_{\mathcal{P}H}(\bigcap\nolimits^{r_E}_{\mathbb{Z}_{(p)}[G/H]}\mathcal{P}^H_{(p)})$$

which satisfies

$$\Delta^{H,p}_{L,S,T}((\wedge_{i\in[d]}b_i)\otimes(\wedge_{i\in[d]}b_i^*)) = (\wedge_{a\in[d]\setminus Z_E}T_H(b_a)^*\circ\phi^H)(\wedge_{i\in[d]}T_H(b_i))$$

where ϕ^H denotes the restriction of ϕ to $P^H_{(p)}$. In the case that H is the trivial subgroup of G we abbreviate $\Delta^{H,p}_{L,S,T}$ to $\Delta^p_{L,S,T}$.

Proposition 7.15. *Fix a normal subgroup H of G for which the field $E = L^H$ is such that $r_E > r_L$. Then for each prime p there is an inclusion $\mathrm{im}(\Delta^p_{L,S,T}) \subseteq \mathrm{F}_{\pi_E}(\bigcap^{r_L}_G\mathcal{O}^\times_{L,S,T})_{(p)}$ and a commutative diagram of homomorphisms of $\xi(\mathbb{Z}_{(p)}[G])$-modules*

$$\begin{tikzcd}
 & \mathrm{F}_{\pi_E}(\bigcap^{r_L}_G\mathcal{O}^\times_{L,S,T})_{(p)} \arrow[dr] & \\
\det_{\mathbb{Z}[G]}(C_{L,S,T})_{(p)} \arrow[ur, "\Delta^p_{L,S,T}"] \arrow[dr, "\Delta^{H,p}_{L,S,T}"'] & & \mathrm{F}_{\pi_E}(\bigcap^{r_L}_G\mathcal{P})_{H,(p)} \\
 & \mathrm{F0}_{\mathcal{P}H}(\bigcap^{r_E}_\Gamma\mathcal{O}^\times_{E,S,T})_{(p)}. \arrow[ur, "(\mathrm{Rec}_{\mathcal{P},H})_p"'] &
\end{tikzcd}$$

Here the unlabeled arrow denotes the composite

$$\mathrm{F}_{\pi_E}(\bigcap\nolimits^{r_L}_G\mathcal{O}^\times_{L,S,T})_{(p)} \to \mathrm{F}_{\pi_E}(\bigcap\nolimits^{r_L}_G\mathcal{P})_{(p)} \to \mathrm{F}_{\pi_L}(\bigcap\nolimits^{r}_G\mathcal{P})_{H,(p)}$$

of the natural inclusion and projection maps.

Proof. The inclusion $\mathrm{im}(\Delta^p_{L,S,T}) \subseteq \mathrm{F}_{\pi_E}(\bigcap^{r_L}_G\mathcal{O}^\times_{L,S,T})_p$ is an easy consequence of Lemma 7.10. Given the description of $\mathrm{Rec}^{\mathcal{P}}_H$ in §7.4.1, the commutativity of the diagram is then verified by means of an explicit, and straightforward, diagram chase. □

8. Higher non-abelian Stark elements

In this section we fix data $L/K, G, S$ and T as in §7.

8.1. **The general set-up.** To describe the general set-up we fix a subring Λ of $\mathbb{Q}$ and a Λ-order $\mathcal{A}$ in a semisimple $\mathbb{Q}$-algebra A. We assume that $\mathcal{A}$ satisfies the conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ discussed in §4.2.2 (with R replaced by Λ)

In the sequel we also assume to be given a finitely generated $(\mathcal{A}, \mathbb{Z}[G])$-bimodule Π which satisfies all of the following conditions.

(Π_1) Π is a locally-free $\mathcal{A}$-module.

(Π_2) The association $W \mapsto W \otimes_{\mathcal{A}} \Pi$ induces an injection from the set of isomorphism classes of simple right $A_{\mathbb{C}}$-modules to the set of isomorphism classes of simple right $\mathbb{C}[G]$-modules.

(Π_3) Condition (Π_2) is true with Π replaced by the dual lattice $\check{\Pi} := \mathrm{Hom}_{\mathcal{A}}(\Pi, \mathcal{A})$, regarded as a $(\mathcal{A}, \mathbb{Z}[G])$-bimodule by means of the anti-involutions $\iota_{\mathcal{A}}$ and $\iota_{\#}$.

Remark 8.1. Under condition (Π_2) one obtains a bijection Π_* from the set of Wedderburn components Wed_A of $A_{\mathbb{C}}$ to a subset Υ_Π of $\widehat{G}$ and hence a commutative diagram of abelian groups

$$\begin{array}{ccccc} K_1(\mathbb{Q}[G]) & \xrightarrow{\mathrm{Nrd}_{\mathbb{Q}[G]}} & \zeta(\mathbb{Q}[G])^\times & \xrightarrow{\subset} & \zeta(\mathbb{C}[G])^\times = \prod_{\widehat{G}} \mathbb{C}^\times \\ \mu^1_\Pi \downarrow & & & & \downarrow \iota_\Pi \\ K_1(A) & \xrightarrow{\mathrm{Nrd}_A} & \zeta(A)^\times & \xrightarrow{\subset} & \zeta(A_{\mathbb{C}})^\times = \prod_{\mathrm{Wed}_A} \mathbb{C}^\times . \end{array} \tag{32}$$

Here μ^1_Π sends the class of an automorphism α of a finitely generated left $\mathbb{Q}[G]$-module V to the class of the induced automorphism $\mathrm{id}_\Pi \otimes_{\mathbb{Z}[G]} \alpha$ of $\Pi \otimes_{\mathbb{Z}[G]} V$ and ι_Π sends each element $(z_\chi)_{\chi \in \widehat{G}}$ to $(z_{\Pi_*(C)})_{C \in \mathrm{Wed}_A}$.

We particularly have in mind the following two sorts of examples of this sort of data.

Example 8.2. If the algebra A is a direct factor of $\mathbb{Q}[G]$ then for any homomorphism of rings $\kappa : \mathbb{Z}[G] \to \mathcal{A}$ one can set $\mathcal{A}_\kappa := \mathcal{A}$ and $\Pi_\kappa := \mathcal{A}$. In all cases the lattice Π_κ satisfies the conditions (Π_1), (Π_2) and (Π_3) and Υ_{Π_κ} is the subset of $\widehat{G}$ comprising characters which occur in $A_{\mathbb{C}}$. The order $\mathcal{A}$ satisfies both conditions ($\mathcal{A}_1$) and ($\mathcal{A}_2$) in the context of Example 4.2(i) and also if $\mathcal{A}$ is a regular Λ-algebra of dimension one.

Example 8.3. Let ρ be a representation of the form $G \to \mathrm{GL}_{\rho(1)}(\mathcal{O}_\rho)$ for a finite extension $\mathcal{O}_\rho$ of Λ. Set $\mathcal{A}_\rho := \mathcal{O}_\rho$ and $\Pi_\rho := \mathcal{O}_\rho^{\rho(1)}$, regarded as an $(\mathcal{O}_\rho, \mathbb{Z}[G])$-bimodule via ρ. Then this data satisfies the conditions ($\mathcal{A}_1$), ($\mathcal{A}_2$), (Π_1), (Π_2) and (Π_3) and one has $\Upsilon_{\Pi_\rho} = \{\rho\}$.

8.2. **The associated functors.** The bimodule Π fixed above gives rise to functors $M \mapsto {}^\Pi M$ and $M \mapsto {}_\Pi M$ from the category of left G-modules to the category of left $\mathcal{A}$-modules by setting

$${}^\Pi M := H^0(G, \Pi \otimes_{\mathbb{Z}} M) \quad \text{and} \quad {}_\Pi M := H_0(G, \Pi \otimes_{\mathbb{Z}} M) = \Pi \otimes_{\mathbb{Z}[G]} M,$$

where the left action of G on the tensor product is via $g(\pi \otimes m) = (\pi)g^{-1} \otimes g(m)$. These functors are respectively left and right exact.

If N is any G-module for which $\Pi \otimes_{\mathbb{Z}} N$ is a cohomologically-trivial G-module, then the map

$$\mathrm{Tr}_{\Pi,N} : H_0(G, \Pi \otimes_{\mathbb{Z}} N) \to H^0(G, \Pi \otimes_{\mathbb{Z}} N)$$

induced by sending each element $\pi \otimes n$ of $\Pi \otimes_{\mathbb{Z}} N$ to $\sum_{g \in G} g(\pi \otimes n)$ is an isomorphism of $\mathcal{A}$-modules.

In particular, if $\iota : M \to P$ belongs to a strict family $\mathcal{P}$ of locally-free pre-envelopes of M, then the upper left hand arrow in the commutative diagram

$$\begin{tikzcd} {}^{\Pi}P & {}_{\Pi}P \arrow[l, "\mathrm{Tr}_{\Pi,P}"'] \arrow[r, "\subset"] & \mathbb{Q}\cdot {}_{\Pi}P \\ {}^{\Pi}M \arrow[u, "{}^{\Pi}\iota"] & {}_{\Pi}M \arrow[l, "\mathrm{Tr}_{\Pi,M}"] \arrow[u, "{}_{\Pi}\iota"'] \arrow[r] & \mathbb{Q}\cdot {}_{\Pi}M \arrow[u, "\mathbb{Q}\cdot{}_{\Pi}\iota"'] \end{tikzcd}$$

is bijective, where we abbreviate the tensor product '$\mathbb{Q}\otimes_{\mathbb{Z}}$' to '$\mathbb{Q}\cdot$'. In addition, the commutativity of this diagram combines with the bijectivity of $\mathrm{Tr}_{\Pi,\mathbb{Q}\cdot M}$ to imply that $\mathrm{im}((\mathrm{Tr}_{\Pi,P})^{-1}\circ {}^{\Pi}\iota)$ is contained in the image of the injective map $\mathbb{Q}\cdot {}_{\Pi}\iota$ and so for any $\mathcal{A}$-submodule N of M we may define an $\mathcal{A}$-submodule of $\mathbb{Q}\cdot {}_{\Pi}M$ by setting

$$\text{(33)} \qquad {}^{\Pi}_{\mathcal{P}}N := ((\mathbb{Q}\cdot {}_{\Pi}\iota)^{-1}\circ (\mathrm{Tr}_{\Pi,P})^{-1}\circ {}^{\Pi}\iota)(N).$$

This construction induces a faithful exact covariant functor from the category of $\mathcal{A}$-submodules of M to the category of $\mathcal{A}$-submodules of $\mathbb{Q}\cdot {}_{\Pi}M$. It is straightforward to see that this functor is independent of the choice of representative ι of the strict family $\mathcal{P}$ and hence, as the notation suggests, depends only on Π and $\mathcal{P}$.

8.3. **Higher derivatives of equivariant L-series.** For any set of data L, S, Π as above we write $\Delta^{\Pi}_{L,S,T}$ for the set of surjective homomorphisms of $\mathcal{A}$-modules

$$\pi : {}_{\Pi}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L) \to Y_\pi$$

in which Y_π is locally-free and then set

$$r^{\Pi}_{L,S} := \max\{\mathrm{rk}_{\mathcal{A}}(Y_\pi)\}_{\pi\in\Delta^{\Pi}_{L,S,T}}.$$

In the sequel we shall write M_{tf} for the quotient of an $\mathcal{A}$-module M by its R-torsion submodule M_{tor}.

Remark 8.4. If ${}_{\Pi}X_{L,S} = ({}_{\Pi}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L))_{\mathrm{tf}}$ is a locally-free $\mathcal{A}$-module, as is automatically the case if $\mathcal{A}$ is a Dedekind domain, then it is clear that $r^{\Pi}_{L,S} = \mathrm{rk}_{\mathcal{A}}({}_{\Pi}X_{L,S})$. In general, if one fixes any place v in S, then ${}_{\Pi}Y_{L,S\setminus\{v\}}$ is a quotient of ${}_{\Pi}X_{L,S}$ and so the exact sequence (27) implies that $r^{\Pi}_{L,S} \geq |S^{\Pi}_v|$ with

$$S^{\Pi}_v := \{v' \in S\setminus\{v\} : \text{ the } \mathcal{A}\text{-module } H_0(G_{v'},\Pi)_{\mathrm{tf}} \text{ is both non-zero and locally-free}\}$$

where $G_{v'}$ denotes the decomposition subgroup in G of some choice of place of L above v'. Note also that, in the setting of Example 8.2, the $\mathcal{A}_\kappa$-module $H_0(G_{v'},\Pi_\kappa)_{\mathrm{tf}}$ is non-zero and locally-free if κ sends each element of $G_{v'}$ to the identity of A.

For an irreducible character χ of G, we denote by $L_{S,T}(\chi,z)$ the S-truncated T-modified Artin L-function for χ. For a non-negative integer r with $r \leq \mathrm{ord}_{s=0}L_{S,T}(\chi,z)$, we set

$$L^{r}_{S,T}(\chi,0) := \lim_{z\to 0} z^{-r}L_{S,T}(\chi,z).$$

For each π in $\Delta^{\Pi}_{L,S,T}$ we then define an element of $\zeta(\mathbb{C}[G])$ by setting

$$\theta^{\pi}_{S,T}(0) := \sum_{\chi\in\Upsilon_\Pi} e_\chi L^{r_\pi(\chi)}_{S,T}(\chi,0).$$

Here, as before, e_χ denotes the primitive central idempotent $\frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1}$, and we have also set

$$r_\pi(\chi) := \dim_{\mathbb{C}}(W_\chi \otimes_{A_{\mathbb{C}}} (\mathbb{C}\cdot Y_\pi))$$

where for each χ in Υ_Π we choose a simple right $A_{\mathbb{C}}$-module W_χ for which the associated $\mathbb{C}[G]$-module $W_\chi \otimes_{A_{\mathbb{C}}} (\mathbb{C}\cdot\Pi)$ has character χ.

We also define a central idempotent of A by setting

$$e_\pi := \sum e$$

where the sum runs over all primitive central idempotents e of A for which $e(\mathbb{Q}\cdot\ker(\pi))$ vanishes.

Lemma 8.5. *Fix π in $\Delta^{\Pi}_{L,S,T}$. Then one has $\theta^{\pi}_{S,T}(0) = e_\pi\theta^{\pi}_{S,T}(0)$ and, in addition, $\theta^{\pi}_{S,T}(0) = 0$ unless $\mathrm{rk}_{\mathcal{A}}(Y_\pi) = r^{\Pi}_{L,S}$.*

Proof. For each χ in $\widehat{G}$ set $V_\chi := W_\chi \otimes_{A_{\mathbb{C}}} (\mathbb{C}\cdot\Pi)$. Then, since this module has character χ, an analysis of the functional equation of Artin L-series shows that the order of vanishing at $z=0$ of the meromorphic function $L_{S,T}(\chi,z)$ is equal to

$$\mathrm{ord}_{z=0}L_{S,T}(\chi,z) = \dim_{\mathbb{C}}(H^0(G,\mathrm{Hom}_{\mathbb{C}}(V_{\check{\chi}},\mathbb{C}\cdot X_{L,S}))) \tag{34}$$

(for details see, for example, [49, Chap. I, Prop. 3.4]). Taken together with the natural isomorphisms of vector spaces

$$\begin{aligned} H^0(G,\mathrm{Hom}_{\mathbb{C}}(V_{\check{\chi}},\mathbb{C}\cdot X_{L,S})) &\cong H_0(G,\mathrm{Hom}_{\mathbb{C}}(V_{\check{\chi}},\mathbb{C}\cdot X_{L,S})) \cong H_0(G, V_\chi\otimes_{\mathbb{C}}\mathbb{C}\cdot X_{L,S}) \\ &\cong V_\chi \otimes_{\mathbb{C}[G]} (\mathbb{C}\cdot X_{L,S}) \cong W_\chi \otimes_{A_{\mathbb{C}}} (\mathbb{C}\cdot {}_{\Pi}X_{L,S}) \end{aligned}$$

(where the last isomorphism follows from the definition of ${}_{\Pi}X_{L,S}$) this shows that

$$\mathrm{ord}_{z=0}L_{S,T}(\chi,z) = \dim_{\mathbb{C}}(W_\chi \otimes_{A_{\mathbb{C}}} (\mathbb{C}\cdot {}_{\Pi}X_{L,S})) = r_\pi(\chi) + \dim_{\mathbb{C}}(W_\chi\otimes_{A_{\mathbb{C}}}\ker(\pi)).$$

This formula implies, in particular, that if the space $W_\chi\otimes_{A_{\mathbb{C}}}\ker(\pi)$ does not vanish, then $L^{r_\pi(\chi)}_{S,T}(\chi,0) = 0$. This in turn implies the claimed equality $\theta^{\pi}_{S,T}(0) = e_\pi\theta^{\pi}_{S,T}(0)$ since each character χ in Υ_Π for which $W_\chi\otimes_{A_{\mathbb{C}}}\ker(\pi)$ vanishes corresponds (via the projection ι_Π in (32)) to a unique primitive central idempotent e of A for which $e(\mathbb{Q}\cdot\ker(\pi))$ vanishes.

The second claim of the lemma also follows from the same argument. This is because if $\mathrm{rk}_{\mathcal{A}}(Y_\pi) \neq r^{\Pi}_{L,S}$, then $\mathrm{rk}_{\mathcal{A}}(Y_\pi) < r^{\Pi}_{L,S}$ so that $\mathbb{Q}\cdot\ker(\pi)$ contains a submodule isomorphic to A and hence $W_\chi\otimes_{A_{\mathbb{C}}}\ker(\pi)$ does not vanish for any χ in Υ_Π. $\square$

The following examples show that in several cases the elements $\theta^{\pi}_{S,T}(0)$ recover elements that occur in the theory of refined Stark conjectures.

Example 8.6. Fix a normal subgroup H of G and write $\kappa = \kappa_H$ for the natural projection map $\mathbb{Z}[G]\to\mathbb{Z}[G/H]$. Then, in the notation of Example 8.2, one has $\mathcal{A}_\kappa = \mathbb{Z}[G/H]$ and $\Upsilon_{\mathbb{Q}[G/H]} = \widehat{G/H} \subseteq \widehat{G}$, whilst Remark 8.4 implies that, for any fixed place v_0 of S, one has $r^{\Pi_\kappa}_{L,S} \geq |\Sigma^H_{v_0}|$ with $\Sigma^H_{v_0}$ the set of places in $S\setminus\{v_0\}$ which split completely in L^H/K. In this setting we write

$$\pi^{G/H,v_0}_{L/K,S,T} : {}_{\Pi_\kappa}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L) \cong \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L^H) \to X_{L^H,S} \to Y_{L^H,\Sigma^H_{v_0}}$$

for the composite surjection of G/H-modules, where the first arrow is as in (27) and the second is the natural projection. Then, after abbreviating $\pi^{G/H,v_0}_{L/K,S,T}$ to π, one finds that $\theta^{\pi}_{S,T}(0)$ coincides with the 'higher-order Stickelberger element' $\theta^{(r')}_{L^H/K,S,T}(0)$ defined by the first author in [7] with $r' := |\Sigma^H_{v_0}|$.

Example 8.7. In the setting of Example 8.3 there is a natural composite surjection of $\mathcal{O}_\rho$-modules

$$\pi^{\rho}_{L/K,S,T} : {}_{\Pi_\rho}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L) \to ({}_{\Pi_\rho}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L))_{\mathrm{tf}} \cong ({}_{\Pi_\rho}X_{L,S})_{\mathrm{tf}},$$

where the isomorphism is induced by applying the (right exact) functor $M \mapsto {}_{\Pi_\rho}M$ to the exact sequence (27). Then, abbreviating $\pi^{\rho}_{L/K,S,T}$ to π, one finds that $\theta^{\pi}_{S,T}(0) = e_\rho L^*_{S,T}(\rho, 0)$, where we write $L^*_{S,T}(\rho, 0)$ for the leading term of $L_{S,T}(\rho, z)$ at $z = 0$.

8.4. **The definition of higher non-abelian Stark elements.** In the sequel we write

$$R_{L,S} : \mathbb{R} \cdot \mathcal{O}^{\times}_{L,S} \to \mathbb{R} \cdot X_{L,S} \tag{35}$$

for the isomorphism of $\mathbb{R}[G]$-modules which at each $u \in \mathcal{O}^{\times}_{L,S}$ satisfies

$$R_{L,S}(u) = -\sum_{w \in S_L} \log|u|_w \cdot w,$$

where $|\cdot|_w$ denotes the normalised absolute value at w.

We fix π in $\Delta^{\Pi}_{L,S,T}$ and, following Lemma 8.5, assume the locally-free $\mathcal{A}$-module Y_π has rank $r := r^{\Pi}_{L,S}$. We then choose a subset $\underline{b} = \{b_i\}_{1 \le i \le r}$ of Y_π which spans a full free $\mathcal{A}$-submodule.

The definition of the idempotent e_π implies that $R_{L,S}$ induces an isomorphism of $\zeta(e_\pi A_{\mathbb{C}})$-modules

$$\lambda^{\pi}_{L,S} : e_\pi\big({\textstyle\bigwedge}^r_{A_{\mathbb{C}}} \mathbb{C} \cdot {}_{\Pi}\mathcal{O}^{\times}_{L,S}\big) \xrightarrow{\sim} e_\pi\big({\textstyle\bigwedge}^r_{A_{\mathbb{C}}} \mathbb{C} \cdot Y_\pi\big)$$

and, since $\theta^{\pi}_{S,T}(0) = e_\pi \theta^{\pi}_{S,T}(0)$ (by Lemma 8.5), one has $\theta^{\pi}_{S,T}(0) \cdot \wedge^{i=r}_{i=1} b_i \in \mathrm{im}(\lambda^{\pi}_{L,S})$. We may therefore make the following definition.

Definition 8.8. For any subset $\underline{b}$ as above the 'higher non-abelian Stark element (relative to $\underline{b}$)' in $e_\pi(\bigwedge^r_{A_{\mathbb{C}}} \mathbb{C} \cdot {}_{\Pi}\mathcal{O}^{\times}_{L,S})$ is obtained by setting

$$\epsilon^{\pi}_{\underline{b}} := (\lambda^{\pi}_{L,S})^{-1}(\theta^{\pi}_{S,T}(0) \cdot \wedge^{i=r}_{i=1} b_i).$$

The following examples show that this definition provides a common generalization of constructions that have been made in the literature.

Example 8.9. Assume S contains all places which ramify in L/K. Fix v_0 in S and abbreviate the homomorphism $\pi^{G,v_0}_{L/K,S,T}$ defined in Example 8.6 to π. For each v in $\Sigma^{\{\mathrm{id}\}}_{v_0}$ fix a place w_v of L above v and set $\underline{b} := \{w_v : v \in \Sigma^{\{\mathrm{id}\}}_{v_0}\}$. Then the data $(\underline{b}, \pi)$ is suitable to be used in Definition 8.8 and so we may set

$$\epsilon^{v_0}_{L/K,S,T} := \epsilon^{\pi}_{\underline{b}}.$$

We refer to this element as the 'non-abelian Rubin-Stark element' with respect to the data $L/K, S, T$ and v_0. If v_0 does not split completely in L/K, then $\Sigma_{v_0}^{\{\mathrm{id}\}}$ and hence also $\epsilon_{L/K,S,T}^{v_0}$ is independent of v_0. In addition, if we set $r := |\Sigma_{v_0}^{\{\mathrm{id}\}}|$, assume G is abelian and for each $\psi \in \widehat{G}$ take the elements $v_{\psi,j}$ that are used in the definition (1) of exterior powers to be e_ψ, then the discussion of Example 8.6 implies that $\epsilon_{L/K,S,T}^{v_0}$ coincides with the element $\epsilon_{S,T}$ of $\mathbb{R} \cdot \wedge_{\mathbb{Z}[G]}^r \mathcal{O}_{L,S}^\times$ that occurs in the statement of the Rubin-Stark Conjecture [44, Conj. B′].

Example 8.10. As a generalization of Example 8.9 one can assume S contains all places which ramify in L/K and, in addition, that for some fixed place v_0 in S there exists a non-negative integer r such that if for every character χ in $\widehat{G}$ we set $L_\chi := L^{\ker(\chi)}$, then the set $\Sigma_{v_0}^\chi$ of places in $S \setminus \{v_0\}$ which split completely in L_χ/K has cardinality at least r. The elements $\epsilon_{L_\chi/K,S,T}^{v_0}$ are then defined as in Example 8.9 and can be combined to give an element

$$\epsilon_{L/K,S,T}^{v_0,*} := \sum_{\chi \in \widehat{G}} \frac{1}{|\ker(\chi)|^{r\chi(1)}} e_\chi(\epsilon_{L_\chi/K,S,T}^{v_0})$$

of $\wedge_{\mathbb{R}[G]}^r \mathbb{R} \cdot \mathcal{O}_{L,S}^\times$. If G is abelian and we choose $v_{\psi,j} = e_\psi$ for each index j (as in Example 8.9), then [19, Prop. 4.7] implies that this element coincides with the element $\varepsilon_{L/K,S,T,r}$ that occurs in the 'generalized Rubin-Stark Conjecture' formulated by Emmons and Popescu in [19, Conj. 3.8].

Example 8.11. Assume the notation and hypotheses of Examples 8.3 and 8.7. Abbreviate the homomorphism $\pi_{L/K,S,T}^\rho$ to π, set $\mathcal{O} := \mathcal{O}_\rho$ and write E for the quotient field of $\mathcal{O}$ and r for the rank of the (locally-free) $\mathcal{O}$-module $({}_{\Pi_\rho}X_{L,S})_{\mathrm{tf}}$. Let $\underline{b} = \{b_i\}_{1\le i\le r}$ be any subset of $({}_{\Pi_\rho}X_{L,S})_{\mathrm{tf}}$ that is linearly independent over $\mathcal{O}$. Then the data $(\underline{b}, \pi)$ can be used in Definition 8.8 and, in this case, the elements $\epsilon_{\underline{b}}^\pi$ are related to those that occur in the refined Stark conjecture of [7, Conj. 2.6.1]. In particular, if $r = 1$ and ρ is non-trivial, then there is a unique place v in S for which $H^0(G, \Pi_\rho)$ is infinite for any choice of place w of L above v and, for suitable primitive idempotents f_ρ of $E[G]$, the elements $\epsilon_{\{|G|\cdot f_\rho(w)\}}^\pi$ recover the elements that are studied in Stark's original articles [47, 48] and the subsequent article of Chinburg [14]. For details of these connections see the discussion of §13.2.

Part III: Conjectures and Results

9. Statement of the conjectures

We continue to fix data $L/K, G, S$ and T as in §7 and §8.

9.1. **Non-abelian zeta elements and the central conjecture.**

In this first section we define the key notion of 'non-abelian zeta element', formulate our central conjecture concerning these elements and describe some basic properties of this conjecture.

9.1.1. The equivariant L-function associated to the data $L/K, S$ and T is defined by setting

$$\theta_{L/K,S,T}(z) := \sum_{\chi\in\widehat{G}} L_{S,T}(\check{\chi}, z)e_\chi,$$

where $\check{\chi}$ is the contragredient of χ.

The leading term of $\theta_{L/K,S,T}(z)$ at $z = 0$ is then defined by setting

$$\theta^*_{L/K,S,T}(0) := \sum_{\chi\in\widehat{G}} L^*_{S,T}(\check{\chi}, 0)e_\chi$$

and is easily shown to belong to $\zeta(\mathbb{R}[G])^\times$.

The Dirichlet regulator isomorphism (35) combines with the general construction (22) to give a canonical isomorphism of $\zeta(\mathbb{C}[G])$-modules

$$\vartheta_{\mathbb{C}\otimes_\mathbb{R} R_{L,S}} : \mathrm{det}_{\mathbb{C}[G]}(\mathbb{C}\cdot C_{L,S,T}) \to \zeta(\mathbb{C}[G]).$$

In the sequel we denote this isomorphism by $\lambda_{L,S}$.

It can be shown that the full pre-image under $\lambda_{L,S}$ of the subspace $\zeta(\mathbb{R}[G])$ of $\zeta(\mathbb{C}[G])$ is equal to $\mathrm{det}_{\mathbb{R}[G]}(\mathbb{R}\cdot C_{L,S,T}) = \mathbb{R}\cdot\mathrm{det}_{\mathbb{Z}[G]}(C_{L,S,T})$ and this fact allows us to make the following definition.

Definition 9.1. The 'zeta element of $\mathbb{G}_m$ relative to $L/K, S$ and T' is the pre-image $z_{L/K,S,T}$ in $\mathbb{R}\cdot\mathrm{det}_{\mathbb{Z}[G]}(C_{L,S,T})$ of $\theta^*_{L/K,S,T}(0)$ under the isomorphism $\lambda_{L,S}$.

This notion is a natural extension of that introduced (in the setting of abelian extensions L/K) in [11].

The central conjecture that we make concerning these non-abelian zeta elements is the following.

Conjecture 9.2. *$z_{L/K,S,T}$ is a locally-primitive basis of* $\mathrm{det}_{\mathbb{Z}[G]}(C_{L,S,T})$.

Remark 9.3. If G is abelian, then $\xi(\mathbb{Z}[G]) = \mathbb{Z}[G]$ (see Lemma 3.2) and Corollary 6.18 implies that every basis of the $\mathbb{Z}[G]$-module $\mathrm{det}_{\mathbb{Z}[G]}(C_{L,S,T})$ is automatically a primitive basis. Conjecture 9.2 is thus equivalent in this case to asserting that $z_{L/K,S,T}$ is a basis of the $\mathbb{Z}[G]$-module $\mathrm{det}_{\mathbb{Z}[G]}(C_{L,S,T})$ and so recovers the central conjecture of [11].

9.1.2. In this section we record several properties of Conjecture 9.2 that will be useful in the sequel.

To state our first result we fix a bimodule Π as in §8.1 and write $z^\Pi_{L/K,S,T}$ for the image of $z_{L/K,S,T}$ under the projection map

$$\begin{aligned}(36)\quad \mathbb{R}\cdot\mathrm{det}_{\mathbb{Z}[G]}(C_{L,S,T}) &\subset \prod_{\chi\in\widehat{G}} \mathrm{det}_\mathbb{C}(\mathrm{Hom}_{\mathbb{C}[G]}(V_\chi, \mathbb{C}\cdot C_{L,S,T}))\\ &\to \prod_{\chi\in\Upsilon_\Pi} \mathrm{det}_\mathbb{C}(\mathrm{Hom}_{\mathbb{C}[G]}(V_\chi, \mathbb{C}\cdot C_{L,S,T})) \cong \mathbb{C}\cdot\mathrm{det}_\mathcal{A}(\Pi\otimes^\mathbb{L}_{\mathbb{Z}[G]} C_{L,S,T})\end{aligned}$$

where the isomorphism is induced by the bijection $\Pi_* : \mathrm{Wed}_A \to \Upsilon_\Pi$ described in Remark 8.1. We also write $\Upsilon^{\mathrm{symp}}_\Pi$ for the subset of Υ_Π comprising (irreducible) characters χ for which the complex algebra $(\Pi_*)^{-1}(\chi)$ is induced from a Wedderburn component of $A_\mathbb{R}$ that is a matrix ring over the division ring of real quaternions.

Proposition 9.4. *Assume Conjecture 9.2 is valid.*

(i) *Then* $z^{\Pi}_{L/K,S,T}$ *is a locally-primitive basis of* $\det_{\mathcal{A}}(\Pi\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C_{L,S,T})$.

(ii) *If* $L^*(\chi,0)$ *is positive for all characters* χ *in* $\Upsilon^{\rm symp}_{\Pi}$, *then* $z^{\Pi}_{L/K,S,T}$ *is a primitive basis of* $\det_{\mathcal{A}}(\Pi\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C_{L,S,T})$.

Proof. We choose a representative $P\to F$ of the complex $C_{L,S,T}$ as in Proposition 7.8 and note that $\Pi\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C_{L,S,T}$ is then represented by the induced complex of $\mathcal{A}$-modules $\Pi\otimes_{\mathbb{Z}[G]} P\to\Pi\otimes_{\mathbb{Z}[G]} F$.

For each prime p the $\mathbb{Z}_p[G]$-modules P_p and F_p are free of the same rank, e say, and so we may fix bases $\{b_{0,i}\}_{1\le i\le e}$ and $\{b_{1,i}\}_{1\le i\le e}$. For each p we can also fix a basis $\{\pi_j\}_{1\le j\le k}$ of the free $\mathcal{A}_p$-module Π_p.

Then the sets $\{\pi_j\otimes_{\mathbb{Q}_p[G]} b_{a,i}\}_{1\le j\le k,1\le i\le e}$ are $\mathcal{A}_p$-bases of $(\Pi\otimes_{\mathbb{Z}[G]} P)_p$ (for $a=0$) and $(\Pi\otimes_{\mathbb{Z}[G]} F)_p$ (for $a=1$) and, writing π_Π for the projection map (36) one has

$$\pi_A((\wedge_{i=1}^{i=e}b_{0,i})\otimes(\wedge_{i=1}^{i=e}b_{1,i}^*))=(\wedge_{i=1}^{i=e}(\wedge_{j=1}^{j=k}\pi_j\otimes b_{0,i}))\otimes(\wedge_{i=1}^{i=e}(\wedge_{j=1}^{j=k}(\pi_j\otimes b_{1,i})^*))$$

where the exterior powers and tensor products on the left hand side are taken over $\mathbb{Q}_p[G]$ and on the right hand side over A_p.

The assertion of claim (i) now follows because $\pi_A(z_{L/K,S,T})=z^{\Pi}_{L/K,S,T}$.

Claim (ii) is proved in §11.3. □

Remark 9.5. Conjecture 9.2 implies that for all primes p the zeta element $z_{L/K,S,T}$ is a primitive $\xi(\mathbb{Z}_{(p)}[G])$-basis of $\det_{\mathbb{Z}[G]}(C_{L,S,T})_{(p)}=\det_{\mathbb{Z}_{(p)}[G]}(\mathbb{Z}_{(p)}\otimes_{\mathbb{Z}} C_{L,S,T})$. In the sequel we shall refer to this assertion for any fixed prime p as the 'p-component of Conjecture 9.2'. We will also refer to the explicit predictions in Proposition 9.4 concerning $z^{\Pi}_{L/K,S,T}$ as the 'Π-component of Conjecture 9.2'.

We next record a dual version of Conjecture 9.2 that will be useful in the sequel. To do this we set

$$C^*_{L,S,T}:=R\operatorname{Hom}_{\mathbb{Z}}(C_{L,S,T},\mathbb{Z})[-1],$$

endowed with the natural contragredient action of G. Then $C^*_{L,S,T}$ belongs to $D^{\rm lf,0}(\mathbb{Z}[G])$, is acyclic outside degrees zero and one and there is an identification

$$\mathbb{C}\cdot H^a(C^*_{L,S,T})=\operatorname{Hom}_{\mathbb{C}}(H^{1-a}(C_{L,S,T}),\mathbb{C})$$

for $a=0$ and $a=1$. One can therefore define an isomorphism of $\zeta(\mathbb{C}[G])$-modules

$$\lambda^*_{L,S}:\det_{\mathbb{C}[G]}(\mathbb{C}\cdot C^*_{L,S,T})\to\zeta(\mathbb{C}[G])$$

just as for $\lambda_{L,S}$ except that the role of $R_{L,S}$ is now played by its linear dual $\operatorname{Hom}_{\mathbb{R}}(R_{L,S},\mathbb{R})$.

Definition 9.6. The 'dual zeta element of $\mathbb{G}_m$ relative to $L/K,S$ and T' is the pre-image $z^*_{L/K,S,T}$ in $\mathbb{R}\cdot\det_{\mathbb{Z}[G]}(C^*_{L,S,T})$ of $\iota_\#(\theta^*_{L/K,S,T}(0))$ under the isomorphism $\lambda^*_{L,S}$.

Lemma 9.7. *Conjecture 9.2 is valid if and only if* $z^*_{L/K,S,T}$ *is a locally-primitive* $\xi(\mathbb{Z}[G])$*-basis of* $\det_{\mathbb{Z}[G]}(C^*_{L,S,T})$.

Proof. This is a straightforward consequence of a following (easy) fact concerning linear duals.

Let $\lambda : V \to W$ be an isomorphism of free rank one $\zeta(\mathbb{C}[G])$-modules. Fix basis elements v and w of V and W and write v^* and w^* for the corresponding dual bases of $\mathrm{Hom}_{\zeta(\mathbb{C}[G])}(V, \zeta(\mathbb{C}[G]))$ and $\mathrm{Hom}_{\zeta(\mathbb{C}[G])}(W, \zeta(\mathbb{C}[G]))$. Then if x is the element of $\zeta(\mathbb{C}[G])^\times$ defined by the equality $\lambda(v) = x \cdot w$ in W one has

$$\mathrm{Hom}_{\zeta(\mathbb{C}[G])}(\lambda, \zeta(\mathbb{C}[G]))(w^*) = \iota_\#(x) \cdot v^*$$

in $\mathrm{Hom}_{\zeta(\mathbb{C}[G])}(W, \zeta(\mathbb{C}[G]))$. □

9.1.3. To end this section we record a simple functorial property of zeta elements which is unconditionally true and will be used in the sequel.

Lemma 9.8. *Fix a normal subgroup H of G and identify $X_{L^H,S}$ with a submodule of $X_{L,S}$ in the manner described at the beginning of § 7.2.2. Then the zeta element $z_{L^H/K,S,T}$ is equal to the image of $z_{L/K,S,T}$ under the homomorphism*

$$\mathbb{R} \cdot \det_{\mathbb{Z}[G]}(C_{L,S,T}) \to \mathbb{R} \cdot \zeta(\mathbb{Z}[G/H]) \otimes_{\zeta(\mathbb{Z}[G])} \det_{\mathbb{Z}[G]}(C_{L,S,T}) \cong \mathbb{R} \cdot \det_{\mathbb{Z}[G/H]}(C_{L^H,S,T})$$

induced by applying Proposition 6.8(iii) to the canonical isomorphism $\mathbb{Z}[G/H] \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C_{L,S,T} \cong C_{L^H,S,T}$ described in Lemma 7.4(ii).

Proof. This is verified by a concrete computation that uses the following two facts. The image of $\theta^*_{L/K,S,T}(0)$ under the natural projection map $\zeta(\mathbb{R}[G]) \to \zeta(\mathbb{R}[G/H])$ is equal to $\theta^*_{L^H/K,S,T}(0)$ and, with respect to the stated identification of $X_{L^H,S}$ with a submodule of $X_{L,S}$, the argument of Tate in [49, Chap. I, §6.5] shows that there is a commutative diagram of G-modules

$$\begin{array}{ccc} \mathcal{O}^\times_{L,S,T} & \xrightarrow{R_{L,S}} & \mathbb{R} \cdot X_{L,S} \\ \uparrow & & \subseteq\uparrow \\ \mathcal{O}^\times_{L^H,S,T} & \xrightarrow{R_{L^H,S}} & \mathbb{R} \cdot X_{L^H,S}, \end{array}$$

in which the left hand vertical arrow is the natural inclusion. □

9.2. **Statement of explicit consequences.** In this section we describe several concrete consequences of Conjecture 9.2.

To do this we shall use the hypotheses and notation of §8. In particular we fix a surjective homomorphism of $\mathcal{A}$-modules

$$\pi : {}_{\Pi}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L) \to Y_\pi$$

in $\Delta^{\Pi}_{L,S,T}$ and assume (as we may, following Lemma 8.5) that $\mathrm{rk}_{\mathcal{A}}(Y_\pi) = r^{\Pi}_{L,S}$.

In addition, we write χ_Π for the A-valued character of (the free A-module) $\mathbb{Q} \cdot \Pi$ and set

$$\mathrm{pr}_\Pi := \sum_{g \in G} \chi_\Pi(g) \otimes g^{-1} \in A[G].$$

For any additive homomorphism $\epsilon : A \to \mathbb{Q}$ we also write ϵ_G for the homomorphism $A[G] \to \mathbb{Q}[G]$ sending each element $\sum_{g\in G} a_g g$ to $\sum_{g\in G} \epsilon(a_g) g$.

Theorem 9.9. *Fix π in $\Delta^{\Pi}_{L,S,T}$ as above, set $r := r^{\Pi}_{L,S}$ and fix a set $\underline{b} = \{b_i\}_{1\le i\le r}$ which spans a full free $\mathcal{A}$-submodule $Y_{\underline{b}}$ of Y_π.*

Then if the Π-component of Conjecture 9.2 is valid so are all of the following claims.

(i) *The transpose Selmer group ${}_{\Pi}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$ has a quadratic $\mathcal{A}$-module presentation bundle $h^{\Pi}_{L/K,S,T}$ for which*

$$\mathrm{FI}^0_{\mathcal{A}}(Y_\pi/Y_{\underline{b}})^{-1}\{(\wedge_{i=1}^{i=r}\varphi_i)(\epsilon^{\pi}_{\underline{b}}) : \varphi_i \in \mathrm{Hom}_{\mathcal{A}}({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T}, \mathcal{A})\} = \mathrm{FI}^r_{\mathcal{A}}(h^{\Pi}_{L/K,S,T}).$$

(ii) *If $L^*(\chi,0)$ is positive for every χ in $\Upsilon^{\mathrm{symp}}_{\Pi}$, then ${}_{\Pi}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$ has a quadratic $\mathcal{A}$-module presentation $h^{\Pi,\mathrm{global}}_{L/K,S,T}$ for which*

$$\mathrm{FI}^0_{\mathcal{A}}(Y_\pi/Y_{\underline{b}})^{-1}\{(\wedge_{i=1}^{i=r}\varphi_i)(\epsilon^{\pi}_{\underline{b}}) : \varphi_i \in \mathrm{Hom}_{\mathcal{A}}({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T}, \mathcal{A})\} = \mathrm{FI}^r_{\mathcal{A}}(h^{\Pi,\mathrm{global}}_{L/K,S,T}).$$

(iii) *Let S' be any subset of S with $S^{\infty}_K \subseteq S'$ and such that the composite homomorphism*

$$({}_{\Pi}X_{L,S'})_{\mathrm{tf}} \to ({}_{\Pi}X_{L,S})_{\mathrm{tf}} \cong {}_{\Pi}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)_{\mathrm{tf}} \to Y_\pi$$

has finite cokernel, where the first arrow is induced by the inclusion $X_{L,S'} \subseteq X_{L,S}$ and the last arrow by π.

Then for any elements a of $\mathfrak{A}(\mathcal{A}) \cdot \mathrm{Ann}_{\mathcal{A}}(\mathrm{Tor}^G_1(\Pi, \mathcal{O}^{\times}_{L,S',T}))$ and $(\varphi_i)_{1\le i\le r}$ of $\mathrm{Hom}_{\mathcal{A}}({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T}, \mathcal{A})^r$ and any additive homomorphism $\epsilon : \mathcal{A} \to \mathbb{Z}$ one has

$$\epsilon_G(a(\wedge_{i=1}^{i=r}\varphi_i)(\epsilon^{\pi}_{\underline{b}}) \cdot \mathrm{pr}_{\Pi}) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S'}(L)).$$

To state the first corollary of this result we specialize to the setting of Example 8.9. We recall, in particular, that in this context V_L denotes the subset of $S \setminus \{v_0\}$ comprising places which split completely in L/K.

Corollary 9.10. *Let the place v_0 be chosen as in Example 8.9 and set $V := V_L$ and $r := |V|$. Then if Conjecture 9.2 is valid for L/K so are all of the following claims.*

(i) *The transpose Selmer group $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$ has a quadratic G-module presentation bundle $h_{L/K,S,T}$ for which*

$$\xi(\mathbb{Z}[G])\{(\wedge_{i=1}^{i=r}\varphi_i)(\epsilon^{v_0}_{L/K,S,T}) : \varphi_i \in \mathrm{Hom}_G(\mathcal{O}^{\times}_{L,S,T}, \mathbb{Z}[G])\} = \mathrm{FI}^r_{\mathbb{Z}[G]}(h_{L/K,S,T}).$$

(ii) *If $L^*(\chi,0)$ is positive for every irreducible symplectic character χ of G, then $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$ has a G-module presentation $h^{\mathrm{global}}_{L/K,S,T}$ for which*

$$\xi(\mathbb{Z}[G])\{(\wedge_{i=1}^{i=r}\varphi_i)(\epsilon^{v_0}_{L/K,S,T}) : \varphi_i \in \mathrm{Hom}_G(\mathcal{O}^{\times}_{L,S,T}, \mathbb{Z}[G])\} = \mathrm{FI}^r_{\mathbb{Z}[G]}(h^{\mathrm{global}}_{L/K,S,T}).$$

(iii) *For any a in $\mathfrak{A}(G)$ and $(\varphi_i)_{1\le i\le r}$ in $\mathrm{Hom}_G(\mathcal{O}^{\times}_{L,S,T}, \mathbb{Z}[G])^r$ one has*

$$a \cdot (\wedge_{i=1}^{i=r}\varphi_i)(\epsilon^{v_0}_{L/K,S,T}) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S'}(L))$$

with $S' := S^{\infty}_K \cup V_L \cup \{v_0\}$.

(iv) (a) *For every prime p there exists an element ζ of $\mathrm{det}_{\mathbb{Z}[G]}(C_{L,S,T})_{(p)}$ with the following property: for every normal subgroup H of G, with $E := L^H$, one has $\Delta^{H,p}_{L,S,T}(\zeta) = \epsilon^{v_0}_{E/K,S,T}$ with $\Delta^{H,p}_{L,S,T}$ the homomorphism defined in §7.5.*

(b) *For all normal subgroups H of G, with $E = L^H$, one has*
$$\rho_{\pi,H}(\epsilon^{v_0}_{L/K,S,T}) = \mathrm{Rec}^{\mathcal{P}}_H(\epsilon^{v_0}_{E/K,S,T})$$
with $\rho_{\pi,H}$ the natural projection $\mathrm{F}_{\pi_E}(\bigcap^r_G \mathcal{O}^\times_{L,S,T}) \to \mathrm{F}_{\pi_E}(\bigcap^r_G \mathcal{P}_{L,S,T})_H$.

Remark 9.11. The predictions in Corollary 9.10 incorporate simultaneous refinements and generalizations of several well-known conjectures. For example, claim (i) implies $\epsilon^{v_0}_{L/K,S,T}$ belongs to $\bigcap^r_G \mathcal{O}^\times_{L,S,T}$ and Remark 3.4 implies this containment is a natural generalization (to non-abelian extensions) of the Rubin-Stark Conjecture for L/K. Taken together, claims (i), (ii) and (iii) constitute a strong refinement of the central conjecture (Conjecture 2.4.1) formulated by the first author in [7] (and hence also, by specializing to the case that $r = 0$, of the 'non-abelian Brumer Conjecture' formulated by Nickel in [41]). Proposition 7.3(iii) implies that claim (iv) provides a natural generalization to non-abelian extensions of the congruence relations between derivatives of Dirichlet L-series that were formulated independently by the second author [45] and by Mazur and Rubin [39].

By specializing Theorem 9.9 to the setting of Example 8.11 one obtains a refinement of the general refined Stark conjecture of [7, Conj. 2.6.1] (and hence of the earlier conjecture formulated by the first author in [6]). For more details see Remark 13.2.

In the next result we make this refinement explicit in the context of the algebraic units that were first discussed by Stark [48] and later conjectured to exist by Chinburg in [14, Conj. 1].

In this regard we recall from Example 8.11 that if ψ is non-trivial and such that $L_{S,T}(\psi, z)$ vanishes to order one at $z = 0$, then there is a unique place v_1 in S for which the module $H^0(G_{w_1}, \Pi_\psi)$ is infinite, where G_{w_1} is the decomposition subgroup in G of any fixed place w_1 of K above v_1.

Corollary 9.12. *Assume K is a number field, ψ has degree two, $L_{S,T}(\psi, z)$ vanishes to order one at $z = 0$, $|S| > 1$ and the unique place v_1 in S for which $H^0(G_{w_1}, \Pi_\psi)$ is not finite is archimedean. Fix an embedding of L in $\mathbb{C}$ corresponding to w_1 and use this to regard L as a subfield of $\mathbb{C}$.*

If the Π_ψ-component of Corollary 9.2 is valid, then the element
$$\epsilon_{S,T,\psi} := \exp(-\sum_{\gamma \in G_{E_\psi/\mathbb{Q}}} L'_{S,T}(\psi^\gamma, 0))$$
is a real unit in L which has all of the following properties:-

(i) *either $\epsilon_{S,T,\psi}$ or $-\epsilon_{S,T,\psi}$ is congruent to 1 modulo all of the places in T_L;*
(ii) *$|\epsilon_{S,T,\psi}|_w = 1$ if w is any place of L that does not lie above v_1;*
(iii) *for $g \in G$ one has $-\log|g^{-1}(\epsilon_{S,T,\psi})|_{w_1} = \sum_{\gamma \in G_{E_\psi/\mathbb{Q}}}(\psi^\gamma(g) + \psi^\gamma(g\tau))L'_{S,T}(\psi^\gamma, 0)$ with τ the (unique) non-trivial element of G_{w_1};*
(iv) *for ϕ in $\mathrm{Hom}_G(\mathcal{O}^\times_L, \mathbb{Z}[G])$ one has $2^{-2}|G|^3\phi(\epsilon_{S,T,\psi}) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T(L))$.*

Remark 9.13. The discussion following [6, Prop. 3.3] shows that properties (i), (ii) and (iii) in Corollary 9.12 imply $\epsilon_{S,T,\psi}$ is an algebraic unit of the sort first discussed by Stark [48] and later conjectured to exist by Chinburg in [14, Conj. 1]. A containment of the form stated in Corollary 9.12(iv) was first predicted in [7, Prop. 12.2.1], but this earlier conjecture is weaker than claim (iv) in that it uses the factor $|G|^4$ in place of $|G|^3$ and $\mathrm{Cl}(L)$ in place of the larger group $\mathrm{Cl}^T(L)$. In particular, by showing that the result of [7, Prop. 12.2.1] is not best possible, Corollary 9.12 now answers the question raised explicitly in [7, Rem. 12.2.2].

10. Statement of the main evidence

In this section we state the main evidence that we have in support of Conjecture 9.2 and the consequences of it that are described in §9.2.

To do this we refer to the 'leading term conjecture' formulated by the first author in [7, §6.1, Conj. LTC(L/K)]. This conjecture is an equality in the relative algebraic K-group $K_0(\mathbb{Z}[G],\mathbb{R}[G])$.

For any bimodule Π as in §8.1 we use the homomorphism

$$\mu_\Pi^{\mathrm{rel}} : K_0(\mathbb{Z}[G],\mathbb{R}[G]) \to K_0(\mathcal{A},A_{\mathbb{C}}) \tag{37}$$

that is induced by the functor $\Pi\otimes_{\mathbb{Z}[G]} -$.

Theorem 10.1. *Let Π be a bimodule as in §8.1. Then the Π-component of Conjecture 9.2 is valid if and only if the equality of LTC(L/K) is valid modulo* $\ker(\mu_\Pi)$.

By specializing this result to the setting of Example 8.9 one obtains the following immediate consequence. We note that, in view of Remark 9.11 and the fact that we now make no assumptions on the torsion subgroup of $\mathcal{O}_K^\times$, this consequence constitutes a strong refinement of the main result (Theorem 4.1.1) of [7].

Corollary 10.2. *If LTC(L/K) is valid, then Conjecture 9.2, and hence also claims (i), (ii), (iii) and (iv) of Corollary 9.10, are valid.*

Remark 10.3. In the context of Example 8.10 the methods used below will also allow one to prove a generalization of Corollary 10.2 that refines the conjectures that are formulated by Erickson and Stark [21, Conj. 4.1], by Emmons and Popescu [19, Conj. 3.8] and by Vallieres [50, Conj. 4.16]. For details see the forthcoming work of Livingstone-Boomla.

If L is abelian over $\mathbb{Q}$, then for any subfield K the validity of LTC(L/K) has been proved by Greither and the first author [10] and by Flach [23]. If L has positive characteristic, then the validity of LTC(L/K) was also recently proved in [8, Cor. 1.3]. Corollary 10.2 therefore leads directly to the following result.

Corollary 10.4. *Conjecture 9.2, and hence also claims (i), (ii), (iii) and (iv) of Corollary 9.10, are valid if L is either an abelian extension of $\mathbb{Q}$ or has positive characteristic.*

Aside from the cases considered in Corollary 10.4 there are also special classes of non-abelian Galois extensions of number fields for which LTC(L/K) has been verified and so Corollary 10.2 implies the validity of Conjecture 9.10. For details of these cases (that are due to several different authors) we refer the reader to the comprehensive survey given by Johnston and Nickel in [31, §4.3] and to their results in [31, §4.6].

Corollary 10.2 can also be combined with previous work of the first author to relate Conjecture 9.2 to earlier conjectures of Gross and of Serre and Tate and hence obtain the following supporting evidence.

In the sequel we write E^{cyc} for the cyclotomic $\mathbb{Z}_p$-extension of a number field E and $\mu_p(E)$ for the p-adic μ-invariant of the extension E^{cyc}/E. We recall that Iwasawa has conjectured in [28] that $\mu_p(E)$ should always vanish.

Corollary 10.5. *Let L be a CM Galois extension of a totally real field K with group G. Let p be an odd prime which either does not divide $|G|$ or is such that $\mu_p(L)$ vanishes.*

(i) *If p is tamely ramified in L/K and the p-adic Stark Conjecture at $s = 1$ of Serre and Tate is valid for all totally even characters of G, then the p-part of Conjecture 9.2 is valid after taking plus parts.*

(ii) *If the p-adic Gross-Stark Conjecture is valid for all totally odd characters of G, then the p-part of Conjecture 9.2 is valid after taking minus parts.*

The proof of Corollary 10.5(i) shows that the condition that p is tamely ramified can actually be replaced by the more general hypothesis that the 'local epsilon constant conjecture' of Breuning [3] is valid for all extensions that are obtained by p-adically completing L/K. For more details see §13.3 (and, in particular, Remark 13.4).

Corollary 10.5(ii) combines with the results of Ventullo [51] to obtain unconditional verifications of the p-part of Conjecture 9.2 in the technically most difficult case that L/K is a non-abelian extension of number fields, p divides $|G|$ and the relevant p-adic L-series possess trivial zeroes. For example, in §13.4 we will use this approach to prove the following result.

Corollary 10.6. *Let L/K be a finite CM Galois extension for which G is the semi-direct product of an abelian group A by a supersolvable group. Assume that the field F^A is totally real and has at most one p-adic place which splits completely in F/F^+ and that $\mu_p(F^P)$ vanishes where P is any given subgroup of G of p-power order. Then the p-component of Conjecture 9.2 is valid after taking minus parts.*

Example 10.7. If, in the notation of Corollary 10.6, the field F^P is abelian over $\mathbb{Q}$, then $\mu_p(F^P)$ vanishes by Ferrero-Washington [22]. In particular, if in any such case the field F^A has only one p-adic place, then Corollary 10.6 implies the unconditional validity of Conjecture 9.2 after p-localization and taking minus parts. It is straightforward to describe families of non-abelian extensions which satisfy these hypotheses.
(i) Let F be a real quadratic field in which p does not split and assume that L is a CM abelian extension of F of exponent $2p^n$ for some natural number n. One can then set $K = \mathbb{Q}$ and $A = G_{L/F}$ and assume that L/K is Galois with (generalised) dihedral Galois group. Then $L^A = F$, P is normal in G and the quotient group G/P is abelian, as required.
(ii) Let E be a totally real A_4 extension of $\mathbb{Q}$ with the property that 3 does not split in its unique cubic subfield. Then for any imaginary quadratic field F the field $L := EF$ is a CM Galois extension of $\mathbb{Q}$ and $G_{L/\mathbb{Q}}$ is of the form $A \rtimes \mathbb{Z}/3$ with $A := \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ (where $\mathbb{Z}/3$ acts trivially on one copy of $\mathbb{Z}/2$ and cyclically permutes the non-trivial elements in the remaining factor $\mathbb{Z}/2 \times \mathbb{Z}/2$) and so is abelian-by-cyclic. The field L^A is then the unique cubic subfield of L and so is totally real with only one 3-adic place and the field $E_1 := L^A F$ is abelian over $\mathbb{Q}$ so $\mu_3(E_1)$ vanishes. One can also show that if $\mu_3(E_2)$ vanishes for any given quadratic extension E_2 of E_1 in L, then $\mu_3(L)$ also vanishes and so Corollary 10.6 applies to the extension $L/\mathbb{Q}$.

Remark 10.8. A similar approach to that used above also allows one to construct many concrete examples in which all of the hypotheses of [9, Cor. 3.3] are satisfied by characters that are both faithful and of arbitrarily large degree. For all such examples one therefore obtains a p-adic analytic construction of p-units that both generate non-abelian Galois extensions of totally real fields and also encode explicit structural information about ideal class groups, thereby extending and refining the p-adic analytic approach to Hilbert's twelfth problem this is described for linear p-adic characters by Gross in [27, Prop. 3.14]. In the same way one deduces that these examples verify a natural p-adic analogue of a question of Stark in [48] and a conjecture of Chinburg in [14] which were both formulated in the setting of characters of degree two. For more details see Remark 13.5.

Finally we note that Theorem 10.1 combines with the argument of [7, §12.1] to obtain the following more concrete result.

Corollary 10.9. *A character ψ in $\widehat{G}$ validates the Strong-Stark Conjecture if and only if the T_ψ-component of Conjecture 9.2 is valid. In particular, for any such character the result of Corollary 9.12 is valid unconditionally.*

In the context of this result recall that Tate has proved that any rational valued character validates the Strong-Stark Conjecture [49, Ch. II, Th. 6.8]. Note also that all characters of G are rational valued if G is a symmetric group of any degree or a group of exponent two or the quaternion group of order 8.

This shows, in particular, that Corollary 10.9 has the following unconditional consequence.

Corollary 10.10. *Assume K is a number field, ψ is rational valued and of degree two, $r_S(\psi) = 1$, $|S| > 1$ and the unique place v_1 in S for which $V_\psi^{G_{w_1}}$ does not vanish is archimedean. Write τ for the non-trivial element of G_{w_1} and fix an embedding of L in $\mathbb{C}$ corresponding to w_1 which is used to regard L as a subfield of $\mathbb{C}$.*

Then $\epsilon_{S,T,\psi} := \exp(-L'_{S,T}(\psi, 0))$ is a real unit in L with the following properties.

(i) *Either $\epsilon_{S,T,\psi}$ or $-\epsilon_{S,T,\psi}$ is congruent to 1 modulo all of the places in T_L.*
(ii) *$|\epsilon_{S,T,\psi}|_w = 1$ if w is any place of K that does not lie above v_1.*
(iii) *For $g \in G$ one has $-\log|g^{-1}(\epsilon_{S,T,\psi})|_{w_1} = (\psi(g) + \psi(g\tau))L'_{S,T}(\psi, 0)$.*
(iv) *For $\phi \in \mathrm{Hom}_G(\mathcal{O}_L^\times, \mathbb{Z}[G])$ one has $2^{-2}|G|^3\phi(\epsilon_{S,T,\psi}) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T(L))$.*

11. Zeta elements and the leading term conjecture

In this section we explicitly compute the zeta element $z_{L/K,S,T}$ and then use this description to prove Theorem 10.1. We also complete the proof of Proposition 9.4.

11.1. **Explicit computation of the zeta element.**

We fix a representative of $C_{L,S,T}$ of the form $P^0 \to P$ as in Proposition 7.8 so that there is a tautological exact sequence

$$0 \to \mathcal{O}_{L,S,T}^\times \xrightarrow{\iota} P^0 \xrightarrow{\phi} P \xrightarrow{\varpi} \mathcal{S}_{S,T}^{\mathrm{tr}}(\mathbb{G}_m/L) \to 0. \tag{38}$$

Since $C_{L,S,T}$ belongs to $D^{\mathrm{lf},0}(\mathbb{Z}[G])$ we may also, and will, assume that $\mathbb{Q} \cdot P^0 = \mathbb{Q} \cdot P$.

Since the algebra $\mathbb{Q}[G]$ is semisimple there exist $\mathbb{Q}[G]$-equivariant sections ι_1 and ι_2 to the surjections $\mathbb{Q}\cdot P = \mathbb{Q}\cdot P^0 \to \mathbb{Q}\cdot \mathrm{im}(\phi)$ and $\mathbb{Q}\cdot P \to \mathbb{Q}\cdot \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$ that are induced by ϕ and ϖ respectively. We thereby obtain a direct sum decomposition of $\mathbb{R}[G]$-modules

$$\mathbb{R}\cdot P = (\mathbb{R}\cdot \mathcal{O}^\times_{L,S,T}) \oplus (\mathbb{R}\otimes_\mathbb{Q} \iota_1)(\mathbb{R}\cdot \mathrm{im}(\phi))$$

and we use this to define $\langle \phi, \iota_1, \iota_2\rangle$ to be the unique automorphism of the (free) $\mathbb{R}[G]$-module $\mathbb{R}\cdot P$ that is equal to $(\mathbb{R}\otimes_\mathbb{Q} \iota_2)\circ R_{L,S}$ on $\mathbb{R}\cdot\mathcal{O}^\times_{L,S,T}$ and to $\mathbb{R}\otimes_\mathbb{Z}\phi$ on $(\mathbb{R}\otimes_\mathbb{Q}\iota_1)(\mathbb{R}\cdot\mathrm{im}(\phi))$.

Lemma 11.1. *For any choice of $\mathbb{Q}[G]$-basis $\{b_i\}_{1\le i\le e}$ of $\mathbb{Q}\cdot P^0 = \mathbb{Q}\cdot P$ one has*

$$z_{L/K,S,T} = \theta^*_{L/K,S,T}(0)\mathrm{Nrd}_{\mathbb{R}[G]}(\langle \phi,\iota_1,\iota_2\rangle)^{-1}\cdot (\wedge_{i=1}^{i=e} b_i)\otimes(\wedge_{i=1}^{i=e} b_i^*).$$

Proof. An explicit computation shows the isomorphism $\lambda_{L,S}$ defined at the beginning of §9.1 sends the element $(\wedge_{i=1}^{i=e} b_i)\otimes(\wedge_{i=1}^{i=e} b_i^*)$ of $\mathbb{R}\cdot\det_{\mathbb{Z}[G]}(C_{L,S,T})$ to $\mathrm{Nrd}_{\mathbb{R}[G]}(\langle\phi,\iota_1,\iota_2\rangle)$ in $\zeta(\mathbb{R}[G])$.

Given this, the stated equality is an immediate consequence of the definition of $z_{L/K,S,T}$. □

This computation leads to the following reinterpretation of Conjecture 9.2.

Proposition 11.2. *Conjecture 9.2 is valid if and only if for every prime p there exists a unit u_p of $\mathbb{Z}_p[G]$ with $\theta^*_{L/K,S,T}(0) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(u_p)\mathrm{Nrd}_{\mathbb{R}[G]}(\langle\phi,\iota_1,\iota_2\rangle)$.*

If the G-module P in (38) is free, then this condition is satisfied if and only if there is an element u of $K_1(\mathbb{Z}[G])$ with $\theta^*_{L/K,S,T}(0) = \mathrm{Nrd}_{\mathbb{Q}[G]}(u)\mathrm{Nrd}_{\mathbb{R}[G]}(\langle\phi,\iota_1,\iota_2\rangle)$.

Proof. Fix a prime p. Then the $\mathbb{Z}_{(p)}[G]$-module $P^0_{(p)}$ is free of the same rank as $P_{(p)}$ and so, after making an appropriate adjustment to the map in (38) one can assume both that $P^0_{(p)} = P_{(p)}$ and that the elements $\{b_i\}_{1\le i\le e}$ chosen in Lemma 11.1 give a $\mathbb{Z}_{(p)}[G]$-basis of this module.

In addition, if $\{b_i'\}_{1\le i\le e}$ and $\{b_i''\}_{1\le i\le e}$ are any other $\mathbb{Z}_{(p)}[G]$-bases of $P_{(p)}$, then Corollary 2.9 implies there exists an invertible matrix U_p over $\mathbb{Z}_{(p)}[G]$ with

$$(\wedge_{i=1}^{i=e} b_i')\otimes(\wedge_{i=1}^{i=e}(b_i'')^*) = \mathrm{Nrd}_{\mathbb{Q}[G]}(U_p)\cdot(\wedge_{i=1}^{i=e} b_i)\otimes(\wedge_{i=1}^{i=e} b_i^*).$$

Given this, the formula of Lemma 11.1 shows that $z_{L/K,S,T}$ is a primitive $\mathbb{Z}_{(p)}[G]$-basis of $\det_{\mathbb{Z}[G]}(C_{L,S,T})_{(p)}$ if and only if $\theta^*_{L/K,S,T}(0)\mathrm{Nrd}_{\mathbb{R}[G]}(\langle\phi,\iota_1,\iota_2\rangle)^{-1}$ is equal to $\mathrm{Nrd}_{\mathbb{Q}[G]}(U_p)$ for an invertible matrix U_p over $\mathbb{Z}_{(p)}[G]$.

This implies the first claim because, as $\mathbb{Z}_{(p)}[G]$ is semi-local, any such element $\mathrm{Nrd}_{\mathbb{Q}[G]}(U_p)$ is equal to $\mathrm{Nrd}_{\mathbb{Q}[G]}(u_p)$ for some unit u_p of $\mathbb{Z}_{(p)}[G]$.

If P^0 is free, then it is isomorphic to P and so, after adjusting the map d in (38), we can assume that $P^0 = P$ and that the elements $\{b_i\}_{1\le i\le e}$ in Lemma 11.1 constitute a $\mathbb{Z}[G]$-basis of this module. The rest of the argument now proceeds as above. □

11.2. **The proof of Theorem 10.1.** Before proving Theorem 10.1 we quickly recall the statement of $\mathrm{LTC}(L/K)$.

Let R be an integral domain of characteristic zero with field of fractions F and $\mathfrak{A}$ an R-order in a finite dimensional semisimple F-algebra A. For any extension field E of F we write A_E for the (semisimple) E-algebra $E\otimes_F A$ and $K_0(\mathfrak{A}, A_E)$ for the relative algebraic

K-group of the inclusion $\mathfrak{A} \subset A_E$. This group is functorial in the pair $(\mathfrak{A}, A_E)$ and also sits in a long exact sequence of relative K-theory

$$K_1(\mathfrak{A}) \xrightarrow{\partial^2_{\mathfrak{A},A_E}} K_1(A_E) \xrightarrow{\partial^1_{\mathfrak{A},A_E}} K_0(\mathfrak{A}, A_E) \xrightarrow{\partial^0_{\mathfrak{A},A_E}} K_0(\mathfrak{A}). \tag{39}$$

We recall that to each pair (C, t) comprising an object C of $D^{\mathrm{lf},0}(\mathfrak{A})$ for which $E \otimes_R C$ is acyclic outside degrees a and $a+1$ for some integer a and an isomorphism of A_E-modules $t : E \otimes_R H^a(C) \cong E \otimes_R H^{a+1}(C)$ one can define a canonical 'refined Euler characteristic' $\chi_{\mathfrak{A}}(C, t)$ in $K_0(\mathfrak{A}, A_E)$.

In the case $R \subset \mathbb{Q}$ and $E \subseteq \mathbb{R}$ we write

$$\delta_{\mathfrak{A},A_E} : \zeta(A_E)^\times \to K_0(\mathfrak{A}, A_E)$$

for the natural 'extended boundary homomorphism' and recall that in this case the connecting homomorphism $\partial^1_{\mathfrak{A},A_E}$ in (39) factors as

$$\partial^1_{\mathfrak{A},A_E} = \delta_{\mathfrak{A},A_E} \circ \mathrm{Nrd}_{A_E} \tag{40}$$

where Nrd_{A_E} denotes the homomorphism $K_1(A_E) \to \zeta(A_E)^\times$ induced by taking reduced norms. If $\mathfrak{A} = \mathbb{Z}[G]$ and $A_E = \mathbb{R}[G]$, then we often abbreviate $\partial^i_{\mathfrak{A},A_E}$ and $\delta_{\mathfrak{A},A_E}$ to ∂^i_G and δ_G respectively.

In the following result we use, for any bimodule Π as in §8.1, the projection maps μ^{rel}_Π from (37) and ι_Π from (32).

Proposition 11.3. *Let Π be a bimodule as in §8.1. Then the equality of* LTC*(L/K) is valid modulo* $\ker(\mu^{\mathrm{rel}}_\Pi)$ *if and only if one has*

$$\delta_{\mathcal{A},A_\mathbb{R}}(\iota_\Pi(\theta^*_{L/K,S,T}(0))) = \chi_{\mathcal{A}}(\Pi \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C_{L,S,T}, R^\Pi_{L,S})$$

in $K_0(\mathcal{A}, A_\mathbb{R})$, where $R^\Pi_{L,S}$ denotes the restriction of $R_{L,S}$ to $\mathbb{R} \cdot {}_\Pi\mathcal{O}^\times_{L,S,T}$.

Proof. Set $C := C_{L,S,T}$. If one takes Π to be $\mathbb{Z}[G]$, regarded as a $(\mathbb{Z}[G], \mathbb{Z}[G])$-bimodule in the obvious way, then μ^{rel}_Π and ι_Π are bijective and so we are required to prove that $\mathrm{LTC}(L/K)$ is valid if and only if $\delta_G(\theta^*_{L/K,S,T}(0))) = \chi_{\mathbb{Z}[G]}(C, R_{L,S})$. This equivalence is proved by a natural (non-abelian) extension of the argument used to prove [11, Prop. 3.4].

To deduce the general case from this one need only use the commutative diagram

$$\begin{array}{ccc} \zeta(\mathbb{R}[G])^\times & \xrightarrow{\delta_G} & K_0(\mathbb{Z}[G], \mathbb{R}[G]) \\ {\scriptstyle \mu_\Pi}\downarrow & & \downarrow{\scriptstyle \iota^{\mathrm{rel}}_\Pi} \\ \zeta(A_\mathbb{R})^\times & \xrightarrow{\delta_{\mathcal{A},A_\mathbb{R}}} & K_0(\mathcal{A}, A_\mathbb{R}) \end{array}$$

(which follows from the commutativity of (32) and the naturality of connecting homomorphisms in relative K-theory) and the fact that the map μ^{rel}_Π sends $\chi_{\mathbb{Z}[G]}(C, R_{L,S})$ to $\chi_{\mathcal{A}}(\Pi \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C, R^\Pi_{L,S})$. □

We now prove Theorem 10.1 by comparing the equality in Proposition 11.3 to the conditions in Proposition 11.2.

To do this we recall that any isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$ induces a homomorphism of abelian groups $j_{0,*} : K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G])$ and that, as p and j vary, the diagonal homomorphism

$$K_0(\mathbb{Z}[G], \mathbb{R}[G]) \xrightarrow{(j_{0,*})_{p,j}} \prod_{p,j} K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G])$$

is injective (cf. [7, §9.1]). We further recall that for each prime p and isomorphism j there is a commutative diagram

$$\begin{array}{ccc} \zeta(\mathbb{R}[G])^\times & \xrightarrow{\delta_G} & K_0(\mathbb{Z}[G], \mathbb{R}[G]) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j_{0,*}} \\ \zeta(\mathbb{C}_p[G])^\times & \xrightarrow{\Delta_{G,p}} & K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G]) \end{array}$$

where j_* is the natural inclusion induced by j and $\Delta_{G,p}$ the composite of the inverse of the (bijective) reduced norm map $K_1(\mathbb{C}_p[G]) \to \zeta(\mathbb{C}_p[G])^\times$ and the connecting homomorphism $K_1(\mathbb{C}_p[G]) \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G])$ in (39) (see, for example, [7, (28)]).

This shows that the equality in Proposition 11.3 is valid if and only if for all primes p and all isomorphisms j one has an equality

$$\Delta_{G,p}(j_*(\theta^*_{L/K,S,T}(0))) = \chi_{\mathbb{Z}_p[G]}(C_{L,S,T,p}, \mathbb{C}_p \otimes_{\mathbb{R},j} R_{L,S}).$$

In addition, since $C_{L,S,T,p}$ is represented by the p-completion of $P^0 \xrightarrow{\phi} P$ and the $\mathbb{Z}_p[G]$-modules P^0_p and P_p are isomorphic, the result of [7, Lem. A.1] implies that

$$\chi_{\mathbb{Z}_p[G]}(C_{L,S,T,p}, \mathbb{C}_p \otimes_{j,\mathbb{R}} R_{L,S}) = \Delta_{G,p}(\mathrm{Nrd}_{\mathbb{Q}[G]}(\langle \phi, \iota_1, \iota_2 \rangle)).$$

By comparing the last two equalities one finds that the equality of Proposition 11.3 is valid if and only if for all primes p and isomorphisms j one has

$$j_*(\theta^*_{L/K,S,T}(0)) \cdot \mathrm{Nrd}_{\mathbb{Q}[G]}(\langle \phi, \iota_1, \iota_2 \rangle)^{-1} \in \ker(\Delta_{G,p}) = \mathrm{Nrd}_{\mathbb{Q}[G]}(\mathbb{Z}_{(p)}[G]^\times)$$

(where the equality follows from the exactness of (39)).

The result of Theorem 10.1 now follows by comparing this reinterpretation of $\mathrm{LTC}(L/K)$ with the result of Proposition 11.2.

11.3. **The proof of Proposition 9.4.** In this section we complete the proof of Proposition 9.4 by proving claim (ii). To do this we set ${}_\Pi C := \Pi \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C_{L,S,T}$.

We first note that Proposition 9.4(i) combines with the argument given in the previous section to imply that the validity of Conjecture 9.2 implies that in $K_0(\mathcal{A}, A_\mathbb{R})$ one has

$$\delta_{\mathcal{A},A_\mathbb{R}}(\iota_\Pi(\theta^*_{L/K,S,T}(0))) = \chi_{\mathcal{A}}({}_\Pi C, R^\Pi_{L,S}).$$

Next we note that the hypothesis on leading terms made in claim (ii) implies $L^*_{S,T}(\psi, 0)$ is positive for every χ in Υ_Π and hence, by the Hasse-Schilling Maass Norm Theorem, that $\iota_\Pi(\theta^*_{L/K,S,T}(0))$ belongs to $\mathrm{im}(\mathrm{Nrd}_{A_\mathbb{R}})$.

From the equality (40) and the exactness of (39), it thus follows that $\delta_{A_\mathbb{R}}(\iota_\Pi(\theta^*_{L/K,S,T}(0)))$ belongs to $\ker(\partial^0_{\mathcal{A},\mathbb{R}})$, or equivalently that $\partial^0_{\mathcal{A},A_\mathbb{R}}(\chi_{\mathcal{A}}({}_\Pi C, R^\Pi_{L,S}))$ vanishes. Since ${}_\Pi C$ belongs to $D^{\mathrm{lf},0}(\mathcal{A})$ this last fact combines with Proposition 6.17(iii) to imply that the $\xi(\mathcal{A})$-lattice $\det_{\mathcal{A}}({}_\Pi C)$ has a primitive basis, z' say.

Combining now the results of Propositions 9.4(i) and 6.19(ii) we deduce that Conjecture 9.2 implies $z^{\Pi}_{L/K,S,T} = \lambda \cdot z'$ with λ an element of $\mathrm{Nrd}_A(K_1(A))$ that belongs to $\mathrm{Nrd}_{A_p}(K_1(\mathcal{A}_p))$ for every prime p. From Lemma 11.4 below it follows that λ belongs to $\mathrm{Nrd}_A(K_1(\mathcal{A}))$ and hence, by Proposition 6.19(i), that $z^{\Pi}_{L/K,S,T}$ is a primitive basis of $\det_{\mathcal{A}}({}_\Pi C)$, as claimed.

This completes the proof of Proposition 9.4.

Lemma 11.4. *An element of* $\mathrm{Nrd}_A(K_1(A))$ *belongs to* $\mathrm{Nrd}_A(K_1(\mathcal{A}_{(p)}))$ *for every prime* p *if and only if it belongs to* $\mathrm{Nrd}_A(K_1(\mathcal{A}))$.

The same assertion is also true with each group $\mathrm{Nrd}_A(K_1(\mathcal{A}_{(p)}))$ *replaced by* $\mathrm{Nrd}_{A_p}(K_1(\mathcal{A}_p))$.

Proof. The relevant cases of the exact sequence (39) give rise to an exact commutative diagram of abelian groups

$$\begin{array}{ccccccc} K_1(\mathcal{A}) & \xrightarrow{\iota_{\mathcal{A}}} & K_1(A) & \longrightarrow & K_0(\mathcal{A},A) & & \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \iota} & & \\ \prod_p K_1(\mathcal{A}_{(p)}) & \xrightarrow{(\iota_{\mathcal{A}_{(p)}})_p} & \prod'_p K_1(A) & \longrightarrow & \bigoplus_p K_0(\mathcal{A}_{(p)},A) & \longrightarrow & 0 \end{array}$$

in which the vertical arrows are the natural diagonal maps and $\prod'_p$ denotes the restricted direct product (over p) of the groups $K_1(A)$ with respect to the subgroups $\mathrm{im}(\iota_{\mathcal{A}_{(p)}})$.

Since the map ι is injective (see the discussion following [16, (49.12)]) this diagram implies that an element of $K_1(A)$ belongs to $\mathrm{im}(\iota_{\mathcal{A}_{(p)}})$ for every prime p if and only if it belongs to $\mathrm{im}(\iota_{\mathcal{A}})$.

This fact implies the claimed equality since Nrd_A is injective (cf. [16, §45A]) and, by definition, one has $\mathrm{Nrd}_A(K_1(\mathcal{A})) = \mathrm{Nrd}_A(\mathrm{im}(\iota_{\mathcal{A}}))$ and $\mathrm{Nrd}_A(K_1(\mathcal{A}_{(p)})) = \mathrm{Nrd}_A(\mathrm{im}(\iota_{\mathcal{A}_{(p)}}))$.

To prove the same result with each group $\mathrm{Nrd}_A(K_1(\mathcal{A}_{(p)}))$ replaced by $\mathrm{Nrd}_{A_p}(K_1(\mathcal{A}_p))$ one argues in just the same way after replacing the lower row of the diagram by the corresponding exact sequence

$$\prod_p K_1(\mathcal{A}_p) \xrightarrow{(\iota_{\mathcal{A}_p})_p} \prod_p{}' K_1(A_p) \to \bigoplus_p K_0(\mathcal{A}_p, A_p) \to 0$$

and noting that Nrd_{A_p} is injective and the natural map $K_0(\mathcal{A}_{(p)}, A) \to K_0(\mathcal{A}_p, A_p)$ bijective. □

Remark 11.5. Fix a representative of $C_{L,S,T}$ of the form $P^0 \to P$ as in Proposition 7.8. Then $\partial^0_G(\chi_{\mathbb{Z}[G]}(C_{L,S,T}, R_{L,S}))$ is equal to $[P^0] - [P]$. Thus, if Conjecture 9.2 is valid, then the argument used above to prove Proposition 9.4(ii) implies that if $L^*(\chi, 0)$ is positive for every irreducible symplectic character χ of G, then $[P^0] = [P]$ in $K_0(\mathbb{Z}[G])$. Since (by assumption) the free G-module P has rank at least two, this equality combines with the Bass Cancellation Theorem to imply P^0 is isomorphic to P. This shows that in such cases the family of pre-envelopes $\mathcal{P}_{L,S,T}$ should be free. In particular, in any such case one can take all of the surjective homomorphisms $\pi_{L,S,T,p}$ that occur in Proposition 7.1 to be induced by a single surjective homomorphism of G-modules $P \to \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$ constructed as in Lemma 7.7.

Remark 11.6. The (conjectural) equality

$$\partial^0_G(\chi_{\mathbb{Z}[G]}(C_{L,S,T}, R_{L,S})) = \partial^0_G(\delta_G(\theta^*_{L/K,S,T}(0)))$$

discussed in Remark 11.5 has been shown (by Flach and the first author) to be equivalent to the central conjecture formulated by Chinburg in [15].

12. The proof of Theorem 9.9

12.1. **An explicit formula for higher non-abelian Stark elements.** As a first step we show Conjecture 9.2 implies an explicit formula for the element $\epsilon^{\pi}_{\underline{b}}$ that occurs in Theorem 9.9.

To do this we fix a prime p and assume, just as in the proof of Proposition 11.2, that the module P^0 in (38) is such that $P^0_{(p)} = P_{(p)}$.

Lemma 12.1. *There exists an exact sequence of $\mathcal{A}_{(p)}$-modules*

$$0 \to ({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})_{(p)} \xrightarrow{j_{\Pi,p}} ({}_{\Pi}P)_{(p)} \xrightarrow{({}_{\Pi}\phi)_{(p)}} ({}_{\Pi}P)_{(p)} \xrightarrow{({}_{\Pi}\varpi)_{(p)}} ({}_{\Pi}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L))_{(p)} \to 0$$

in which one has $\mathbb{Q}\cdot j_{\Pi,p} = \mathbb{Q}\cdot({}_{\Pi}\iota)_{(p)}$.

Proof. Since the functors $M \to {}^{\Pi}M$ and $M \to {}_{\Pi}M$ are respectively left and right exact, the exact sequence (38) gives rise to an exact commutative diagram of $\mathcal{A}_p$-modules

$$\begin{array}{ccccccccc}
0 & \to & ({}^{\Pi}\mathcal{O}^{\times}_{L,S,T})_{(p)} & \xrightarrow{({}^{\Pi}\iota)_{(p)}} & ({}^{\Pi}P)_{(p)} & \xrightarrow{({}^{\Pi}\phi)_{(p)}} & ({}^{\Pi}P)_{(p)} & & \\
 & & & & \uparrow{\scriptstyle(\mathrm{Tr}_{\Pi,P})_{(p)}} & & \uparrow{\scriptstyle(\mathrm{Tr}_{\Pi,P})_{(p)}} & & \\
 & & & & ({}_{\Pi}P)_{(p)} & \xrightarrow{({}_{\Pi}\phi)_{(p)}} & ({}_{\Pi}P)_{(p)} & \xrightarrow{({}_{\Pi}\varpi)_{(p)}} & ({}_{\Pi}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L))_{(p)} \to 0
\end{array}$$

in which the vertical maps are bijective since P is a free G-module.

Next we note that the injection ι in (38) belongs to the family of pre-envelopes $\mathcal{P} = \mathcal{P}_{L,S,T}$ that is constructed in Proposition 7.1. The claimed exact sequence is therefore obtained by combining the above diagram with the fact that the definition (33) of ${}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T}$ implies the existence of a commutative diagram of $\mathcal{A}_{(p)}$-modules

$$\begin{array}{ccc}
({}^{\Pi}\mathcal{O}^{\times}_{L,S,T})_{(p)} & \xrightarrow{({}^{\Pi}\iota)_{(p)}} & ({}^{\Pi}P)_{(p)} \\
\uparrow & & \uparrow{\scriptstyle(\mathrm{Tr}_{\Pi,P})_{(p)}} \\
({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})_{(p)} & \xrightarrow{j_{\Pi,p}} & ({}_{\Pi}P)_{(p)}
\end{array} \tag{41}$$

in which one has $\mathbb{Q}\cdot j_{\Pi,p} = \mathbb{Q}\cdot({}_{\Pi}\iota)_{(p)}$ and the left hand vertical arrow is bijective. □

To proceed we now fix a basis $\underline{b}_p = \{b_{p,i}\}_{1\le i\le r}$ of the free $\mathcal{A}_{(p)}$-module $Y_{\pi,(p)}$. Then, since the composite homomorphism of $\mathcal{A}_{(p)}$-modules

$${}_{\Pi}\varpi' : ({}_{\Pi}P)_{(p)} \xrightarrow{({}_{\Pi}\varpi)_{(p)}} ({}_{\Pi}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L))_{(p)} \xrightarrow{\pi_{(p)}} Y_{\pi,(p)}$$

is surjective, we can choose an $\mathcal{A}_{(p)}$-basis $\underline{\hat{b}}_p = \{\hat{b}_{i,p}\}_{1\le i\le d}$ of $({}_\Pi P)_{(p)}$ with

$$
{}_\Pi\varpi'(\hat{b}_{i,p}) = \begin{cases} b_i, & \text{if } 1\le i\le r \\ 0, & \text{if } r<i\le d. \end{cases} \tag{42}
$$

We write $\underline{\hat{b}}^* = \{\hat{b}_i^*\}_{1\le i\le d}$ for the corresponding dual basis of $\mathrm{Hom}_A(\mathbb{Q}\cdot {}_\Pi P, A)$. For each index i we then also set $({}_\Pi\phi)_i := \hat{b}_i^*\circ({}_\Pi\phi) \in \mathrm{Hom}_A(\mathbb{Q}\cdot {}_\Pi P, A)$.

We can now state the main result of this section. In this result we use the map $j_{\Pi,p}$ in Lemma 12.1 to regard $\bigwedge_A^r(\mathbb{Q}\cdot_{\mathcal{P}}^{\Pi}\mathcal{O}_{L,S,T}^\times)$ as a subspace of $\bigwedge_A^r(\mathbb{Q}\cdot_\Pi P)$.

Theorem 12.2. *If the p-component of Conjecture 9.2 is valid, then for any $\mathcal{A}_{(p)}$-basis $\underline{\hat{b}}_p$ of $({}_\Pi P)_{(p)}$ which satisfies (42) there exists a unit u_p of $\mathbb{Z}_{(p)}[G]$ with*

$$
\epsilon_{\underline{b}}^\pi := \mathrm{Nrd}_{\mathbb{Q}[G]}(u_p)\cdot(\bigwedge_{a=r+1}^{a=d}({}_\Pi\phi)_a)(\bigwedge_{c=1}^{c=d}\hat{b}_{c,p}) \in {\bigwedge}_A^r(\mathbb{Q}\cdot_\Pi P).
$$

Proof. We claim first that the element

$$
\epsilon'_{\underline{b}} := (\bigwedge_{a=r+1}^{a=d}({}_\Pi\phi)_a)(\bigwedge_{c=1}^{c=d}\hat{b}_{c,p})
$$

is stable under multiplication by the idempotent e_π. To show this it is enough to show that $e'(\epsilon'_{\underline{b}})=0$ for every primitive idempotent e' of $A_\mathbb{C}$ that is orthogonal to e_π. But for any such e' the surjective map $e'(\mathbb{C}\cdot {}_\Pi X_{L,S})\to e'(\mathbb{C}\cdot Y_\pi)$ is not bijective, and so the exact sequence in Lemma 12.1 implies

$$
\begin{aligned}
\dim_\mathbb{C}(e'(\mathbb{C}\cdot\mathrm{im}({}_\Pi\phi))) &= \dim_\mathbb{C}(e'(\mathbb{C}\cdot {}_\Pi P)) - \dim_\mathbb{C}(e'(\mathbb{C}\cdot {}_\Pi X_{L,S}))\\
&< \dim_\mathbb{C}(e'(\mathbb{C}\cdot {}_\Pi P)) - \dim_\mathbb{C}(e'(\mathbb{C}\cdot Y_\pi))\\
&= \dim_\mathbb{C}(e'(A_\mathbb{C})^d) - \dim_\mathbb{C}(e'(A_\mathbb{C})^r)\\
&= (d-r)\dim_\mathbb{C}(e'A_\mathbb{C}),
\end{aligned}
$$

and hence that the space $e'(\mathrm{im}(\bigwedge_{a=r+1}^{a=d}({}_\Pi\phi)_a))$ vanishes.

We note next that, since the map $e_\pi(\mathbb{Q}\cdot {}_\Pi\mathcal{S}_{S,T}^{\mathrm{tr}}(\mathbb{G}_m/L))\to e_\pi(\mathbb{Q}\cdot Y_{\underline{b}})$ induced by π is bijective, our choice of basis $\underline{\hat{b}}$ satisfying (42) implies both that $\{e_\pi(\hat{b}_{i,p})\}_{r<i\le d}$ is a basis of the $e_\pi A$-module $e_\pi(\mathbb{Q}\cdot\mathrm{im}({}_\Pi d))$ and that $e_\pi(\mathbb{Q}\cdot {}^\Pi\mathcal{O}_{L,S,T}^\times)$ is the kernel of the map

$$
e_\pi(\mathbb{Q}\cdot {}^\Pi P)\xrightarrow{({}_\Pi\phi)_c}\prod_{r<c\le d}\mathbb{Q}\cdot A.
$$

Applying Proposition 2.10 in this context we may therefore deduce that there is a containment

$$
\epsilon'_{\underline{b}} = e_\pi(\epsilon'_{\underline{b}}) \in e_\pi\bigwedge_A^r(\mathbb{Q}\cdot {}^\Pi\mathcal{O}_{L,S,T}^\times). \tag{43}
$$

We next consider the composite homomorphism of $\mathbb{R}[G]$-modules

$$\tilde{R}^{\pi}_{L,S} : \mathbb{R}\cdot_{\Pi} P \to \mathbb{R}\cdot_{\Pi}\mathcal{O}^{\times}_{L,S,T} \to \mathbb{R}\cdot_{\Pi} X_{L,S} \to \mathbb{R}\cdot Y_{\pi} = \mathbb{R}\cdot Y_{\underline{b}},$$

where the first arrow is induced by the section ι_1 chosen just after (38), the second is $(\mathbb{R}\cdot\Pi)\otimes_{\mathbb{R}[G]} R_{L,S}$ and the third is induced by π. For each integer i with $1 \le i \le r$ we also write $\tilde{R}^{\pi,i}_{L,S} : \mathbb{R}\cdot_{\Pi} F \to \mathbb{R}\cdot A$ for the composite $b_i^{*}\circ\tilde{R}^{\pi}_{L,S}$.

Then the inclusion (43) implies that in $A_{\mathbb{R}}$ one has

$$\begin{aligned}
\lambda^{\pi}_{L,S}(\epsilon'_{\underline{b}}) &= e_{\pi}(\lambda^{\pi}_{L,S}(\epsilon'_{\underline{b}}))\\
&= e_{\pi}((\bigwedge\nolimits^{r}_{A_{\mathbb{R}}} R^{\pi}_{L,S})(\epsilon'_{\underline{b}}))\\
&= e_{\pi}((\bigwedge_{i=1}^{i=r} R^{\pi,i}_{L,S})(\epsilon'_{\underline{b}})\cdot\bigwedge_{i=1}^{i=r} b_i)\\
&= e_{\pi}((((\bigwedge_{A_{\mathbb{C}_p}}^{r} R^{\pi,i}_{L,S})\wedge_{A_{\mathbb{C}_p}}(\bigwedge_{a=r+1}^{a=d}({}_{\Pi}\phi)_a))(\bigwedge_{c=1}^{c=d}\hat{b}_{c,p})\cdot(\bigwedge_{i=1}^{i=r} b_i))\\
&= e_{\pi}\mathrm{Nrd}_{A_{\mathbb{R}}}(M(\phi,\iota_1,\hat{\underline{b}}))\cdot(\bigwedge_{i=1}^{i=r} b_i).
\end{aligned}\tag{44}$$

Here the second equality follows from the definition of $\lambda^{\pi}_{L,S}$, the third and fourth are clear and the last follows from Proposition 2.6 with $M(\phi,\iota_1,\hat{\underline{b}})$ the matrix in $\mathrm{M}_d(A_{\mathbb{R}})$ defined by setting

$$M(\phi,\iota_1,\hat{\underline{b}})_{ij} = \begin{cases} R^{\pi,j}_{L,S}(\hat{b}_i), & \text{if } 1\le i\le d,\ 1\le j\le r\\ ({}_{\Pi}\phi)_j(\hat{b}_i), & \text{if } 1\le i\le d,\ r<j\le d.\end{cases}$$

Now if we assume, as we may, that the section ι_2 chosen just after (38) sends b_i to $\hat{b}_i$ for each i with $1\le i\le r$, then $e_{\pi}M(\phi,\iota_1,\hat{\underline{b}})$ is the matrix of $e_{\pi}(\Pi_{\mathbb{R}}\otimes_{\mathbb{R}[G]}\langle\phi,\iota_1,\iota_2\rangle)$ with respect to the basis $e_{\pi}\hat{\underline{b}}$ of $e_{\pi}({}_{\Pi}F)_{\mathbb{R}}$ over $e_{\pi}A_{\mathbb{R}}$.

Proposition 11.2 therefore shows that the p-component of Conjecture 9.2 implies the existence of a unit u_p of $\mathbb{Z}_{(p)}[G]$ with

$$\begin{aligned}
e_{\pi}\mathrm{Nrd}_{A_{\mathbb{R}}}(M(\phi,\iota_1,\hat{\underline{b}})) &= e_{\pi}\mathrm{Nrd}_{A_{\mathbb{R}}}(\langle\phi,\iota_1,\iota_2\rangle)\\
&= \mathrm{Nrd}_{\mathbb{Q}[G]}(u_p)^{-1}\cdot e_{\pi}\theta^{*}_{L/K,S,T}(0)\\
&= \mathrm{Nrd}_{\mathbb{Q}[G]}(u_p)^{-1}\cdot\theta^{\pi}_{S,T}(0).
\end{aligned}$$

Multiplying (44) by $\mathrm{Nrd}_{\mathbb{Q}[G]}(u_p)$ therefore gives an equality

$$\lambda^{\pi}_{L,S}(\mathrm{Nrd}_{\mathbb{Q}[G]}(u_p)\epsilon'_{\underline{b}}) = \theta^{\pi}_{S,T}(0)\cdot\bigwedge_{i=1}^{i=r} b_i.$$

The claimed formula for $\epsilon^{\pi}_{\underline{b}}$ now follows directly by comparing this equality to Definition 8.8 and noting that the map $\lambda^{\pi}_{L,S}$ is injective. $\square$

12.2. **The proof of Theorem 9.9(i).** We now deduce Theorem 9.9(i) from the formula in Theorem 12.2.

We start with a useful observation.

Lemma 12.3. *The validity of claims (i) and (ii) of Theorem 9.9 is independent of the choice of elements* $\underline{b}$.

Proof. Let $\underline{b'} = \{b'_i\}_{1\le i\le r}$ be any other set of elements that spans a full free $\mathcal{A}$-submodule $Y_{\underline{b'}}$ of Y_π of rank r.

Then, after replacing each element b'_i by $N \cdot b'_i$ for any large enough integer N, we may assume $Y_{\underline{b'}} \subseteq Y_{\underline{b}}$.

In this case each module in the short exact sequence of $\mathcal{A}$-modules

$$0 \to Y_{\underline{b}}/Y_{\underline{b'}} \to Y_\pi/Y_{\underline{b'}} \to Y_\pi/Y_{\underline{b}} \to 0$$

has a quadratic G-module presentation bundle and so the results of Proposition 5.8(iii) and (iv) together imply an equality of invertible $\xi(\mathcal{A})$-modules

$$\mathrm{FI}^0_{\mathcal{A}}(Y_\pi/Y_{\underline{b'}})^{-1} = (\mathrm{FI}^0_{\mathcal{A}}(Y_{\underline{b}}/Y_{\underline{b'}}) \cdot \mathrm{FI}^0_{\mathcal{A}}(Y_\pi/Y_{\underline{b'}}))^{-1} = \mathrm{Nrd}_A(M_{\underline{b'},\underline{b}})^{-1}\mathrm{FI}^0_{\mathcal{A}}(Y_\pi/Y_{\underline{b'}})^{-1}$$

with $M_{\underline{b'},\underline{b}}$ the matrix, with respect to the basis $\underline{b}$, of the endomorphism φ of $Y_{\underline{b}}$ which sends each element b_i to b'_i.

In addition, in terms of this notation, Corollary 2.9 implies that

$$\bigwedge_{i=1}^{i=r} b'_i = \bigwedge_{i=1}^{i=r} \varphi(b_i) = \mathrm{Nrd}_A(M_{\underline{b'},\underline{b}}) \cdot \bigwedge_{i=1}^{i=r} b_i$$

and hence also that $\epsilon_{\underline{b'}} = \mathrm{Nrd}_A(M_{\underline{b'},\underline{b}}) \cdot \epsilon_{\underline{b}}$.

The claimed independence is now clear. □

To prove Theorem 9.9(i) it therefore suffices to show that for each prime p the $\mathcal{A}_{(p)}$-module ${}_\Pi\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L})_{(p)}$ has a presentation $h^{\Pi,p}_{L/K,S,T}$ for which, for a suitable set $\underline{b}$ as in Theorem 9.9, one has

$$\mathrm{FI}^0_{\mathcal{A}}(Y_\pi/Y_{\underline{b}})^{-1}_{(p)}\{(\bigwedge_{a=1}^{a=r}\varphi_a)(\epsilon_{\underline{b}}) : \varphi_i \in \mathrm{Hom}_{\mathcal{A}_{(p)}}((^{\Pi}_{\mathcal{P}}\mathcal{O}^\times_{L,S,T})_{(p)}, \mathcal{A}_{(p)})\} = \mathrm{FI}^r_{\mathcal{A}_{(p)}}(h^{\Pi,p}_{L/K,S,T}). \tag{45}$$

To show this we take the presentation $h^{\Pi,p}_{L/K,S,T}$ to be defined by the exact sequence in Lemma 12.1. Since the cokernel of the map $j_{\Pi,p}$ in the latter sequence is $\mathbb{Z}_{(p)}$-torsion-free our assumption $(\mathcal{A}_2)$ on the order $\mathcal{A}$ combines with Lemma 4.4(ii) to imply that we may choose a lift $\hat{\varphi}_i$ of each φ_i in $\mathrm{Hom}_{\mathcal{A}_{(p)}}((^{\Pi}_{\mathcal{P}}\mathcal{O}^\times_{L,S,T})_{(p)}, \mathcal{A}_{(p)})$ through the map

$$\mathrm{Hom}_{\mathcal{A}_{(p)}}(({}_\Pi P)_{(p)}, \mathcal{A}_{(p)}) \to \mathrm{Hom}_{\mathcal{A}_{(p)}}((^{\Pi}_{\mathcal{P}}\mathcal{O}^\times_{L,S,T})_{(p)}, \mathcal{A}_{(p)})$$

that is induced by restriction through $j_{\Pi,p}$.

Next we use Roiter's Lemma to choose a set $\underline{b}$ as in Theorem 9.9 for which $Y_{\underline{b},(p)} = Y_{\pi,(p)}$. This implies that

$$\mathrm{FI}^0_{\mathcal{A}}(Y_\pi/Y_{\underline{b}})^{-1}_{(p)} = \xi(\mathcal{A})_{(p)} \tag{46}$$

and also that Theorem 12.2 can be applied in this setting. In particular, assuming the validity of the p-component of Conjecture 9.2, the latter result combines with Corollary 2.9 to imply that

$$(47)\qquad \begin{aligned}(\bigwedge_{a=1}^{a=r}\varphi_a)(\epsilon_{\underline{b}}) &= \mathrm{Nrd}_{\mathbb{Q}[G]}(u_p)\cdot(\bigwedge_{a=1}^{a=r}\hat{\varphi}_a)((\bigwedge_{a=r+1}^{a=d}(\pi\phi)_i)(\bigwedge_{c=1}^{c=d}\hat{b}_{c,p}))\\ &= \mathrm{Nrd}_{\mathbb{Q}[G]}(u_p)\cdot((\bigwedge_{a=1}^{a=r}\hat{\varphi}_a)\wedge(\bigwedge_{a=r+1}^{a=d}(\pi\phi)_i))(\bigwedge_{c=1}^{c=d}\hat{b}_{c,p})\\ &= \mathrm{Nrd}_{\mathbb{Q}[G]}(u_p)\cdot\mathrm{Nrd}_A(N(\phi,\{\hat{\varphi}_i\},\hat{\underline{b}}))\end{aligned}$$

where $N(\phi,\{\hat{\varphi}_i\},\hat{\underline{b}})$ is the matrix in $\mathrm{M}_d(A)$ defined by setting

$$N(\psi,\{\hat{\varphi}_i\},\hat{\underline{b}})_{ij} = \begin{cases}\hat{\varphi}_j(\hat{b}_i), & \text{if } 1\le i\le d,\ 1\le j\le r\\ (\pi\phi)_j(\hat{b}_i), & \text{if } 1\le i\le d,\ r<j\le d.\end{cases}$$

Now the condition (42) implies that the first r columns of the matrix of the endomorphism $\pi\phi$ with respect to the basis $\hat{\underline{b}}_p$ are equal to 0. This implies that, as the maps φ_i vary over $\mathrm{Hom}_{\mathcal{A}_{(p)}}(({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})_{(p)},\mathcal{A}_{(p)})$, the matrices $N(\phi,\{\hat{\varphi}_i\},\hat{\underline{b}})$ account for all of the matrices which both occur in the definition of $\mathrm{FI}^r_{\mathcal{A}_{(p)}}(h^{\Pi,p}_{L/K,S,T})$ and have non-zero reduced norm.

The formula (47) therefore implies that

$$\xi(\mathcal{A})_{(p)}\cdot\{(\bigwedge_{a=1}^{a=r}\varphi_a)(\epsilon_{\underline{b}}) : \varphi_i\in\mathrm{Hom}_{\mathcal{A}_{(p)}}(({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})_{(p)},\mathcal{A}_{(p)})\} = \mathrm{FI}^r_{\mathcal{A}_{(p)}}(h^{\Pi,p}_{L/K,S,T})$$

and this then combines with (46) to give the required equality.

This proves Theorem 9.9(i).

12.3. **The proof of Theorem 9.9(ii).** In this section we prove Theorem 9.9(ii).

Under the hypotheses of this claim the argument of §11.3 implies that the complex $\Pi\otimes^{\mathbb{L}}_{\mathbb{Z}[G]}C_{L,S,T}$ is isomorphic to a complex of finitely generated free $\mathcal{A}$-modules of the form $P'\xrightarrow{\phi'}P'$ where the first module occurs in degree zero.

The same argument as in Lemma 12.1 then gives an exact sequence of $\mathcal{A}$-modules of the form

$$0\to{}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T}\to P'\xrightarrow{\phi'}P'\to{}_{\Pi}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)\to 0.$$

This sequence constitutes a presentation $h^{\Pi,\mathrm{global}}_{L/K,S,T}$ of the $\mathcal{A}$-module ${}_{\Pi}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$.

With this choice of presentation, the result of Theorem 9.9(ii) can be proved by the same argument as in §12.2 but with the role of the exact sequence in Lemma 12.1 now being played by the p-localization of the exact sequence above.

12.4. **The proof of Theorem 9.9(iii).** In this section we prove Theorem 9.9(iii), the notation and hypotheses of which we assume.

For any left $\mathcal{A}$-module M we also write M^* for the linear dual $\mathrm{Hom}_{\mathcal{A}}(M,\mathcal{A})$, regarded as a left $\mathcal{A}$-module via the action $(a\theta)(m) := \theta(m)\iota_{\mathcal{A}}(a)$.

12.4.1. We start by making two useful reductions.

We write $\rho_{S,S'}$ for the natural restriction map $({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})^* \to ({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S',T})^*$.

Proposition 12.4. *It suffices to prove Theorem 9.9(iii) for elements $\Phi = \bigwedge_{1\le i\le r}\varphi_i$ where the set $\{\rho_{S,S'}(\varphi_i)\}_{1\le i\le r}$ spans a free A-module of rank r.*

Proof. Fix any subset $\{\varphi_i\}_{1\le i\le r}$ of $({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})^*$. Choose a natural number m that is sufficiently large to ensure $m\cdot\mathfrak{A}(\mathcal{A}) \subseteq |\mathrm{Cl}^T_{S'}(L)|\cdot\mathcal{A}$ and then apply Lemma 12.5 below to the integer m. This gives an integer n_m which one can substitute into the result of Lemma 12.6 below to obtain a subset $\{\varphi'_i\}_{1\le i\le r}$ of $({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})^*$ for which one has

$$(\wedge_{1\le i\le r}\varphi_i)(\epsilon_{\underline{b}}) \equiv (\wedge_{1\le i\le r}\varphi'_i)(\epsilon_{\underline{b}}) \text{ modulo } m\cdot\mathcal{A}.$$

This congruence implies that for any element a of $\mathfrak{A}(\mathcal{A})$ and any additive homomorphism $\epsilon : \mathcal{A}\to\mathbb{Z}$ one has

$$\epsilon_G(a\cdot(\wedge_{i=1}^{i=r}\varphi_i)(\epsilon_{\underline{b}})\cdot\mathrm{pr}_\Pi) \equiv \epsilon_G(a\cdot(\wedge_{i=1}^{i=r}\varphi_i)(\epsilon_{\underline{b}})\cdot\mathrm{pr}_\Pi) \text{ modulo } \epsilon(m\cdot\mathfrak{A}(\mathcal{A}))\cdot\mathbb{Z}[G].$$

Since our choice of m implies $\epsilon(m\cdot\mathfrak{A}(\mathcal{A})) \subseteq \epsilon(|\mathrm{Cl}^T_{S'}(L)|\cdot\mathcal{A}) \subseteq |\mathrm{Cl}^T_{S'}(L)|\cdot\mathbb{Z}[G]$, this congruence shows that Theorem 9.9(iii) is true for the exterior power $\wedge_{1\le i\le r}\varphi_i$ if and only if it is true for the exterior power $\wedge_{1\le i\le r}\varphi'_i$.

This proves the claimed result since the set $\{\rho_{S,S'}(\varphi'_i)\}_{1\le i\le r}$ spans a free A-module of rank r as a consequence of Lemma 12.6(ii). □

Lemma 12.5. *For any integer m there exists an integer n_m with the following property. If $\{\varphi_i\}_{1\le i\le r}$ and $\{\varphi'_i\}_{1\le i\le r}$ are subsets of $({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})^*$ with $\varphi_i\equiv\varphi'_i$ modulo $n_m\cdot({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})^*$ for all integers i, then $(\wedge_{1\le i\le r}\varphi_i)(\epsilon_{\underline{b}}) \equiv (\wedge_{1\le i\le r}\varphi'_i)(\epsilon_{\underline{b}})$ modulo $m\cdot\mathcal{A}$.*

Proof. In this argument we fix a product decomposition $A=\prod_{i\in I}A_i$ and associated data $e_i, E, A'_i, V_i, d_i, \mathcal{O}, T_i$ and v_j as in the proof of Proposition 3.5(iii).

With respect to these choices each element $\wedge_{1\le a\le r}\varphi_a$ belongs to the submodule

$$\Lambda := \bigoplus_{i\in I}\bigwedge_{\mathcal{O}}^{rd_i}(T_i\otimes_{(\mathcal{A}'_i)^{\mathrm{op}}}\mathrm{Hom}_{\mathcal{A}'_i}(e_i(\mathcal{O}\cdot({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})),\mathcal{A}'_i))$$

of $E\cdot\bigwedge^r_{A^{\mathrm{op}}}\mathrm{Hom}_A(\mathbb{Q}\cdot{}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T}, A)$. In addition, since Λ is a finitely generated $\mathcal{O}$-module and the index of $\mathcal{O}\cdot\mathcal{A}$ in $\prod_{i\in I}\mathcal{A}'_i$ is finite there exists an integer n_m with the property that for every λ in $n_m\cdot\Lambda$ one has $\lambda(\epsilon_{\underline{b}})\in m(\mathcal{O}\cdot\mathcal{A})$.

Now, by the stated assumptions, there is a congruence in Λ of the form

$$\wedge_{1\le i\le r}\varphi_i \equiv \wedge_{1\le i\le r}\varphi'_i \pmod{n_m\cdot\Lambda}$$

and hence (by the above argument) a congruence in $E\cdot A$ of the form

$$(\wedge_{1\le i\le r}\varphi_i)(\epsilon_{\underline{b}}) \equiv (\wedge_{1\le i\le r}\varphi'_i)(\epsilon_{\underline{b}}) \text{ modulo } m(\mathcal{O}\cdot\mathcal{A}).$$

This then implies the claimed congruence since each element $\wedge_{1\le i\le r}\varphi_i$ and $\wedge_{1\le i\le r}\varphi'_i$ belongs to A and one has $A\cap m(\mathcal{O}\cdot\mathcal{A}) = m\cdot\mathcal{A}$. □

Lemma 12.6. *For each integer i with $1\le i\le r$ let φ_i be an element of $({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})^*$. Then for any given integer n there is a subset $\{\varphi'_i : 1\le i\le r\}$ of $({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})^*$ which satisfies the following properties.*

(i) *For each i one has $\varphi'_i \equiv \varphi_i$ modulo $n \cdot ({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})^*$.*

(ii) *The $\mathcal{A}$-submodule of $({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S',T})^*$ generated by $\{\rho_{S,S'}(\varphi'_i)\}_{1\le i\le r}$ is free of rank r.*

Proof. The explicit choice of S' in Theorem 9.9(iii) implies that we may choose a free $\mathcal{A}$-submodule $\mathcal{F}$ of $({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{K,S',T})^*$ of rank r. We then choose a subset $\{f_i\}_{1\le i\le r}$ of $({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})^*$ for which $\{\rho_{S,S'}(f_i)\}_{1\le i\le r}$ is an A-basis of $\mathbb{Q}\cdot\mathcal{F}$. For any integer m we set $\varphi_{i,m} := \varphi_i + mnf_i$ and note it suffices to show that for any sufficiently large m the elements $\{\rho_{S,S'}(\varphi_{i,m}) : 1 \le i \le r\}$ are linearly independent over A.

Consider the composite homomorphism of $\mathcal{A}$-modules $\mathcal{F} \to \mathbb{Q}({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S',T})^* \to \mathbb{Q}\mathcal{F}$ where the first arrow sends each $\rho_{S,S'}(f_i)$ to $\rho_{S,S'}(\varphi_{i,m})$ and the second is induced by a choice of A-equivariant section to the projection $\mathbb{Q}({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S',T})^* \to \mathbb{Q}(({}^{\Pi}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S',T})^*/\mathcal{F})$. Then, with respect to the basis $\{\rho_{S,S'}(f_i) : 1 \le i \le r\}$, this linear map is represented by a matrix of the form $M + mnI_r$ for a matrix M in $\mathrm{M}_r(A)$ that is independent of m. In particular, if m is large enough to ensure that $-mn$ is not an eigenvalue of the image of M in any of the simple algebra components of $\mathrm{M}_r(A_{\mathbb{C}})$, then the composite homomorphism is injective and so the elements $\{\rho_{S,S'}(\varphi_{i,m}) : 1 \le i \le r\}$ are linearly independent over A, as required. □

Next we note that it suffices to prove the displayed containment in Theorem 9.9(iii) after localization at each prime p. At each prime p one can then make the following reduction.

Proposition 12.7. *It suffices to prove the p-localization of Theorem 9.9(iii) for sets $\underline{b}$ such that $Y_{\underline{b},(p)} = Y_{\pi,(p)}$.*

Proof. Note first that a set $\underline{b}$ satisfying the given conditions exists by virtue of Roiter's Lemma (just as in the proof of Theorem 9.9(i) given above).

Let $\underline{b}' = \{b'_i\}_{1\le i\le r}$ be any other set chosen as in Theorem 9.9 . Since $Y_{\underline{b}',(p)} \subseteq Y_{\pi,(p)} = Y_{\underline{b},(p)}$ there exists a matrix M in $\mathrm{M}_r(\mathcal{A}_{(p)})$ with $b'_i = \sum_{j=1}^{j=r} M_{ij}b_j$ for all i and j.

Corollary 2.9 then implies that $\wedge_{i=1}^{i=r} b'_i = \mathrm{Nrd}_A(M) \cdot \wedge_{j=1}^{j=r} b_i$ and hence also $\Phi(\epsilon_{\underline{b}'}) = \mathrm{Nrd}_A(M) \cdot \Phi(\epsilon_{\underline{b}})$. For any a in $\mathfrak{A}(\mathcal{A})$ one therefore has

$$a \cdot \epsilon_G(\Phi(\epsilon_{\underline{b}'}) \cdot \mathrm{pr}_\Pi) = a' \cdot \epsilon_G(\Phi(\epsilon_{\underline{b}}) \cdot \mathrm{pr}_\Pi).$$

with $a' := \mathrm{Nrd}_A(M)a$.

To prove the claimed result it is thus enough to note the argument of Lemma 5.6(iii) shows that a' belongs to $\mathfrak{A}(\mathcal{A})_{(p)}$. □

We end this section by establishing some basic properties of the complex $C^*_{L,S,T}$ that occurs in Definition 9.6.

Lemma 12.8. *The complex ${}_{\Pi}C^*_{L,S,T} := \Pi \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C^*_{L,S,T}$ belongs to $D^{\mathrm{p}}(\mathcal{A})$ and is acyclic outside degrees zero and one. There are natural identifications $H^0({}_{\Pi}C^*_{L,S,T}) = ({}_{\check{\Pi}}X_{L,S})^*$ and $H^1({}_{\Pi}C^*_{L,S,T}) = {}_{\Pi}\mathcal{S}_{S,T}(\mathbb{G}_m/L)$ and a natural surjection ${}_{\Pi}\mathcal{S}_{S,T}(\mathbb{G}_m/L) \to ({}^{\check{\Pi}}_{\mathcal{P}}\mathcal{O}^{\times}_{L,S,T})^*$.*

Proof. Set $C := C_{L,S,T}$ and $C^* := R\,\mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})[-2]$.

Then Proposition 7.8 implies C is represented by a complex $C^\bullet$ of finitely generated projective G-modules $P^0 \xrightarrow{\phi} P$, where the first module is placed in degree one, and hence that ${}_{\Pi}C^*$ is represented by the complex ${}_{\Pi}\,\mathrm{Hom}_{\mathbb{Z}}(C^\bullet, \mathbb{Z})[-2]$. This shows, in particular, that ${}_\Pi C^*$ belongs to $D^{\rm p}(\mathcal{A})$ and is acyclic outside degrees zero and one and that $H^1({}_\Pi C^*)$ identifies with ${}_\Pi H^1(C^*)$. The claimed identification $H^1({}_\Pi C^*) = {}_\Pi \mathcal{S}_{S,T}(\mathbb{G}_m/L)$ is thus induced from the identification $H^1(C^*) = \mathcal{S}_{S,T}(\mathbb{G}_m/L)$ described in [11, Prop. 2.4(iii)].

To prove the remaining claims we use the fact that ${}_\Pi C^*$ is naturally isomorphic to the complex $R\operatorname{Hom}_{\mathcal{A}}({}_{\check\Pi}C, \mathcal{A})[-2] \cong \operatorname{Hom}_{\mathcal{A}}(\check\Pi \otimes_{\mathbb{Z}[G]} C^\bullet, \mathcal{A})[-2]$. This is true because for any finitely generated projective G-module Q there are natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathcal{A}}({}_{\check\Pi}Q, \mathcal{A}) &= \operatorname{Hom}_{\mathcal{A}}(H_0(G, \check\Pi \otimes_{\mathbb{Z}} Q), \mathcal{A}) \cong H^0(G, \operatorname{Hom}_{\mathcal{A}}(\check\Pi \otimes_{\mathbb{Z}} Q, \mathcal{A}))\\ &\cong H_0(G, \operatorname{Hom}_{\mathcal{A}}(\check\Pi \otimes_{\mathbb{Z}} Q, \mathcal{A})) \cong H_0(G, \Pi \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})) \cong {}_\Pi(\operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})),\end{aligned}$$

where the second isomorphism is induced by the action of T_G (and the fact that the G-module $\operatorname{Hom}_{\mathcal{A}}(\check\Pi \otimes_{\mathbb{Z}} Q, \mathcal{A}) \cong \Pi \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})$ is cohomologically-trivial).

This description of ${}_\Pi C^*$ leads to a spectral sequence

$$\operatorname{Ext}^a_{\mathcal{A}}(H^b({}_{\check\Pi}C), \mathcal{A}) \Rightarrow H^{a-b}({}_\Pi C^*)$$

which combines with the properties of $\mathcal{A}$ described in Remark 4.3(i) and (ii) to imply that in each degree i there is a natural exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(H^{-i+1}({}_{\check\Pi}C), A/\mathcal{A}) \to H^i({}_\Pi C^*) \to H^{-i}({}_{\check\Pi}C)^* \to 0$$

This sequence combines with the description of the groups $H^i({}_{\check\Pi}C)$ given by the exact sequence in Lemma 12.1 (with Π replaced by $\check\Pi$) to give the claimed identification $H^0({}_\Pi C^*) = ({}_{\check\Pi}X_{L,S})^*$ and surjection ${}_\Pi \mathcal{S}_{S,T}(\mathbb{G}_m/L) = H^1({}_\Pi C^*) \to ({}^{\check\Pi}_{\mathcal{P}}\mathcal{O}^\times_{L,S,T})^*$. □

12.4.2. We now turn to the proof of Theorem 9.9(iii).

To do this we fix an exterior product of homomorphisms Φ as in Proposition 12.4. We then choose a lift $\widetilde\varphi_i$ of each homomorphism φ_i through the surjective map ${}_\Pi \mathcal{S}_{S,T}(\mathbb{G}_m/L) \to ({}^{\Pi}_{\mathcal{P}}\mathcal{O}^\times_{L,S,T})^*$ in Lemma 12.8 and write $\mathcal{E}_\Phi$ for the $\mathcal{A}$-module generated by $\{\widetilde\varphi_i\}_{1\le i\le r}$.

Proposition 12.9. *Fix an exterior product Φ as in (12.4). Assume the set $\underline{b}$ in Theorem 9.9 is such that $Y_{\underline{b},(p)} = Y_{\pi,(p)}$.*

Then the $\mathcal{A}_{(p)}$-module $({}_\Pi \mathcal{S}_{S,T}(\mathbb{G}_m/L)/\mathcal{E}_\Phi)_{(p)}$ has a quadratic presentation and, if Conjecture 9.2 is valid, then the element $\Phi(\epsilon_{\underline{b}})^\#$ belongs to $\operatorname{FI}^0_{\mathcal{A}}({}_\Pi \mathcal{S}_{S,T}(\mathbb{G}_m/L)/\mathcal{E}_\Phi)_{(p)}$.

Proof. We use the existence of an exact triangle in $D^{\rm p}(\mathcal{A}_{(p)})$ of the form

$$\mathcal{A}^{\oplus r,\bullet}_{(p)} \xrightarrow{\theta} \Pi_{(p)} \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C^*_{L,S,T} \xrightarrow{\theta'} C^\bullet_{(p)} \to \mathcal{A}^{\oplus r,\bullet}_{(p)}[1]. \tag{48}$$

Here $\mathcal{A}^{\oplus r,\bullet}_{(p)}$ denotes the complex $\mathcal{A}^{\oplus r}_{(p)}[0] \oplus \mathcal{A}^{\oplus r}_{(p)}[-1]$ and, writing $\{c_i\}_{1\le i\le r}$ for the canonical basis of $\mathcal{A}^{\oplus r}_{(p)}$, the morphism θ is uniquely specified by the condition that for each i one has

$$H^i(\theta)(c_i) = \begin{cases} b_i^* \in Y^*_{\underline{b},(p)} = Y^*_{\pi,(p)} \subset ({}_\Pi X_{L,S})^*_{(p)} = H^0(\Pi_{(p)} \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C^*_{L,S,T}), & \text{if } i = 0\\ \widetilde\varphi_i \in {}_\Pi \mathcal{S}_{S,T}(\mathbb{G}_m/L)_{(p)} = H^1(\Pi_{(p)} \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C^*_{L,S,T}), & \text{if } i = 1.\end{cases}$$

With this definition the long exact cohomology sequence of the triangle (48) implies $C^\bullet_{(p)}$ is acyclic outside degrees zero and one and induces identifications

$$H^i(C^\bullet_{(p)}) = \begin{cases} \ker(\epsilon_{\Pi,L,S,p})^*, & \text{if } i=0 \\ ({}_\Pi\mathcal{S}_{S,T}(\mathbb{G}_m/L)/\mathcal{E}_\Phi)_{(p)}, & \text{if } i=1, \end{cases}$$

where $\epsilon_{\Pi,L,S,p}$ denotes the (split) surjection of $\mathcal{A}_{(p)}$-modules $({}_\Pi X_{L,S})_{(p)} \to Y_{\pi,(p)}$. Thus, since $H^0(C^\bullet_{(p)})$ is $\mathbb{Z}_{(p)}$-torsion-free, the same reasoning as used in the proof of Proposition 7.8 shows that the complex $C^\bullet_{(p)}$ is represented by a complex

$$P \xrightarrow{\delta} P \tag{49}$$

where P is a finitely generated free $\mathcal{A}_{(p)}$-module and the first term is placed in degree zero. This shows, in particular, that the $\mathcal{A}_{(p)}$-module $({}_\Pi\mathcal{S}_{S,T}(\mathbb{G}_m/L)/\mathcal{E}_\Phi)_{(p)}$ has a quadratic presentation, as claimed.

We now write e for the idempotent e_π of $\zeta(A)$ that is defined just prior to Lemma 8.5. Then the definition of e combines with the above descriptions to imply that the spaces $e(\mathbb{Q}\cdot H^0(C^\bullet_{(p)}))$ and $e(\mathbb{Q}\cdot H^1(C^\bullet_{(p)}))$ vanish and so we may choose a commutative diagram of $A_\mathbb{C}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (A_\mathbb{C})^{\oplus r} & \xrightarrow{H^0(\theta)} & \mathbb{C}\cdot H^0(\Pi_{(p)}\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C^*_{L,S,T}) & \xrightarrow{H^0(\theta')} & \mathbb{C}\cdot H^0(C^\bullet_{(p)}) & \longrightarrow & 0 \\ & & \lambda_1\downarrow & & \lambda_2\downarrow & & \lambda_3\downarrow & & \\ 0 & \longrightarrow & (A_\mathbb{C})^{\oplus r} & \xrightarrow{H^1(\theta)} & \mathbb{C}\cdot H^1(\Pi_{(p)}\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C^*_{L,S,T}) & \xrightarrow{H^1(\theta')} & \mathbb{C}\cdot H^1(C^\bullet_{(p)}) & \longrightarrow & 0 \end{array} \tag{50}$$

in which the vertical maps are bijective and such that $e\lambda_2 = e((\mathbb{C}\cdot\Pi)\otimes_{\mathbb{R}[G]} R^*_{L,S})$.

The commutativity of the left hand square in (50) implies that

$$e\cdot\wedge_{1\le i\le r}R^{\Pi,*}_{L,S}(b_i^*) = \mathrm{Nrd}_{eA_\mathbb{C}}(e\lambda_1)\cdot\Phi$$

with $R^{\Pi,*}_{L,S} := (\mathbb{C}\cdot\Pi)\otimes_\mathbb{R} R^*_{L,S}$. Thus, if Conjecture 9.2 is valid, then one has

$$\begin{aligned} \Phi(\epsilon_{\underline{b}}) &= \mathrm{Nrd}_{eA_\mathbb{C}}(e\lambda_1)^{-1}\cdot(\wedge_{1\le i\le r}R^{\Pi,*}_{L,S}(b_i^*))(\epsilon_{\underline{b}}) \\ &= \mathrm{Nrd}_{eA_\mathbb{C}}(e\lambda_1)^{-1}\cdot(\wedge_{1\le i\le r}R^{\Pi,*}_{L,S}(b_i^*))(\theta^\pi_{S,T}(0)\cdot\wedge_{1\le i\le r}(R^\Pi_{L,S})^{-1}(b_i)) \\ &= \mathrm{Nrd}_{eA_\mathbb{C}}(e\lambda_1)^{-1}\cdot\theta^\pi_{S,T}(0) \\ &= \mathrm{Nrd}_A(u)\mathrm{Nrd}_A(\delta) \\ &\in \mathrm{FI}^0_{\mathcal{A}_{(p)}}(({}_\Pi\mathcal{S}_{S,T}(\mathbb{G}_m/L)/\mathcal{E}_\Phi)_{(p)}) = \mathrm{FI}^0_{\mathcal{A}}({}_\Pi\mathcal{S}_{S,T}(\mathbb{G}_m/L)/\mathcal{E}_\Phi)_{(p)} \end{aligned}$$

where u belongs to $\mathcal{A}^\times_{(p)}$, δ is the morphism that occurs in the complex (49), the second equality follows directly from the definition of $\epsilon_{\underline{b}}$, the fourth directly from the result of Lemma 12.10 below and the containment from the definition of the zero-th noncommutative Fitting invariant of the $\mathcal{A}_{(p)}$-module $\mathrm{cok}(\delta) = H^1(C^\bullet_{(p)}) = ({}_\Pi\mathcal{S}_{S,T}(\mathbb{G}_m/L)/\mathcal{E}_\Phi)_{(p)}$. □

Proposition 12.9 combines with Proposition 5.8 to imply that for any a' of $\mathfrak{A}(\mathcal{A})$ one has

$$a'\cdot\Phi(\epsilon_{\underline{b}}) \in \mathrm{Ann}_{\mathcal{A}_{(p)}}(({}_\Pi\mathcal{S}_{S,T}(\mathbb{G}_m/L)/\mathcal{E}_\Phi)_{(p)}) \tag{51}$$

and we shall now show that Theorem 9.9(iii) is a consequence of this containment.

To do this we recall that there exists a natural surjective homomorphism of G-modules $f : \mathcal{S}_{S,T}(\mathbb{G}_m/L) \to \mathcal{S}_{S',T}(\mathbb{G}_m/L)$ (see [11, Prop. 2.4(ii)]). Since, by our choice of Φ, the lattice $(\mathcal{E}_\Phi)_{(p)}$ is disjoint from the kernel of the induced surjection of $\mathcal{A}_{(p)}$-modules $f_{\Pi,p} : {}_\Pi\mathcal{S}_{S,T}(\mathbb{G}_m/L)_{(p)} \to {}_\Pi\mathcal{S}_{S',T}(\mathbb{G}_m/L)_{(p)}$ one obtains a surjection of $\mathcal{A}_{(p)}$-modules

$$({}_\Pi\mathcal{S}_{S,T}(\mathbb{G}_m/L)/\mathcal{E}_\Phi)_{(p)} \twoheadrightarrow {}_\Pi\mathcal{S}_{S',T}(\mathbb{G}_m/L)_p/f_{\Pi,p}((\mathcal{E}_\Phi)_{(p)}). \tag{52}$$

In addition, the exact sequence of G-modules (26) induces an exact sequence of $\mathcal{A}_{(p)}$-modules

$$\mathrm{Tor}_1^G(\Pi, \mathcal{O}^\times_{L,S',T})_{(p)} \to {}_\Pi(\mathrm{Cl}^T_{S'}(L)^\vee)_{(p)} \to {}_\Pi\mathcal{S}_{S',T}(\mathbb{G}_m/L)_{(p)}/f_{\Pi,p}((\mathcal{E}_\Phi)_{(p)}).$$

and this sequence combines with the surjection (52) and containment (51) to imply that

$$a\cdot\Phi(\epsilon_{\underline{b}}) \in \mathrm{Ann}_{\mathcal{A}_p}({}_\Pi(\mathrm{Cl}^T_{S'}(L)^\vee)_{(p)}) \tag{53}$$

for all a in $\mathfrak{A}(\mathcal{A})\cdot\mathrm{Ann}_{\mathcal{A}}(\mathrm{Tor}_1^G(\Pi, \mathcal{O}^\times_{L,S',T}))$.

Applying Lemma 12.11(ii) below in this case we then deduce $\epsilon_G(\iota_{\mathcal{A}}(a\Phi(\epsilon_{\underline{b}}))\cdot \mathrm{pr}_{\Pi^*})^\#$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S'}(L)^\vee)_{(p)}$, and hence that

$$\epsilon_G(\iota_{\mathcal{A}}(a\Phi(\epsilon_{\underline{b}}))\cdot \mathrm{pr}_{\Pi^*}) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S'}(L))_{(p)},$$

as required to complete the proof of Theorem 9.9(iii).

12.4.3. In this section we prove two results that were used in §12.4.2.

Lemma 12.10. *If conjecture 9.2 is valid, then there exists a unit u of $\mathcal{A}_{(p)}$ such that*

$$\mathrm{Nrd}_{eA_\mathbb{C}}(e\lambda_1)^{-1}\cdot\theta^\pi_{S,T}(0) = \mathrm{Nrd}_A(u)\mathrm{Nrd}_A(\delta)$$

where δ is the morphism that occurs in the representative (49) of $C^\bullet_{(p)}$.

Proof. Write η for the image of $\iota_\#(\theta^*_{L/K,S,T}(0))$ under the map $\iota_\Pi : \zeta(\mathbb{C}[G])^\times \to \zeta(A_\mathbb{C})^\times$ defined in §8.1. Then the diagram (50) combines with Proposition 6.11 to imply that for any given primitive $\xi(\mathcal{A})_{(p)}$-bases $\underline{x}_1$ of $\det_{\mathcal{A}_{(p)}}(\mathcal{A}^{\oplus r,\bullet}_{(p)})$ and $\underline{x}_3$ of $\det_{\mathcal{A}_{(p)}}(C^\bullet_{(p)})$ there exists a primitive $\mathcal{A}_{(p)}$-basis $\underline{x}_2$ of $\det_{\mathcal{A}_{(p)}}(\Pi_{(p)}\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C^*_{L,S,T})$ such that if one defines elements $c_{\eta,1}$ and $c_{\eta,2}$ of $\zeta(A_\mathbb{C})^\times$ by the equalities $\vartheta^{-1}_{\lambda_j}(\eta) = c_{\eta,j}\cdot\underline{x}_j$ for $j = 1, 2$, then one has

$$\vartheta^{-1}_{\lambda_3}(1_A) = c_{\eta,2}c^{-1}_{\eta,1}\cdot\underline{x}_3$$

with 1_A denoting the identity element of A.

The explicit structure of $\mathcal{A}^{\oplus r,\bullet}_{(p)}$ implies $\vartheta^{-1}_{\lambda_1}(\eta) = \eta\cdot\mathrm{Nrd}_{A_\mathbb{C}}(\lambda_1)^{-1}\cdot(\wedge^{i=r}_{i=1}c_i\otimes\wedge^{j=r}_{j=1}c^*_j)$ and hence (by Corollary 2.9) that $c_{\eta,1}$ is equal to $\eta\cdot\mathrm{Nrd}_{A_\mathbb{C}}(\lambda_1)^{-1}\mathrm{Nrd}_A(u_1)$ for some unit u_1 of $\mathcal{A}_{(p)}$.

In addition, if we assume the validity of Conjecture 9.2, then Lemma 9.7 implies that $\mathrm{Nrd}_{A_\mathbb{C}}((\Pi\otimes_{\mathbb{C}[G]}R_{L,S})\circ\lambda_2^{-1})\cdot\vartheta^{-1}_{\lambda_2}(\eta)$ is a primitive basis of $\det_{\mathcal{A}_{(p)}}(\Pi_p\otimes^{\mathbb{L}}_{\mathbb{Z}[G]}C^*_{L,S,T})$ and hence that $c_{\eta,2} = \mathrm{Nrd}_{A_\mathbb{C}}((\Pi\otimes_{\mathbb{C}[G]}R_{L,S})\circ\lambda_2^{-1})\cdot\mathrm{Nrd}_A(u_2)$ for some unit u_2 of $\mathcal{A}_{(p)}$.

The above displayed equality therefore implies that

$$\vartheta^{-1}_{\lambda_3}(1_A) = \eta^{-1}\mathrm{Nrd}_A(u)\mathrm{Nrd}_{A_\mathbb{C}}(\lambda_1)\mathrm{Nrd}_{A_\mathbb{C}}((\Pi\otimes_{\mathbb{C}[G]}R_{L,S})\circ\lambda_2^{-1})\cdot x_3$$

with $u := u_2u_1^{-1} \in \mathcal{A}_{(p)}^\times$.

We now multiply this equality by the idempotent $e := e_\pi$. Noting that $e\eta = \theta^\pi_{S,T}(0)^\#$ and $\mathrm{Nrd}_{A_\mathbb{C}}((\Pi \otimes_{\mathbb{C}[G]} R_{L,S}) \circ \lambda_2^{-1})) = e$ we thereby obtain equalities

$$\vartheta^{-1}_{e\lambda_3}(e) = e(\vartheta^{-1}_{\lambda_3}(1_A)) = \theta^\pi_{S,T}(0)^{-1}\mathrm{Nrd}_A(u)\mathrm{Nrd}_{A_\mathbb{C}}(\lambda_1) \cdot e(x_3).$$

Next we note that the complex $e(\mathbb{Q} \cdot C^\bullet_{(p)})$ is acyclic and hence that one can choose the primitive basis x_3 of $\det_{\mathcal{A}_{(p)}}(C^\bullet_{(p)})$ such that $\vartheta^{-1}_{e\lambda_3}(e) = \mathrm{Nrd}_{\mathbb{C}\cdot Ae}(Ae \otimes_\mathcal{A} \delta)^{-1} \cdot e(x_3)$. Given this, the last displayed equality implies the claimed formula except for the fact that the term $\mathrm{Nrd}_A(\delta)$ is replaced by $\mathrm{Nrd}_{\mathbb{C}\cdot Ae}(Ae \otimes_\mathcal{A} \delta)$.

It thus suffices to note that $\mathrm{Nrd}_A(\delta)$ is equal to $\mathrm{Nrd}_{\mathbb{C}\cdot Ae}(Ae\otimes_\mathcal{A}\delta)$ since if e' is any primitive idempotent of $\zeta(A)$ that is orthogonal to e, then the definition of e implies that the space $e'(\mathbb{Q} \cdot H^0(C^\bullet_{(p)})) = e'(\mathbb{Q} \cdot \ker(\delta))$ does not vanish and hence that $e' \cdot \mathrm{Nrd}_A(\delta) = 0$. $\square$

In the next result we use the involutions $\iota_\mathcal{A}$ on $\mathcal{A}$ and $\iota_\#$ on $\mathbb{Z}[G]$ to endow the linear dual $\Pi^* := \mathrm{Hom}_\mathbb{Z}(\Pi, \mathbb{Z})$ with the structure of an $(\mathcal{A}, \mathbb{Z}[G])$-bimodule that is locally-free over $\mathcal{A}$.

Lemma 12.11. *Let M be a finite G-module and $\epsilon : \mathcal{A} \to \mathbb{Z}$ an additive homomorphism.*

(i) *For any a in $\mathrm{Ann}_\mathcal{A}({}^\Pi M)$ one has $\epsilon_G(a \cdot \mathrm{pr}_\Pi) \in \mathrm{Ann}_{\mathbb{Z}[G]}(M)$.*

(ii) *For any a in $\mathrm{Ann}_\mathcal{A}({}_\Pi M)$ one has $\epsilon_G(\iota_\mathcal{A}(a) \cdot \mathrm{pr}_{\Pi^*})^\# \in \mathrm{Ann}_{\mathbb{Z}[G]}(M)$.*

Proof. We write n for the rank of the free A-module $\mathbb{Q} \cdot \Pi$. After localizing at a prime p we fix a basis $\{\pi_i : i \in [n]\}$ of the free $\mathcal{A}_{(p)}$-module $\Pi_{(p)}$ and write $\rho_{\Pi,p} : G \to \mathrm{GL}_n(A)$ for the corresponding representation that arises from the action of G on A.

For each m in M and each index i the element $T_i(m) := \sum_{g\in G} g(\pi_i \otimes m)$ belongs to ${}^\Pi M$ and so one has $a(T_i(m)) = 0$. In $\Pi_{(p)} \otimes_\mathbb{Z} M$ one therefore has

$$\begin{aligned}0 = a(T_i(m)) = \sum_{g\in G} a\pi_i g^{-1} \otimes g(m) &= \sum_{g\in G}\sum_{j=1}^{j=n} a\rho_{\Pi,p}(g^{-1})_{ij}\pi_j \otimes g(m)\\ &= \sum_{j=1}^{j=n}\left(a(\sum_{g\in G}\rho_{\Pi,p}(g^{-1})_{ij})\pi_j \otimes g(m)\right)\end{aligned}$$

and hence also, since $\{\pi_i : i \in [n]\}$ is an $\mathcal{A}_{(p)}$-basis of $\Pi_{(p)}$, equalities in $\mathcal{A} \otimes_\mathbb{Z} M$

$$a \cdot \sum_{g\in G} \rho_{\Pi,p}(g^{-1})_{ij} \otimes_\mathbb{Z} g(m) = 0.$$

By applying the homomorphism $\epsilon \otimes \mathrm{id} : \mathcal{A}_{(p)} \otimes M \to M_{(p)}$ this implies

$$\epsilon(a \cdot \sum_{g\in G} \rho_{\Pi,p}(g^{-1})_{ij})g(m) = 0,$$

or equivalently that each element $c_p(a)_{ij} := \epsilon(a\cdot\sum_{g\in G} \rho_{\Pi,p}(g^{-1})_{ij})g$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(M)_{(p)}$.

In particular, the element

$$\sum_{i=1}^{i=n} c_p(a)_{ii} = \epsilon(a \cdot \sum_{g\in G}\left(\sum_{i=1}^{i=n}\rho_{\Pi,p}(g^{-1})_{ii}\right))g = \epsilon(a \cdot \sum_{g\in G}\chi_\Pi(g^{-1}))g = \epsilon_G(a \cdot \mathrm{pr}_\Pi)$$

belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(M)_{(p)}$. Since this is true for all primes p it follows that $\epsilon_G(a\cdot \mathrm{pr}_\Pi)$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(M)$, as required to prove claim (i).

We now derive claim (ii) as a consequence of claim (i). To do this we note that there is a natural isomorphism of $\mathcal{A}$-modules $H^0(G,\Pi\otimes_{\mathbb{Z}} M)^\vee \cong H_0(G,\Pi^*\otimes_{\mathbb{Z}} M^\vee)$, and hence an equality $\mathrm{Ann}_{\mathcal{A}}({}_\Pi M) = \iota_{\mathcal{A}}(\mathrm{Ann}_{\mathcal{A}}({}^{\Pi^*}(M^\vee)))$. Applying claim (i) in this context therefore implies that $\epsilon_G(\iota_{\mathcal{A}}(a)\cdot \mathrm{pr}_{\Pi^*})$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(M^\vee)$ for every a in $\mathrm{Ann}_{\mathcal{A}}({}_\Pi M)$.

This in turn implies claim (ii) because for any finite G-module N one has $\mathrm{Ann}_{\mathbb{Z}[G]}(M^\vee) = \iota_\#(\mathrm{Ann}_{\mathbb{Z}[G]}(M))$. □

13. The proofs of Corollaries 9.10, 9.12, 10.5 and 10.6

13.1. **The proof of Corollary 9.10.** In this section we derive Corollary 9.10 as a consequence of Theorems 9.9 and 12.2.

13.1.1. To prove claims (i), (ii) and (iii) of Corollary 9.10 we simply apply Theorem 9.9 in the setting of Example 8.9.

To be more precise, we take π to be the homomorphism $\pi^{G,v_0}_{L/K,S,T}$, S' to be $S_K^\infty\cup V_L\cup\{v_0\}$, $\underline{b}$ to be the (ordered) set of places of L specified in Example 8.9, Π to be $\mathbb{Z}[G]$, endowed with its natural structure as $(\mathbb{Z}[G],\mathbb{Z}[G])$-bimodule and the order $\mathcal{A}$ to be $\mathbb{Z}[G]$.

In this case the functor $N\mapsto {}^{\Pi}_{\mathcal{P}}N$ described in §8.2 is canonically isomorphic to the identity functor and the module ${}_\Pi(Y_{L,\Sigma})/\Pi_{\underline{b}}$ in Theorem 9.9 vanishes so that $\mathrm{FI}^0_{\mathcal{A}}({}_\Pi(Y_{L,\Sigma})/\Pi_{\underline{b}})^{-1}$ is equal to $\xi(\mathbb{Z}[G])$.

Given these observations, claims (i) and (ii) of Corollary 9.10 follow directly from the statement of Theorem 9.9(i) and (ii) in this case.

In addition, in this case one has $\chi_\Pi(g)=g$ for all $g\in G$ and so, taking $\epsilon:\mathbb{Z}[G]\to\mathbb{Z}$ to be projection onto the coefficient of the identity element of G then for x in $\mathbb{Q}[G]$ one has

$$\epsilon_G(x\cdot \mathrm{pr}_\Pi) = \epsilon_G(\sum_{g\in G} xg^{-1}\otimes g) = \sum_{g\in G}\epsilon(xg^{-1})g = x.$$

Hence, since $\mathrm{Tor}_1^G(\Pi,\mathcal{O}^\times_{L,S',T})$ vanishes in this case, the statement of Theorem 9.9(iii) for this choice of ϵ directly implies the claim of Corollary 9.10(iii).

13.1.2. We now prove Corollary 9.10(iv).

If $r_E=r$, then we need only prove the assertion of Corollary 9.10(iv)(b) and the explicit definition of $\mathrm{Rec}^{\mathcal{P}}_H$ in this case (given at the beginning of §7.4.1) means that this statement is a direct consequence of Lemma 9.8.

We therefore assume in the sequel that $r_E>r$. In this case the group $\mathrm{F}_{\pi_E}(\bigcap^r_G\mathcal{P}_{L,S,T})_H$ is finite (by Proposition 7.3(i)) and hence the assertion of Corollary 9.10(iv)(b) is true if it is true after p-localization at each prime p. The commutative diagram in Proposition 7.15 therefore implies that it suffices to prove the first assertion of Corollary 9.10(iv) for every prime p and to do this we use the explicit formula of Theorem 12.2.

We note that for each normal subgroup H of G the identification of $X_{L^H,S}$ with $T_H(X_{L,S})$ that is described in §7.2.2 has two important consequences. Firstly, it implies that the

$\underline{b} = \{b_i\}_{1\le i\le d}$ of the free G-module P fixed in Lemma 7.7 is such that

$$\varrho_E(\varpi^H(T_H(b_i))) = \begin{cases} w_{i,E}, & \text{if } 1 \le i \le n, \\ 0, & \text{if } n < i \le d. \end{cases} \tag{54}$$

Secondly, it implies that the section ι_2 chosen just after (38) is such that for every integer i that belongs to Z_E (so v_i splits in E/K) one has

$$\sigma_2(w_{i,E}) = \sigma_2(T_H(w_{i,L})) = T_H(\sigma_2(w_{i,L})) = T_H(b_i). \tag{55}$$

Having noted these facts, we now apply the formula of Theorem 12.2 with Π taken to be $\mathbb{Z}[G/H]$, regarded as a $(\mathbb{Z}[G/H], \mathbb{Z}[G])$-bimodule in the natural way.

In this case the functor $M \mapsto {}^{\Pi}M$ identifies with the fixed-point functor $M \mapsto H^0(H, M)$. In particular, the property (54) implies that we can take the basis $\hat{\underline{b}}_{(p)}$ of $({}_{\Pi}P)_{(p)}$ that is used in Theorem 12.2 to be $\{T_H(b)\}_{1\le i\le d}$ and the property (55) implies that the section σ_2 used in the proof of Theorem 12.2 can be chosen to be the restriction of a fixed section as chosen just after (38).

After recalling the definition of $\epsilon^{v_0}_{E/K,S,T}$ that is given in Example 8.9, this case of the formula of Theorem 12.2 therefore implies that for each normal subgroup H of G one has

$$\epsilon^{v_0}_{E/K,S,T} := \mathrm{Nrd}_{\mathbb{Q}[G]}(u_p) \cdot (\bigwedge_{a=r_E+1}^{a=d} \phi^H_a)(\bigwedge_{c=1}^{c=d} T_H(b_c)) \in {\bigwedge}^{r_E}_{\mathbb{Q}[G/H]}(\mathbb{Q}\cdot P^H)$$

where u_p is a unit of $\mathbb{Z}_{(p)}[G]$ that is independent of H.

This equality is in turn clearly equivalent to an equality

$$\Delta^{H,p}_{L,S,T}(\zeta) = \epsilon^{v_0}_{E/K,S,T}$$

where the map $\Delta^{H,p}_{L,S,T}$ is as defined in §7.5 and we have set

$$\zeta := \mathrm{Nrd}_{\mathbb{Q}[G]}(u_p) \cdot \wedge_{c=1}^{c=d} b_c \otimes \wedge_{c=1}^{c=d} b_c^* \in \det_{\mathbb{Z}_{(p)}[G]}(C_{L,S,T,(p)}) = \det_{\mathbb{Z}[G]}(C_{L,S,T})_{(p)}.$$

This fact completes the proof of Corollary 9.10.

13.2. **Higher Chinburg-Stark elements and the proof of Corollary 9.12.** In this section we discuss Theorem 9.9 in the setting of Example 8.11 and then prove Corollary 9.12.

We use the notation of Example 8.11. In addition, we identify each character ψ in $\widehat{G}$ with a corresponding representation $G \to \mathrm{GL}_{\psi(1)}(\mathcal{O}_\psi)$ and abbreviate the associated functors $M \mapsto {}^{\Pi_\psi}M$ and $M \to {}_{\Pi_\psi}M$ to $M \to M^\psi$ and $M \to M_\psi$ respectively.

Proposition 13.1. *Fix θ in $\mathbb{C}\cdot\mathrm{Hom}_G(\mathcal{O}^\times_{L,S,T}, X_{L,S})$. Then, for each non-trivial character ψ in $\widehat{G}$ the containment of Theorem 9.9(iii) for the homomorphism $\pi^\psi_{L/K,S,T}$ implies that*

$$|G|\cdot \mathrm{tr}_{E_\psi/\mathbb{Q}}(L^*_{S,T}(\check{\psi},0)\det_{\mathbb{C}}(|G|m\cdot(R^{-1}_{L,S}\circ\theta)^\psi)\cdot \mathrm{pr}_\psi) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S'}(L)),$$

where m is any element of $\mathcal{O}_\psi$ for which $m\cdot\theta(\mathcal{O}^{\times,\psi}_{L,S,T}) \subseteq (X_{L,S,\psi})_{\mathrm{tf}}$ and S' any subset of S as described in Theorem 9.9(iii).

Proof. We set $U := \mathcal{O}^\times_{L,S,T}$, $X := X_{L,S}$ and $\mathcal{O} := \mathcal{O}_\psi$ and write E for the field of fractions of $\mathcal{O}$.

It suffices to prove the displayed containment after p-localization. To do this we fix a prime p and a subset $\underline{b} = \{b_i\}_{1\le i\le r}$ of $(X_\psi)_{\rm tf}$ which gives an $\mathcal{O}_{(p)}$-basis of $(X_\psi)_{{\rm tf},(p)}$. For each integer i in $[r]$ we set $\theta_{b_i} := b_i^*\circ\theta$. Then for any element n of $\mathcal{O}$ one has

$$L^*_{S,T}(\check{\psi},0)\cdot {\rm det}_{\mathbb{C}}(n\cdot (R^{-1}_{L,S}\circ\theta)^\psi) = (\wedge_{i=1}^{i=r}(n\cdot\theta_{b_i}))(\epsilon_{\underline{b}}). \tag{56}$$

We next give an explicit interpretation of the diagram (41). To do this we write e_G for the trivial idempotent of G and note that for any G-module N the decomposition

$$E\cdot(\Pi_\psi\otimes_{\mathbb{Z}} N) = e_G(E\cdot(\Pi_\psi\otimes_{\mathbb{Z}} N))\oplus (1-e_G)(E\cdot(\Pi_\psi\otimes_{\mathbb{Z}} N))$$

induces an identification of

$$E\cdot({}^{\Pi_\psi}N) := H^0(G, E\cdot(\Pi_\psi\otimes_{\mathbb{Z}} N)) = H_0(G, E\cdot(\Pi_\psi\otimes_{\mathbb{Z}} N)) =: E\cdot({}_{\Pi_\psi}N).$$

With respect to this identification the diagram (41) implies that

$${\rm Hom}_{\mathcal{O}}({}^{\Pi_\psi}_{\mathcal{P}}U,\mathcal{O}) = |G|\cdot {\rm Hom}_{\mathcal{O}}(U^\psi,\mathcal{O})\subset {\rm Hom}_E(E\cdot U^\psi, E).$$

In particular, for any integer m with $m\cdot\theta(U^\psi)\subseteq X_{\psi,{\rm tf}}$ one has $|G|m\cdot\rho_{b_i}\in {\rm Hom}_{\mathcal{O}}({}^{\Pi_\psi}_{\mathcal{P}}U,\mathcal{O})_{(p)}$ for each index i. In this case one also has $\mathfrak{A}(\mathcal{O}) = \mathcal{O}$ and so can take $a = |G|$ in the statement of Theorem 9.9(iii).

The claimed result now follows directly by combining (56) with $n = |G|m$ together with the result of Theorem 9.9(iii) for the data $\pi = \pi^\psi_{L/K,S,T}$, $\mathcal{A} = \mathcal{O}, \Pi = \Pi_\psi, \varphi_i = |G|m\cdot\rho_{b_i}$ and with ϵ taken to be the trace map ${\rm tr}_{E/\mathbb{Q}}$. $\square$

Remark 13.2. The containment in Proposition 13.1 is finer than the prediction made in [7, Conj. 2.6.1] in that the term $\psi(1)^{-1}|G|^{3+r}e_\psi = |G|^{2+r}{\rm pr}_\psi$ that occurs in the latter conjecture is here replaced by $|G|^{1+r}{\rm pr}_\psi$ and the group ${\rm Cl}_{S'}(L)$ that occurs in loc. cit. is here replaced by ${\rm Cl}^T_{S'}(L)$.

Turning now to the proof of Corollary 9.12, claims (i), (ii) and (iii) follow directly by comparing the first assertion of Corollary 10.9 with the results of [7, Th. 4.3.1(ii) and Prop. 12.2.1].

The annihilation statement of Corollary 9.12(iv) is however finer than that of [7, Prop. 12.2.1]. The key point in its proof is that the hypotheses on ψ and S that are made in Corollary 9.12 imply ${\rm dim}_{E_\psi}(E_\psi\cdot X_{L,S_K^\infty,\psi}) = 1$ and hence that the set $S' = S_K^\infty$ satisfies the hypothesis of Theorem 9.9(iii) (and therefore also Proposition 13.1) with respect to the homomorphism $\pi = \pi^\psi_{L/K,S,T}$.

More precisely, to prove Corollary 9.12(iv) one need only make the following two changes to the proof of [7, Prop. 12.2.1(iv)]: the use of the containment [7, (44)] is replaced by the stronger containment discussed in Remark 13.3 below (with $r = 1$ and $S' = S_K^\infty$) and the use of [7, Lem. 11.1.2(i)] is replaced by an application of Lemma 12.11(ii) with $\Pi = \Pi_\psi$.

Remark 13.3. In terms of the notation used in the proof of Proposition 13.1, the $\mathcal{O}_{(p)}$-modules $U^\psi_{(p)}$ and $(X_\psi)_{{\rm tf},(p)}$ are both free of rank r and so one can choose the homomorphism

θ such that $\theta(U^{\psi}_{(p)}) = (X_\psi)_{\mathrm{tf},(p)}$. For such a θ the equality (56) with $n = 1$ combines with the containment (53) to imply that

$$|G|^{1+r}(\wedge_{i=1}^{i=r}\theta_{b_i})(\epsilon_{\underline{b}}) = |G|L^*_{S,T}(\check{\psi},0)\cdot \det_{\mathbb{C}}(|G|\cdot(R^{-1}_{L,S}\circ\theta)^{\psi}) \in \mathrm{Ann}_{\mathcal{O}}(\mathrm{Cl}^T_{S'}(L)_\psi)_{(p)}.$$

This implies, by choice of θ, that $|G|^{1+r}\epsilon_{\underline{b}}$ belongs to $\mathrm{Ann}_{\mathcal{O}}(\mathrm{Cl}^T_{S'}(L)_\psi)\cdot(\wedge^r_{\mathcal{O}}U^{\psi})_{(p)}$ and hence also that

$$\begin{aligned}|G|^{1+r}L^*_{S,T}(\check{\psi},0)\cdot(\wedge^r_{\mathcal{O}}X_{\psi,\mathrm{tf}})_{(p)} &= \mathcal{O}_{(p)}\cdot|G|^{1+r}L^*_{S,T}(\check{\psi},0)\wedge_{i=1}^{i=r}b_i = \mathcal{O}_{(p)}\cdot(\wedge^r_{\mathbb{C}}R_{L,S})(|G|^{1+r}\epsilon_{\underline{b}})\\ &\subseteq \mathrm{Ann}_{\mathcal{O}}(\mathrm{Cl}^T_{S'}(L)_\psi)\cdot(\wedge^r_{\mathbb{C}}\mathrm{R}_{L,S})(\wedge^r_{\mathcal{O}}U^{\psi})_{(p)}.\end{aligned}$$

Since this is true for all primes p it is a refinement of the containment [7, (44)] (in which $\mathrm{Cl}_{S'}(L)_\psi$ rather than $\mathrm{Cl}^T_{S'}(L)_\psi$ occurs).

This proof is also much simpler than that in loc. cit. since, amongst other things, it avoids any use of the constructions of Ritter and Weiss in [42].

13.3. **The proof of Corollary 10.5.** The proof of Theorem 10.1 shows that the claims of Corollary 10.5 will follow if we can show the stated hypotheses imply the validity of the p-component of $\mathrm{LTC}(L/K)$ after taking plus and minus parts respectively.

It is shown in [7, §9.1] that the p-component of $\mathrm{LTC}(L/K)$ is valid after taking plus parts provided that all of the following conditions are satisfied: the p-adic Stark Conjecture at $s = 1$ of Serre and Tate is valid for all p-adic characters of G; if p divides $|G|$, then the μ-invariant of L^{cyc}/L vanishes; the p-component of a certain element $T\Omega^{\mathrm{loc}}(\mathbb{Q}(0)_L,\mathbb{Z}[G])$ of $K_0(\mathbb{Z}[G],\mathbb{R}[G])$ vanishes.

Claim (i) therefore follows from the fact that the p-component of $T\Omega^{\mathrm{loc}}(\mathbb{Q}(0)_L,\mathbb{Z}[G])$ vanishes if the 'local epsilon constant conjecture' of Breuning [3] is valid for all extensions obtained by p-adically completing L/K (this follows from [3, Th. 4.1]) and that [3, Th. 3.6] shows the latter condition to be satisfied if p is tamely ramified in L/K.

It now only remains to note that [9, Cor. 3.8] shows that the hypothesis of Corollary 10.5(ii) combines with the observation made in Remark 15.5 below to imply the validity of the p-component of $\mathrm{LTC}(L/K)$ after taking minus parts.

Remark 13.4. Breuning's local epsilon constant conjecture has also been verified for certain classes of wildly ramified extensions of local fields (see, for example, Breuning [2] and Bley and Cobbe [1]). All such results can be combined with the above argument to derive corresponding generalizations of Corollary 10.5(i).

13.4. **The proof of Corollary 10.6.** We finally prove Corollary 10.6. To do this it suffices to show that the given hypotheses imply that the hypotheses of Corollary 10.5(ii) are satisfied.

The stated hypotheses imply the vanishing of $\mu_p(F)$ because (as is well-known) if $\mu_p(E)$ vanishes for some number field E, then Nakayama's Lemma implies that $\mu_p(E')$ vanishes for any p-power degree Galois extension E' of E.

The key point regarding the p-adic Gross-Stark Conjecture (for more details of which see Remark 15.5 below) is that if G is a finite group of the form $A\rtimes Q$ with A abelian and Q supersolvable, then for any irreducible $\mathbb{Q}_p^c$-valued character ρ of G there exists a subgroup

A_ρ of G which contains A and a linear $\mathbb{Q}_p^c$-valued character ρ' of A_ρ such that $\rho = \mathrm{Ind}_{A_\rho}^G(\rho')$ (for a proof of this fact see [46, II-22, Exercice] and the argument of [46, II-18]).

In particular, since the inductivity properties of p-adic Artin L-series implies that the Gross-Stark Conjecture is true for ρ is and only if it is true for ρ' we can assume (after replacing L/K by L/L^{A_ρ} and ρ by ρ') that ρ is linear.

In addition, one knows, by assumption, that the field L^{A_ρ} has at most one p-adic place which splits completely in L/L^+ and, under this hypothesis, the validity of the Gross-Stark Conjecture is known by results of Gross [27, Prop. 2.13], of Darmon, Dasgupta and Pollack [17] and of Ventullo [51].

Remark 13.5. We can now also give more details of the sort of examples discussed in Remark 10.8.

To do this we fix a totally real field E and a cyclic CM extension E' of E in which precisely one p-adic place v of E splits completely and no other place of E that ramifies in $E/\mathbb{Q}$ splits completely. We let k be any subfield of E for which the restriction of v has absolute degree one and write F for the Galois closure of E' over k. Then F is a CM field and for any faithful linear character ψ' of $G_{E'/E}$ the character $\psi := \mathrm{Ind}_{G_{F/E}}^{G_{F/k}}(\mathrm{Inf}_{G_{E'/E}}^{G_{F/E}}(\psi'))$ of $G_{F/k}$ is irreducible, totally odd, faithful and of degree $[E:k]$. Further, the functoriality of p-adic L-functions under induction and inflation combines with the result of [27, Prop. 2.13] and [51, Th. 1] to imply that ψ validates all of the hypotheses of [9, Cor. 3.3] with S taken to be the union of all places of k that are either archimedean, p-adic or ramify in E/k and v_1 the place of k below v.

Part IV: The p-adic theory

In the remainder of the article we fix an odd prime p and a finite CM Galois extension L of a totally real number field K with group G.

In this context we shall introduce a natural generalization of the 'p-adic Gross-Rubin-Stark elements' that are defined (in the setting of abelian extensions L/K) by the first author in [9] and a natural p-adic analogue of the zeta element of $\mathbb{G}_m$ relative to L/K from Definition 9.1.

Using these elements we then explain how the approach of [9] leads to a natural analogue for L/K of the theory we discussed in earlier sections in which the roles of Dirichlet regulators and Artin L-series are respectively replaced by Gross's p-adic regulators and the Deligne-Ribet p-adic Artin L-series of the totally even p-adic characters of G (as discussed by Greenberg in [25]).

We also prove that the central conjecture of this p-adic theory is valid modulo Iwasawa's conjecture on the vanishing of cyclotomic μ-invariants, and even in some interesting cases unconditionally (see Remark 16.3), and derive several explicit consequences of this result.

14. Higher non-abelian p-adic Stark elements

We write $\mathrm{Ir}_p(G)$ for the set of irreducible $\mathbb{C}_p$-valued characters of G. Then, with τ denoting the (unique) non-trivial element of G_{L/L^+} we write e_- for the central idempotent

$(1-\tau)/2$ of $\mathbb{Q}[G]$ and let $\mathrm{Ir}_p^{\pm}(G)$ denote the subsets of $\mathrm{Ir}_p(G)$ comprising characters for which one has $\chi(\tau)=\pm\chi(1)$. For any G-module M we also write $M^{\pm}$ for the G-submodule $\{m\in M:\tau(m)=\pm m\}$.

14.1. **Equivariant p-adic regulator maps and L-series.**

14.1.1. For each place w of L Gross defines in [26, §1] a local p-adic absolute value $||\cdot||_{w,p}$ on $L_w^{\times}$ by means of the commutative diagram

$$\begin{tikzcd} L_w^{\times} \arrow[rr, "||\cdot||_{w,p}"] \arrow[dr, "r_w"'] & & \mathbb{Z}_p^{\times} \\ & G_{L_w^{\mathrm{ab}}/L_w} \arrow[ur, "\chi_w^{-1}"'] & \end{tikzcd} \tag{57}$$

where L_w^{ab} denotes the maximal abelian extension of L_w in L_w^c, r_w the reciprocity map of local class field theory and χ_w the p-adic cyclotomic character.

For any finite set of places Σ of K that contains both S_K^{∞} and the set S_K^p of all p-adic places of K, we write

$$R_{L,\Sigma}^p:\mathcal{O}_{L,\Sigma,p}^{\times,-}\to Y_{L,\Sigma,p}^-$$

for the homomorphism of $\mathbb{Z}_p[G]$-modules that sends each u in $\mathcal{O}_{L,\Sigma}^{\times,-}$ to $\sum_{\pi\in\Sigma_L}\log_p||u||_{\pi,p}\cdot\pi$.

We write $\mathrm{Ir}_p^{\mathrm{ss}}(G)$ for the subset of $\mathrm{Ir}_p^-(G)$ comprising those characters ρ for which the induced homomorphism $R_{L,S}^{p,\rho}:=\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check\rho},\mathbb{C}_p\cdot R_{L,S}^p)$ is injective and we then obtain an idempotent of $\zeta(\mathbb{Q}_p[G])$ by setting

$$e_{\mathrm{ss}}:=\sum_{\rho\in\mathrm{Ir}_p^{\mathrm{ss}}(G)}e_\rho.$$

Remark 14.1.
(i) In [27, Conj. 1.15] Gross conjectures each homomorphism $R_{L,\Sigma}^{p,\rho}$ to be injective. If valid, this conjecture would imply $\mathrm{Ir}_p^{\mathrm{ss}}(G)=\mathrm{Ir}_p^-(G)$ and hence that $e_{\mathrm{ss}}=e_-$.
(ii) Set $\Sigma_0:=S_K^{\infty}\cup S_K^p$ and for each ρ in $\mathrm{Ir}_p^-(G)$ also $r_{\Sigma_0,\rho}:=\dim_{\mathbb{C}_p}(\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check\rho},\mathbb{C}_p\cdot Y_{L,S})$. If $r_{\Sigma_0,\rho}=0$, then the injectivity of $R_{L,S}^{p,\rho}$ is obvious. Excluding this case, however, the injectivity of $R_{L,S}^{p,\rho}$ has so far only been verified in the case that $r_{\Sigma_0,\rho}=1$ in which case Gross has shown (in [27, Prop. 2.13]) that it follows from Brumer's p-adic version of Baker's theorem. In general, one knows (from [9, Th. 4.2]) that this injectivity is equivalent to the semisimplicity of a natural Iwasawa module.

14.1.2. We now fix a $\mathbb{Z}_p$-order $\mathcal{A}_p$ in a semisimple $\mathbb{Q}_p$-algebra A_p and assume that $\mathcal{A}_p$ satisfies the conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ discussed in §4.2.2 (with R replaced by $\mathbb{Z}_p$).

We also assume to be given a finitely generated $(\mathcal{A}_p,\mathbb{Z}_p[G])$-bimodule Π_p which is free over $\mathcal{A}_p$ and satisfies the obvious (p-adic) analogues of the conditions (Π_2) and (Π_3) in §8.1. We write Υ_{Π_p} for the associated subset of $\mathrm{Ir}_p(G)$ (obtained by the same method as in Remark 8.1) and for each character ρ in Υ_{Π_p} we write W_ρ for the corresponding simple right $A_{\mathbb{C}_p}$-module.

We fix a surjective homomorphism of $\mathcal{A}_p$-modules

$$\pi_p : \Pi_p \otimes_{\mathbb{Z}_p[G]} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)_p \to Y_{\pi_p}$$

in which Y_{π_p} is free. For each character ρ in Υ_{Π_p} we set $r_{\pi_p}(\rho) := \dim_{\mathbb{C}_p}(W_\rho \otimes_{A_p} Y_{\pi_p})$.

For any such homomorphism π_p we then define a $\zeta(\mathbb{C}_p[G])$-valued meromorphic function of a p-adic variable z by setting

$$\theta^{\pi_p}_{L/K,S,T}(z) := \sum_\rho e_\rho \cdot z^{-r_{\pi_p}(\rho)} L_{p,S,T}(\check{\rho} \cdot \omega_K, z),$$

where ρ runs over $\mathrm{Ir}^-_p(G) \cap \Upsilon_{\Pi_p}$. Here ω_K denotes the Teichmüller character $G_K \to \mathbb{Z}_p^\times$ and for any finite set of places Σ of K containing $S_K^\infty \cup S_K^p$ and any representation ρ of $\mathrm{Ir}^+_p(G)$ we write $L_{p,\Sigma}(\rho, s)$ for the Σ-truncated Deligne-Ribet p-adic Artin L-series of ρ, as discussed by Greenberg in [25].

We write e_{π_p} for the idempotent of $\zeta(\mathbb{Q}_p[G])$ obtained by summing e_ρ over all the subset Υ_{π_p} of $\mathrm{Ir}^-_p(G)$ comprising characters ρ for which the space $\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\rho}}, \mathbb{C}_p {\cdot} \ker(\pi_p))$ vanishes and we then obtain a further idempotent of $\zeta(\mathbb{Q}_p[G])$ by setting

$$e^{\mathrm{ss}}_{\pi_p} := e_{\mathrm{ss}} \cdot e_{\pi_p}.$$

Lemma 14.2. *For each homomorphism π_p as above the following claims are valid.*

(i) *$\theta^{\pi_p}_{L/K,S,T}(z)$ is p-adic holomorphic at $z = 0$.*

(ii) *$\theta^{\pi_p}_{L/K,S,T}(0)$ belongs to $e^{\mathrm{ss}}_{\pi_p} \cdot \zeta(\mathbb{Q}_p[G])^\times$.*

Proof. Claim (i) is equivalent to asserting that for each ρ in $\mathrm{Ir}^+_p(G)$ the order of vanishing at $z = 0$ of $L_{p,S,T}(\check{\rho} \cdot \omega_K, z)$ is at least $r_\pi(\rho)$. The key point in proving this is that [9, Th. 3.1(i)] shows this order of vanishing to be at least $\dim_{\mathbb{C}_p}(\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\rho}}, \mathbb{C}_p \cdot X_{L,S}))$ and, given this fact, the required inequality is proved by the same argument as in Lemma 8.5.

Next we note it is clear $\theta^{\pi_p}_{L/K,S,T}(0)$ belongs to $\zeta(\mathbb{Q}_p[G])$. In addition, one has $e^{\mathrm{ss}}_{\pi_p} = \sum_\rho e_\rho$ where ρ runs over $\mathrm{Ir}^{\mathrm{ss}}_p(G) \cap \Upsilon_{\pi_p}$ and the argument of claim (i) implies $e_\rho \theta^{\pi_p}_{L/K,S,T}(0)$ vanishes unless ρ belongs to Υ_{π_p}.

To prove claim (ii) it is therefore enough to show that for each ρ in Υ_{π_p} the element $e_\rho \theta^{\pi_p}_{L/K,S,T}(0)$ is non-zero precisely when ρ belongs to $\mathrm{Ir}^{\mathrm{ss}}_p(G)$. This is in turn equivalent to proving that for each ρ in Υ_{π_p} the order of vanishing of $L_{p,S,T}(\check{\rho} \cdot \omega_K, z)$ at $z = 0$ is equal to $r_\pi(\rho)$ precisely when ρ belongs to $\mathrm{Ir}^{\mathrm{ss}}_p(G)$. This is true because for each ρ in Υ_{π_p} one has $r_\pi(\rho) = \dim_{\mathbb{C}_p}(\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\rho}}, \mathbb{C}_p \cdot X_{L,S}))$ and, by [9, Th. 3.1(ii)], one knows that the order of vanishing of $L_{p,S,T}(\check{\rho} \cdot \omega_K, z)$ at $z = 0$ is equal to $\dim_{\mathbb{C}_p}(\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\rho}}, \mathbb{C}_p \cdot X_{L,S}))$ precisely when ρ belongs to $\mathrm{Ir}^{\mathrm{ss}}_p(G)$. $\square$

14.2. **The definition of higher non-abelian p-adic Stark elements.** For each homomorphism of $\mathcal{A}_p$-modules π_p as above we write r_{π_p} for the $\mathcal{A}_p$-rank of Y_{π_p} and then choose an ordered $\mathcal{A}_p$-basis $\underline{b} = \{b_i\}_{1 \le i \le r_{\pi_p}}$ of Y_{π_p}.

For any such basis Lemma 14.2(ii) implies that the element $\theta^{\pi_p}_{L/K,S,T}(0) \cdot \wedge_{i=1}^{i=r_{\pi_p}} b_i$ belongs to $e^{\mathrm{ss}}_{\pi_p}(\bigwedge^{r_{\pi_p}}_{A_p}(\mathbb{Q}_p \cdot \Pi_p \otimes_{A_p} Y_{\pi_p})$ and so enables us to make the following definition.

Definition 14.3. *For any basis $\underline{b}$ as above the 'higher non-abelian p-adic Stark element (relative to $\underline{b}$)' is the unique element $\epsilon_{\underline{b}}^{\pi_p}$ of $e_{\pi_p}^{\rm ss}(\bigwedge_{A_p}^{r_{\pi_p}}(\mathbb{Q}_p\cdot\Pi_p\otimes_{\mathbb{Z}_p[G]}\mathcal{O}_{L,S,p}^{\times,-}))$ which satisfies*

$$\lambda_{L,S}^{\pi_p}(\epsilon_{\underline{b}}^{p,\pi}) = \theta_{L/K,S,T}^{\pi_p}(0)\cdot\wedge_{i=1}^{i=r}b_i,$$

where $\lambda_{L,S}^{\pi_p}$ denotes the isomorphism $e_{\pi_p}^{\rm ss}(\mathbb{Q}_p\cdot\Pi_p\otimes_{\mathbb{Q}_p[G]}\lambda_{L,S}^p)$.

Example 14.4. For each place v in S we fix a place w_v of L above v and write $\overline{w_v}$ for its complex conjugate. We also write V_L for the subset of S comprising places which split completely in L/K, set $r_L := |V_L|$ and note that, as L is a CM extension of K, the $\mathbb{Z}_p[G]^-$-module $(Y_{L,V_L,p})^-$ is free of rank r_L with basis $\underline{b} := \{w_v - \overline{w_v} : v \in V_L\}$. Finally we write π_p for the natural surjective homomorphism of $\mathbb{Z}_p[G]^-$-modules $\mathcal{S}_{S,T}^{\rm tr}(\mathbb{G}_m/L)_p^- \to (Y_{L,V_L,p})^-$.

Then the data $(\underline{b},\pi_p)$ is suitable to be used in Definition 14.3 (with $\mathcal{A}_p = \Pi_p = \mathbb{Z}_p[G]^-$) and so we may set

$$\epsilon_{L/K,S,T}^{p} := \epsilon_{\underline{b}}^{\pi_p}.$$

We refer to this element as the 'non-abelian (p-adic) Gross-Rubin-Stark element' with respect to the data $L/K,S,T$ and note that it constitutes a natural generalization of the Gross-Rubin-Stark elements that are defined (in the setting of abelian extensions) by the first author in [9, §3.5]. The conjectural link between these elements and the non-abelian Rubin-Stark elements defined in Example 8.9 is described in Proposition 15.6(i) below.

In this context we also define a 'r_L-th order non-abelian p-adic Stickelberger series' by setting

$$\theta_{L/K,S,T}^{p,(r_L)}(z) := \theta_{L/K,S,T}^{\pi_p}(z) = \sum_{\rho\in{\rm Ir}_p^-(G)} e_\rho\cdot z^{-\rho(1)r_L}L_{p,S,T}(\check{\rho}\cdot\omega_K,z).$$

15. Statement of the conjectures

15.1. **Non-abelian p-adic zeta elements and determinant modules.** In the sequel we set $\mathbb{Z}_p[G]^{\rm ss} := \mathbb{Z}_p[G]e_{\rm ss}$.

For any object C of $D^{\rm lf,0}(\mathbb{Z}_p[G])$ we write $C^{\rm ss}$ for the associated object $\mathbb{Z}_p[G]^{\rm ss}\otimes_{\mathbb{Z}_p[G]}^{\mathbb{L}} C$ of $D^{\rm lf,0}(\mathbb{Z}_p[G]^{\rm ss})$ and we note that in each degree i there is a canonical identification of $\mathbb{Q}_p$-spaces $\mathbb{Q}_p\cdot H^i(C^{\rm ss}) = e_{\rm ss}(\mathbb{Q}_p\cdot H^i(C))$.

15.1.1. We write κ_w for the residue field of each place w in T_L. Then for each such place w the complex $R\Gamma_{\rm \acute{e}t}(\kappa_w,\mathbb{Z}_p(1))$ is acyclic outside degree one and there exists a natural morphism in $D^{\rm lf,0}(\mathbb{Z}_p[G])$

$$R\Gamma_{\rm \acute{e}t}(\mathcal{O}_{L,S},\mathbb{Z}_p(1)) \xrightarrow{\theta_{L,T}} \bigoplus_{w\in T_L} R\Gamma_{\rm \acute{e}t}(\kappa_w,\mathbb{Z}_p(1)) \tag{58}$$

for which $H^1(\theta_{L,T})$ is induced by the natural projection maps $\mathcal{O}_{L,S}^\times \to \kappa_w^\times$ (for more details see [9, Lem. 4.3]).

We write $R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ for the mapping fibre of $\theta_{L,T}$ and $H_T^i(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ for the cohomology of this complex in degree i and we note that the long exact cohomology sequence associated to the definition of mapping fibre induces canonical identifications

$$H^i_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))^- \cong \begin{cases} \mathcal{O}^{\times,-}_{L,S,T,p}, & \text{if } i=1 \\ \mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_m/L)^-_p, & \text{if } i=2 \\ 0, & \text{otherwise.} \end{cases} \tag{59}$$

In view of this description, the definition of the idempotent $e_{\rm ss}$ implies that the homomorphism $R^p_{L,S}$ restricts to give an isomorphism of $\mathbb{Q}_p[G]^{\rm ss}$-modules

$$R^{p,{\rm ss}}_{L,S} : \mathbb{Q}_p\cdot H^1(R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))^{\rm ss}) \cong \mathbb{Q}_p\cdot H^2(R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))^{\rm ss}).$$

This isomorphism then combines with the general construction (22) to give a canonical isomorphism of $\zeta(\mathbb{Q}_p[G])^{\rm ss}$-modules

$$\vartheta_{\mathbb{Q}_p\otimes_{\mathbb{Z}_p}R^{p,{\rm ss}}_{L,S}} : {\rm det}_{\mathbb{Q}_p[G]^{\rm ss}}(\mathbb{Q}_p\cdot R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))^{\rm ss}) \to \zeta(\mathbb{Q}_p[G]^{\rm ss})$$

and in the sequel we denote this isomorphism by $\lambda^p_{L,S}$.

Finally we note that the leading term $\theta^{p,*}_{L/K,S,T}(0)$ at $z=0$ of the function

$$\theta^p_{L/K,S,T}(z) := \sum_{\rho\in{\rm Ir}_p^-(G)} e_\rho\cdot L_{p,S,T}(\check\rho\cdot\omega_K,z)$$

belongs to $\zeta(\mathbb{Q}_p[G]^-)^\times$ and hence that $\theta^{p,*}_{L/K,S,T}(0)e_{\rm ss}$ belongs to $\zeta(\mathbb{Q}_p[G]^{\rm ss})^\times$.

Definition 15.1. The 'p-adic zeta element of $\mathbb{G}_m$ relative to $L/K, S$ and T' is the preimage $z^p_{L/K,S,T}$ in $\mathbb{Q}_p\cdot{\rm det}_{\mathbb{Z}_{(p)}[G]^{\rm ss}}(R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))^{\rm ss})$ of the element $\theta^{p,*}_{L/K,S,T}(0)e_{\rm ss}$ under the isomorphism $\lambda^p_{L,S}$.

These elements constitute a natural p-adic analogue of the zeta elements from Definition 9.1 and the central conjecture that we make concerning them is the following analogue of Conjecture 9.2.

Conjecture 15.2. $\xi(\mathbb{Z}_p[G]^{\rm ss})\cdot z^p_{L/K,S,T} = {\rm det}_{\mathbb{Z}_p[G]^{\rm ss}}(R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))^{\rm ss})$.

Remark 15.3.
(i) Conjecture 15.2 constitutes a natural generalization of a conjecture formulated for abelian extensions L/K by the first author in [9, Conj. 3.6].
(ii) For any normal subgroup H of G for which the corresponding intermediate field $F := L^H$ is CM, there is a natural descent isomorphism in $D^{{\rm lf},0}(\mathbb{Z}_p[G/H]^{\rm ss})$ of the form

$$\mathbb{Z}_p[G/H]^{\rm ss}\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]^{\rm ss}} R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))^{\rm ss} \cong R\Gamma_T(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\rm ss}.$$

It is straightforward to show that the p-adic zeta elements $z^p_{L/K,S,T}$ and $z^p_{F/K,S,T}$ satisfy the same functoriality property with respect to this isomorphism as was observed for the corresponding zeta elements in Lemma 9.8. The validity of Conjecture 15.2 for L/K therefore implies its validity for F/K.

15.1.2. In this section we explain the precise connections between Conjectures 9.2 and 15.2 and between the Gross-Rubin-Stark and Rubin-Stark elements that are respectively defined in Examples 14.4 and 8.9.

Before stating these results we give a precise statement of the weak p-adic Gross-Stark Conjecture.

To do this we fix an injective homomorphism of G-modules $\phi : \mathcal{O}^\times_{F,S} \to X_{F,S}$. Noting that the scalar extension $\mathbb{C}_p \otimes \phi$ is bijective, for each ρ in $\mathrm{Ir}^-_p(G)$ we then define a $\mathbb{C}_p$-valued $\mathscr{L}$-invariant by setting

$$\mathscr{L}_S(\phi,\rho) := \det_{\mathbb{C}_p}((\mathbb{C}_p \otimes_{\mathbb{Z}} \phi)^{-1} \circ (\mathbb{C}_p \otimes_{\mathbb{Z}_p} R^p_{L,S}) \mid \mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\rho}}, \mathbb{C}_p \cdot \mathcal{O}^\times_{L,S})).$$

In a similar way, for each field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ and each ρ in $\mathrm{Ir}^-_p(G)$ we define a $\mathbb{C}$-valued regulator by setting

$$R_S(\phi, \rho^{j^{-1}}) := j^{-1}(\det_{\mathbb{C}_p}((\mathbb{C}_p \otimes_{\mathbb{Z}} \phi)^{-1} \circ (\mathbb{C}_p \otimes_{\mathbb{R},j} R_{L,S}) \mid \mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\rho}}, \mathbb{C}_p \cdot \mathcal{O}^\times_{L,S}))).$$

The following conjecture is formulated by Gross in [27, Conj. 2.12] and is commonly referred to as the 'weak p-adic Gross-Stark Conjecture'.

Conjecture 15.4. *Fix a character ρ in $\mathrm{Ir}^-_p(G)$, an isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$ and an injective homomorphism of G-modules $\phi : \mathcal{O}_{F,S} \to X_{F,S}$. Then one has*

$$L^{r_{S,\rho}}_{p,S,T}(\check{\rho}\cdot\omega_K, 0)\mathscr{L}_S(\phi,\rho) = j(L^{r_{S,\rho}}_{S,T}(\check{\rho}^{j^{-1}}, 0) R_\Sigma(\phi, \rho^{j^{-1}})),$$

with $r_{S,\rho} := \dim_{\mathbb{C}_p}(\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\rho}}, \mathbb{C}_p \cdot \mathcal{O}^\times_{L,S}))$.

Remark 15.5. The equality of Conjecture 15.4 and the prediction that $e_{\mathrm{ss}} = e_-$ (as recalled in Remark 14.1(i)) together constitute the 'p-adic Gross-Stark Conjecture' that occurs in the statement of Corollary 10.5(ii).

We can now state the main result of this section.

Proposition 15.6. *If Conjecture 15.4 is valid for L/K, then so are the following claims.*

(i) $\epsilon^p_{L/K,S,T} = e_{\mathrm{ss}}(\epsilon^{V_L}_{L/K,S,T})$ *with V_L the set of places splitting completely in L/K.*

(ii) *If Conjecture 15.2 is also valid for L/K, then $e_{\mathrm{ss}} \cdot z_{L/K,S,T}$ is a primitive basis of* $\det_{\mathbb{Z}_p[G]}(\mathbb{Z}_p[G]^{\mathrm{ss}} \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C_{L,S,T})$.

Proof. We set $r = r_L$, write $\mathrm{Ir}^{-,r}(G)$ for the subset of $\mathrm{Ir}^-(G)$ comprising characters for which $L_{S,T}(\check{\psi}, z)$ vanishes to order $r\cdot\psi(1)$ at $z = 0$ and write e_r for the associated idempotent $\sum_{\psi\in\mathrm{Ir}^{-,r}(G)} e_\psi$ of $\zeta(\mathbb{Q}[G])$.

Then the explicit definitions of $\epsilon^p_{L/K,S,T}$ and $\epsilon^{V_L}_{L/K,S,T}$ combine with the arguments of Lemma 8.5 and 14.2(ii) to imply $\epsilon^{V_L}_{L/K,S,T} = e_r(\epsilon^{V_L}_{L/K,S,T})$ and $\epsilon^p_{L/K,S,T} = e_{\mathrm{ss}}e_r(\epsilon^p_{L/K,S,T})$, whilst a direct comparison of these definitions shows that for any fixed isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$ one has

$$\epsilon^p_{L/K,S,T} = v \cdot e_r e_{\mathrm{ss}}(\epsilon^{V_L}_{L/K,S,T}) \tag{60}$$

with v the element

$$(\theta^{p,(r)}_{L/K,S,T}(0)e_{\mathrm{ss}}) j_*(\theta^{(r)}_{L/K,S,T}(0)e_r e_{\mathrm{ss}})^{-1} \cdot \mathrm{Nrd}_{\mathbb{C}_p[G]^{\mathrm{ss}}}((\mathbb{C}_p \otimes_{\mathbb{Q}_p} \lambda^p_{L,S}) \circ e_{\mathrm{ss}}(\mathbb{C}_p \otimes_{\mathbb{R},j} \lambda_{L,S})^{-1})$$

of $\zeta(e_r\mathbb{C}_p[G]^{\rm ss})^\times$, where j_* denotes the ring isomorphism $\mathbb{C}[G]\cong\mathbb{C}_p[G]$ induced by j.

Set $\mathrm{Ir}_p^{-,r}(G):=\{\psi^{j^{-1}}:\psi\in\mathrm{Ir}^{-,r}(G)\}$. Then for each ρ in $\mathrm{Ir}_p^{\rm ss}(G)\cap\mathrm{Ir}_p^{-,r}(G)$ one has both

$$e_\rho(\theta^{p,(r)}_{L/K,S,T}(0)e_{\rm ss})j_*(\theta^{(r)}_{L/K,S,T}(0)e_re_{\rm ss})^{-1}=(L^{r_{S,\rho}}_{p,S,T}(\check\rho\omega_K,0)\cdot j(L^{r_{S,\rho}}_{S,T}(\check\rho^{j^{-1}},0))^{-1})e_\rho$$

and

$$\begin{aligned}
&e_\rho(\mathrm{Nrd}_{\mathbb{C}_p[G]^{\rm ss}}((\mathbb{C}_p\otimes_{\mathbb{Q}_p}\lambda^p_{L,S})\circ e_{\rm ss}(\mathbb{C}_p\otimes_{\mathbb{R},j}\lambda_{L,S})^{-1}))\\
=&\det_{\mathbb{C}_p}((\mathbb{C}_p\otimes_{\mathbb{R},j}R_{L,S})^{-1}\circ(\mathbb{C}_p\otimes_{\mathbb{Z}_p}R^p_{L,S})\mid\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check\rho},\mathbb{C}_p\cdot\mathcal{O}^\times_{L,S}))e_\rho\\
=&\det_{\mathbb{C}_p}((\mathbb{C}_p\otimes_{\mathbb{R},j}R_{L,S})^{-1}\circ(\mathbb{C}_p\otimes_{\mathbb{Z}}\phi)\mid\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check\rho},\mathbb{C}_p\cdot\mathcal{O}^\times_{L,S}))\\
&\qquad\times\det_{\mathbb{C}_p}((\mathbb{C}_p\otimes_{\mathbb{Z}}\phi)^{-1}\circ(\mathbb{C}_p\otimes_{\mathbb{Z}_p}R^p_{L,S})\mid\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check\rho},\mathbb{C}_p\cdot\mathcal{O}^\times_{L,S}))\\
=&j(R_S(\phi,\rho^{j^{-1}}))^{-1}\mathscr{L}_S(\phi,\rho)e_\rho.
\end{aligned}$$

Taken together with the last two displayed formulas the validity of Conjecture 15.4 for L/K implies that

$$v=\sum_{\rho\in\mathrm{Ir}_p^{\rm ss}(G)\cap\mathrm{Ir}_p^{-,r}(G)}e_\rho(v)=\sum_{\rho\in\mathrm{Ir}_p^{\rm ss}(G)\cap\mathrm{Ir}_p^{-,r}(G)}e_\rho=e_re_{\rm ss}$$

and so claim (i) follows directly from the equality (60).

To prove claim (ii) we recall (from [9]) that the Artin-Verdier Duality Theorem gives a canonical isomorphism in $D^{\rm lf,0}(\mathbb{Z}_p[G])$ of the form

$$R\Gamma_T(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^-\cong(\mathbb{Z}_p\otimes_{\mathbb{Z}}C_{L,S,T})^-\tag{61}$$

which in turn induces an identification of $\xi(\mathbb{Z}_{(p)}[G]^{\rm ss})$-lattices

$$\det_{\mathbb{Z}_{(p)}[G]^{\rm ss}}(R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))^{\rm ss})=\det_{\mathbb{Z}_p[G]^{\rm ss}}(\mathbb{Z}_p[G]^{\rm ss}\otimes^{\mathbb{L}}_{\mathbb{Z}[G]}C_{L,S,T}).\tag{62}$$

Taking account of this identification, Proposition 6.19(ii) implies the claimed result will follow if we can show Conjecture 15.4 implies that $z^p_{L/K,S,T}$ and $e_{\rm ss}\cdot z_{L/K,S,T}$ differ by multiplication by an element of $\mathrm{Nrd}_{\mathbb{Q}_p[G]^{\rm ss}}(K_1(\mathbb{Z}_p[G]^{\rm ss}))$.

Now an explicit comparison of the definitions of $z^p_{L/K,S,T}$ and $e_{\rm ss}\cdot z_{L/K,S,T}$ shows that

$$z^p_{L/K,S,T}=v'\cdot(1\otimes_{\mathbb{R},j}(e_{\rm ss}\cdot z_{L/K,S,T}))$$

with v' the element

$$(\theta^{p,*}_{L/K,S,T}(0)e_{\rm ss})j_*(\theta^*_{L/K,S,T}(0)e_{\rm ss})^{-1}\cdot\mathrm{Nrd}_{\mathbb{C}_p[G]^{\rm ss}}((\mathbb{C}_p\otimes_{\mathbb{Q}_p}\lambda^p_{L,S})\circ e_{\rm ss}(\mathbb{C}_p\otimes_{\mathbb{R},j}\lambda_{L,S})^{-1})$$

of $\zeta(\mathbb{C}_p[G]^{\rm ss})^\times$.

In addition, the same computation as above shows that Conjecture 15.4 implies $v'=e_{\rm ss}$ and this element clearly belongs to $\mathrm{Nrd}_{\mathbb{Q}_p[G]^{\rm ss}}(K_1(\mathbb{Z}_p[G]^{\rm ss}))$, as required. $\square$

15.2. **p-adic Stark elements, Fitting invariants and reciprocity maps.** The same sort of arguments that are used to derive explicit consequences of Conjecture 9.2 in §9.2 can be used to show that Conjecture 15.2 implies a range of explicit consequences concerning the elements $\epsilon_{\underline{b}}^{\pi_p}$ from Definition 14.3. For brevity, however, we only consider these consequences in the setting of Example 14.4.

In claim (iii) of the next result we use the strict family of pre-envelopes $\mathcal{P} = \mathcal{P}_{L,S,T}$ and surjective bundle of G-module morphisms $\pi : \mathcal{P} \to \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)$ that are constructed in Proposition 7.1. We also recall that in the setting of Example 14.4 V_F is equal to the set of places in S which split completely in F/K.

Conjecture 15.7. *For each Galois CM extension F of K in L set $r_F := |V_F|$.*

(i) *The transpose Selmer group $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/L)^-_p$ has a quadratic $\mathbb{Z}_p[G]$-module presentation $h^{p,-}_{L/K,S,T}$ for which*

$$\xi(\mathbb{Z}_p[G])\{(\wedge_{i=1}^{i=r}\varphi_i)(\epsilon^p_{L/K,S,T}) : \varphi_i \in \mathrm{Hom}_{\mathbb{Z}_p[G]}(\mathcal{O}^{\times,-}_{L,S,T,p}, \mathbb{Z}_p[G])\} = \mathrm{FI}^{r_L}_{\mathbb{Z}_p[G]}(h^{p,-}_{L/K,S,T}).$$

(ii) *For any a in $\mathfrak{A}(G)$ and $(\varphi_i)_{1\le i\le r_L}$ in $\mathrm{Hom}_G(\mathcal{O}^\times_{L,S,T}, \mathbb{Z}[G])^{r_L}$ one has*

$$a \cdot (\wedge_{i=1}^{i=r_L}\varphi_i)(\epsilon^p_{L/K,S,T}) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S_K^\infty \cup V_L}(L))_p.$$

(iii) *For all normal subgroups H of G, with $E = L^H$, one has*

$$\rho_{\pi,H}(\epsilon^p_{L/K,S,T}) = \mathrm{Rec}^{\mathcal{P}}_H(\epsilon^p_{E/K,S,T})$$

with $\mathcal{P} = \mathcal{P}_{L,S,T}$ and $\rho_{\pi,H}$ the natural projection $\mathrm{F}_{\pi_E}(\bigcap^r_G \mathcal{O}^\times_{L,S,T}) \to \mathrm{F}_{\pi_E}(\bigcap^r_G \mathcal{P})_H$.

Remark 15.8. If L/K validates the p-adic Gross-Stark conjecture (as recalled in Remark 15.5), then Proposition 15.6(i) implies $\epsilon^p_{L/K,S,T} = e_-(\epsilon^{V_L}_{L/K,S,T})$ and hence that Conjecture 15.7 recovers the 'minus components' of the p-parts of the properties of Rubin-Stark elements that are derived from Conjecture 9.2 in Corollary 9.10.

16. The main result

16.1. **Statement of the main result.** In the next result we state the main evidence that we can currently offer in support of Conjectures 15.2 and 15.7.

This result relies on an explicit cohomological construction of approximations to the Gross-Rubin-Stark elements defined above and shows, in particular, that if the p-adic μ-invariant $\mu_p(L)$ of L^{cyc}/L vanishes (as is conjectured to be the case by Iwasawa), then Conjecture 15.2 is valid and Conjecture 15.7 is valid provided that $e_{\mathrm{ss}} = e_-$ (as is conjectured by Gross).

In claim (ii) of this result we use the notation of Example 14.4.

Theorem 16.1. *If p is such that $\mu_p(L)$ vanishes, then the following claims are valid.*

(i) *Conjecture 15.2 is valid.*

(ii) *For each Galois CM extension E of K in L, with $H := G_{L/E}$, there exists an element $\epsilon^{p,\mathrm{coh}}_{E/K,S,T}$ of $\bigwedge^{r_F}_{\mathbb{Q}_p[G/H]}(\mathbb{Q}_p \cdot \mathcal{O}^\times_{E,S})^-$ which has all of the following properties.*

(a) *One has*

$$(\bigwedge\nolimits^{r_E}_{\mathbb{Q}_p[G/H]}(\mathbb{Q}_p\cdot R^p_{E,S}))(\epsilon^{p,\mathrm{coh}}_{E/K,S,T})=\theta^{p,(r_E)}_{E/K,S,T}(0)\cdot\wedge_{v\in V_E}(w_{v,E}-\overline{w_{v,E}})$$

and hence $\epsilon^p_{E/K,S,T}=e_{\mathrm{ss}}\cdot\epsilon^{p,\mathrm{coh}}_{E/K,S,T}$.

(b) $\epsilon^{p,\mathrm{coh}}_{E/K,S,T}$ *belongs to* $\mathrm{F}_{\pi_{E,p}}(\bigcap^{r_E}_{\mathbb{Z}_p[G/H]}\mathcal{O}^{\times,-}_{E,S,T,p})$.

(c) *The transpose Selmer group* $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_m/E)^-_p$ *has a quadratic* $\mathbb{Z}_p[G/H]$*-module presentation* $h^{p,-}_{E/K,S,T}$ *for which*

$$\xi(\mathbb{Z}_p[G/H])\{(\wedge_{i=1}^{i=r_E}\varphi_i)(\epsilon^{p,\mathrm{coh}}_{E/K,S,T}):\varphi_i\in\mathrm{Hom}_{\mathbb{Z}_p[G/H]}(\mathcal{O}^{\times,-}_{E,S,T,p},\mathbb{Z}_p[G/H])\}=\mathrm{FI}^{r_E}_{\mathbb{Z}_p[G/H]}(h^{p,-}_{E/K,S,T}).$$

(d) *For any* a *in* $\mathfrak{A}(G/H)_p$ *and* $(\varphi_i)_{1\le i\le r_E}$ *in* $\mathrm{Hom}_{\mathbb{Z}_p[G/H]}(\mathcal{O}^{\times,-}_{E,S,T,p},\mathbb{Z}_p[G/H])^{r_E}$ *one has*

$$a\cdot(\wedge_{i=1}^{i=r_E}\varphi_i)(\epsilon^{p,\mathrm{coh}}_{E/K,S,T})\in\mathrm{Ann}_{\mathbb{Z}_p[G/H]}(\mathrm{Cl}^T_{S_K^\infty\cup V_E}(E)).$$

(e) *There exists an element* ζ_p^{coh} *of* $\det_{\mathbb{Z}_p[G]}(R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))^-)$ *with the property that for every* E *and* H *as above one has*

$$\Delta^{H,p,-}_{L,S,T}(\zeta_p^{\mathrm{coh}})=\epsilon^{p,\mathrm{coh}}_{E/K,S,T}$$

where we write $\Delta^{H,p,-}_{L,S,T}$ *for the composite of the identification (62) and the homomorphism* $(\Delta^{H,p}_{L,S,T})^-$ *defined in §7.5.*

(f) *For every* F *and* H *as above one has*

$$\rho_{\pi,H}(\epsilon^{p,\mathrm{coh}}_{L/K,S,T})=\mathrm{Rec}^{\mathcal{P}}_H(\epsilon^{p,\mathrm{coh}}_{E/K,S,T})$$

with $\rho_{\pi,H}$ *the natural projection* $\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G\mathcal{O}^\times_{L,S,T})\to\mathrm{F}_{\pi_E}(\bigcap^{r_L}_G\mathcal{P}_{L,S,T})_H$.

Remark 16.2.
(i) As a refinement of Theorem 16.1(ii)(a) our methods will show that, given the family $\mathcal{P}_{L,S,T,p}$, the elements $\epsilon^{p,\mathrm{coh}}_{E/K,S,T}$ can be collectively specified uniquely up to scalar multiplication by a single element in the kernel of the natural projection $K_1(\mathbb{Z}_p[G])\to K_1(\mathbb{Z}_p[G]^{\mathrm{ss}})$.
(ii) The argument of Proposition 7.3(iii) implies Theorem 16.1(ii)(e) constitutes a natural generalization of a result proved (for abelian extensions) in [9, Cor. 3.12].

Remark 16.3. In [32] Johnston and Nickel identify families of extensions L/K for which one can prove the main conjecture of non-commutative p-adic Iwasawa theory for the extension L^{cyc}/K without assuming that $\mu_p(L)$ vanishes (or that p does not divide $[L:K]$). In all such cases our method shows that the assertions of Theorem 16.1, and in particular therefore Conjecture 15.2, are valid unconditionally.

16.2. **The proof of Theorem 16.1.** To prove claim (i) we note that the same argument as used to prove Theorem 10.1 shows that Conjecture 15.2 is equivalent to an equality in $K_0(\mathbb{Z}_p[G]^{\mathrm{ss}},\mathbb{Q}_p[G]^{\mathrm{ss}})$ of the form

$$\delta_{\mathbb{Z}_p[G]^{\mathrm{ss}}}(\theta^{p,*}_{L/K,S,T}(0)e_{\mathrm{ss}})=\chi_{\mathbb{Z}_p[G]^{\mathrm{ss}}}(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\mathrm{ss}},R^{p,\mathrm{ss}}_{L/K})$$

where $\delta_{\mathbb{Z}_p[G]^{\rm ss}}$ denotes the composite homomorphism

$$\zeta(\mathbb{Q}_p[G]^{\rm ss})^\times \to K_1(\mathbb{Q}_p[G]^{\rm ss}) \to K_0(\mathbb{Z}_p[G]^{\rm ss}, \mathbb{Q}_p[G]^{\rm ss})$$

where the first arrow is the inverse of the (bijective) reduced norm map and the second is the canonical connecting homomorphism of relative K-theory and $\chi_{\mathbb{Z}_p[G]^{\rm ss}}(-,-)$ is the refined Euler characteristic discussed in §11.2. Given this equivalence, claim (i) is an immediate consequence of [9, Th. 3.6].

To prove claim (ii) we assume (as we may) that the ordering of S made in §7.1 is such that v_0 is archimedean. In particular, with this choice v_0 doesn't split completely in any CM extension E of K and so the subset V_E of S defined in Example 14.4 coincides with the set defined (using the same notation) in §7.1.

We then fix a pair $(\varpi, \underline{b})$ belonging to the class $\mathcal{C}_{S,T}(L/K)$ that is defined in Lemma 7.7 and used in §7.2.3 to construct the family $\mathcal{P}_{L,S,T}$.

We recall that ϖ is a surjective homomorphism of G-modules $\varpi : P \to \mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_m/L)$, where P is free of rank d, and $\underline{b} = \{b_i\}_{1\le i\le d}$ is an ordered G-basis of P. We set $\varpi_p^- := (\mathbb{Z}_p \otimes_{\mathbb{Z}} \varpi)^-$ and for each index i also $b_i^- := (1-\tau)b_i$.

Just as in §12.1 we then fix an exact sequence of $\mathbb{Z}_p[G]^-$-modules

$$0 \to \mathcal{O}^{\times,-}_{L,S,T,p} \to P_p^- \xrightarrow{\phi_p^-} P_p^- \xrightarrow{\varpi_p^-} \mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_m/L)_p^- \to 0$$

which, in view of the isomorphism (61), induces an identification between $R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))^-$ and the complex $P_p^- \xrightarrow{\phi_p^-} P_p^-$, where the first module is placed in degree one.

Now claim (i) combines with the argument of Proposition 11.2(ii) to imply the existence of an element u_p' of $K_1(\mathbb{Z}_p[G]^{\rm ss})$ which satisfies

$$\theta^{p,*}_{L/K,S,T}(0)e_{\rm ss} = \mathrm{Nrd}_{\mathbb{Q}_p[G]^{\rm ss}}(u_p')\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\langle \phi_p^-, \iota_1, \iota_2\rangle) \tag{63}$$

for any choice of $\mathbb{Q}_p[G]$-sections ι_1 and ι_2 to ϕ_p^- and ϖ_p^-. By Bass's Theorem (cf. [34, Chap. 7, (20.9)]) we can then fix a pre-image u_p of u_p' under the natural projection map $K_1(\mathbb{Z}_p[G]) \to K_1(\mathbb{Z}_p[G]^{\rm ss})$.

As in §7.3.1, we write Z_E for the subset of $[n]$ comprising the r_E integers i for which v_i belongs to V_E (and so splits completely in E/K). We then set

$$\epsilon^{p,\rm coh}_{E/K,S,T} := \mathrm{Nrd}_{\mathbb{Q}_p[G]}(u_p)\cdot(\bigwedge_{a\in[d]\setminus Z_E}(\phi_p^-)_a)(\bigwedge_{c\in[d]} T_H(b_c^-)) \in {\bigwedge}^{r_E}_{\mathbb{Q}_p[G/H]}(\mathbb{Q}_p\cdot P_p^{H,-}).$$

Noting that $(1-e_{\rm ss})({\bigwedge}^{r_E}_{\mathbb{Q}_p[G/H]}(\mathbb{Q}_p\cdot R^p_{E,S}))$ is the zero map one has

$$({\bigwedge}^{r_E}_{\mathbb{Q}_p[G/H]}(\mathbb{Q}_p\cdot R^p_{E,S}))(\epsilon^{p,\rm coh}_{L/K,S,T}) = ({\bigwedge}^{r_E}_{\mathbb{Q}_p[G/H]}(\mathbb{Q}_p\cdot R^p_{E,S}))(e_{\rm ss}\cdot\epsilon^{p,\rm coh}_{L/K,S,T})$$

and, given this fact, the equality (63) combines with the argument used in the proof of Theorem 12.2 to imply the displayed equality in claim (ii)(a).

The final assertion of claim (ii)(a) then follows directly from the fact that the definition of $e_{\rm ss}$ implies that the map $\lambda^p_{E,S} = e_{\rm ss}({\bigwedge}^{r_E}_{\mathbb{Q}_p[G/H]}(\mathbb{Q}_p\cdot R^p_{E,S}))$ is injective.

Given the above explicit definition of the elements $\epsilon^{p,\rm coh}_{E/K,S,T}$ the remaining assertions of claim (ii) can be proved by mimicking the arguments used in §12 and §13.1 to deduce

Corollary 9.10 from the explicit formula for the Rubin-Stark elements $\epsilon^{v_0}_{E/K,S,T}$ that is given by Theorem 12.2. However, since this process is routine, we shall for brevity leave the detailed derivation to an interested reader.

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