ON NON-COMMUTATIVE IWASAWA THEORY AND DERIVATIVES OF EULER SYSTEMS

TO THE MEMORY OF JAN NEKOVÁŘ, FRIEND AND COLLEAGUE

Abstract. We use the theory of reduced determinant functors of Burns and Sano [24] to give a new, computationally useful, description of the relative K_0-groups of orders in finite dimensional separable algebras that need not be commutative. By combining this approach with a canonical generalization to non-commutative algebras of the notion of 'zeta element' introduced by Kato [51], we then formulate, for each odd prime p, a natural main conjecture of non-commutative p-adic Iwasawa theory for $\mathbb{G}_m$ over arbitrary number fields. This conjecture predicts a simple relation between a canonical Rubin-Stark non-commutative Euler system that we introduce and the compactly supported p-adic cohomology of $\mathbb{Z}_p$ and is shown to simultaneously extend both the higher rank (commutative) main conjecture for $\mathbb{G}_m$ formulated by Burns, Kurihara and Sano [19] and the K-theoretical formalism of main conjectures in non-commutative Iwasawa theory developed by Ritter and Weiss [72] and by Coates, Fukaya, Kato, Sujatha and Venjakob [27]. In particular, via these links we obtain strong evidence in support of the conjecture in the setting of Galois CM extensions of totally real fields. Our approach also leads to the formulation over arbitrary number fields of a precise conjectural 'higher derivative formula' for the Rubin-Stark non-commutative Euler system that is shown to recover upon appropriate specialisation the classical Gross-Stark Conjecture for Deligne-Ribet p-adic L-functions. We then show that this conjectural derivative formula can be combined with the main conjecture of non-commutative p-adic Iwasawa theory to give a strategy for obtaining evidence in support of the equivariant Tamagawa Number Conjecture for $\mathbb{G}_m$ over arbitrary finite Galois extensions of number fields, thereby obtaining a wide-ranging generalization of the main result of [19].

Contents

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1. Introduction

1.1. **Background and main results.** The theory of determinant functors for complexes of modules over commutative noetherian rings was developed by Knudsen and Mumford in [55], with later clarifications provided by Knudsen in [54], in both cases following initial suggestions of Grothendieck. It was subsequently shown by Deligne in [33] that for any exact category (in the sense of Quillen [71]) there exists a universal determinant functor that takes values in an associated category of 'virtual objects'.

Such determinant functors have hitherto played a key role in the formulation with respect to non-commutative coefficient rings of arithmetic special value conjectures, including the equivariant Tamagawa Number Conjecture that was formulated by Burns and Flach [17], following the seminal work of Bloch and Kato [4] and of Fontaine and Perrin-Riou [36].

In an earlier article [24], we constructed a canonical family of extensions of the determinant functor of Grothendieck, Knudsen and Mumford in the setting of the derived categories of perfect complexes over orders in finite dimensional separable algebras that need not be commutative. These 'reduced determinant functors' are of a more explicit, and concrete, nature than are the virtual objects used in [17] and, after establishing certain necessary results in integral representation theory, we were able, as a first application, to use them to develop a theory of non-commutative Euler systems in the setting of p-adic representations that are defined over arbitrary number fields and are endowed with the action of an arbitrary Gorenstein $\mathbb{Z}_p$-order.

In this article we consider the question of whether the general approach developed in [24] can be expected to have concrete consequences for the study of special value conjectures relative to non-commutative coefficient rings. In particular, we answer this question affirmatively in the special case of the equivariant Tamagawa Number Conjecture for $\mathbb{G}_m$, or eTNC($\mathbb{G}_m$) for short in the rest of this introduction, relative to arbitrary Galois extensions of number fields.

As some motivation for focusing on this case, we recall that eTNC($\mathbb{G}_m$) is both known to strengthen many classical refinements of Stark's seminal conjectures on the leading terms at zero of Artin L-series (see, for example, [14]) and can also, via the philosophy described by Huber and Kings in [45, §3.3] and by Fukaya and Kato in [37, §2.3.5], be seen to play an important role in the study of the most general case of the equivariant Tamagawa Number Conjecture (in fact, in [52, Ch. I, § 3.3] Kato even refers to this special case of the conjecture as 'the universal case').

However, in order to directly apply the constructions of [24] to the study of eTNC($\mathbb{G}_m$), we must first prove several intermediate results which we feel may themselves be of independent interest. To better orientate the reader through the article, therefore, we shall separate the remainder of this discussion according to these intermediate steps.

1.1.1. *Zeta elements and relative K-theory.* In the first part of the article (comprising §2 to §4) we prove some results in relative K-theory.

These results are established in the setting of an order $\mathcal{A}$ that is defined over a Dedekind domain R and spans a (finite-dimensional) separable algebra A over the quotient field of R. Recalling that the 'Whitehead order' $\xi(\mathcal{A})$ of $\mathcal{A}$ is a canonical R-order in the centre of A that is defined in [24], we shall here introduce natural notions of 'locally-primitive' and 'primitive' bases over $\xi(\mathcal{A})$ for the reduced determinant lattices of perfect complexes of $\mathcal{A}$-modules.

By using these notions, we shall then give (in Theorem 4.8) a concrete, and computationally useful, interpretation of the sort of equalities in relative K_0-groups that have hitherto underpinned the formulation of refined special value conjectures relative to non-commutative coefficient rings.

In particular, in this way we shall obtain an explicit reinterpretation of eTNC($\mathbb{G}_m$) in terms of a natural non-commutative generalization of the notion of 'zeta element' introduced by Kato in [51] (for details see Remark 4.9).

We recall that the notion of zeta element plays a key role in the strategy to investigate eTNC($\mathbb{G}_m$) over abelian extensions of arbitrary number fields that is developed by Burns, Kurihara and Sano in [18, 19]. In a little more detail, then, one of the main aims of the present article will be to use the techniques introduced in [24] to extend the latter strategy to the setting of arbitrary Galois extensions of number fields.

1.1.2. *Non-commutative Euler systems.* After reviewing in §5 relevant properties of the canonical Selmer modules and modified étale cohomology complexes that are constructed in [18], our next step will be to adapt, and in appropriate ways refine, aspects of the theory of non-commutative Euler systems that is developed in [25].

To this end, in §6 we introduce a notion of '(higher rank) non-commutative (pre-)Euler system for $\mathbb{G}_m$' and use a detailed analysis of the compactly supported p-adic cohomology of $\mathbb{Z}_p$ to give an unconditional construction of such systems over any number field. In this way we are able, for example, to strengthen the main result of [25] concerning the existence of 'extended cyclotomic Euler systems' over $\mathbb{Q}$ (for details see Theorem 6.19).

More generally, for any Galois extension of number fields E/K, we use the values at zero of higher derivatives of the Artin L-series of complex characters of $\mathrm{Gal}(E/K)$ to unconditionally define a 'non-commutative Rubin-Stark element' $\varepsilon^{\mathrm{RS}}_{E/K}$ for E/K. These elements belong, a priori, to the real vector spaces spanned by an appropriate reduced exterior power bidual of unit groups and we show that, for any fixed set Σ of archimedean places of K, as E varies over the finite Galois extensions of K (in some fixed algebraic closure) in which all places in Σ split completely, the elements $\varepsilon^{\mathrm{RS}}_{E/K}$ constitute a pre-Euler system $\varepsilon^{\mathrm{RS}}_{K,\Sigma}$ of rank $|\Sigma|$ for $\mathbb{G}_m$ over K.

Such systems $\varepsilon^{\mathrm{RS}}_{K,\Sigma}$ will be referred to as 'Rubin-Stark pre-Euler systems for K' and play a fundamental role in our approach. In preparation for such applications, we formulate, as Conjecture 6.8, a natural generalization to non-abelian Galois extensions of the central conjecture formulated by Rubin in [74] and show that this 'non-commutative Rubin-Stark Conjecture' implies that the systems $\varepsilon^{\mathrm{RS}}_{K,\Sigma}$ are non-commutative Euler (rather than just pre-Euler) systems of rank $|\Sigma|$ for $\mathbb{G}_m$ over K.

1.1.3. *Non-commutative Iwasawa theory.* In §7 and §8 we then develop certain aspects of non-commutative Iwasawa theory that are necessary for our approach.

As a first step, in §7 we formulate a 'main conjecture of higher rank non-commutative p-adic Iwasawa theory' for $\mathbb{G}_m$ over an arbitrary number field K. This prediction is stated as Conjecture 7.4 and uses the formalism of primitive bases developed in §4 to express a precise connection between the non-commutative Rubin-Stark pre-Euler system $\varepsilon^{\mathrm{RS}}_{K,\Sigma}$ and the reduced determinant of the compactly supported p-adic cohomology of $\mathbb{Z}_p$ over a compact p-adic Lie extension of K in which all places in Σ split completely.

Conjecture 7.4 can be seen to simultaneously generalize both the higher rank main conjecture of (commutative) Iwasawa theory that is formulated by Burns, Kurihara and Sano in [19] and the standard formulation of main conjectures in non-commutative Iwasawa theory following the approaches of Ritter and Weiss in [72] and of Coates, Fukaya, Kato, Sujatha and Venjakob in [27]. In particular, in an important special case (in which Σ is empty) our approach allows us to deduce the validity of Conjecture 7.4 by using the known validity, due independently to Ritter and Weiss [73], and to Kakde [50], of the main conjecture of non-commutative Iwasawa theory for totally real fields.

In §8, we then prove the existence over arbitrary compact p-adic Lie extensions of K of a distinguished family of resolutions of the compactly supported p-adic cohomology of $\mathbb{Z}_p$. This family of resolutions has two important roles in the present article and will also have further applications elsewhere (cf. §1.1.6).

In the remainder of §8 we use these resolutions to introduce a natural notion of 'semisimplicity' for the Selmer module of $\mathbb{G}_m$ over p-adic Lie extensions of K of rank one. This notion provides our theory with an appropriate generalization of the hypothesis that a finitely generated torsion module over the classical Iwasawa algebra should be 'semisimple at zero' (which is a standard assumption that arises in relation to descent computations in contexts in which the associated p-adic L-functions have trivial zeroes). In addition, we show that, in all relevant cases, our notion of semisimplicity specializes to recover the generalization, due to Jaulent [47], of the 'Order of Vanishing Conjecture' for p-adic Artin L-series at zero that was originally independently conjectured (for CM fields) by Gross [41] and Kuz'min [57].

1.1.4. *The Generalized Gross-Stark Conjecture.* As the final preparatory step in our approach, in §9 we formulate a generalization to arbitrary Galois extensions of number fields of the (p-adic) Gross-Stark Conjecture. We recall that the latter conjecture was originally formulated by Gross [41] in the setting of CM Galois extensions of totally real fields and has been unconditionally verified for all odd p by Dasgupta, Kakde and Ventullo in [29].

To formulate our conjecture we first use the resolutions constructed in §8 to introduce, under appropriate hypotheses, a canonical notion of the 'value of a higher derivative' of the non-commutative pre-Euler system $\varepsilon^{\mathrm{RS}}_{K,\Sigma}$. By comparing the reduced exterior products of certain natural Bockstein maps with those of canonical 'valuation' maps, we also construct a canonical '$\mathscr{L}$-invariant' homomorphism between the exterior power biduals (of differing ranks) of unit groups.

Then, in any given setting, the 'Generalized Gross-Stark Conjecture' of Conjecture 9.7 predicts an explicit formula for the value of an appropriate higher derivative of $\varepsilon^{\mathrm{RS}}_{K,\Sigma}$ in terms of the image under the relevant $\mathscr{L}$-invariant map of a non-commutative Rubin-Stark element of an appropriately higher rank.

This conjectural derivative formula encodes families of significant, and even sometimes explicit, integral relations between the non-commutative Rubin-Stark elements of differing ranks that are defined relative to finite Galois extensions of K. In this way, the conjecture therefore also encodes information about families of fine integral relations between the values at zero of higher derivatives (of different orders) of the Artin L-series defined over K.

For example, in the setting of CM extensions of totally real fields, we can show that the 'odd component' of the derivative formula in Conjecture 9.7 precisely recovers the classical Gross-Stark Conjecture. In this special case, therefore, we can thereby derive the unconditional validity of Conjecture 9.7 as a consequence of the main result of Dasgupta et al [29].

1.1.5. eTNC($\mathbb{G}_m$). In §10 we finally establish a concrete link between the results obtained in earlier sections and the conjecture eTNC($\mathbb{G}_m$).

Before doing so, however, we start by reviewing what is presently known concerning eTNC($\mathbb{G}_m$) over non-abelian Galois extensions and, at the same time, clarify aspects of the results of Burns in [16].

Under a suitable semisimplicity hypothesis, we are then able to establish over arbitrary finite Galois extensions of number fields a precise link between the non-commutative higher rank Iwasawa main conjecture (of Conjecture 7.4), the explicit derivative formula given by the Generalized Gross-Stark Conjecture (of Conjecture 9.7) and the reinterpretation of eTNC($\mathbb{G}_m$) in terms of zeta elements that is presented in §4. The latter result is stated explicitly as Theorem 10.15 and constitutes our desired generalization to arbitrary Galois extensions of the main result of Burns, Kurihara and Sano in [19].

The result of Theorem 10.15 sheds new light on the essential nature of eTNC($\mathbb{G}_m$) over arbitrary Galois extensions and also, more concretely, presents a strategy for obtaining evidence for it beyond the case of CM-extensions of totally real fields. It is therefore to be hoped that, in the same way the main result of [19] has motivated subsequent work and led to significant arithmetic results (see, for example, the recent articles of Bley and Hofer [3] and of Bullach and Hofer [12]), the strategy presented here will lead to concrete new evidence for eTNC($\mathbb{G}_m$) for families of non-abelian Galois extensions. For example, even in the case of non-abelian CM extensions of totally real fields, Theorem 10.15 already gives a significant simplification of the proofs of results of Burns in [16] (see, for example, Remark 10.17).

1.1.6. *Other connections.* To end the introduction, we note that the techniques developed here, and in the earlier articles [24] and [25] of Burns and Sano, can also be shown to have consequences for the formulation and study of special value conjectures over non-commutative coefficient rings beyond the special cases that we focus on in the present article.

As an example, in work of Burns, Puignau, Sano and Seo [22] it is shown that the distinguished family of resolutions constructed in §8 can be used to define canonical 'non-commutative Artin-Bockstein maps' that extend the classical reciprocity maps of local class field theory to non-abelian Galois extensions (of local fields) and thereby to formulate a generalization to arbitrary finite Galois extensions of the 'refined class number formula conjecture for $\mathbb{G}_m$' (or, as it is also often known, the 'Mazur-Rubin-Sano Conjecture'). We recall that the latter conjecture was independently conjectured for abelian extensions by Mazur and Rubin in [63] and by Sano in [75] and has played a key role in the study of eTNC($\mathbb{G}_m$) over such extensions. We further note that its natural generalization to arbitrary Galois extensions (as formulated precisely in [22, Conj. 5.1]) essentially constitutes a refined version 'at finite level' of the Iwasawa-theoretic Generalised Gross-Stark Conjecture that we formulate here as Conjecture 9.7.

One can also change focus from $\mathbb{G}_m$ to abelian varieties. We recall that, in this direction, the article [61] of Macias Castillo and Tsoi already gives interesting applications of the algebraic constructions made in [24] to the study of the Hasse-Weil-Artin L-series of dihedral twists of elliptic curves over general number fields. In addition, in [20] Burns, Kurihara and Sano have established an analogue of the main result of [19] that is relevant to the study of the Birch-Swinnerton-Dyer Conjecture for elliptic curves over $\mathbb{Q}$ (see, in particular, [20, Th. 7.6 and Rem. 7.7]). In light of this, it would be interesting to know if the general approach developed here can be adapted to shed further light on concrete relations between the Birch-Swinnerton-Dyer Conjecture and the GL_2 Main Conjecture of Iwasawa theory for elliptic curves without complex multiplication of Coates et al [27].

Finally, and in a much more general setting, the theory of non-commutative Euler systems developed in [25] can be seen to play an important role in relation to the strategies described by Huber and Kings [45, §3.3] and Fukaya and Kato [37, §2.3.5] to study the general case of the equivariant Tamagawa number conjecture. In particular, given a motive M defined over $\mathbb{Q}$, one can 'twist' in an appropriate sense the family of cyclotomic non-commutative Euler systems that is constructed unconditionally in Theorem 6.19 of the present article in order to obtain analytic families of classical, commutative Euler systems (of suitable rank) for lattices $T_p(M)$ in the p-adic realisation of M. Then, by applying the general theory of higher rank Euler systems of Burns, Sakamoto and Sano [23] to these systems, one can study the Selmer modules of $T_p(M)$. This important aspect of our theory will be discussed elsewhere.

1.2. **General notation.** For the reader's convenience, we collect together some general notation that will be used throughout the article.

For each ring R, we write $\zeta(R)$ for its centre and R^{op} for the corresponding opposite ring (so that $\zeta(R) = \zeta(R^{\mathrm{op}})$). By an R-module we shall, unless explicitly stated otherwise, mean a left R-module. The transpose of a matrix M over R is denoted by M^{tr}.

We write $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at a prime number p. We write A_p for the pro-p completion of an abelian group, or complex of abelian groups, A (so that $\mathbb{Z}_p$ is the ring of p-adic integers) and use similar notation for morphisms. We often abbreviate $E \otimes_{\mathbb{Z}} A$ to $E \cdot A$ for a field E.

We fix an algebraic closure $\mathbb{Q}_p^c$ of the field of p-adic rationals $\mathbb{Q}_p$ and a completion $\mathbb{C}_p$ of $\mathbb{Q}_p^c$. For a finite group Γ we write $\mathrm{Ir}(\Gamma)$ and $\mathrm{Ir}_p(\Gamma)$ for the respective sets of irreducible $\mathbb{C}$-valued and $\mathbb{C}_p$-valued characters of Γ.

If A is a Γ-module, then we endow linear duals such as $A^\vee := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$ with the natural contragredient action of Γ.

For a Galois extension of fields E/F we often abbreviate $\mathrm{Gal}(E/F)$ to $G_{E/F}$.

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Part I: Reduced determinants and relative K-theory

The equivariant Tamagawa Number Conjecture is formulated in [17] in terms of Deligne's categories of virtual objects and takes the form of an equality in the relative algebraic K_0-group of an appropriate extension of rings.

Alternative approaches to the formulation of conjectures in such groups were subsequently developed by Breuning and Burns [11] using a theory of 'equivariant Whitehead torsion', by Fukaya and Kato [37] using a theory of 'localized K_1-groups', by Muro, Tonks and Witte [64] using Waldhausen K-theory and by Braunling [6, 7, 8] using a theory of 'equivariant Haar measures' (and see also the associated article [9] of Braunling, Henrard and van Roosmalen).

In this first part of the article (comprising §2 to §4), our main aim is to explain how the theory of 'reduced determinant functors' that is developed in [24] can also be used to give a new approach to the formulation of such conjectures.

The key contribution that we shall actually make in these sections is to introduce a notion of '(locally-)primitive basis' in the setting of reduced determinants and to show that this notion is both well-defined and functorially well-behaved on the reduced determinants that arise from objects in suitable derived categories (for details see §3).

To relate this construction to relative algebraic K-theory, we shall also introduce a natural generalization of the notion of 'zeta element' that was first used in a (commutative) arithmetic setting by Kato in [51] (cf. Definition 4.6 and Example 4.7).

Our main result in this regard (Theorem 4.8) is then entirely K-theoretic in nature and perhaps of some independent interest. However, it also provides a natural, and very concrete, interpretation of the equalities that underlie several existing refined special value conjectures in arithmetic (cf. Remark 4.9). In addition, and more importantly, it can also be computationally useful in such contexts.

In particular, in later sections of this article, we shall find that this approach leads to a more direct formulation of main conjectures in non-commutative Iwasawa theory, to a natural generalization of the classical Gross-Stark Conjecture, to improvements in the descent formalism that relates such conjectures to the relevant cases of eTNC($\mathbb{G}_m$) and thereby to the derivation of further concrete evidence in support of eTNC($\mathbb{G}_m$) itself.

2. Reduced determinants

For the convenience of the reader, we shall first review relevant facts concerning the theory of reduced determinant functors developed by Burns and Sano in [24].

To do so we fix a Dedekind domain R with field of fractions F of characteristic zero. We also fix a finite dimensional separable F-algebra A and an R-order $\mathcal{A}$ in A.

2.1. **Reduced exterior powers.**

We write the Wedderburn decomposition of A as

$$A = \prod\nolimits_{i\in I} A_i,$$

where I is a finite index set and each algebra A_i is of the form $\mathrm{M}_{n_i}(D_i)$ for a suitable natural number n_i and a division ring D_i with $F \subseteq \zeta(D_i)$.

We next choose a field extension E of F that, for every index i, contains $\zeta(D_i)$ and splits D_i and then fix associated isomorphisms

$$D_i \otimes_{\zeta(D_i)} E \cong \mathrm{M}_{m_i}(E)$$

for a suitable natural number m_i.

For each index i we also fix an indecomposable idempotent e_i of $\mathrm{M}_{m_i}(E)$. Then the direct sum V_i of n_i-copies of $e_i\mathrm{M}_{m_i}(E)$ is a simple left $A_{i,E} := A_i \otimes_{\zeta(A_i)} E$-module.

Each finitely generated A-module M decomposes as a direct sum

$$M = \bigoplus\nolimits_{i\in I} M_i,$$

where M_i denotes the A_i-module $A_i \otimes_A M$.

For each non-negative integer r we then define the r-th reduced exterior power of M over A by setting

$$\bigwedge\nolimits_A^r M := \bigoplus\nolimits_{i\in I}\bigwedge\nolimits_E^{rd_i}(V_i^*\otimes_{A_{i,E}} M_{i,E}), \tag{2.1.1}$$

with $d_i := \dim_E(V_i) = n_im_i$, $M_{i,E} := M_i \otimes_{\zeta(A_i)} E$, and $V_i^* := \mathrm{Hom}_E(V_i, E)$. This construction depends on E, but is independent of each space V_i up to isomorphism.

To discuss linear duals we note that $\mathrm{Hom}_A(M, A)$ has a natural structure as left A^{op}-module and we consider its exterior powers over A^{op}. We also note that, for each index i, the space V_i^* is a simple left $A_{i,E}^{\mathrm{op}}$-module, and that its dual V_i^{**} is canonically isomorphic to V_i. In this case, the definition above therefore gives

$$\bigwedge\nolimits_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M,A) = \bigoplus\nolimits_{i\in I}\bigwedge\nolimits_E^{rd_i}(V_i\otimes_{A_{i,E}^{\mathrm{op}}} \mathrm{Hom}_{A_{i,E}}(M_{i,E}, A_{i,E})). \tag{2.1.2}$$

For each integer s with $0 \le s \le r$ there are then natural duality pairings

$$\bigwedge\nolimits_A^r M \times \bigwedge\nolimits_{A^{\mathrm{op}}}^s \mathrm{Hom}_A(M,A) \to \bigwedge\nolimits_A^{r-s} M, \quad (m,\varphi)\mapsto \varphi(m). \tag{2.1.3}$$

To make this pairing explicit we fix, for each index i, an E-basis $\{v_{ij} : 1 \le j \le m_i\}$ of $e_i\mathrm{M}_{m_i}(E)$. If A_i is commutative, and hence a field (so that $m_i = 1$), then we always take v_{i1} to be identity element of E. The (lexicographically-ordered) set

$$\underline{\varpi}(i) := \{\varpi_{iaj} : 1 \le a \le n_i, 1 \le j \le m_i\}$$

is then an E-basis of V_i, where ϖ_{iaj} is the element of V_i that is equal to v_{ij} in its a-th component and is zero in all other components.

For any subsets $\{m_a\}_{1\le a\le r}$ of M and $\{\varphi_a\}_{1\le a\le r}$ of $\mathrm{Hom}_A(M, A)$ we then set

$$\wedge_{a=1}^{a=r} m_a := \big(\wedge_{1\le a\le r}(\wedge_{x\in\underline{\varpi}(i)} x^*\otimes m_{ai})\big)_{i\in I} \in \bigwedge\nolimits_A^r M \tag{2.1.4}$$

and

$$\wedge_{a=1}^{a=r}\varphi_a := \big(\wedge_{1\le a\le r}(\wedge_{x\in\underline{\varpi}(i)} x\otimes \varphi_{ai})\big)_{i\in I} \in \bigwedge\nolimits_{A^{\mathrm{op}}}^r \mathrm{Hom}_A(M,A). \tag{2.1.5}$$

Here we write m_{ai} and φ_{ai} for the projections of m_a to $M_{i,E}$ and of φ_a to $\mathrm{Hom}_{A_{i,E}}(M_{i,E}, A_{i,E})$ and $\{x^* : x \in \underline{\varpi}(i)\}$ for the basis of V_i^* that is dual to $\underline{\varpi}(i)$.

These constructions clearly depend on the collection ϖ of ordered bases $\{\underline{\varpi}(i) : i \in I\}$ and so should strictly be written as '$\wedge_\varpi$' rather than '$\wedge$'. However, for simplicity, we prefer not to indicate this dependence, since in practice it does not cause difficulties.

For example, in the sequel we will often use the following fact (from [24, Lem. 4.13]) that is independent of the choice of ϖ: if M is a free A-module of rank r with basis $\{b_a\}_{1\le a\le r}$, then for each φ in $\mathrm{End}_A(M)$ one has

$$\wedge_{a=1}^{a=r}(\varphi(b_a)) = \mathrm{Nrd}_{\mathrm{End}_A(M)}(\varphi)\cdot(\wedge_{a=1}^{a=r} b_a) \in \bigwedge\nolimits_A^r M. \tag{2.1.6}$$

Finally, we recall that, with these definitions, for subsets $\{m_b\}_{1\le b\le r}$ of M and $\{\varphi_a\}_{1\le a\le r}$ of $\mathrm{Hom}_A(M, A)$, the pairing (2.1.3) sends $\big(\wedge_{b=1}^{b=r} m_b, \wedge_{a=1}^{a=r}\varphi_a\big)$ to the element

$$(\wedge_{a=1}^{a=r}\varphi_a)(\wedge_{b=1}^{b=r} m_b) = \mathrm{Nrd}_{A^{\mathrm{op}}}((\varphi_a(m_b))_{1\le a,b\le r}) \in \zeta(A). \tag{2.1.7}$$

Remark 2.1. Let Γ be a finite group and F a subfield of $\mathbb{C}$. For each character χ in $\mathrm{Ir}(\Gamma)$ we write n_χ for the exponent of the quotient group $\Gamma/\ker(\chi)$ and F_χ for the field generated over F by a primitive n_χ-th root of unity. Then, following Brauer [5], we may fix a representation

$$\rho_\chi : \Gamma \to \mathrm{GL}_{\chi(1)}(F_\chi)$$

of character χ. In particular, if F' is any extension of F that contains F_χ for every χ in $\mathrm{Ir}(\Gamma)$, then the induced F'-linear ring homomorphisms $\rho_{\chi,*} : F'[\Gamma] \to \mathrm{M}_{\chi(1)}(F')$ combine to give an isomorphism of F'-algebras

$$F'[\Gamma] \cong \prod\nolimits_{\chi\in\mathrm{Ir}(\Gamma)} \mathrm{M}_{\chi(1)}(F').$$

This shows that F' is a splitting field for $F[\Gamma]$, that the spaces

$$V_\chi := (F')^{\chi(1)},$$

considered as the first columns of the respective χ-components $\mathrm{M}_{\chi(1)}(F')$ of $F'[\Gamma]$, are a complete set of representatives of the isomorphism classes of simple $F'[\Gamma]$-modules and that the standard F'-basis of $(F')^{\chi(1)}$ constitutes an ordered F'-basis of V_χ.

In this way, a fixed choice of representations

$$\{\rho_\chi\}_{\chi\in\mathrm{Ir}(\Gamma)}$$

as above gives rise, for each finitely generated $F[\Gamma]$-module M, to a canonical normalization of the constructions (2.1.1), (2.1.2), (2.1.4) and (2.1.5). We further note that, for each non-negative integer r, the resulting $\zeta(F[\Gamma])$-module ${\bigwedge}^r_{F[\Gamma]} M$ is then finitely generated and behaves functorially under change of F.

2.2. **Reduced Rubin lattices.**

2.2.1. In the sequel we write $\mathrm{Spec}(R)$ for the set of all prime ideals of R and $\mathrm{Spm}(R)$ for the subset $\mathrm{Spec}(R) \setminus \{(0)\}$ of maximal ideals.

For $\mathfrak{p}$ in $\mathrm{Spec}(R)$ we write $R_{(\mathfrak{p})}$ for the localization of R at $\mathfrak{p}$ and for an R-module M set $M_{(\mathfrak{p})} := R_{(\mathfrak{p})} \otimes_R M$. For $\mathfrak{p}$ in $\mathrm{Spm}(R)$ we also write $R_\mathfrak{p}$ for the completion of R at $\mathfrak{p}$ and for an R-module M set $M_\mathfrak{p} := R_\mathfrak{p} \otimes_R M$.

A finitely generated module M over an R-order $\mathcal{A}$ is said to be 'locally-free' if $M_{(\mathfrak{p})}$ is a free $\mathcal{A}_{(\mathfrak{p})}$-module for all $\mathfrak{p}$ in $\mathrm{Spec}(R)$. We recall that for $\mathfrak{p}$ in $\mathrm{Spm}(R)$, the $\mathcal{A}_{(\mathfrak{p})}$-module $M_{(\mathfrak{p})}$ is free if and only if the $\mathcal{A}_\mathfrak{p}$-module $M_\mathfrak{p}$ is free (this follows as an easy consequence of Maranda's Theorem - see, for example, [28, Prop. (30.17)]).

We write $\mathrm{Mod}^{\mathrm{lf}}(\mathcal{A})$ for the category of (finitely generated) locally-free $\mathcal{A}$-modules. For M in $\mathrm{Mod}^{\mathrm{lf}}(\mathcal{A})$ the rank of the free $\mathcal{A}_{(\mathfrak{p})}$-module $M_{(\mathfrak{p})}$ is independent of $\mathfrak{p}$ and equal to the rank of the free A-module $M_F := M_{(0)} = F \otimes_R M$. We refer to this common rank as the 'rank' of M and denote it $\mathrm{rk}_\mathcal{A}(M)$.

Remark 2.2. Localization is an exact functor and so a locally-free $\mathcal{A}$-module is projective. There are important cases in which the converse is also true.
(i) If $\mathcal{A} = R$, then every finitely generated torsion-free $\mathcal{A}$-module M is locally-free, with $\mathrm{rk}_\mathcal{A}(M)$ equal to the dimension of the F-space spanned by M.

(ii) If G is a finite group for which no prime divisor of $|G|$ is invertible in R and $\mathcal{A} = R[G]$ then, by a fundamental result of Swan [80] (see also [28, Th. (32.11)]), a finitely generated projective $\mathcal{A}$-module is locally-free. For any such module M the product $\mathrm{rk}_{R[G]}(M) \cdot |G|$ is equal to the dimension of the F-space spanned by M.
(iii) There are also several classes of order $\mathcal{A}$ for which a finitely generated projective $\mathcal{A}$-module is locally-free if and only if it spans a free A-module. This is the case, for example, if $\mathcal{A}$ is commutative (cf. [28, Prop. 35.7]), or if $\mathcal{A}_{(\mathfrak{p})}$ is a maximal $R_{(\mathfrak{p})}$-order in A for all $\mathfrak{p}$ in $\mathrm{Spm}(R)$ (cf. [28, Th. 26.24(iii)]), or if $\mathcal{A} = R[G]$ for any finite group G (cf. [28, Th. 32.1]).

2.2.2. We use the R-submodule of $\zeta(A)$ defined by

$$\xi(\mathcal{A}) := R \cdot \left\{ \mathrm{Nrd}_A(M) : M \in \bigcup\nolimits_{n \geq 0} \mathrm{M}_n(\mathcal{A}) \right\}.$$

For each $\mathfrak{p}$ in $\mathrm{Spm}(R)$, we similarly define an $R_{(\mathfrak{p})}$-submodule $\xi(\mathcal{A}_{(\mathfrak{p})})$ of $\zeta(A)$ and an $R_{\mathfrak{p}}$-submodule $\xi(\mathcal{A}_{\mathfrak{p}})$ of $\zeta(A_{\mathfrak{p}})$ and we recall that

$$\xi(\mathcal{A}) = \bigcap\nolimits_{\mathfrak{p} \in \mathrm{Spm}(R)} \xi(\mathcal{A}_{(\mathfrak{p})})$$

by [22, Lem. 2.2]. Further, by [24, Lem. 3.2], one knows that $\xi(\mathcal{A})$ is an R-order in $\zeta(A)$ (referred to as the 'Whitehead order' of $\mathcal{A}$ in loc. cit.), that $\xi(\mathcal{A}) = \mathcal{A}$ if and only if $\mathcal{A}$ is commutative and that, in general, one has

$$\xi(\mathcal{A})_{(\mathfrak{p})} = \xi(\mathcal{A}_{(\mathfrak{p})}) = \zeta(A) \cap \xi(\mathcal{A}_{\mathfrak{p}})$$

for every $\mathfrak{p}$ in $\mathrm{Spm}(R)$.

For a finitely generated $\mathcal{A}$-module M, and a non-negative integer r, the r-th reduced Rubin lattice of the $\mathcal{A}$-module M is then defined (in [24, Def. 4.15]) to be the $\xi(\mathcal{A})$-submodule of ${\bigwedge}_A^r M_F$ obtained by setting

$$\bigcap\nolimits_{\mathcal{A}}^r M := \left\{ a \in {\bigwedge}_A^r M_F : (\wedge_{i=1}^{i=r} \varphi_i)(a) \in \xi(\mathcal{A}) \text{ for all } \varphi_1, \ldots, \varphi_r \in \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}) \right\}.$$

This module is finitely generated as a consequence of [24, Th. 4.19(ii)]. The latter result also shows that, whilst the definition (2.1.5) of reduced exterior products depends on the choices of ordered bases ϖ, the lattice $\bigcap_{\mathcal{A}}^r M$ is independent, in a natural sense, of these choices (which are therefore not explicitly indicated in our notation).

We next note that the argument of [24, Lem. 4.16] implies injectivity of the canonical 'evaluation' homomorphism of $\xi(\mathcal{A})$-modules

$$\mathrm{ev}_M^r : \bigcap\nolimits_{\mathcal{A}}^r M \to \prod\nolimits_{\underline{\varphi}} \xi(\mathcal{A}); \;\; x \mapsto \big((\wedge_{j=1}^{j=r} \varphi_j)(x)\big)_{\underline{\varphi}}, \tag{2.2.1}$$

where in the direct product $\underline{\varphi} = (\varphi_1, \ldots, \varphi_r)$ runs over all elements of $\mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A})^r$.

We further recall from [24, Prop. 5.6] that if M belongs to $\mathrm{Mod}^{\mathrm{lf}}(\mathcal{A})$, with $r := \mathrm{rk}_{\mathcal{A}}(M)$, then $\bigcap_{\mathcal{A}}^r M$ is an invertible $\xi(\mathcal{A})$-module with the property that

$$\bigcap\nolimits_{\mathcal{A}}^r M = \bigcap\nolimits_{\mathfrak{p} \in \mathrm{Spm}(R)} \big(\bigcap\nolimits_{\mathcal{A}}^r M\big)_{(\mathfrak{p})}$$

and so can be explicitly computed via its localizations as follows: if for each $\mathfrak{p}$ in $\mathrm{Spm}(R)$ one fixes an ordered $\mathcal{A}_{(\mathfrak{p})}$-basis $\underline{b}_\mathfrak{p} = \{b_{\mathfrak{p},j}\}_{1\le j\le r}$ of $M_{(\mathfrak{p})}$, then

$$(2.2.2) \qquad \big({\textstyle\bigcap}^r_{\mathcal{A}} M\big)_{(\mathfrak{p})} = {\textstyle\bigcap}^r_{\mathcal{A}_{(\mathfrak{p})}} M_{(\mathfrak{p})} = \xi(\mathcal{A}_{(\mathfrak{p})}) \cdot \wedge_{j=1}^{j=r} b_{\mathfrak{p},j}.$$

In particular, if M is a free $\mathcal{A}$-module, with ordered basis $\{b_j\}_{1\le j\le r}$, then one can take $b_{\mathfrak{p},i} = b_i$ for all i with $1 \le i \le r$ and so the $\xi(\mathcal{A})$-module

$$(2.2.3) \qquad {\textstyle\bigcap}^r_{\mathcal{A}} M = \xi(\mathcal{A}) \cdot \wedge_{j=1}^{j=r} b_j$$

is free of rank one (with basis $\wedge_{j=1}^{j=r} b_j$).

Remark 2.3. For other examples of the explicit computation of Whitehead orders and reduced Rubin lattices see [24, Exam. 3.4, Exam. 3.5 and Rem. 4.16].

2.3. **Reduced determinant functors.**

2.3.1. We write $\mathsf{D}(\mathcal{A})$ for the derived category of (left) $\mathcal{A}$-modules. We also write $\mathsf{C}^{\mathrm{lf}}(\mathcal{A})$ for the category of bounded complexes of modules in $\mathrm{Mod}^{\mathrm{lf}}(\mathcal{A})$ and $\mathsf{D}^{\mathrm{lf}}(\mathcal{A})$ for the full triangulated subcategory of $\mathsf{D}(\mathcal{A})$ comprising complexes that are isomorphic to a complex in $\mathsf{C}^{\mathrm{lf}}(\mathcal{A})$.

We write $\mathrm{K}_0^{\mathrm{lf}}(\mathcal{A})$ for the Grothendieck group of $\mathrm{Mod}^{\mathrm{lf}}(\mathcal{A})$ and recall that each object C of $\mathsf{D}^{\mathrm{lf}}(\mathcal{A})$ gives rise to a canonical 'Euler characteristic' element $\chi_{\mathcal{A}}(C)$ in $\mathrm{K}_0^{\mathrm{lf}}(\mathcal{A})$ (for details see [24, §5.1.3]).

We also write $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$ for the kernel of the homomorphism $\mathrm{K}_0^{\mathrm{lf}}(\mathcal{A}) \to \mathbb{Z}$ that is induced by sending each M in $\mathrm{Mod}^{\mathrm{lf}}(\mathcal{A})$ to $\mathrm{rk}_{\mathcal{A}}(M)$.

We then write $\mathsf{C}^{\mathrm{lf},0}(\mathcal{A})$ for the subcategory of $\mathsf{C}^{\mathrm{lf}}(\mathcal{A})$ comprising complexes C for which $\chi_{\mathcal{A}}(C)$ belongs to $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$ and $\mathsf{D}^{\mathrm{lf},0}(\mathcal{A})$ for the full triangulated subcategory of $\mathsf{D}^{\mathrm{lf}}(\mathcal{A})$ comprising complexes C for which $\chi_{\mathcal{A}}(C)$ belongs to $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$. (The latter condition is equivalent to requiring that C is isomorphic in $\mathsf{D}(\mathcal{A})$ to an object of $\mathsf{C}^{\mathrm{lf},0}(\mathcal{A})$).

Remark 2.4. Assume $\mathcal{A}$ is such that, for all $\mathfrak{p}$ in $\mathrm{Spec}(R)$, the reduced projective class group $\mathrm{SK}_0(\mathcal{A}_{(\mathfrak{p})})$ of the $R_{(\mathfrak{p})}$-order $\mathcal{A}_{(\mathfrak{p})}$ vanishes (as is the case, for example, for all of the orders discussed in Remark 2.2). Then, in this case, $\mathsf{D}^{\mathrm{lf},0}(\mathcal{A})$ is naturally equivalent to the full triangulated subcategory of the derived category $\mathsf{D}^{\mathrm{perf}}(\mathcal{A})$ of perfect complexes of $\mathcal{A}$-modules that comprises all (perfect) complexes whose Euler characteristic in $\mathrm{K}_0(\mathcal{A})$ belongs to the subgroup $\mathrm{SK}_0(\mathcal{A})$. A proof of this fact is given in [24, Lem. 5.2].

2.3.2. In the next result we record relevant properties of the reduced determinant functors constructed in [24, §5].

To state this result we recall that, in terms of the notation used in §2.1, the 'reduced rank' of a module M in $\mathrm{Mod}^{\mathrm{lf}}(\mathcal{A})$ is defined to be the vector

$$\mathrm{rr}_{\mathcal{A}}(M) := \big(\mathrm{rk}_{\mathcal{A}}(M) \cdot d_i\big)_{i\in I},$$

where each natural number d_i is defined as in §2.1. By using [24, Lem. 5.1], this vector is regarded as a locally-constant function on $\mathrm{Spec}(\xi(\mathcal{A}))$.

We also write $\mathsf{D}^{\rm lf}(\mathcal{A})_{\rm Isom}$ for the subcategory of $\mathsf{D}^{\rm lf}(\mathcal{A})$ in which morphisms are restricted to be isomorphisms and $\mathcal{P}(\xi(\mathcal{A}))$ for the category of graded invertible $\xi(\mathcal{A})$-modules. We recall that each object of $\mathcal{P}(\xi(\mathcal{A}))$ is a pair

$$X = (X^{\rm u}, {\rm gr}(X))$$

comprising an 'ungraded part' $X^{\rm u}$ that is a locally-free, rank one, $\xi(\mathcal{A})$-module and a 'grading' ${\rm gr}(X)$ that is a locally-constant function on ${\rm Spec}(\xi(\mathcal{A}))$.

Finally, we note that the concept of 'extended determinant functor' originates in [55] and is recalled precisely in [24, Def. 5.12].

Then the following result is an immediate consequence of [24, Th. 5.4].

Theorem 2.5. *For each set of ordered bases ϖ as in §2.1, there exists a canonical extended determinant functor*

$${\rm d}_{\mathcal{A},\varpi} : \mathsf{D}^{\rm lf}(\mathcal{A})_{\rm Isom} \to \mathcal{P}(\xi(\mathcal{A}))$$

that has all of the following properties.

(i) *For each exact triangle*

$$C' \to C \to C'' \to C'[1]$$

in $\mathsf{D}^{\rm lf}(\mathcal{A})$ there exists a canonical isomorphism in $\mathcal{P}(\xi(\mathcal{A}))$

$${\rm d}_{\mathcal{A},\varpi}(C') \otimes {\rm d}_{\mathcal{A},\varpi}(C'') \xrightarrow{\sim} {\rm d}_{\mathcal{A},\varpi}(C)$$

that is functorial with respect to isomorphisms of triangles.

(ii) *If C belongs to $\mathsf{D}^{\rm lf}(\mathcal{A})$ is such that every module $H^i(C)$ also belongs to $\mathsf{D}^{\rm lf}(\mathcal{A})$, then there exists a canonical isomorphism in $\mathcal{P}(\xi(\mathcal{A}))$*

$${\rm d}_{\mathcal{A},\varpi}(C) \cong \bigotimes_{i\in\mathbb{Z}} {\rm d}_{\mathcal{A},\varpi}(H^i(C))^{(-1)^i}$$

that is functorial with respect to quasi-isomorphisms.

(iii) *For each P in ${\rm Mod}^{\rm lf}(\mathcal{A})$ one has*

$${\rm d}_{\mathcal{A},\varpi}(P) = \big({\bigcap}_{\mathcal{A}}^{{\rm rk}_{\mathcal{A}}(P)} P, {\rm rr}_{\mathcal{A}}(P)\big),$$

where the reduced Rubin lattice ${\bigcap}_{\mathcal{A}}^{{\rm rk}_{\mathcal{A}}(P)} P$ is defined with respect to ϖ.

(iv) *The restriction of ${\rm d}_{\mathcal{A},\varpi}$ to $\mathsf{D}^{{\rm lf},0}(\mathcal{A})_{\rm Isom}$ is independent of the choice of ϖ.*

Remark 2.6. The approach of Deligne in [33, §4] constructs a 'universal determinant functor' for the exact category ${\rm Mod}^{\rm lf}(\mathcal{A})$, with values in an associated commutative Picard category $\mathcal{V}^{\rm lf}(\mathcal{A})$ of 'virtual objects'. In particular, in this way each determinant functor ${\rm d}_{\mathcal{A},\varpi}$ constructed as in Theorem 2.5 naturally induces a strongly monoidal functor

$$\phi^{\rm lf}_{\mathcal{A},\varpi} : \mathcal{V}^{\rm lf}(\mathcal{A}) \to \mathcal{P}(\xi(\mathcal{A})).$$

It is known that, in most cases, this functor $\phi^{\rm lf}_{\mathcal{A},\varpi}$ is not an equivalence of commutative Picard categories. For more details see [24, Rems. 5.5 and 5.8].

Remark 2.7. For any free rank one $\zeta(A)$-module W we set

$$W^1 := W \quad \text{and} \quad W^{-1} := \mathrm{Hom}_{\zeta(A)}(W, \zeta(A)),$$

with the linear dual regarded as a (free, rank one) $\zeta(A)$-module via the natural composition action. For each basis element w of W we set $w^1 := w$ and write w^{-1} for the (unique) basis element of W^{-1} that sends w to 1. For any invertible $\xi(\mathcal{A})$-module $\mathcal{L}$ we similarly define invertible $\xi(\mathcal{A})$-modules by setting

$$\mathcal{L}^1 := \mathcal{L} \quad \text{and} \quad \mathcal{L}^{-1} := \mathrm{Hom}_{\xi(\mathcal{A})}(\mathcal{L}, \xi(\mathcal{A})).$$

For any complex $P^\bullet$ in $\mathsf{C}^{\mathrm{lf}}(\mathcal{A})$ of the form

$$\cdots \to P^a \xrightarrow{d^a} P^{a+1} \to \cdots, \tag{2.3.1}$$

we then set

$$\mathrm{d}^\diamond_{\mathcal{A},\varpi}(P^\bullet) := \bigotimes\nolimits_{a\in\mathbb{Z}} \big(\bigcap\nolimits_{\mathcal{A}}^{\mathrm{rk}_{\mathcal{A}}(P^a)} P^a\big)^{(-1)^a},$$

where the tensor product is taken over $\xi(\mathcal{A})$, and

$$\mathrm{rr}_{\mathcal{A}}(P^\bullet) := \sum\nolimits_{a\in\mathbb{Z}} (-1)^a \cdot \mathrm{rr}_{\mathcal{A}}(P^a) = \Big(\sum\nolimits_{a\in\mathbb{Z}} (-1)^a \cdot \mathrm{rk}_{\mathcal{A}}(P^a)\Big) \cdot (d_i)_{i\in I}. \tag{2.3.2}$$

Then claims (i) and (iii) of Theorem 2.5 combine to give a canonical identification

$$\mathrm{d}_{\mathcal{A},\varpi}(P^\bullet) = \big(\mathrm{d}^\diamond_{\mathcal{A},\varpi}(P^\bullet), \mathrm{rr}_{\mathcal{A}}(P^\bullet)\big)$$

so that $\mathrm{d}_{\mathcal{A},\varpi}(P^\bullet)^{\mathrm{u}} = \mathrm{d}^\diamond_{\mathcal{A},\varpi}(P^\bullet)$.

3. Primitive and locally-primitive bases

In this section we introduce natural notions of 'primitive basis' and 'locally-primitive basis' in the setting of the functors that are discussed in Theorem 2.5.

In the sequel we always regard the set of ordered bases ϖ that occurs in Theorem 2.5 as fixed and use the following abbreviations for the associated reduced determinant functors

$$\mathrm{d}_{\mathcal{A}}(-) := \mathrm{d}_{\mathcal{A},\varpi}(-) \quad \text{and} \quad \mathrm{d}^\diamond_{\mathcal{A}}(-) := \mathrm{d}^\diamond_{\mathcal{A},\varpi}(-).$$

3.1. **Primitive bases.**

3.1.1. We write $\mathrm{Mod}^{\mathrm{f}}(\mathcal{A})$ for the full subcategory of $\mathrm{Mod}^{\mathrm{lf}}(\mathcal{A})$ comprising (finitely generated) free $\mathcal{A}$-modules and $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ for the full subcategory of $\mathsf{C}^{\mathrm{lf}}(\mathcal{A})$ comprising complexes $P^\bullet$ in which every term P^a belongs to $\mathrm{Mod}^{\mathrm{f}}(\mathcal{A})$.

Fix $P^\bullet$ in $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ and for each integer a set $r_a := \mathrm{rk}_{\mathcal{A}}(P^a)$. Then the equality (2.2.3) (with M and r taken to be each P^a and r_a) implies that the $\xi(\mathcal{A})$-module $\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet)$ defined in Remark 2.7 is free of rank one.

Further, if for each a we fix an ordered A-basis

$$\underline{b}_a = \{b_{a,j}\}_{1\le j\le r_a}$$

of P^a_F, then we obtain an element of the $\zeta(A)$-module $\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet)_F$ by setting

$$\Upsilon(\underline{b}_\bullet) := \bigotimes\nolimits_{a\in\mathbb{Z}} (\wedge_{j=1}^{j=r_a} b_{a,j})^{(-1)^a}. \tag{3.1.1}$$

In particular, if each $\underline{b}_a$ is an $\mathcal{A}$-basis of P^a, then (2.2.3) implies that $\Upsilon(\underline{b}_\bullet)$ is a basis of the $\xi(\mathcal{A})$-module $\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet)$.

This fact motivates us to make the following definition.

Definition 3.1. For any complex $P^\bullet$ in $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ we shall say that a basis element b of the (free, rank one) $\xi(\mathcal{A})$-module $\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet)$ is 'primitive' if it is equal to $\Upsilon(\underline{b}_\bullet)$ for some choice of ordered bases $\underline{b}_a$ of the $\mathcal{A}$-modules P^a.

The key observation that we shall make about this definition is that it extends naturally to give a well-defined concept on objects of $\mathsf{D}^{\mathrm{lf}}(\mathcal{A})$.

To state the precise result we recall that for any integer d greater than or equal to the stable range $\mathrm{sr}(\mathcal{A})$ of $\mathcal{A}$, the natural homomorphism

$$\mathrm{GL}_d(\mathcal{A}) \to \mathrm{K}_1(\mathcal{A}) \tag{3.1.2}$$

is surjective (cf. [28, Th. (40.42)]). We further recall Bass has shown that $\mathrm{sr}(\mathcal{A})$ is equal to one if R is local, and hence $\mathcal{A}$ is semi-local, and that $\mathrm{sr}(\mathcal{A})$ is equal to two in all other cases. (For more details see [28, Th. (40.31)] and [28, Th. (40.41)] respectively.)

Proposition 3.2. *Let $\lambda : P_1^\bullet \to P_2^\bullet$ be a quasi-isomorphism in $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ and assume that $P_1^\bullet$ and $P_2^\bullet$ each have a term of rank at least $\mathrm{sr}(\mathcal{A})$. Set $r := \mathrm{rr}_{\mathcal{A}}(P_1^\bullet) = \mathrm{rr}_{\mathcal{A}}(P_2^\bullet)$.*

Then an element b of $\mathrm{d}^\diamond_{\mathcal{A}}(P_1^\bullet)$ is a primitive basis of $\mathrm{d}^\diamond_{\mathcal{A}}(P_1^\bullet)$ if and only if the image of (b, r) under $\mathrm{d}_{\mathcal{A}}(\lambda)$ is equal to (b', r) for a primitive basis b' of $\mathrm{d}^\diamond_{\mathcal{A}}(P_2^\bullet)$.

The proof of this fact uses several results concerning the functorial behaviour of primitive bases with respect to the constructions that underlie the proof of [24, Th. 5.4]. In this regard we recall that the arguments in loc. cit. adapt earlier arguments of Burns and Flach in [17, §2] and so rely on explicit constructions made by Knudsen and Mumford in [55].

3.1.2. We first establish several useful technical results.

Lemma 3.3. *Let $P^\bullet$ be a complex in $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ of the form (2.3.1) for which there exists an integer a with $\mathrm{rk}_{\mathcal{A}}(P^a) \geq \mathrm{sr}(\mathcal{A})$.*

Let b be a primitive basis of $\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet)$. Then any other element b' of $\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet)$ is a primitive basis of $\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet)$ if and only if $b' = u \cdot b$ with u in $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$.

Proof. In each degree s set $r_s := \mathrm{rk}_{\mathcal{A}}(P^s)$. Then (2.1.6) implies that for any choices of ordered $\mathcal{A}$-bases $\{b_{sj}\}_{1\leq j\leq r_s}$ and $\{b'_{sj}\}_{1\leq j\leq r_s}$ of P^s there exists a matrix U_s in $\mathrm{GL}_{r_s}(\mathcal{A})$ such that

$$\wedge_{j=1}^{j=r_s} b'_{sj} = \mathrm{Nrd}_A(U_s) \cdot \wedge_{j=1}^{j=r_s} b_{sj}.$$

Since $\mathrm{Nrd}_A(U_s)$ is a unit of $\xi(\mathcal{A})$, this observation (in each degree s) implies that the stated condition is necessary.

To prove sufficiency we assume that b is a primitive basis of $\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet)$ and that $b' = u \cdot b$ with u in $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$.

Since, by assumption, $r_a \geq \mathrm{sr}(\mathcal{A})$ the homomorphism (3.1.2) (with $d = r_a$) is surjective and so there exists a matrix $u_a = (u_{a,tw})_{1\leq t,w\leq r_a}$ in $\mathrm{GL}_{r_a}(\mathcal{A})$ with $\mathrm{Nrd}_A(u_a)^{(-1)^a} = u$.

We then fix ordered bases $\underline{b}_s := \{b_{s,t}\}_{1\leq t\leq r_s}$ of the $\mathcal{A}$-modules P^s such that $b = \Upsilon(\underline{b}_\bullet)$ and write $\underline{b}'_s := \{b'_{s,t}\}_{1\leq t\leq r_s}$ for the ordered basis of each $\mathcal{A}$-module P^s that is obtained by setting $b'_{s,t} := b_{s,t}$ if $s \neq a$ and $b'_{a,t} := \sum_{w=1}^{w=r_a} u_{a,tw} b_{a,w}$. Then the equality (2.1.6) implies that

$$b' = u \cdot b = \mathrm{Nrd}_A(u_a)^{(-1)^a} \cdot \Upsilon(\underline{b}_\bullet) = \Upsilon(\underline{b}'_\bullet),$$

and hence that b' is a primitive basis of $\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet)$, as required. □

Lemma 3.4. *Let*

$$0 \to P_1^\bullet \to P_2^\bullet \to P_3^\bullet \to 0$$

be a short exact sequence in $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ *and write*

$$\Delta : \mathrm{d}_{\mathcal{A}}(P_1^\bullet) \otimes \mathrm{d}_{\mathcal{A}}(P_3^\bullet) \cong \mathrm{d}_{\mathcal{A}}(P_2^\bullet)$$

for the isomorphism induced by Theorem 2.5(i). Set $r_j := \mathrm{rr}_{\mathcal{A}}(P_j^\bullet)$ *for* $j = 1, 2, 3$.

Then $r_1 + r_3 = r_2$ *and, for any primitive bases* x_1 *of* $\mathrm{d}^\diamond_{\mathcal{A}}(P_1^\bullet)$ *and* x_3 *of* $\mathrm{d}^\diamond_{\mathcal{A}}(P_3^\bullet)$, *there exists a primitive basis* x_2 *of* $\mathrm{d}^\diamond_{\mathcal{A}}(P_2^\bullet)$ *such that*

$$\Delta((x_1, r_1) \otimes (x_3, r_3)) = (x_2, r_2).$$

Proof. The given exact sequence induces in each degree a a (split) short exact sequence in $\mathrm{Mod}^{\mathrm{f}}(\mathcal{A})$ of the form

$$0 \to P_1^a \to P_2^a \xrightarrow{\phi^a} P_3^a \to 0. \tag{3.1.3}$$

These sequences imply that $\mathrm{rk}_{\mathcal{A}}(P_1^a) + \mathrm{rk}_{\mathcal{A}}(P_3^a) = \mathrm{rk}_{\mathcal{A}}(P_2^a)$ and so the claimed equality $r_1 + r_3 = r_2$ follows directly from the definition (2.3.2) of each term $\mathrm{rr}_{\mathcal{A}}(P_j^\bullet)$.

In addition, if one sets $r_j^a = \mathrm{rr}_{\mathcal{A}}(P_j^a)$ for $j = 1, 2, 3$, then the above sequences combine with Remark 2.7 to show that the remaining claim is valid provided that for any primitive bases x_1^a of $\mathrm{d}^\diamond_{\mathcal{A}}(P_1^a)$ and x_3^a of $\mathrm{d}^\diamond_{\mathcal{A}}(P_3^a)$, there exists a primitive basis x_2^a of $\mathrm{d}^\diamond_{\mathcal{A}}(P_2^a)$ such that the isomorphism

$$\Delta^a : \mathrm{d}_{\mathcal{A}}(P_1^a) \otimes \mathrm{d}_{\mathcal{A}}(P_3^a) \cong \mathrm{d}_{\mathcal{A}}(P_2^a)$$

induced by (3.1.3) sends $(x_1^a, r_1^a) \otimes (x_3^a, r_3^a)$ to (x_2^a, r_2^a).

For this we fix an $\mathcal{A}$-module section σ to ϕ^a and note that, if x_j corresponds to the ordered $\mathcal{A}$-basis $\{b_{js}\}_{1 \le s \le r_j^a}$ of P_j^a for each $j = 1, 3$, then one obtains an ordered $\mathcal{A}$-basis $\{b_s^\sigma\}_{1 \le s \le r_2^a}$ of P_2^a by setting

$$b_s^\sigma := \begin{cases} b_s^1, & \text{if } 1 \le s \le r_1^a, \\ \sigma(b^3_{s - r_1^a}), & \text{if } r_1^a < s \le r_2^a. \end{cases}$$

Then $\wedge_{j=1}^{j=r_2^a} b_j^\sigma$ is a primitive basis of $\mathrm{d}^\diamond_{\mathcal{A}}(P_2^a) = \bigcap_{\mathcal{A}}^{r_2^a} P_2^a$ and so the required result is true because the isomorphism Δ^a is defined (in [24]) by the condition that

$$\Delta^a\left((\wedge_{s=1}^{s=r_1^a} b_s^1, r_1^a) \otimes (\wedge_{t=1}^{t=r_3^a} b_t^3, r_3^a) \right) = (\wedge_{j=1}^{j=r_2^a} b_j^\sigma, r_2^a).$$

□

Lemma 3.5. *Let* $P^\bullet$ *be an acyclic complex in* $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ *and* b *a primitive basis of* $\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet)$. *Then* $\mathrm{rr}_{\mathcal{A}}(P^\bullet) = 0$ *and the canonical isomorphism* $\mathrm{d}_{\mathcal{A}}(P^\bullet) \cong (\xi(\mathcal{A}), 0)$ *sends* $(b, 0)$ *to* $(u, 0)$ *for some element* u *of* $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$.

Proof. The acyclicity of $P^\bullet$ implies that $\sum_{a\in\mathbb{Z}}(-1)^a\cdot \mathrm{rk}_{\mathcal{A}}(P^a) = 0$ and hence also that $\mathrm{rr}_{\mathcal{A}}(P^\bullet) = 0$, as claimed.

Regarding the second claim, we note that the 'only if' part of Lemma 3.3 (the argument for which does not require the existence of an integer a with $\mathrm{rk}_{\mathcal{A}}(P^a) \geq \mathrm{sr}(\mathcal{A})$) reduces us to proving the existence of a primitive basis b of $\mathrm{d}^{\diamond}_{\mathcal{A}}(P^\bullet)$ such that the canonical isomorphism $\mathrm{d}_{\mathcal{A}}(P^\bullet) \cong (\xi(\mathcal{A}), 0)$ sends $(b, 0)$ to $(u, 0)$ for some element u of $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$.

We first prove this in the special case that there exists an integer a such that P^i vanishes for all $i \notin \{a, a+1\}$. In this situation the acyclicity of $P^\bullet$ implies it has the form $\mathcal{A}^t \xrightarrow{\theta} \mathcal{A}^t$ for a suitable natural number t and isomorphism of $\mathcal{A}$-modules θ. In particular, if we write b for the primitive basis of $\mathrm{d}^{\diamond}_{\mathcal{A}}(P^\bullet)$ that corresponds to the standard basis of $\mathcal{A}^t$ (in both degrees a and $a+1$), then (2.1.6) implies that the canonical isomorphism $\mathrm{d}_{\mathcal{A}}(P^\bullet) \cong (\xi(\mathcal{A}), 0)$ sends $(b, 0)$ to $(\mathrm{Nrd}_A(M_\theta), 0)$, where M_θ is the matrix of θ with respect to the standard basis of $\mathcal{A}^t$. The required result is therefore true since $\mathrm{Nrd}_A(M_\theta)$ belongs to $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$.

Turning now to the general case, we write a and a' for the least and greatest integers m for which P^m is non-zero. Then, if necessary by taking the direct sum of $P^\bullet$ with a suitable collection of complexes of the form $\mathcal{A} \xrightarrow{\mathrm{id}} \mathcal{A}$ (and applying Lemma 3.4), we can assume that $\mathrm{rk}_{\mathcal{A}}(\ker(d^j)) \geq \mathrm{sr}(\mathcal{A})$ for each j with $a < j < a'$.

Then, since $P^\bullet$ is acyclic, the Bass Cancellation Theorem (cf. [28, Th. (41.20)]) combines with an easy downward induction on j to imply, firstly that each $\mathcal{A}$-module $\mathrm{im}(d^j) = \ker(d^{j+1})$ is free and hence that there is an isomorphism of $\mathcal{A}$-modules $P^j \cong \ker(d^j)\oplus\mathrm{im}(d^j)$, and secondly that each module $P^j/\ker(d^j) \cong \mathrm{im}(d^j)$ is free.

Now write $P_1^\bullet$ for the complex $P^a \xrightarrow{d^a} \mathrm{im}(d^a)$ where the first term is placed in degree a, and ι for the natural inclusion of complexes $P_1^\bullet \to P^\bullet$. Then there is a tautological short exact sequence of acyclic complexes

$$0 \to P_1^\bullet \xrightarrow{\iota} P^\bullet \to \mathrm{cok}(\iota) \to 0 \tag{3.1.4}$$

in $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ and, by applying Lemma 3.4 to this sequence, one can use an induction on the number of non-zero modules P^j to reduce to the special case (that P^j vanishes except in two consecutive degrees) that was considered earlier. □

3.1.3. We can now prove Proposition 3.2.

For $j \in \{1, 2\}$ and each integer a we set $r_j^a := \mathrm{rk}_{\mathcal{A}}(P_j^a)$. For each j and a we then fix an ordered $\mathcal{A}$-basis $\underline{b}_{j,a} := \{b_{j,ak}\}_{1\leq k\leq r_j^a}$ of P_j^a and write x_j for the associated primitive basis $\Upsilon(\underline{b}_{j,\bullet})$ of $\mathrm{d}^{\diamond}_{\mathcal{A}}(P_j^\bullet)$.

Then Lemma 3.3 implies that the stated claim is true if and only if there exists an element u of $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$ such that

$$\mathrm{d}_{\mathcal{A}}(\lambda)((x_1, r)) = (u\cdot x_2, r) \tag{3.1.5}$$

with $r := \mathrm{rr}_{\mathcal{A}}(P_1^\bullet) = \mathrm{rr}_{\mathcal{A}}(P_2^\bullet)$. To prove this we shall adapt an argument of Knudsen and Mumford from [55, proof of Th. 1] (see also [38, Chap. III, §3, Lem.]).

For this purpose we recall that the mapping cylinder of λ is the complex $Z_\lambda^\bullet$ that has $Z_\lambda^a = P_1^a\oplus P_2^a\oplus P_1^{a+1}$ in each degree a and is such that, with respect to these decompositions, the differential in degree a is represented by the matrix

$$\begin{pmatrix} d_1^a & 0 & -1 \\ 0 & d_2^a & \lambda^{a+1} \\ 0 & 0 & -d_1^{a+1} \end{pmatrix}$$

where d_j^a denotes the differential of $P_j^\bullet$ in degree a.

Then, with this notation there are quasi-isomorphisms $\lambda_1 : P_1^\bullet \to Z_\lambda^\bullet$, $\lambda_2 : P_2^\bullet \to Z_\lambda^\bullet$ and $\lambda_2' : Z_\lambda^\bullet \to P_2^\bullet$ in $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ with (in the obvious notation)

$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \lambda_2' = \begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix}.$$

In addition, one checks that $\lambda_2' \circ \lambda_1 = \lambda$ and $\lambda_2' \circ \lambda_2 = \mathrm{id}_{P_2^\bullet}$ and that both of the complexes $\mathrm{cok}(\lambda_1)$ and $\mathrm{cok}(\lambda_2)$ are acyclic objects of $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$. It follows that, for each i in $\{1, 2\}$ there are natural isomorphisms

$$\mathrm{d}_{\mathcal{A}}(\lambda_i)' : \mathrm{d}_{\mathcal{A}}(P_i^\bullet) \cong \mathrm{d}_{\mathcal{A}}(P_i^\bullet) \otimes \mathrm{d}_{\mathcal{A}}(\mathrm{cok}(\lambda_i)) \cong \mathrm{d}_{\mathcal{A}}(Z_\lambda^\bullet),$$

where the first map is induced by the acyclicity of $\mathrm{cok}(\lambda_i)$ and the second by the tautological short exact sequence

$$0 \to P_i^\bullet \xrightarrow{\lambda_i} Z_\lambda^\bullet \to \mathrm{cok}(\lambda_i) \to 0.$$

By applying Lemma 3.5 to the first isomorphism in this composite, and then Lemma 3.4 to the second, one deduces that there exists an element u_i of $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$ and a primitive basis y_i of $\mathrm{d}^\diamond_{\mathcal{A}}(Z_\lambda^\bullet)$ such that

$$\mathrm{d}_{\mathcal{A}}(\lambda_i)'((x_i, r)) = ((u_i \cdot y_i, r)).$$

On the other hand, the explicit construction of $\mathrm{d}_{\mathcal{A}}(\lambda)$ in [24] implies directly that

$$\mathrm{d}_{\mathcal{A}}(\lambda) = (\mathrm{d}_{\mathcal{A}}(\lambda_2)')^{-1} \circ \mathrm{d}_{\mathcal{A}}(\lambda_1)'$$

and hence that

$$\mathrm{d}_{\mathcal{A}}(\lambda)((x_1, r)) = ((u_1 u_2^{-1} u_3) \cdot x_2, r)$$

where u_3 is the element of $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$ that is defined by (Lemma 3.3 and) the equality

$$y_1 = u_3 \cdot y_2.$$

In particular, since the element $u := u_1 u_2^{-1} u_3$ belongs to $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$, this proves the required equality (3.1.5) (for this value of u), and hence completes the proof of Proposition 3.2. □

3.2. **Locally-primitive bases.** We can now make the key definitions of §3.

Definition 3.6. Let C be an object of $\mathsf{D}^{\mathrm{lf}}(\mathcal{A})$ and b an element of $\mathrm{d}_{\mathcal{A}}(C)_F$.
(i) We say that b is a 'primitive basis' of the $\xi(\mathcal{A})$-module $\mathrm{d}_{\mathcal{A}}(C)$ if C is isomorphic in $\mathsf{D}^{\mathrm{lf}}(\mathcal{A})$ to a complex $P^\bullet$ in $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ with the property that in some degree a one has $\mathrm{rk}_{\mathcal{A}}(P^a) \geq \mathrm{sr}(\mathcal{A})$ and, with respect to the induced identification $\mathrm{d}_{\mathcal{A}}(C) \cong (\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet), \mathrm{rr}_{\mathcal{A}}(P^\bullet))$, the element b corresponds to $(b', \mathrm{rr}_{\mathcal{A}}(P^\bullet))$ for some primitive basis b' of $\mathrm{d}^\diamond_{\mathcal{A}}(P^\bullet)$.
(ii) We say that b is a 'locally-primitive basis' of the $\xi(\mathcal{A})$-module $\mathrm{d}_{\mathcal{A}}(C)$ if for all $\mathfrak{p}$ in $\mathrm{Spm}(R)$ the image of b in the $\mathfrak{p}$-completion $\mathrm{d}_{\mathcal{A}}(C)_{F,\mathfrak{p}} = \mathrm{d}_{\mathcal{A}_\mathfrak{p}}(C_\mathfrak{p})_{F_\mathfrak{p}}$ of $\mathrm{d}_{\mathcal{A}}(C)_F$ is a primitive basis of the $\xi(\mathcal{A}_\mathfrak{p})$-module $\mathrm{d}_{\mathcal{A}_\mathfrak{p}}(C_\mathfrak{p})$.

(iii) We say that b is a 'generically-primitive basis' of the $\xi(\mathcal{A})$-module $\mathrm{d}_{\mathcal{A}}(C)$ if its image in $\mathrm{d}_{\mathcal{A}}(C)_F = \mathrm{d}_A(C_F)$ is a primitive basis of the $\zeta(A)$-module $\mathrm{d}_A(C_F)$.

The arguments of §3.1 imply that the notion of (locally-)primitive basis is intrinsic to objects of $\mathsf{D}^{\mathrm{lf}}(\mathcal{A})$ and further that, in this setting, it has the following useful functorial properties.

Proposition 3.7. *Let C be an object of $\mathsf{D}^{\mathrm{lf}}(\mathcal{A})$.*

(i) *If C is acyclic, then the canonical isomorphism*

$$\mathrm{d}_{\mathcal{A}}(C) \cong (\xi(\mathcal{A}), 0)$$

coming from Theorem 2.5 sends any primitive basis of $\mathrm{d}_{\mathcal{A}}(C)$ to an element of $(\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A})), 0)$.

(ii) *Let b be a primitive basis of $\mathrm{d}_{\mathcal{A}}(C)$. Then an element b' of $\mathrm{d}_{\mathcal{A}}(C)_F$ is a primitive basis of $\mathrm{d}_{\mathcal{A}}(C)$ if and only if $b' = u \cdot b$ for some u in $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$.*

(iii) *Let*

$$C_1 \to C \to C_3 \to C_1[1]$$

be an exact triangle in $\mathsf{D}^{\mathrm{lf}}(\mathcal{A})$. Then for any primitive bases, respectively locally-primitive bases, x_1 of $\mathrm{d}_{\mathcal{A}}(C_1)$ and x_3 of $\mathrm{d}_{\mathcal{A}}(C_3)$, the canonical isomorphism

$$\mathrm{d}_{\mathcal{A}}(C_1) \otimes \mathrm{d}_{\mathcal{A}}(C_3) \cong \mathrm{d}_{\mathcal{A}}(C)$$

coming from Theorem 2.5 sends $x_1 \otimes x_3$ to a primitive basis, respectively locally-primitive basis, of $\mathrm{d}_{\mathcal{A}}(C)$.

Proof. The key point is Proposition 3.2 implies that if b is a primitive basis of $\mathrm{d}^{\diamond}_{\mathcal{A}}(P_1^\bullet)$ for any complex $P_1^\bullet$ in $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ that is both isomorphic in $\mathsf{D}^{\mathrm{lf}}(\mathcal{A})$ to C and such that in some degree a one has $\mathrm{rk}_{\mathcal{A}}(P_1^a) \geq \mathrm{sr}(\mathcal{A})$, then it also corresponds to a primitive basis of $\mathrm{d}^{\diamond}_{\mathcal{A}}(P_2^\bullet)$ for any other such complex $P_2^\bullet$ in $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$.

Given this fact, the assertions of claims (i), (ii) and (iii) follow directly from the respective results of Lemmas 3.5, 3.3 and 3.4. □

We next clarify the links between the varying notions of basis considered above.

Proposition 3.8. *Let C be an object of $\mathsf{D}^{\mathrm{lf}}(\mathcal{A})$ and b an element of $\mathrm{d}_{\mathcal{A}}(C)_F$. Then the following claims are valid.*

(i) *If b is a primitive basis of $\mathrm{d}_{\mathcal{A}}(C)$, then it is also both a locally-primitive and generically-primitive basis of $\mathrm{d}_{\mathcal{A}}(C)$.*

(ii) *If $\mathrm{d}_{\mathcal{A}}(C)$ has a primitive basis, then the converse of claim (i) is true.*

(iii) *If b is an locally-primitive basis of $\mathrm{d}_{\mathcal{A}}(C)$, then it is a $\xi(\mathcal{A})$-basis of $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}$.*

(iv) *Assume $\mathrm{d}_{\mathcal{A}}(C)$ has a primitive basis. Then the $\xi(\mathcal{A})$-module $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}$ is free and the following claims are valid.*

  (a) *Every basis of $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}$ corresponds to a primitive basis of $\mathrm{d}_{\mathcal{A}}(C)$ if and only if $\xi(\mathcal{A})^\times = \mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$.*

  (b) *Every basis of $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}$ corresponds to a locally-primitive basis of $\mathrm{d}_{\mathcal{A}}(C)$ if and only if $\xi(\mathcal{A})^\times$ is the full pre-image of the direct product of $\mathrm{Nrd}_{A_{\mathfrak{p}}}(\mathrm{K}_1(\mathcal{A}_{\mathfrak{p}}))$ over $\mathfrak{p}$ in $\mathrm{Spm}(R)$ under the diagonal map*

$$\zeta(A)^\times \to \prod_{\mathfrak{p}}\zeta(A_{\mathfrak{p}})^\times = \prod_{\mathfrak{p}}\mathrm{Nrd}_{A_{\mathfrak{p}}}(\mathrm{K}_1(A_{\mathfrak{p}})).$$

Proof. Claim (i) follows easily by a direct comparison of the respective definitions that are given in Definition 3.6.

To prove claim (ii), we fix a primitive basis b of $\mathrm{d}_{\mathcal{A}}(C)$. Then, in view of Proposition 3.7(ii), it is enough to show that if b' is both a generically-primitive and locally-primitive basis of $\mathrm{d}_{\mathcal{A}}(C)$, then one has $b' = u\cdot b$ for some u in $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$.

Now, as b' is a generically-primitive basis, Proposition 3.7(ii) implies the existence of a (unique) element v of $\mathrm{Nrd}_A(\mathrm{K}_1(A))$ such that $b' = v\cdot b$ in $\mathrm{d}_A(C_F)$. Then, since b' is a locally-primitive basis, we can deduce from Proposition 3.7(ii) (and the uniqueness of v) that, for every $\mathfrak{p}$ in $\mathrm{Spm}(R)$, the image of v in $\mathrm{Nrd}_{A_{\mathfrak{p}}}(\mathrm{K}_1(A_{\mathfrak{p}}))$ belongs to $\mathrm{Nrd}_{A_{\mathfrak{p}}}(\mathrm{K}_1(\mathcal{A}_{\mathfrak{p}}))$. But then, by a general result of K-theory (which, for convenience, we defer to Lemma 4.3(ii) below), this implies that v belongs to $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$, as required.

To prove claim (iii) we note that, for each $\mathfrak{p}$ in $\mathrm{Spm}(R)$, the given element b is a basis of the $\xi(\mathcal{A}_{\mathfrak{p}})$-module $\mathrm{d}_{\mathcal{A}_{\mathfrak{p}}}(C_{\mathfrak{p}})^{\mathrm{u}} = \mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}_{\mathfrak{p}}$. This fact implies, firstly, that for every such $\mathfrak{p}$ one has

$$\big(\zeta(A)\cdot b\big)_{\mathfrak{p}} = \zeta(A_{\mathfrak{p}})\cdot b = \big(\xi(\mathcal{A}_{\mathfrak{p}})\cdot b\big)_{F_{\mathfrak{p}}} = \big(\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}_F\big)_{\mathfrak{p}}$$

and hence that $\zeta(A)\cdot b = \mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}_F$. Then, upon applying the general result of [28, Prop. (4.21)(vi)], one deduces the required equality by noting that

$$\begin{aligned}\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}} &= \mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}_F \cap \bigcap\nolimits_{\mathfrak{p}}\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}_{\mathfrak{p}}\\ &= (\zeta(A)\cdot b)\cap\bigcap\nolimits_{\mathfrak{p}}(\xi(\mathcal{A}_{\mathfrak{p}})\cdot b)\\ &= \big(\zeta(A)\cap\bigcap\nolimits_{\mathfrak{p}}\xi(\mathcal{A}_{\mathfrak{p}})\big)\cdot b\\ &= \xi(\mathcal{A})\cdot b.\end{aligned}$$

Here, in each intersection $\mathfrak{p}$ runs over $\mathrm{Spm}(R)$, and the last equality is valid because

$$\zeta(A)\cap\bigcap\nolimits_{\mathfrak{p}}\xi(\mathcal{A}_{\mathfrak{p}}) = \bigcap\nolimits_{\mathfrak{p}}\big(\zeta(A)\cap\xi(\mathcal{A})_{\mathfrak{p}}\big) = \bigcap\nolimits_{\mathfrak{p}}\xi(\mathcal{A})_{(\mathfrak{p})} = \xi(\mathcal{A}).$$

Finally, to prove claim (iv), we fix a primitive basis b of $\mathrm{d}_{\mathcal{A}}(C)$ and an arbitrary element b' of $\mathrm{d}_{\mathcal{A}}(C)_F$. Then one has $b' = x\cdot b$ for a unique element $x = x_{b',b}$ of $\zeta(A)$.

Now, since b is a basis of the $\xi(\mathcal{A})$-module $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}$ one knows that b' is a basis of this module if and only if $x\in\xi(\mathcal{A})^\times$.

In a similar way, the result of Proposition 3.7(ii) implies that b' is a primitive basis, respectively locally-primitive basis, of $\mathrm{d}_{\mathcal{A}}(C)$ if and only if one has $x\in\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$, respectively the image of x in $\zeta(A_{\mathfrak{p}})^\times = \mathrm{Nrd}_{A_{\mathfrak{p}}}(\mathrm{K}_1(A_{\mathfrak{p}}))$ belongs to $\mathrm{Nrd}_{A_{\mathfrak{p}}}(\mathrm{K}_1(\mathcal{A}_{\mathfrak{p}}))$ for every $\mathfrak{p}$ in $\mathrm{Spm}(R)$.

The assertions in claim (iv) follow directly from these observations. □

3.3. **Primitive bases and Euler characteristics.** The result of Proposition 3.7(iii) leaves open the problem of whether there are conditions on $\mathcal{A}$ under which the freeness of $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}$ as a $\xi(\mathcal{A})$-module can itself imply the existence of a locally-primitive basis, or even a primitive basis, of $\mathrm{d}_{\mathcal{A}}(C)$.

To shed light on this problem, in this section we reinterpret the conditions that $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}$ is free, that $\mathrm{d}_{\mathcal{A}}(C)$ has a locally-primitive basis and that $\mathrm{d}_{\mathcal{A}}(C)$ has a primitive basis in terms of the Euler characteristic $\chi_{\mathcal{A}}(C)$.

As necessary preparation for this result, we shall also make some general observations concerning class groups of orders.

3.3.1. We note first that the argument of [28, Rem. (49.11)(iv)] shows that $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$ is naturally isomorphic to the 'locally-free classgroup' $\mathrm{Cl}(\mathcal{A})$ of $\mathcal{A}$, as defined in [28, (49.10)].

We recall $\mathrm{Cl}(\mathcal{A})$ is finite, that it is equal to the set of stable isomorphism classes $[I]$ of locally-free, rank one, $\mathcal{A}$-submodules I of A and that its addition is defined by setting

$$[I_1] + [I_2] := [I_3]$$

whenever there is an isomorphism of $\mathcal{A}$-modules $I_1 \oplus I_2 \cong I_3 \oplus \mathcal{A}$. (Recall from [28, Rem. (49.11)(i)] that $\mathcal{A}$-modules I_1 and I_2 are stably-isomorphic if and only if $I_1 \oplus \mathcal{A}$ is isomorphic to $I_2 \oplus \mathcal{A}$.)

We recall further that if $\mathcal{A}$ is commutative, then $\mathrm{Cl}(\mathcal{A})$ is naturally isomorphic to the multiplicative group of isomorphism classes of locally-free, rank one, $\mathcal{A}$-submodules of A.

Lemma 3.9. *The association $P \mapsto \bigcap_{\mathcal{A}}^{\mathrm{rk}_{\mathcal{A}}(P)} P$ for each P in $\mathrm{Mod}^{\mathrm{lf}}(\mathcal{A})$ induces a well-defined homomorphism of abelian groups*

$$\Delta_{\mathcal{A}} : \mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A}) \to \mathrm{Cl}(\xi(\mathcal{A})).$$

Proof. Since $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$ is naturally isomorphic to $\mathrm{Cl}(\mathcal{A})$, this result is equivalent to the following two claims: firstly, if I_1 and I_2 are any locally-free, rank one, $\mathcal{A}$-modules that are stably-isomorphic, then the $\xi(\mathcal{A})$-modules $\bigcap_{\mathcal{A}}^1 I_1$ and $\bigcap_{\mathcal{A}}^1 I_2$ are isomorphic; secondly, if I_1, I_2 and I_3 are any locally-free, rank one, $\mathcal{A}$-modules for which the $\mathcal{A}$-modules $I_1 \oplus I_2$ and $I_3 \oplus \mathcal{A}$ are isomorphic, then the $\xi(\mathcal{A})$-modules $(\bigcap_{\mathcal{A}}^1 I_1) \otimes_{\xi(\mathcal{A})} (\bigcap_{\mathcal{A}}^1 I_2)$ and $\bigcap_{\mathcal{A}}^1 I_3$ are isomorphic.

To prove the first claim we note that if I_1 and I_2 are stably-isomorphic, then there exists an isomorphism of $\mathcal{A}$-modules

$$I_1 \oplus \mathcal{A} \cong I_2 \oplus \mathcal{A}.$$

Then, since $\bigcap_{\mathcal{A}}^1 \mathcal{A}$ is a free $\xi(\mathcal{A})$-module of rank one (by (2.2.3)), the argument of Lemma 3.4 induces an isomorphism of $\xi(\mathcal{A})$-modules of the required form

$$\begin{aligned}\bigcap\nolimits_{\mathcal{A}}^1 I_1 &\cong (\bigcap\nolimits_{\mathcal{A}}^1 I_1) \otimes_{\xi(\mathcal{A})} (\bigcap\nolimits_{\mathcal{A}}^1 \mathcal{A}) \\ &\cong (\bigcap\nolimits_{\mathcal{A}}^1 I_2) \otimes_{\xi(\mathcal{A})} (\bigcap\nolimits_{\mathcal{A}}^1 \mathcal{A}) \\ &\cong \bigcap\nolimits_{\mathcal{A}}^1 I_2.\end{aligned}$$

The second claim is proved in a similar way since the given isomorphism $I_1 \oplus I_2 \cong I_3 \oplus \mathcal{A}$ induces an isomorphism of $\xi(\mathcal{A})$-modules

$$\begin{aligned}(\bigcap\nolimits_{\mathcal{A}}^{1} I_1) \otimes_{\xi(\mathcal{A})} (\bigcap\nolimits_{\mathcal{A}}^{1} I_2) &\cong \bigcap\nolimits_{\mathcal{A}}^{2} (I_1 \oplus I_2) \\ &\cong \bigcap\nolimits_{\mathcal{A}}^{2} (I_3 \oplus \mathcal{A}) \\ &\cong (\bigcap\nolimits_{\mathcal{A}}^{1} I_3) \otimes_{\xi(\mathcal{A})} (\bigcap\nolimits_{\mathcal{A}}^{1} \mathcal{A}) \\ &\cong \bigcap\nolimits_{\mathcal{A}}^{1} I_3.\end{aligned}$$

□

Remark 3.10. The homomorphism $\Delta_{\mathcal{A}}$ constructed above has a conceptual interpretation. To explain this we recall from Remark 2.6 that the reduced determinant functor $\mathrm{d}_{\mathcal{A},\varpi}$ induces a monoidal functor $\phi^{\mathrm{lf}}_{\mathcal{A},\varpi} : \mathcal{V}^{\mathrm{lf}}(\mathcal{A}) \to \mathcal{P}(\xi(\mathcal{A}))$. The latter functor in turn induces a homomorphism of abelian groups $\pi_0(\phi^{\mathrm{lf}}_{\mathcal{A},\varpi})$ from the Grothendieck group $\mathrm{K}_0^{\mathrm{lf}}(\mathcal{A})$ to the Picard group $\mathrm{Pic}(\xi(\mathcal{A}))$ of the commutative ring $\xi(\mathcal{A})$ (cf. [24, Rem. 5.5]). This Picard group is canonically isomorphic to $\mathrm{Cl}(\xi(\mathcal{A}))$ and, with respect to this identification, $\Delta_{\mathcal{A}}$ is equal to the composite

$$\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A}) \to \mathrm{K}_0^{\mathrm{lf}}(\mathcal{A}) \xrightarrow{\pi_0(\phi^{\mathrm{lf}}_{\mathcal{A},\varpi})} \mathrm{Pic}(\xi(\mathcal{A})) \cong \mathrm{Cl}(\xi(\mathcal{A}))$$

in which the first arrow is the natural inclusion.

We shall next define a canonical subgroup $\ker(\Delta_{\mathcal{A}})^{\mathrm{lp}}$ of the kernel of $\Delta_{\mathcal{A}}$ comprising 'locally-primitive classes'.

To do this we write $J_f(A)$ for the group of finite ideles of the F-algebra A and $\{z_1, z_2\}$ for the standard basis of A^2. For each locally-free, rank one, $\mathcal{A}$-submodule I of A we then write $M(I)$ for the coset of $\prod_{\mathfrak{p}} \mathrm{GL}_2(\mathcal{A}_{\mathfrak{p}})$ in $\mathrm{GL}_2(J_f(A))$ comprising matrices $M = (M_{\mathfrak{p}})_{\mathfrak{p}}$ with the property that for all $\mathfrak{p}$ (in $\mathrm{Spm}(R)$) the set $\{M_{\mathfrak{p}} z_1, M_{\mathfrak{p}} z_2\}$ is an $\mathcal{A}_{\mathfrak{p}}$-basis of $I_{\mathfrak{p}} \oplus \mathcal{A}_{\mathfrak{p}}$.

We write $\mathrm{Nrd}_{J_f(A)}$ for the homomorphism $\mathrm{GL}_2(J_f(A)) \to J_f(\zeta(A))$ that is induced by the product map $\prod_{\mathfrak{p}} \mathrm{Nrd}_{A_{\mathfrak{p}}}$. We also identify $\zeta(A)^\times$ with its image under the natural diagonal embedding $\zeta(A)^\times \to J_f(\zeta(A))$.

Lemma 3.11. *Let* $\ker(\Delta_{\mathcal{A}})^{\mathrm{lp}}$ *denote the subset of* $\mathrm{Cl}(\mathcal{A})$ *comprising elements* $[I]$ *for which the set* $M(I)$ *contains a matrix* M *with* $\mathrm{Nrd}_{J_f(A)}(M) \in \zeta(A)^\times$. *Then* $\ker(\Delta_{\mathcal{A}})^{\mathrm{lp}}$ *is a well-defined subgroup of* $\ker(\Delta_{\mathcal{A}})$.

Proof. The set $X := \ker(\Delta_{\mathcal{A}})^{\mathrm{lp}}$ is well-defined since if $[I] = [J]$, then the $\mathcal{A}$-modules $I \oplus \mathcal{A}$ and $J \oplus \mathcal{A}$ are isomorphic and so there exists a matrix M^* in $\mathrm{GL}_2(A)$ such that N belongs to $M(I)$ if and only if $M^* \cdot N$ belongs to $M(J)$.

To show X is contained in $\ker(\Delta_{\mathcal{A}})$ it is enough to prove that $\bigcap^1_{\mathcal{A}} I$ is a free $\xi(\mathcal{A})$-module whenever $[I]$ belongs to X. This is true since if we fix M in $M(I)$ with $\mathrm{Nrd}_{J_f(A)}(M) \in \zeta(A)^\times$, then the $\xi(\mathcal{A})$-module isomorphisms

$$\begin{aligned}\bigcap\nolimits^1_{\mathcal{A}} I &\cong \bigcap\nolimits^1_{\mathcal{A}} I \otimes_{\xi(\mathcal{A})} \bigcap\nolimits^1_{\mathcal{A}} \mathcal{A}\\ &\cong \bigcap\nolimits^2_{\mathcal{A}} (I \oplus \mathcal{A})\\ &= \xi(\mathcal{A}) \cdot (\mathrm{Nrd}_{J_f(A)}(M) \cdot (z_1 \wedge_A z_2))\end{aligned}$$

imply $\bigcap^1_{\mathcal{A}} I$ is isomorphic to $\xi(\mathcal{A})$.

Finally, to show X is a subgroup of $\ker(\Delta_{\mathcal{A}})$ it is enough to prove the following claims:

(i) If $[I_1]$ belongs to X and I_2 is any locally-free, rank one, $\mathcal{A}$-module for which there exists an isomorphism of $\mathcal{A}$-modules $I_1 \oplus I_2 \cong \mathcal{A} \oplus \mathcal{A}$, then $[I_2]$ belongs to X.

(ii) If $[I_1]$ and $[I_2]$ belong to X, then any isomorphism of $\mathcal{A}$-modules $I_1 \oplus I_2 \cong I_3 \oplus \mathcal{A}$ implies $[I_3]$ belongs to X.

Claim (i) is true since the isomorphism $I_1 \oplus I_2 \cong \mathcal{A} \oplus \mathcal{A}$ implies the existence of a matrix M_* in $\mathrm{GL}_4(A)$ such that if M_1 and M_2 belong to $M(I_1)$ and $M(I_2)$, then in $\mathrm{GL}_4(J_f(A))$ one has

$$M_* \cdot \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \mathrm{Id}_4,$$

where Id_n is the identity matrix in $\mathrm{GL}_n(J_f(A))$, and so

$$\mathrm{Nrd}_{J_f(A)}(M_1) \cdot \mathrm{Nrd}_{J_f(A)}(M_2) = \mathrm{Nrd}_A(M_*)^{-1}$$

belongs to $\zeta(A)^\times$.

In a similar way, the above claim (ii) follows from the fact that the induced isomorphism

$$(I_1 \oplus \mathcal{A}) \oplus (I_2 \oplus \mathcal{A}) \cong (I_3 \oplus \mathcal{A}) \oplus (\mathcal{A} \oplus \mathcal{A})$$

of $\mathcal{A}$-modules implies the existence of a matrix M_*' in $\mathrm{GL}_4(A)$ such that if M_1 and M_2 belongs to $M(I_1)$ and $M(I_2)$, then there exists a matrix M_3 in $M(I_3)$ such that

$$M_*' \cdot \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \begin{pmatrix} M_3 & 0 \\ 0 & \mathrm{Id}_2 \end{pmatrix}$$

in $\mathrm{GL}_4(J_f(A))$, and hence

$$\mathrm{Nrd}_{J_f(A)}(M_3) = \mathrm{Nrd}_A(M_*') \cdot \mathrm{Nrd}_{J_f(A)}(M_1) \cdot \mathrm{Nrd}_{J_f(A)}(M_2).$$

□

3.3.2. We can now state the main result of this section. This result provides explicit criteria in terms of Euler characteristics that determine whether primitive or locally-primitive bases exist.

Theorem 3.12. *Let C be an object of $\mathsf{D}^{\mathrm{lf},0}(\mathcal{A})$. Then $\chi_{\mathcal{A}}(C)$ belongs to $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$ and the following claims are valid.*

(i) $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}$ *is a free $\xi(\mathcal{A})$-module if and only if $\chi_{\mathcal{A}}(C)$ belongs to* $\ker(\Delta_{\mathcal{A}})$.

(ii) $\mathrm{d}_{\mathcal{A}}(C)$ *has a locally-primitive basis if and only if $\chi_{\mathcal{A}}(C)$ belongs to* $\ker(\Delta_{\mathcal{A}})^{\mathrm{lp}}$.

(iii) $\mathrm{d}_{\mathcal{A}}(C)$ *has a primitive basis if and only if $\chi_{\mathcal{A}}(C)$ vanishes.*

Proof. We first fix a complex $P^\bullet$ in $\mathsf{C}^{\rm lf}(\mathcal{A})$ that is isomorphic in $\mathsf{D}^{\rm lf}(\mathcal{A})$ to C. Then, by a standard construction of homological algebra (as, for example, in [32, Rapport, Lem. 4.7]), there exists a quasi-isomorphism of complexes of finitely generated $\mathcal{A}$-modules of the form

$$\theta : Q^\bullet \to P^\bullet,$$

where $Q^\bullet$ is bounded and has the property that if a is the lowest degree of a non-zero module Q^a, then P^j vanishes for all $j < a+2$ and Q^j is a free $\mathcal{A}$-module for all $j > a$. We set $r_j := {\rm rk}_{\mathcal{A}}(Q^j)$ in each degree j and note that, if necessary after replacing $Q^\bullet$ by the direct sum of $Q^\bullet$ and the (acyclic) complex

$$\mathcal{A} \xrightarrow{\rm id} \mathcal{A},$$

where the first term is placed in degree a, we can assume that $r_a \geq 2$, and hence also that $r_a \geq {\rm sr}(\mathcal{A})$.

Now the mapping cone $D^\bullet$ of θ is an acyclic complex for which in each degree j one has $D^j = P^j \oplus Q^{j+1}$. In particular, since D^j belongs to ${\rm Mod}^{\rm lf}(\mathcal{A})$ for all $j \neq a-1$, the acyclicity of $D^\bullet$ combines with the Krull-Schmidt-Azumaya Theorem (as in the argument of [24, Lem. 5.2]) to imply Q^a belongs to ${\rm Mod}^{\rm lf}(\mathcal{A})$, and hence that $Q^\bullet$ belongs to $\mathsf{C}^{\rm lf}(\mathcal{A})$.

To prove claim (i) we now use [28, Prop. (49.3)] to choose an isomorphism of $\mathcal{A}$-modules of the form

$$Q^a \cong (I \oplus \mathcal{A}) \oplus N,$$

where I is locally-free of rank one and N is free of rank $r_a - 2$. Then, since each of the modules Q^j for $j \neq a$ is free, the $\xi(\mathcal{A})$-module ${\rm d}_{\mathcal{A}}(C)^{\rm u}$ is isomorphic to

$$\begin{aligned}(3.3.1)\quad {\rm d}^\diamond_{\mathcal{A}}(Q^\bullet) =&\bigotimes\nolimits_{j\in\mathbb{Z}}\big({\bigcap}^{r_j}_{\mathcal{A}}Q^j\big)^{(-1)^j}\\ \cong&\big({\bigcap}^{2}_{\mathcal{A}}(I\oplus\mathcal{A})\big)^{(-1)^a}\otimes_{\xi(\mathcal{A})}\Big(\big({\bigcap}^{r_a-2}_{\mathcal{A}}N\big)^{(-1)^a}\otimes_{\xi(\mathcal{A})}\bigotimes\nolimits_{j\in\mathbb{Z}\setminus\{a\}}\big({\bigcap}^{r_j}_{\mathcal{A}}Q^j\big)^{(-1)^j}\Big)\\ \cong&\big({\bigcap}^{2}_{\mathcal{A}}(I\oplus\mathcal{A})\big)^{(-1)^a}.\end{aligned}$$

In particular, since the $\xi(\mathcal{A})$-module ${\bigcap}^2_{\mathcal{A}}(I\oplus\mathcal{A})$ is isomorphic to

$$\big({\bigcap}^{1}_{\mathcal{A}}I\big)\otimes_{\xi(\mathcal{A})}\big({\bigcap}^{1}_{\mathcal{A}}\mathcal{A}\big)\cong\big({\bigcap}^{1}_{\mathcal{A}}I\big),$$

the equivalence in claim (i) is a consequence of the fact that $\Delta_{\mathcal{A}}$ sends the element

$$\begin{aligned}(3.3.2)\qquad \chi_{\mathcal{A}}(C) =&\,\chi_{\mathcal{A}}(Q^\bullet)\\ =&\sum\nolimits_{j\in\mathbb{Z}}(-1)^j(Q^j)\\ =&\,(-1)^a(I)+\big((-1)^a(r_a-1)+\sum\nolimits_{j\in\mathbb{Z}\setminus\{a\}}(-1)^j r_j\big)(\mathcal{A})\end{aligned}$$

to $(-1)^a$ times the isomorphism class of ${\bigcap}^1_{\mathcal{A}}I$. This proves claim (i).

Next we note that the isomorphism (3.3.1) implies ${\rm d}_{\mathcal{A}}(C)$ has a locally-primitive basis if and only if the module ${\rm d}^\diamond_{\mathcal{A}}((I\oplus\mathcal{A})[0]) = {\bigcap}^2_{\mathcal{A}}(I\oplus\mathcal{A})$ has a locally-primitive basis. To prove claim (ii) it thus enough to show that ${\bigcap}^2_{\mathcal{A}}(I\oplus\mathcal{A})$ has a locally-primitive basis if and only if the set $M(I)$ contains a matrix M such that ${\rm Nrd}_{J_f(A)}(M)$ belongs to $\zeta(A)^\times$.

To prove this we fix a matrix $M = (M_\mathfrak{p})_\mathfrak{p}$ in $M(I)$ and for each $\mathfrak{p}$ set $b_{1,\mathfrak{p}} := M_\mathfrak{p} \cdot z_1$ and $b_{2,\mathfrak{p}} := M_\mathfrak{p} \cdot z_2$. Then for each $\mathfrak{p}$ (in $\mathrm{Spm}(R)$) the equality (2.1.6) implies

$$b_{1,\mathfrak{p}} \wedge b_{2,\mathfrak{p}} = \mathrm{Nrd}_{A_\mathfrak{p}}(M_\mathfrak{p}) \cdot (z_1 \wedge z_2) = \mathrm{Nrd}_{J_f(A)}(M)_\mathfrak{p} \cdot (z_1 \wedge z_2) \tag{3.3.3}$$

and so the description (2.2.2) implies that

$$\big(\bigcap\nolimits_{\mathcal{A}}^2 (I \oplus \mathcal{A})\big)_\mathfrak{p} = \xi(\mathcal{A})_\mathfrak{p} \cdot (\mathrm{Nrd}_{J_f(A)}(M)_\mathfrak{p} \cdot (z_1 \wedge z_2)).$$

Assume now that $\mu := \mathrm{Nrd}_{J_f(A)}(M)$ belongs to $\zeta(A)^\times$. Then $\mu \cdot (z_1 \wedge z_2)$ is a generator of the $\zeta(A)$-module $\big(\bigcap\nolimits_{\mathcal{A}}^2 (I \oplus \mathcal{A})\big)_F$ and one has $\mu = \mathrm{Nrd}_{J_f(A)}(M)_\mathfrak{p}$ for all $\mathfrak{p}$ in $\mathrm{Spm}(R)$. By the argument of Proposition 3.8(iii), the above displayed equalities therefore imply that $\mu \cdot (z_1 \wedge z_2)$ is a locally-primitive basis of $\bigcap\nolimits_{\mathcal{A}}^2 (I \oplus \mathcal{A})$, as required.

To prove the converse we assume $\bigcap\nolimits_{\mathcal{A}}^2 (I \oplus \mathcal{A})$ has a locally-primitive basis b that corresponds to a choice of ordered $\mathcal{A}_\mathfrak{p}$-basis $\underline{b}_\mathfrak{p}$ of $I_\mathfrak{p} \oplus \mathcal{A}_\mathfrak{p}$ for each $\mathfrak{p}$ in $\mathrm{Spm}(R)$. Then, in this case, the equalities (3.3.3) imply the transition matrices $M_\mathfrak{p}$ from $\{z_1, z_2\}$ to $\underline{b}_\mathfrak{p}$ combine to give a matrix $M = (M_\mathfrak{p})_\mathfrak{p}$ in $\mathrm{GL}_2(J_f(A))$ with the property that

$$b = \mathrm{Nrd}_{J_f(A)}(M)_\mathfrak{p} \cdot (z_1 \wedge z_2)$$

for every $\mathfrak{p}$. These equalities combine to imply that $\mathrm{Nrd}_{J_f(A)}(M)$ belongs to $\zeta(A)^\times$, and this completes the proof of claim (ii).

Turning to claim (iii), we note Proposition 3.2 implies that $\mathrm{d}_\mathcal{A}(C)$ has a primitive basis if and only if C is isomorphic in $\mathsf{D}(\mathcal{A})$ to a complex $K^\bullet$ that belongs to both $\mathsf{C}^{\mathrm{lf},0}(\mathcal{A})$ and $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$ and also has the property that $\mathrm{rk}_\mathcal{A}(K^a) \geq \mathrm{sr}(\mathcal{A})$ in some degree a.

In particular, if such a complex $K^\bullet$ exists, then it is clear that the Euler characteristic $\chi_\mathcal{A}(C) = \chi_\mathcal{A}(K^\bullet)$ vanishes. Conversely, if $\chi_\mathcal{A}(C)$ vanishes, then the sum (3.3.2) also vanishes and so the $\mathcal{A}$-module $Q^a \cong (I \oplus \mathcal{A}) \oplus N$ is stably-free. Then, since $\mathrm{rk}_\mathcal{A}(Q^a) \geq 2$, the Bass Cancellation Theorem implies Q^a is a free $\mathcal{A}$-module of rank at least $\mathrm{sr}(\mathcal{A})$ and hence that the complex $Q^\bullet$ implies $\mathrm{d}_\mathcal{A}(C)$ has a primitive basis, as required. □

The following result describes two useful, and concrete, consequences of Theorem 3.12.

Corollary 3.13.

(i) *If $\mathcal{A}$ is commutative, then, for any object C of $\mathsf{D}^{\mathrm{lf},0}(\mathcal{A})$, an element of $\mathrm{d}_\mathcal{A}(C)_F$ is a primitive basis of $\mathrm{d}_\mathcal{A}(C)$ if and only if it is a basis of $\mathrm{d}_\mathcal{A}(C)^{\mathrm{u}}$ as a $\xi(\mathcal{A})$-module.*

(ii) *If $|\mathrm{Cl}(\mathcal{A})|$ does not divide $|\mathrm{Cl}(\xi(\mathcal{A}))|$, then there exist objects C of $\mathsf{D}^{\mathrm{lf},0}(\mathcal{A})$ for which $\mathrm{d}_\mathcal{A}(C)^{\mathrm{u}}$ is a free $\xi(\mathcal{A})$-module but $\mathrm{d}_\mathcal{A}(C)$ has no primitive basis.*

Proof. For claim (i) we note that, if $\mathcal{A}$ is commutative, then $\xi(\mathcal{A}) = \mathcal{A}$ and the map $\Delta_\mathcal{A}$ identifies with the identity automorphism of $\mathrm{Cl}(\mathcal{A}) = \mathrm{Cl}(\xi(\mathcal{A}))$. The group $\ker(\Delta_\mathcal{A})$ therefore vanishes and so the conditions in Theorem 3.12(i), (ii) and (iii) are equivalent.

In particular, in this case, if $\mathrm{d}_\mathcal{A}(C)^{\mathrm{u}}$ is a free $\xi(\mathcal{A})$-module, then $\mathrm{d}_\mathcal{A}(C)$ has a primitive basis b. In addition, any basis b' of the $\xi(\mathcal{A})$-module $\mathrm{d}_\mathcal{A}(C)^{\mathrm{u}}$ must differ from b by multiplication by an element of $\mathcal{A}^\times = \mathrm{Nrd}_A(\mathcal{A}^\times)$ and so Proposition 3.7(ii) implies b' is also a primitive basis of $\mathrm{d}_\mathcal{A}(C)$. This proves claim (i).

The hypothesis of claim (ii) implies that the group $\ker(\Delta_{\mathcal{A}})$ is not trivial. Fix a non-zero element c of this group and a locally-free, rank one, $\mathcal{A}$-module I that corresponds to c under the isomorphism $\mathrm{Cl}(\mathcal{A}) \cong \mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$.

Then the complex $C := I[0] \oplus \mathcal{A}[-1]$ belongs to $\mathsf{D}^{\mathrm{lf},0}(\mathcal{A})$ and its Euler characteristic $\chi_{\mathcal{A}}(C)$ is equal to c. From Theorem 3.12(i) and (iii) it therefore follows that the $\xi(\mathcal{A})$-module $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}$ is free and that $\mathrm{d}_{\mathcal{A}}(C)$ has no primitive basis. This proves claim (ii). □

4. Relative K-theory and zeta elements

In this section we fix a finite extension F of either $\mathbb{Q}$ or $\mathbb{Q}_p$ for some prime p. If F is a finite extension of $\mathbb{Q}$, then we write $\mathcal{O}_F$ for its ring of integers.

We fix a Dedekind domain R with field of fractions F. We also fix an R-order $\mathcal{A}$ in a finite dimensional separable F-algebra A and, for any extension field $\mathcal{F}$ of F, we consider the (finite dimensional, separable) $\mathcal{F}$-algebra $A_{\mathcal{F}} := \mathcal{F} \otimes_F A$.

4.1. **Relative K-theory.**

4.1.1. We write $\mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}})$ for the relative algebraic K-group of the ring inclusion $\mathcal{A} \subset A_{\mathcal{F}}$.

We recall from [81, p. 215] that this group can be described as a quotient (by certain explicit relations) of the free abelian group on elements (P, g, Q) where P and Q are finitely generated projective $\mathcal{A}$-modules and g an isomorphism of $A_{\mathcal{F}}$-modules $\mathcal{F} \otimes_R P \cong \mathcal{F} \otimes_R Q$.

We further recall that for any extension field $\mathcal{F}'$ of $\mathcal{F}$ there exists a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{K}_1(\mathcal{A}) & \longrightarrow & \mathrm{K}_1(A_{\mathcal{F}'}) & \xrightarrow{\partial_{\mathcal{A},\mathcal{F}'}} & \mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}'}) & \xrightarrow{\partial'_{\mathcal{A},\mathcal{F}'}} & \mathrm{K}_0(\mathcal{A}) \\
\Vert & & {\scriptstyle \iota_{A,\mathcal{F},\mathcal{F}'}}\uparrow & & {\scriptstyle \iota_{\mathcal{A},\mathcal{F},\mathcal{F}'}}\uparrow & & \Vert \\
\mathrm{K}_1(\mathcal{A}) & \longrightarrow & \mathrm{K}_1(A_{\mathcal{F}}) & \xrightarrow{\partial_{\mathcal{A},\mathcal{F}}} & \mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}}) & \xrightarrow{\partial'_{\mathcal{A},\mathcal{F}}} & \mathrm{K}_0(\mathcal{A})
\end{array} \tag{4.1.1}$$

in which the upper and lower rows are the long exact sequences in relative K-theory of the inclusions $\mathcal{A} \subset A_{\mathcal{F}'}$ and $\mathcal{A} \subset A_{\mathcal{F}}$ and the homomorphisms $\iota_{A,\mathcal{F},\mathcal{F}'}$ and $\iota_{\mathcal{A},\mathcal{F},\mathcal{F}'}$ are injective and induced by the inclusion $A_{\mathcal{F}} \subseteq A_{\mathcal{F}'}$. (For more details see [81, Th. 15.5].)

For each prime ideal $\mathfrak{p}$ of R we regard the group $\mathrm{K}_0(\mathcal{A}_{\mathfrak{p}}, A_{\mathfrak{p}})$ as a subgroup of $\mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}})$ by means of the canonical composite injective homomorphism

$$\Big(\bigoplus\nolimits_{\mathfrak{p}\in \mathrm{Spm}(R)} \mathrm{K}_0(\mathcal{A}_{\mathfrak{p}}, A_{\mathfrak{p}})\Big) \cong \mathrm{K}_0(\mathcal{A}, A) \xrightarrow{\iota_{\mathcal{A},F,\mathcal{F}}} \mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}}), \tag{4.1.2}$$

in which the isomorphism is described in the discussion following [28, (49.12)].

We write $p(\mathfrak{p})$ for the residue characteristic of each $\mathfrak{p}$ in $\mathrm{Spm}(R)$. If F is a finite extension of $\mathbb{Q}$ in $\mathbb{C}$, then for each $\mathfrak{p}$ in $\mathrm{Spm}(R)$, we write $\mathrm{Isom}_{\mathfrak{p}}$ for the set of field isomorphisms $j : \mathbb{C} \cong \mathbb{C}_{p(\mathfrak{p})}$ for which the induced embedding $F \subset \mathbb{C} \to \mathbb{C}_p$ induces the prime ideal $\mathfrak{p}$.

4.1.2. Since $A_{\mathcal{F}}$ is semisimple, one can compute in the Whitehead group $\mathrm{K}_1(A_{\mathcal{F}})$ by using the injective homomorphism

$$\mathrm{Nrd}_{A_{\mathcal{F}}} : \mathrm{K}_1(A_{\mathcal{F}}) \to \zeta(A_{\mathcal{F}})^{\times}$$

that is induced by taking reduced norms (see [28, §45.A]). The image of $\mathrm{Nrd}_{A_{\mathcal{F}}}$ is described explicitly by the Hasse-Schilling-Maass Norm Theorem (cf. [28, (7.48)]). We recall, in particular, that this map is bijective if $\mathcal{F}$ is either algebraically closed or a subfield of the completion $\mathbb{C}_p$ of an algebraic closure of $\mathbb{Q}_p$ for any prime p, but that, in general, its cokernel is of exponent 2.

We further recall that A is said to be 'ramified' at an archimedean place v of F if there exists a simple component A' in the Wedderburn decomposition of A that is ramified, as a simple central $\zeta(A')$-algebra, at a place of $\zeta(A')$ lying above v.

In the following result we construct, for suitable fields $\mathcal{F}$, a canonical 'extended boundary homomorphism' $\zeta(A_{\mathcal{F}})^\times \to \mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}})$ (that extends the special case considered by Burns and Flach in [17, §4.2, Lem. 9]).

Proposition 4.1. *Fix an embedding of fields $F \to \mathcal{F}$ with the following property:*

- *if F is a number field, then $\mathcal{F} \subseteq \mathbb{C}$ and the chosen embedding induces an archimedean place v of F in such a way that $\mathcal{F} = F_v$.*

Then there exists a canonical homomorphism of abelian groups

$$\delta_{\mathcal{A},\mathcal{F}} : \zeta(A_{\mathcal{F}})^\times \to \mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}})$$

that has all of the following properties.

(i) *The connecting homomorphism $\partial_{\mathcal{A},\mathcal{F}}$ in (4.1.1) is equal to $\delta_{\mathcal{A},\mathcal{F}} \circ \mathrm{Nrd}_{A_{\mathcal{F}}}$.*

(ii) *The kernel of $\delta_{\mathcal{A},\mathcal{F}}$ comprises all elements of $\zeta(A)^\times$ whose image in $\zeta(A_{\mathfrak{p}})^\times$ belongs to $\mathrm{Nrd}_{A_{\mathfrak{p}}}(\mathrm{K}_1(\mathcal{A}_{\mathfrak{p}}))$ for every $\mathfrak{p}$ in $\mathrm{Spm}(R)$.*

(iii) *For any extension $\mathcal{E}$ of F in $\mathcal{F}$, the full pre-image of $\mathrm{im}(\iota_{\mathcal{A},\mathcal{E},\mathcal{F}})$ under $\delta_{\mathcal{A},\mathcal{F}}$ is equal to $\zeta(A_{\mathcal{E}})^\times$.*

(iv) *If F is a number field, then for all $\mathfrak{p} \in \mathrm{Spm}(R)$ and $j \in \mathrm{Isom}_{\mathfrak{p}}$, there exists a commutative diagram of the form*

$$\begin{array}{ccc} \zeta(A_{\mathcal{F}})^\times & \xrightarrow{\delta_{\mathcal{A},\mathcal{F}}} & \mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}}) \\ {\scriptstyle j'_*}\downarrow & & \downarrow{\scriptstyle j_*} \\ \zeta(\mathbb{C}_{p(\mathfrak{p})} \otimes_{F,j} A)^\times & \xrightarrow{\delta_{\mathcal{A}_{\mathfrak{p}},\mathbb{C}_{p(\mathfrak{p})}}} & \mathrm{K}_0(\mathcal{A}_{\mathfrak{p}}, \mathbb{C}_{p(\mathfrak{p})} \otimes_{F,j} A). \end{array}$$

Here j'_* denotes the embedding induced by the restriction of j to $\mathcal{F}$ and the homomorphism j_* is induced by sending each tuple (P, g, Q) to $(P_{\mathfrak{p}}, \mathbb{C}_{p(\mathfrak{p})} \otimes_{\mathcal{F},j} g, Q_{\mathfrak{p}})$.

(v) *Assume F is a number field, A is unramified at all archimedean places other than v and $\mathrm{Spec}(R)$ is open in $\mathrm{Spec}(\mathcal{O}_F)$. Then, in terms of the notation in claim (iv), for each x in $\zeta(A_{\mathcal{F}})$, the element $\delta_{\mathcal{A},\mathcal{F}}(x)$ is uniquely determined by the elements $\{\delta_{\mathcal{A}_{\mathfrak{p}},\mathbb{C}_p}(j'_*(x))\}_{\mathfrak{p},j}$, as $\mathfrak{p}$ ranges over $\mathrm{Spm}(R)$ and j over $\mathrm{Isom}_{\mathfrak{p}}$. In particular, in this case, the map $\delta_{\mathcal{A},\mathcal{F}}$ is uniquely determined by the commutativity of all diagrams in claim (iv).*

Proof. If either F is a finite extension of $\mathbb{Q}_p$, or $\mathcal{F} = \mathbb{C}$, then the map $\mathrm{Nrd}_{A_{\mathcal{F}}}$ is bijective and we set

$$\delta_{\mathcal{A},\mathcal{F}} := \partial_{\mathcal{A},\mathcal{F}} \circ (\mathrm{Nrd}_{A_{\mathcal{F}}})^{-1}. \tag{4.1.3}$$

Then, with this definition, claim (i) is obvious and claims (ii) and (iii) both follow from the exactness of the relevant case of (4.1.1) and the fact that $\mathrm{Nrd}_{A_{\mathcal{E}}}(\mathrm{K}_1(A_{\mathcal{E}})) = \zeta(A_{\mathcal{E}})^\times$.

Hence, in the rest of the argument, we assume F is a number field and $\mathcal{F} = \mathbb{R}$ arises as the completion of F at a place v.

Then, writing $\prod_{i\in I}F_i$ for the Wedderburn decomposition of $\zeta(A)$ and $\Sigma(i,v)$ for the set of places of each field F_i above v, there is a natural decomposition

$$\zeta(A_{\mathcal{F}})^\times = \prod\nolimits_{i\in I}(F_i\otimes_F F_v)^\times = \prod\nolimits_{i\in I}\prod\nolimits_{w\in\Sigma(i,v)}F_{i,w}^\times. \tag{4.1.4}$$

For any element $x=(x_i)_{i\in I}$ of $\zeta(A_{\mathcal{F}})^\times$, and each index i, we can therefore use the weak approximation theorem to choose an element $\lambda_{i,x}$ of $F_i^\times$ with the property that for each w in $\Sigma(i,v)$ for which $F_{i,w}=\mathbb{R}$, one has $(\lambda_{i,x}x_i)_w>0$. Then, writing λ_x for the element $(\lambda_{i,x})_{i\in I}$ of $\zeta(A)^\times$, the Hasse-Schilling-Maass Norm Theorem implies that the product $\lambda_x x$ belongs to $\mathrm{Nrd}_{A_{\mathcal{F}}}(\mathrm{K}_1(A_{\mathcal{F}}))$.

We may therefore define an element of $\mathrm{K}_0(\mathcal{A},A_{\mathcal{F}})$ by means of the sum

$$\delta_{\mathcal{A},\mathcal{F}}(x) := \partial_{\mathcal{A},\mathcal{F}}\big((\mathrm{Nrd}_{A_{\mathcal{F}}})^{-1}(\lambda_x x)\big) - \sum\nolimits_{\mathfrak{p}\in\mathrm{Spm}(R)}\delta_{\mathcal{A}_\mathfrak{p},F_\mathfrak{p}}(\lambda_x). \tag{4.1.5}$$

Here the sum is regarded as an element of $\mathrm{K}_0(\mathcal{A},A_{\mathcal{F}})$ via the embedding (4.1.2) (this makes sense since the corresponding cases of the long exact sequence in (4.1.1) implies that each element of $\zeta(A)^\times$ belongs to the kernel of $\delta_{\mathcal{A}_\mathfrak{p},F_\mathfrak{p}}$ for almost all $\mathfrak{p}$ in $\mathrm{Spm}(R)$).

Note that if $\lambda'_x=(\lambda'_{i,x})_{i\in I}$ is any other element of $\zeta(A)^\times$ chosen as above (with respect to the same element x), then for each w in $\Sigma(i,v)$ for which $F_{i,w}=\mathbb{R}$, one has

$$(\lambda'_{i,x}(\lambda_{i,x})^{-1})_w = (\lambda'_{i,x}x_i)_w\cdot(\lambda_{i,x}x_i)_w^{-1}>0$$

and hence $\lambda'_x(\lambda_x)^{-1}\in\mathrm{Nrd}_A(\mathrm{K}_1(A))$. This fact implies that $\delta_{\mathcal{A},\mathcal{F}}(x)$ is independent of the choice of λ_x. It is also easily seen that the assignment $x\mapsto\delta_{\mathcal{A},\mathcal{F}}(x)$ is a group homomorphism.

Further, with this explicit definition, the property in claim (i) is clear since for each x in $\mathrm{im}(\mathrm{Nrd}_{A_{\mathcal{F}}})$ one can compute $\delta_{\mathcal{A},\mathcal{F}}(x)$ by taking $\lambda_x=1$ in (4.1.5).

Claim (iii) is also true since for each x in $\zeta(A_{\mathcal{F}})$, and any λ_x in $\zeta(A)^\times$ as fixed in (4.1.5), one has

$$\begin{aligned}\delta_{\mathcal{A},\mathcal{F}}(x)\in\mathrm{im}(\iota_{\mathcal{A},\mathcal{E},\mathcal{F}}) &\Longleftrightarrow \partial_{\mathcal{A},\mathcal{F}}\big((\mathrm{Nrd}_{A_{\mathcal{F}}})^{-1}(\lambda_x x)\big)\in\mathrm{im}(\iota_{\mathcal{A},\mathcal{E},\mathcal{F}})\\ &\Longleftrightarrow (\mathrm{Nrd}_{A_{\mathcal{F}}})^{-1}(\lambda_x x)\in\mathrm{im}(\iota_{A,\mathcal{E},\mathcal{F}})\\ &\Longleftrightarrow \lambda_x x\in\mathrm{Nrd}_{A_{\mathcal{F}}}\big(\mathrm{im}(\iota_{A,\mathcal{E},\mathcal{F}})\big)=\mathrm{Nrd}_{A_{\mathcal{E}}}(\mathrm{K}_1(A_{\mathcal{E}}))\\ &\Longleftrightarrow x\in\zeta(A_{\mathcal{E}})^\times.\end{aligned}$$

Here the first equivalence is true since each term $\delta_{\mathcal{A}_\mathfrak{p},F_\mathfrak{p}}(\lambda)$ in (4.1.5) belongs to the subgroup $\mathrm{im}(\iota_{\mathcal{A},F,\mathcal{F}})$ of $\mathrm{im}(\iota_{\mathcal{A},\mathcal{E},\mathcal{F}})$, the second follows from the exact commutative diagram (4.1.1) (with $\mathcal{F}$ and $\mathcal{F}'$ replaced by $\mathcal{E}$ and $\mathcal{F}$), the third is clear and the fourth is true since $\lambda_x\in\zeta(A)^\times$.

To prove claim (iv) we abbreviate $p(\mathfrak{p})$ to p and write ι_j for the scalar extension map $\mathrm{K}_1(A_{\mathcal{F}})\to\mathrm{K}_1(\mathbb{C}_p\otimes_{F,j}A)$ induced by j. Then the given diagram commutes since for each $x\in\zeta(A_{\mathcal{F}})^\times$, and each λ fixed as in (4.1.5), one has

$$\begin{aligned} j_*\big(\delta_{\mathcal{A},\mathcal{F}}(x)\big) &= j_*\big(\partial_{\mathcal{A},\mathcal{F}}\big((\mathrm{Nrd}_{A_{\mathcal{F}}})^{-1}(\lambda_x x)\big)\big) - j_*\big(\delta_{\mathcal{A}_\mathfrak{p},F_\mathfrak{p}}(\lambda_x)\big)\\ &= \partial_{\mathcal{A}_\mathfrak{p},\mathbb{C}_p}\left(\iota_j(\mathrm{Nrd}_{A_{\mathcal{F}}})^{-1}(\lambda_x x)\right) - \delta_{\mathcal{A}_\mathfrak{p},\mathbb{C}_p}(\lambda_x)\\ &= \delta_{\mathcal{A}_\mathfrak{p},\mathbb{C}_p}\left(j'_*(\lambda_x x)\right) - \delta_{\mathcal{A}_\mathfrak{p},\mathbb{C}_p}(\lambda)\\ &= \delta_{\mathcal{A}_\mathfrak{p},\mathbb{C}_p}\left(j'_*(x)\right). \end{aligned}$$

Here the first equality follows from the formula (4.1.5) and the fact that $j_*(\delta_{\mathcal{A}_\mathfrak{q},F_\mathfrak{q}}(\lambda_x)) = 0$ for all $\mathfrak{q} \in \mathrm{Spm}(R) \setminus \{\mathfrak{p}\}$, the second from commutativity of the relevant case of (4.1.1), the third from the definition (4.1.3) of $\delta_{\mathcal{A}_\mathfrak{p},\mathbb{C}_p}$ and the compatibility of reduced norms under scalar extension and the last equality is clear.

Claim (v) follows directly from the commutativity of the diagrams in claim (iv) and the general result of Lemma 4.2 below.

In a similar way, since $\zeta(A)^\times$ is the full pre-image under $\delta_{\mathcal{A},\mathcal{F}}$ of $\mathrm{im}(\iota_{\mathcal{A},A,\mathcal{F}})$ (by claim (iii)), the assertion of claim (ii) in the number field case follows from Lemma 4.2 and the exactness of the relevant cases of (4.1.1).

This completes the proof of the proposition. □

In the next result we assume the notation and hypotheses of Proposition 4.1(v). For each $\mathfrak{p}$ in $\mathrm{Spm}(R)$ and $j \in \mathrm{Isom}_\mathfrak{p}$ we also set $A_j^c := \mathbb{C}_{p(\mathfrak{p})} \otimes_{F,j} A$.

Lemma 4.2. *Assume F is a number field, A is unramified at all archimedean places other than v and* $\mathrm{Spec}(R)$ *is open in* $\mathrm{Spec}(\mathcal{O}_F)$. *Then the natural diagonal map*

$$\mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}}) \xrightarrow{(j_*)_{\mathfrak{p},j}} \prod\nolimits_{\mathfrak{p}\in\mathrm{Spm}(R)}\prod\nolimits_{j\in\mathrm{Isom}_\mathfrak{p}} \mathrm{K}_0(\mathcal{A}_\mathfrak{p}, A_j^c)$$

is injective.

Proof. We consider the exact sequences that are given by the lower row of (4.1.1) with the pair $(R,\mathcal{F})$ taken to be (R,F), $(R,\mathcal{F})$, $(R_\mathfrak{p},F_\mathfrak{p})$ and $(R_\mathfrak{p},\mathbb{C}_{p(\mathfrak{p})})$ and the maps between these sequences that are induced by the obvious inclusions and by an isomorphism j in $\mathrm{Isom}_\mathfrak{p}$. Then an easy diagram chase gives a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{K}_0(\mathcal{A},A) & \longrightarrow & \mathrm{K}_0(\mathcal{A},A_{\mathcal{F}}) & \longrightarrow & \mathrm{K}_1(A_{\mathcal{F}})/\,\mathrm{K}_1(A) & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{K}_0(\mathcal{A}_\mathfrak{p},A_\mathfrak{p}) & \longrightarrow & \mathrm{K}_0(\mathcal{A}_\mathfrak{p},A_j^c) & \longrightarrow & \mathrm{K}_1(A_j^c)/\,\mathrm{K}_1(A_\mathfrak{p}) & \longrightarrow & 0. \end{array}$$

In view of the isomorphism in (4.1.2) it is therefore enough to show that the natural diagonal map

$$\mathrm{K}_1(A_{\mathcal{F}})/\,\mathrm{K}_1(A) \to \prod\nolimits_{\mathfrak{p}\in\mathrm{Spm}(R)}\prod\nolimits_{j\in\mathrm{Isom}_\mathfrak{p}} \mathrm{K}_1(A_j^c)/\,\mathrm{K}_1(A_\mathfrak{p})$$

is injective. To do this we fix x in $\mathrm{K}_1(A_{\mathcal{F}})$ with $j_*(x) \in \mathrm{K}_1(A_\mathfrak{p}) \subseteq \mathrm{K}_1(A_j^c)$ for all $\mathfrak{p}$ and all $j \in \mathrm{Isom}_\mathfrak{p}$ and must show that x belongs to the subgroup $\mathrm{K}_1(A)$ of $\mathrm{K}_1(A_{\mathcal{F}})$.

We use the (injective) maps $\mathrm{Nrd}_{A_{\mathcal{F}}}$ and Nrd_A to identify $\mathrm{K}_1(A_{\mathcal{F}})$ and $\mathrm{K}_1(A)$ with $\mathrm{Nrd}_{A_{\mathcal{F}}}(\mathrm{K}_1(A_{\mathcal{F}}))$ and $\mathrm{Nrd}_A(\mathrm{K}_1(A))$ respectively. We then fix an F-basis $\{a_\omega : \omega \in \Omega\}$ of $\zeta(A)$ so that $x = \sum_{\omega\in\Omega} c_\omega a_\omega$ with each c_ω in $\mathcal{F}$ and for every $j \in \mathrm{Isom}_\mathfrak{p}$ one has

$$j'_*(x) = \sum\nolimits_{\omega\in\Omega} j(c_\omega)a_\omega \in \zeta(A_\mathfrak{p})^\times, \tag{4.1.6}$$

where j'_* denotes the inclusion $\zeta(A_\mathcal{F})^\times \to \zeta(A^c_\mathfrak{p})^\times$ induced by j.

We now fix ω in Ω and consider the coefficient c_ω. If c_ω was transcendental over F, then there would exist an isomorphism j in $\mathrm{Isom}_\mathfrak{p}$ such that $j(c_\omega) \notin F_\mathfrak{p}$, thereby contradicting (4.1.6). Therefore c_ω is algebraic over F. The fact that $j(c_\omega)$ belongs to $F_\mathfrak{p}$ for all $j \in \mathrm{Isom}_\mathfrak{p}$ then implies that $\mathfrak{p}$ is completely split in the extension $F(c_\omega)/F$. Hence, since $\mathrm{Spec}(R)$ is open in $\mathrm{Spec}(\mathcal{O}_F)$, the Tchebotarov Density Theorem implies $F(c_\omega) = F$.

At this stage, we know that x belongs to both $\mathrm{Nrd}_{A_\mathcal{F}}(\mathrm{K}_1(A_\mathcal{F}))$ and $\zeta(A)^\times$ and so it suffices to show that, under the stated hypotheses, one has

$$\mathrm{Nrd}_{A_\mathcal{F}}(\mathrm{K}_1(A_\mathcal{F})) \cap \zeta(A)^\times = \mathrm{Nrd}_A(\mathrm{K}_1(A)).$$

To verify this we use the decomposition (4.1.4) and, for each $i \in I$, we write $\Sigma(i,v)'$ for the subset of $\Sigma(i,v)$ comprising (archimedean) places at which the algebra $A_i \otimes_{F_i} F_{i,w}$ is ramified (and so $F_{i,w} = \mathbb{R}$). It is then enough to note that the Hasse-Schilling-Maass Norm Theorem implies both that

$$\begin{aligned}&\mathrm{Nrd}_{A_\mathcal{F}}(\mathrm{K}_1(A_\mathcal{F}))\\ &\qquad = \{(x_{i,w})_{i,w} \in \prod\nolimits_{i\in I}\prod\nolimits_{w\in\Sigma(i,v)} F^\times_{i,w} = \zeta(A_\mathcal{F})^\times : x_{i,w} > 0 \;\text{ for all } w \in \Sigma(i,v)'\},\end{aligned}$$

and, as A is unramified at all archimedean places of F other than v, also

$$\mathrm{Nrd}_A(\mathrm{K}_1(A)) = \{(x_i)_i \in \prod\nolimits_{i\in I} F^\times_i = \zeta(A)^\times : x_{i,w} > 0 \;\text{ for all } w \in \Sigma(i,v)'\}.$$

□

4.1.3. In the following result we describe another useful consequence of the long exact sequence of relative K-theory.

Lemma 4.3. *For each element x of $\mathrm{Nrd}_A(\mathrm{K}_1(A))$ the following claims are valid.*

(i) *For each $\mathfrak{p}$ in $\mathrm{Spm}(R)$ one has $x \in \mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}_{(\mathfrak{p})}))$ if and only if the image of x in $\mathrm{Nrd}_{A_\mathfrak{p}}(\mathrm{K}_1(A_\mathfrak{p}))$ belongs to $\mathrm{Nrd}_{A_\mathfrak{p}}(\mathrm{K}_1(\mathcal{A}_\mathfrak{p}))$.*

(ii) *The following conditions are equivalent:*
   (a) *$x \in \mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$.*
   (b) *For all $\mathfrak{p} \in \mathrm{Spm}(R)$, one has $x \in \mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}_{(\mathfrak{p})}))$.*
   (c) *For all $\mathfrak{p} \in \mathrm{Spm}(R)$, the image of x in $\mathrm{Nrd}_{A_\mathfrak{p}}(\mathrm{K}_1(A_\mathfrak{p}))$ belongs to $\mathrm{Nrd}_{A_\mathfrak{p}}(\mathrm{K}_1(\mathcal{A}_\mathfrak{p}))$.*

Proof. The relevant cases of the exact sequence (4.1.1) give rise to an commutative diagram of abelian groups in which all rows are exact

$$\begin{array}{ccccc} \mathrm{K}_1(\mathcal{A}) & \xrightarrow{\iota_{\mathcal{A}}} & \mathrm{K}_1(A) & \longrightarrow & \mathrm{K}_0(\mathcal{A}, A) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\kappa} \\ \prod_{\mathfrak{p}} \mathrm{K}_1(\mathcal{A}_{(\mathfrak{p})}) & \xrightarrow{(\iota_{(\mathfrak{p})})_{\mathfrak{p}}} & \prod'_{\mathfrak{p}} \mathrm{K}_1(A) & \longrightarrow & \bigoplus_{\mathfrak{p}} \mathrm{K}_0(\mathcal{A}_{(\mathfrak{p})}, A) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle(\kappa_{\mathfrak{p}})_{\mathfrak{p}}} \\ \prod_{\mathfrak{p}} \mathrm{K}_1(\mathcal{A}_{\mathfrak{p}}) & \xrightarrow{(\iota_{\mathfrak{p}})_{\mathfrak{p}}} & \prod''_{\mathfrak{p}} \mathrm{K}_1(A_{\mathfrak{p}}) & \longrightarrow & \bigoplus_{\mathfrak{p}} \mathrm{K}_0(\mathcal{A}_{\mathfrak{p}}, A_{\mathfrak{p}}). \end{array}$$

Here $\iota_{(\mathfrak{p})}$ and $\iota_{\mathfrak{p}}$ denote the scalar extension maps $\mathrm{K}_1(\mathcal{A}_{(\mathfrak{p})}) \to \mathrm{K}_1(A)$ and $\mathrm{K}_1(\mathcal{A}_{\mathfrak{p}}) \to \mathrm{K}_1(A_{\mathfrak{p}})$, $\prod'_{\mathfrak{p}}$ the restricted direct product over $\mathfrak{p}$ in $\mathrm{Spm}(R)$ of $\mathrm{K}_1(A)$ with respect to the subgroups $\mathrm{im}(\iota_{(\mathfrak{p})})$ and $\prod''_{\mathfrak{p}}$ the restricted direct product of the groups $\mathrm{K}_1(A_{\mathfrak{p}})$ with respect to $\mathrm{im}(\iota_{\mathfrak{p}})$. In addition, the upper vertical arrows are the natural diagonal maps, the first and second lower vertical maps are induced by the scalar extension maps $\mathrm{K}_1(\mathcal{A}_{(\mathfrak{p})}) \to \mathrm{K}_1(\mathcal{A}_{\mathfrak{p}})$ and $\mathrm{K}_1(A) \to \mathrm{K}_1(A_{\mathfrak{p}})$ and $\kappa_{\mathfrak{p}}$ denotes the scalar extension map $\mathrm{K}_0(\mathcal{A}_{(\mathfrak{p})}, A) \to \mathrm{K}_0(\mathcal{A}_{\mathfrak{p}}, A_{\mathfrak{p}})$.

We recall that each map $\kappa_{\mathfrak{p}}$ is bijective since both groups $\mathrm{K}_0(\mathcal{A}_{(\mathfrak{p})}, A)$ and $\mathrm{K}_0(\mathcal{A}_{\mathfrak{p}}, A_{\mathfrak{p}})$ identify with the Grothendieck group of finitely generated $\mathfrak{p}$-torsion $\mathcal{A}$-modules of finite projective dimension over $\mathcal{A}$ (cf. the discussion in [28, Rem. (40.19)]). From the commutativity of the lower part of the diagram, one can therefore deduce that an element y of $\mathrm{K}_1(A)$ belongs to $\mathrm{im}(\iota_{(\mathfrak{p})})$ if and only if its image in $\mathrm{K}_1(A_{\mathfrak{p}})$ belongs to $\mathrm{im}(\iota_{\mathfrak{p}})$. This implies claim (i) since the maps Nrd_A and $\mathrm{Nrd}_{A_{\mathfrak{p}}}$ are both bijective and, by definition, one has both $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A})) = \mathrm{Nrd}_A(\mathrm{im}(\iota_{\mathcal{A}}))$ and $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}_{(\mathfrak{p})})) = \mathrm{Nrd}_A(\mathrm{im}(\iota_{(\mathfrak{p})}))$.

In a similar way, the result of claim (i) combines with the injectivity of Nrd_A to reduce the proof of claim (ii) to showing that an element y of $\mathrm{K}_1(A)$ belongs to $\mathrm{im}(\iota_{\mathcal{A}})$ if and only if, for every $\mathfrak{p}$ in $\mathrm{Spm}(R)$, it belongs to $\mathrm{im}(\iota_{(\mathfrak{p})})$. It is thus enough to note that this property follows directly from the injectivity of κ and the commutativity of the upper part of the diagram. □

4.2. **Virtual objects and zeta elements.**

4.2.1. In this section we recall the construction of Euler characteristics that underlies the formulation of a range of refined special value conjectures in the literature.

To do this we recall first that, as already mentioned in a special case in Remark 2.6, in [33, §4] Deligne constructs for any category $\mathcal{E}$ that is exact in the sense of Quillen [71, p. 91] a universal determinant functor of the form

$$[-] : \mathcal{E}_{\mathrm{Isom}} \to \mathcal{V}(\mathcal{E}).$$

Here $\mathcal{E}_{\mathrm{Isom}}$ denotes the subcategory of $\mathcal{E}$ in which morphisms are restricted to isomorphisms and $\mathcal{V}(\mathcal{E})$ is the Picard category of 'virtual objects' associated to $\mathcal{E}$.

In the case that $\mathcal{E}$ is the category $\mathrm{Mod}^{\mathrm{proj}}(\Lambda)$ of finitely generated projective left modules over a ring Λ, we write $\mathcal{V}(\mathcal{E})$ as $\mathcal{V}(\Lambda)$.

If now $\Lambda \to \Sigma$ is a ring homomorphism, then the functor $\mathrm{Mod}^{\mathrm{proj}}(\Lambda) \to \mathrm{Mod}^{\mathrm{proj}}(\Sigma)$ that sends each P to $\Sigma \otimes_\Lambda P$ is exact and so, by [33, §4.11], induces a monoidal functor

$$\mathcal{V}(\Lambda) \to \mathcal{V}(\Sigma), \quad L \mapsto L_\Sigma$$

that is unique up to natural isomorphism.

Let $\mathcal{P}_0$ be the Picard category with unique object $\mathbf{1}_{\mathcal{P}_0}$ and trivial automorphism group $\mathrm{Aut}_{\mathcal{P}_0}(\mathbf{1}_{\mathcal{P}_0})$. Then the fibre product diagram

$$\begin{array}{ccc} \mathcal{V}(\Lambda,\Sigma) & \longrightarrow & \mathcal{P}_0 \\ \downarrow & & \downarrow {\scriptstyle \mathbf{1}_{\mathcal{P}_0}\mapsto \mathbf{1}_{\mathcal{V}(\Sigma)}} \\ \mathcal{V}(\Lambda) & \xrightarrow{L\mapsto L_\Sigma} & \mathcal{V}(\Sigma) \end{array}$$

defines a Picard category $\mathcal{V}(\Lambda,\Sigma)$ in which objects are pairs (L,t) with L in $\mathcal{V}(\Lambda)$ and t an isomorphism $L_\Sigma \to \mathbf{1}_{\mathcal{V}(\Sigma)}$ in $\mathcal{V}(\Sigma)$.

It is shown by Breuning and Burns in [11, Lem. 5.1] (following an argument of [17, §2.8, Prop. 2.5]) that there exists a canonical isomorphism of abelian groups

$$\tau_{\Lambda,\Sigma} : K_0(\Lambda,\Sigma) \to \pi_0(\mathcal{V}(\Lambda,\Sigma)).$$

This map sends each element (P,g,Q) in $\mathrm{K}_0(\Lambda,\Sigma)$ to the isomorphism class of the pair comprising $[P]\otimes[Q]^{-1}$ and the composite isomorphism

$$([P]\otimes[Q]^{-1})_\Sigma \longrightarrow [\Sigma\otimes_\Lambda P]\otimes[\Sigma\otimes_\Lambda Q]^{-1} \xrightarrow{[g]\otimes\mathrm{id}} [\Sigma\otimes_\Lambda Q]\otimes[\Sigma\otimes_\Lambda Q]^{-1} \longrightarrow \mathbf{1}_{\mathcal{V}(\Sigma)}.$$

In particular, if α is an automorphism of the Σ-module $\Sigma\otimes_\Lambda P$, and $\langle\alpha\rangle$ its class in $\mathrm{K}_1(\Sigma)$, then one has

$$\tau_{\Lambda,\Sigma}(\partial_{\Lambda,\Sigma}(\langle\alpha\rangle)) = \tau_{\Lambda,\Sigma}((P,\alpha,P)) = [\mathbf{1}_{\mathcal{V}(\Lambda)}, u_\alpha]. \tag{4.2.1}$$

Here $\partial_{\Lambda,\Sigma} : \mathrm{K}_1(\Sigma) \to \mathrm{K}_0(\Lambda,\Sigma)$ is the canonical connecting homomorphism as in (4.1.1), u_α denotes the image of $\langle\alpha\rangle$ under the canonical identification

$$\mathrm{K}_1(\Sigma) \cong \pi_1(\mathcal{V}(\Sigma)) := \mathrm{Aut}_{\mathcal{V}(\Sigma)}(\mathbf{1}_{\mathcal{V}(\Sigma)}),$$

and we write $[L,t]$ for the isomorphism class of a pair (L,t) in $\mathcal{V}(\Lambda,\Sigma)$.

The following definition of Euler characteristic underlies the constructions that are made in [11] and [17].

Definition 4.4. Fix C in $\mathsf{D}^{\mathrm{perf}}(\mathcal{A})$ and a morphism $t : [C_\mathcal{F}] \to \mathbf{1}_{\mathcal{V}(A_\mathcal{F})}$ in $\mathcal{V}(A_\mathcal{F})$. Then $\chi_{\mathcal{A},\mathcal{F}}(C,t)$ denotes the element of $\mathrm{K}_0(\mathcal{A},A_\mathcal{F})$ that $\tau_{\mathcal{A},A_\mathcal{F}}$ sends to $[[C],t]$.

Remark 4.5. The homomorphism $\partial'_{\mathcal{A},\mathcal{F}}$ in (4.1.1) sends each element $\chi_{\mathcal{A},\mathcal{F}}(C,t)$ to the classical Euler characteristic of C in $\mathrm{K}_0(\mathcal{A})$. It is for this reason that the elements $\chi_{\mathcal{A},\mathcal{F}}(C,t)$ are sometimes referred to as 'refined Euler characteristics'.

4.2.2. As in §3, we shall in the sequel abbreviate the functors $\mathrm{d}_{A_\mathcal{F},\varpi}$ and $\mathrm{d}_{\mathcal{A},\varpi}$ to $\mathrm{d}_{A_\mathcal{F}}$ and $\mathrm{d}_\mathcal{A}$ respectively.

The following definition is a natural analogue in our setting of the 'zeta elements' that were introduced (in an arithmetic setting) by Kato in [51].

Definition 4.6. Let C be an object of $\mathsf{D}^{\mathrm{lf},0}(\mathcal{A})$ and λ an isomorphism in $\mathcal{P}(\zeta(A_\mathcal{F}))$ of the form $\mathrm{d}_{A_\mathcal{F}}(C_\mathcal{F}) \cong (\zeta(A_\mathcal{F}),0)$.

Then, for each element x of $\zeta(A_\mathcal{F})^\times$, the 'zeta element' associated to the pair (λ,x) is the unique element $z_{\lambda,x}$ of $\mathrm{d}_{A_\mathcal{F}}(C_\mathcal{F})$ that satisfies

$$\lambda(z_{\lambda,x}) = (x,0)$$

in $(\zeta(A_{\mathcal{F}}), 0)$.

Example 4.7. There are several important examples of zeta elements in the literature.
(i) Assume $F = \mathbb{Q}_p$, $R = \mathbb{Z}_p$ and $\mathcal{F} = \mathbb{C}_p$. Let M be a motive defined over a number field K with coefficients in a finite dimensional semisimple $\mathbb{Q}$-algebra B, and T a Galois-stable lattice in the p-adic realization of M that is projective over some $\mathbb{Z}_p$-order $\mathcal{A}$ in $A := \mathbb{Q}_p \otimes_{\mathbb{Q}} B$. Fix a finite set Π of places of K containing all archimedean and p-adic places and all places at which M has bad reduction. Then the p-adic étale cohomology complex $C(M) := \mathsf{R}\operatorname{Hom}_{\mathcal{A}}(\mathsf{R}\Gamma_c(\mathcal{O}_{K,\Pi}, T), \mathcal{A}[-2])$ belongs to $\mathsf{D}^{\mathrm{f},0}(\mathcal{A})$ and, after fixing an isomorphism of fields $\mathbb{C} \cong \mathbb{C}_p$, there exists a canonical 'period-regulator isomorphism' of the form $\lambda : \mathrm{d}_{A_{\mathbb{C}_p}}(C(M)_{\mathbb{C}_p}) \xrightarrow{\sim} (\zeta(A_{\mathbb{C}_p}), 0)$ (cf. Remark 4.9 below). In particular, if we write x for the leading term $L^*(M, 0)$ in $\zeta(A_{\mathbb{C}})^\times \cong \zeta(A_{\mathbb{C}_p})^\times$ of the L-function of M at $s = 0$, then the element $z_{\lambda,x}$ defined above generalizes the zeta elements defined for abelian extensions by Kato in [51].
(ii) Assume $F = \mathbb{Q}$, $R = \mathbb{Z}$ and $\mathcal{F} = \mathbb{R}$. Let L/K be a finite Galois extension of number fields with group G and set $\mathcal{A} := \mathbb{Z}[G]$ (so that $A_{\mathcal{F}} = \mathbb{R}[G]$). Then, for a suitable choice of complex C, isomorphsm λ and element x, the element $z_{\lambda,x}$ defined above generalizes the 'zeta elements' $z_{L/K,\Pi}$ defined for abelian extensions L/K by Burns, Kurihara and Sano in [18, Def. 3.5] (following Kato [51]). For details, see Remark 6.7 below.

In the next result we interpret zeta elements in terms of the Euler characteristic construction in Definition 4.4.

Before stating this result we note that every object of the category $\mathrm{Mod}^{\mathrm{fg}}(A_{\mathcal{F}})$ of finitely generated $A_{\mathcal{F}}$-modules is projective (as $A_{\mathcal{F}}$ is semisimple) and so the construction of Deligne [33] gives a determinant functor

$$[-] : \mathrm{Mod}^{\mathrm{fg}}(A_{\mathcal{F}})_{\mathrm{Isom}} \to \mathcal{V}(A_{\mathcal{F}}).$$

In particular, the universal nature of this functor implies that, for any choice of ordered bases ϖ as in §2.1, there exists a canonical additive functor

$$\nu = \nu_{A_{\mathcal{F}},\varpi} : \mathcal{V}(A_{\mathcal{F}}) \to \mathcal{P}(\zeta(A_{\mathcal{F}}))$$

(cf. [17, §2.6, Lem. 2]).

This functor sends the virtual object $[M]$, for each M in $\mathrm{Mod}^{\mathrm{fg}}(A_{\mathcal{F}})$, to $\mathrm{d}_{A_{\mathcal{F}},\varpi}(M)$, and $[\alpha]$, for each α in $\pi_1(\mathcal{V}(A_{\mathcal{F}})) \cong \mathrm{K}_1(A_{\mathcal{F}})$, to the automorphism of $(\zeta(A_{\mathcal{F}}), 0)$ that is given by multiplication by $\mathrm{Nrd}_{A_{\mathcal{F}}}(\alpha)$.

Theorem 4.8. *We assume to be given data of the following sort:*

- *an R-order $\mathcal{A}$ in a finite dimensional separable F-algebra A;*
- *an embedding of fields $F \to \mathcal{F}$, as in Proposition 4.1;*
- *an object C of $\mathsf{D}^{\mathrm{lf},0}(\mathcal{A})$;*
- *a morphism $t : [C_{\mathcal{F}}] \to \mathbf{1}_{\mathcal{V}(A_{\mathcal{F}})}$ in $\mathcal{V}(A_{\mathcal{F}})$.*

Then $\nu(t)$ is an isomorphism $\mathrm{d}_{A_{\mathcal{F}}}(C_{\mathcal{F}}) \cong (\zeta(A_{\mathcal{F}}), 0)$ in $\mathcal{P}(\zeta(A_{\mathcal{F}}))$ and, for each element x of $\zeta(A_{\mathcal{F}})^\times$, one can consider the possible equality

$$\delta_{\mathcal{A},\mathcal{F}}(x) \stackrel{?}{=} \chi_{\mathcal{A},\mathcal{F}}(C, t) \tag{4.2.2}$$

in $\mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}})$. In any such situation, the following claims are valid.

(i) *If (4.2.2) is valid, then* $z_{\nu(t),x}$ *is a basis of the* $\xi(\mathcal{A})$*-module* $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}$.
(ii) *Assume* F *is a finite extension of* $\mathbb{Q}_p$, R *is its valuation ring and* $\mathcal{F}$ *is an extension of* F *in* $\mathbb{C}_p$. *Then (4.2.2) is valid if and only if* $z_{\nu(t),x}$ *is a primitive basis of* $\mathrm{d}_{\mathcal{A}}(C)$.

In the remaining claims we assume that F *is a number field, that* A *is unramified at all archimedean places of* F *other than that corresponding to the fixed embedding* $F \to \mathcal{F} \subseteq \mathbb{C}$ *and that* $\mathrm{Spec}(R)$ *is open in* $\mathrm{Spec}(\mathcal{O}_F)$.

(iii) *The equality (4.2.2) is valid if and only if* $z_{\nu(t),x}$ *is a locally-primitive basis of* $\mathrm{d}_{\mathcal{A}}(C)$.
(iv) *If (4.2.2) is valid, then* $\mathrm{d}_{\mathcal{A}}(C)$ *has a primitive basis if and only if there exists an element* x' *of* $\mathrm{Nrd}_{A_{\mathcal{F}}}(\mathrm{K}_1(A_{\mathcal{F}}))$ *of* $\zeta(A_{\mathcal{F}})^{\times}$ *such that, for all* $\mathfrak{p}$ *in* $\mathrm{Spm}(R)$, *and all field isomorphisms* j *in* $\mathrm{Isom}_{\mathfrak{p}}$, *one has* $j'_*(x \cdot x') \in \mathrm{Nrd}_{A_{\mathfrak{p}}}(\mathrm{K}_1(\mathcal{A}_{\mathfrak{p}}))$.
(v) *If (4.2.2) is valid, then* $z_{\nu(t),x}$ *is a primitive basis of* $\mathrm{d}_{\mathcal{A}}(C)$ *if and only if one has* $x \in \mathrm{Nrd}_{A_{\mathcal{F}}}(\mathrm{K}_1(A_{\mathcal{F}}))$.

Remark 4.9. The 'equivariant Tamagawa Number Conjecture' that is formulated by Burns and Flach in [17, Conj. 4(iv)], and hence also various associated 'equivariant leading term conjectures' in the literature, are equalities of the form (4.2.2) for suitable choices of data $\mathcal{A}, \mathcal{F}, x, C$ and t. We briefly mention two concrete applications of Theorem 4.8 in this setting.
(i) If, in the setting of Example 4.7(i), we consider the composite morphism

$$t : [C(M)_{\mathbb{C}_p}] = [\mathbb{C}_p \otimes^{\mathsf{L}}_{\mathbb{Z}_p} \mathsf{R}\Gamma_c(\mathcal{O}_{K,S}, T)]^{-1} \cong \Xi(M)^{-1}_{\mathbb{C}_p} \cong \mathbf{1}_{\mathcal{V}(A_{\mathbb{C}_p})},$$

where $\Xi(M)$ is the 'fundamental line' defined in [17, (29)] and the first and the second isomorphisms are respectively induced by the morphisms $\vartheta_p(M,S)$ and ϑ_∞ in [17, §3.4], then the equivariant Tamagawa Number Conjecture for $(M, \mathcal{A})$ is formulated as the equality $\delta_{\mathcal{A},\mathbb{C}_p}(L^*(M,0)) = \chi_{\mathcal{A},\mathbb{C}_p}(C(M), t)$. Theorem 4.8(ii) therefore implies that this conjecture is valid if and only if the zeta element $z_{\nu(t),L^*(M,0)}$ is a primitive basis of $\mathrm{d}_{\mathcal{A}}(C(M))$.
(ii) In the setting of Example 4.7(i), Theorem 4.8(iii) provides a similar reinterpretation of the 'lifted root number conjecture' of Gruenberg, Ritter and Weiss [43]. For details see Remark 6.7 below.

Remark 4.10. Assume the setting of Theorem 4.8(ii). Then, in this case, the result of Theorem 4.8 combines with Proposition 3.7(ii) to give an equivalence

$$\delta_{\mathcal{A},\mathcal{F}}(x) = \chi_{\mathcal{A},\mathcal{F}}(C,t) \Longleftrightarrow \mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A})) \cdot z_{\nu(t),x} = \mathrm{d}_{\mathcal{A}}(C)^{\mathrm{pb}},$$

where $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{pb}}$ denotes the subset of $\mathrm{d}_{\mathcal{A}}(C)$ comprising all primitive-basis elements.

Remark 4.11. Assume $\mathcal{A}$ is such that a finitely generated $\mathcal{A}$-module is locally-free if and only if it is both projective and spans a free A-module (cf. Remark 2.2(iii)). Then, in this case, it is easily seen that every element of $\mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}})$ is of the form $\chi_{\mathcal{A},\mathcal{F}}(C,t)$ for a suitable choice of data C and t as in Theorem 4.8.

The proof of Theorem 4.8 will occupy the rest of §4.

4.2.3. To prove claim (i) it is enough to show that, for every $\mathfrak{p}$ in $\mathrm{Spm}(R)$, the validity of the image of (4.2.2) under the natural map $\mathrm{K}_0(\mathcal{A}, A_{\mathcal{F}}) \to K_0(\mathcal{A}_{(\mathfrak{p})}, A_{\mathcal{F}})$ implies an equality

$$\xi(\mathcal{A}_{(\mathfrak{p})}) \cdot z_{\nu(t),x} = \mathrm{d}_{\mathcal{A}_{(\mathfrak{p})}}(C_{(\mathfrak{p})})^{\mathrm{u}}.$$

Thus, after fixing $\mathfrak{p}$ and replacing $\mathcal{A}$ by $\mathcal{A}_{(\mathfrak{p})}$ we will assume that R is local (with maximal ideal $\mathfrak{p}$) and C belongs to $\mathsf{C}^{\mathrm{f}}(\mathcal{A})$. For each integer a we then set $r_a := \mathrm{rk}_{\mathcal{A}}(C^a)$ and fix an ordered $\mathcal{A}$-basis $\{b_{as}\}_{1\le s\le r_a}$ of C^a. Taken together, these choices determine an isomorphism in $\mathcal{V}(\mathcal{A})$ of the form

$$\kappa : [C] \cong \bigotimes\nolimits_{a\in\mathbb{Z}}[C^a]^{(-1)^a} \cong \bigotimes\nolimits_{a\in\mathbb{Z}}[\mathcal{A}^{r_a}]^{(-1)^a} \cong [\mathcal{A}]^{\sum_{a\in\mathbb{Z}}(-1)^a r_a} \cong \mathbf{1}_{\mathcal{V}(\mathcal{A})}$$

where the last map is induced by the fact $\sum_{a\in\mathbb{Z}}(-1)^a r_a = 0$ (as C belongs to $\mathsf{C}^{\mathrm{lf},0}(\mathcal{A})$).

This isomorphism in turn induces an isomorphism in $\mathcal{V}(\mathcal{A}, A_{\mathcal{F}})$

$$([C], t) \cong (\kappa([C]), t\circ\kappa^{-1}) = (\mathbf{1}_{\mathcal{V}(\mathcal{A})}, t\circ\kappa^{-1})$$

which combines with (4.2.1) to imply that

$$\chi_{\mathcal{A},\mathcal{F}}(C,t) = \delta_{\mathcal{A},\mathcal{F}}(\epsilon_{t,\kappa}),$$

with $\epsilon_{t,\kappa} := \mathrm{Nrd}_{A_{\mathcal{F}}}(t\circ\kappa^{-1}) \in \zeta(A_{\mathcal{F}})^\times$.

The validity of (4.2.2) is therefore equivalent to an equality $\delta_{\mathcal{A},\mathcal{F}}(\epsilon_{t,\kappa}) = \delta_{\mathcal{A},\mathcal{F}}(x)$, and hence to a containment

$$x\cdot\epsilon_{t,\kappa}^{-1} \in \ker(\delta_{\mathcal{A},\mathcal{F}}). \tag{4.2.3}$$

In particular, if this containment is valid, then Proposition 4.1(ii) implies that $x\cdot\epsilon_{t,\kappa}^{-1}$ belongs to both $\zeta(A)^\times$ and $\mathrm{Nrd}_{A_{\mathfrak{p}}}(\mathrm{K}_1(\mathcal{A}_{\mathfrak{p}})) \subseteq \xi(\mathcal{A}_{\mathfrak{p}})^\times$. In this situation it would therefore follow that $x\cdot\epsilon_{t,\kappa}^{-1}$ belongs to $\zeta(A)\cap\xi(\mathcal{A}_{\mathfrak{p}})^\times = \xi(\mathcal{A})^\times$.

On the other hand, the ordered bases $\{b_{as}\}_{1\le s\le r_a}$ fixed above (for each $a\in\mathbb{Z}$) together determine a primitive basis z' of $\mathrm{d}_{\mathcal{A}}(C)$ that $\nu(\kappa)$ sends to the element $(1,0)$ of $(\zeta(A),0)$. One therefore has $\nu(t)(z') = (\epsilon_{t,\kappa},0)$ and so the definition of $z_{\nu(t),x}$ implies that

$$z_{\nu(t),x} = (x\cdot\epsilon_{t,\kappa}^{-1})\cdot z'. \tag{4.2.4}$$

Thus, if $x\cdot\epsilon_{t,\kappa}^{-1}$ belongs to $\xi(\mathcal{A})^\times$, then one has

$$\begin{aligned}\xi(\mathcal{A})\cdot z_{\nu(t),x} &= \xi(\mathcal{A})\cdot((x\cdot\epsilon_{t,\kappa}^{-1})\cdot z')\\ &= (\xi(\mathcal{A})\cdot(x\cdot\epsilon_{t,\kappa}^{-1}))\cdot z'\\ &= \xi(\mathcal{A})\cdot z'\\ &= \mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}},\end{aligned}$$

as required to prove claim (i).

Claim (ii) of Theorem 4.8 also follows directly from the above argument and the fact that, under the given hypotheses, the equality (4.2.4) combines with Proposition 3.7(ii) to imply that $z_{\nu(t),x}$ is a primitive-basis of $\mathrm{d}_{\mathcal{A}}(C)$ if and only if $x\cdot\epsilon_{t,\kappa}^{-1}$ belongs to $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A})) = \ker(\delta_{\mathcal{A},\mathcal{F}})$.

Remark 4.12. Assume F is a finite extension of $\mathbb{Q}$ and fix a prime ideal $\mathfrak{p}$ in $\mathrm{Spm}(R)$. Then the above argument also shows that if the image of (4.2.2) under the natural localization map $\mathrm{K}_0(\mathcal{A}, \mathcal{A}_{\mathcal{F}}) \to K_0(\mathcal{A}_{(\mathfrak{p})}, \mathcal{A}_{\mathcal{F}})$ is valid, then $z_{\nu(t),x}$ is a basis of the $\xi(\mathcal{A}_{(\mathfrak{p})})$-module $\mathrm{d}_{\mathcal{A}}(C)^{\mathrm{u}}_{(\mathfrak{p})} = \mathrm{d}_{\mathcal{A}_{(\mathfrak{p})}}(C_{(\mathfrak{p})})^{\mathrm{u}}$.

4.2.4. In the remainder of the argument we assume that F is a number field, that A is unramified at all archimedean places of F other than the place v corresponding to $\mathcal{F}$ and that $\mathrm{Spec}(R)$ is open in $\mathrm{Spec}(\mathcal{O}_F)$.

To prove claim (iii) we also use the following notation: for each $\mathfrak{p}$ in $\mathrm{Spm}(R)$ and each isomorphism j in $\mathrm{Isom}_\mathfrak{p}$ we set $\mathcal{A}_j := \mathcal{A}_\mathfrak{p}$, $A_j := A_\mathfrak{p}$, $A_j^c := \mathbb{C}_{p(\mathfrak{p})} \otimes_{F,j} A$ and $C_j := C_\mathfrak{p}$.

Then, in this case, Proposition 4.1(v) implies that the equality (4.2.2) is valid as stated if and only if for all $\mathfrak{p}$ in $\mathrm{Spm}(R)$ and all j in $\mathrm{Isom}_\mathfrak{p}$ it is valid with $\mathcal{A}, \mathcal{F}, x, C$ and t respectively replaced by $\mathcal{A}_j, A_j^c, j_*(x), C_j$ and $t_j := \mathbb{C}_{p(\mathfrak{p})} \otimes_{\mathcal{F},j} t$.

In addition, the argument of claim (ii) implies that (4.2.2) is valid for any such collection of data if and only if $z_{\nu(t_j),j_*(x)}$ is a primitive-basis of $\mathrm{d}_{\mathcal{A}_j}(C_j)$. To deduce claim (iii) it is therefore enough to note $z_{\nu(t_j),j_*(x)}$ is equal to the image of $z_{\nu(t),x}$ under the natural map

$$\mathrm{d}_{A_\mathcal{F}}(C_\mathcal{F}) \to \mathbb{C}_{p(\mathfrak{p})} \otimes_{\mathcal{F},j} \mathrm{d}_{A_\mathcal{F}}(C_\mathcal{F}) = \zeta(A_j^c) \otimes_{\xi(\mathcal{A}_j)} \mathrm{d}_{\mathcal{A}_j}(C_j).$$

To prove claim (iv) we note Theorem 3.12(iii) implies that $\mathrm{d}_\mathcal{A}(C)$ has a primitive-basis if and only if $\chi_\mathcal{A}(C)$ vanishes. In addition, since Remark 4.5 implies

$$\chi_\mathcal{A}(C) = \partial'_{\mathcal{A},\mathcal{F}}(\chi_{\mathcal{A},\mathcal{F}}(C,t)) = \partial'_{\mathcal{A},\mathcal{F}}(\delta_{\mathcal{A},\mathcal{F}}(x)),$$

the result of Proposition 4.1(i) combines with the exactness of the lower row of (4.1.1) to imply $\chi_\mathcal{A}(C)$ vanishes if and only if there exists an element x' of $\mathrm{Nrd}_{A_\mathcal{F}}(\mathrm{K}_1(A_\mathcal{F}))$ such that $x \cdot x'$ belongs to $\ker(\delta_{\mathcal{A},\mathcal{F}})$. The result of claim (iv) therefore follows directly from the description of $\ker(\delta_{\mathcal{A},\mathcal{F}})$ given in Proposition 4.1(ii).

Finally, to prove claim (v) we abbreviate $z_{\nu(t),x}$ to z. We first assume z is a primitive-basis of $\mathrm{d}_\mathcal{A}(C)$. In this case we can assume C belongs to $\mathsf{C}^\mathrm{f}(\mathcal{A})$ and hence that z arises from a choice of (ordered) $\mathcal{A}$-bases of each module C^a. Then, just as in the proof of claim (i), this choice of bases determines an isomorphism $\kappa : [C] \cong \mathbf{1}_{\mathcal{V}(\mathcal{A})}$ in $\mathcal{V}(\mathcal{A})$ with the property that $\nu(\kappa)$ sends z to the element $(1,0)$ of $(\zeta(A_\mathcal{F}),0)$. The definition of z therefore implies that

$$\begin{aligned}(x,0) &= \nu(t)(z)\\ &= \nu(t\circ\kappa^{-1})(\nu(\kappa)(z))\\ &= \mathrm{Nrd}_{A_\mathcal{F}}(t\circ\kappa^{-1})\cdot(1,0)\\ &= (\mathrm{Nrd}_{A_\mathcal{F}}(t\circ\kappa^{-1}),0)\end{aligned}$$

and hence that x belongs to $\mathrm{Nrd}_{A_\mathcal{F}}(\mathrm{K}_1(A_\mathcal{F}))$, as required.

To prove the converse we assume x belongs to $\mathrm{Nrd}_{A_\mathcal{F}}(\mathrm{K}_1(A_\mathcal{F}))$. In this case, the result of claim (iv) (with x' taken to be x^{-1}) implies that $\mathrm{d}_\mathcal{A}(C)$ has a primitive-basis z'.

Since z is assumed to be a locally-primitive basis of $\mathrm{d}_\mathcal{A}(C)$, Proposition 3.7(ii) therefore implies that $z = y \cdot z'$ for some element y of $\xi(\mathcal{A})^\times$ that belongs to $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}_\mathfrak{p}))$ for all $\mathfrak{p}$ in $\mathrm{Spm}(R)$ and further that z is a primitive-basis of $\mathrm{d}_\mathcal{A}(C)$ if and only if y belongs to $\mathrm{Nrd}_A(\mathrm{K}_1(\mathcal{A}))$. It therefore follows from Lemma 4.3(ii) that z is a primitive-basis of $\mathrm{d}_\mathcal{A}(C)$ if y belongs to $\mathrm{Nrd}_A(\mathrm{K}_1(A))$.

In addition, the same argument as used above shows that the image under $\nu(t)$ of the primitive-basis z' is equal to $(u,0)$ for some u in $\mathrm{Nrd}_{A_\mathcal{F}}(\mathrm{K}_1(A_\mathcal{F}))$ and hence that

$$\begin{aligned}(y,0) &= \nu(t)(z)\cdot\nu(t)(z')^{-1}\\ &= x\cdot\nu(t)(z')^{-1}\\ &= (x\cdot u^{-1},0).\end{aligned}$$

In particular, because x is now assumed to belong to $\mathrm{Nrd}_{A_{\mathcal{F}}}(\mathrm{K}_1(A_{\mathcal{F}}))$, it follows that the element $y = x\cdot u^{-1}$ belongs to both $\zeta(A)^\times$ and $\mathrm{Nrd}_{A_{\mathcal{F}}}(\mathrm{K}_1(A_{\mathcal{F}}))$.

To deduce that y belongs to $\mathrm{Nrd}_A(\mathrm{K}_1(A))$, and hence complete the proof of claim (v), it is thus enough to note that, under the present hypotheses, the Hasse-Schilling-Maass Norm Theorem implies (via the argument at the end of the proof of Lemma 4.2) that $\zeta(A)^\times \cap \mathrm{Nrd}_{A_{\mathcal{F}}}(\mathrm{K}_1(A_{\mathcal{F}})) = \mathrm{Nrd}_A(\mathrm{K}_1(A))$.

This then completes the proof of Theorem 4.8.

Part II: Arithmetic over non-abelian Galois extensions

In the remainder of the article we shall make some technical improvements to the theory of non-commutative Euler systems introduced by Burns and Sano in [24] and then combine these strengthened results with the K-theoretic techniques developed in §3 and §4 to improve aspects of the theory of leading term conjectures over arbitrary Galois extensions.

In particular, in this way we shall formulate both a natural main conjecture of higher-rank non-commutative Iwasawa theory for $\mathbb{G}_m$ over arbitrary number fields and a precise 'derivative formula' for the 'non-commutative Rubin-Stark Euler system' that generalizes to arbitrary Galois extensions of number fields the classical Gross-Stark Conjecture.

We shall also obtain strong evidence in support of both of these conjectures in important special cases and establish a precise link between them and the equivariant Tamagawa Number Conjecture for $\mathbb{G}_m$ over arbitrary Galois extensions, thereby generalising the main result of Burns, Kurihara and Sano in [19].

5. Integral arithmetic cohomology and Selmer modules

As a convenience for the reader, in this section we shall first recall some basic facts about the arithmetic modules and complexes that will play a key role in our theory.

Throughout, we fix a finite Galois extension L/K of global fields.

5.1. **Selmer modules.**

5.1.1. For a finite set of places Π of K and an extension E of K we write Π_E for the set of places of E lying above those in Π, $Y_{E,\Pi}$ for the free abelian group on the set Π_E and $X_{E,\Pi}$ for the submodule of $Y_{E,\Pi}$ comprising elements whose coefficients sum to zero.

For each place v of K we fix a place w_v of L above v and for each intermediate field E of L/K we write $w_{v,E}$ for the restriction of w_v to E. For each non-archimedean place w of E we write κ_w for its residue field and $\mathrm{N}w$ for its absolute norm.

We write S_K^∞ for the set of archimedean places of K (so that $S_K^\infty = \emptyset$ unless K is a number field). For an extension E of K we write $S_{\mathrm{ram}}(E/K)$ for the set of primes of K that ramify in E.

If Π is non-empty and (in the number field case) contains S_K^∞, then we write $\mathcal{O}_{E,\Pi}$ for the subring of E comprising elements integral at all places outside Π_E and $\mathcal{O}_{E,\Pi}^\times$ for the unit group of $\mathcal{O}_{E,\Pi}$. (If $\Pi = S_K^\infty$, then we abbreviate $\mathcal{O}_{E,\Pi}$ to $\mathcal{O}_E$.)

In this case, for any finite set of places Π' of K that is disjoint from Π, we write $\mathcal{O}_{E,\Pi,\Pi'}^\times$ for the (finite index) subgroup of $\mathcal{O}_{E,\Pi}^\times$ consisting of those elements congruent to 1 modulo all places in Π'_E. In addition, we write $\mathrm{Cl}_\Pi^{\Pi'}(E)$ for the ray class group of $\mathcal{O}_{E,\Pi}$ modulo $\prod_{w\in\Pi'_E} w$ (that is, the quotient of the group of fractional $\mathcal{O}_E$-ideals whose supports are coprime to all places in $(\Pi\cup\Pi')_E$ by the subgroup of principal ideals with a generator congruent to 1 modulo all places in Π'_E).

If E/K is Galois, we set $G_{E/K} := \mathrm{Gal}(E/K)$ and note each of the groups $Y_{E,\Pi}, X_{E,\Pi}$, $\mathcal{O}_{E,\Pi}^\times$, $\mathcal{O}_{E,\Pi,\Pi'}^\times$ and $\mathrm{Cl}_\Pi^{\Pi'}(E)$ has a natural action of $G_{E/K}$. In this case, for a non-archimedean place v of K we also fix a lift Fr_v to $G_{E/K}$ of the Frobenius automorphism of $w_{v,E}$.

5.1.2. If Π contains S_K^∞, then the '(Π-relative Π'-trivialized) integral dual Selmer group for $\mathbb{G}_m$ over E' is defined in [18, Def. 2.1] (where the notation $\mathcal{S}_{\Pi,\Pi'}(\mathbb{G}_m/E)$ is used) by setting

$$\mathrm{Sel}_\Pi^{\Pi'}(E) := \mathrm{cokernel}\Big(\prod\nolimits_{w\notin(\Pi\cup\Pi')_E}\mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(E_{\Pi'}^\times,\mathbb{Z})\Big),$$

where $E_{\Pi'}^\times$ is the group $\{a\in E^\times : \mathrm{ord}_w(a-1)>0 \text{ for all } w\in\Pi'_E\}$ and the arrow denotes the homomorphism that sends $(x_w)_w$ to the map $(a\mapsto \sum_{w\notin(\Pi\cup\Pi')_E}\mathrm{ord}_w(a)x_w)$.

We recall from loc. cit. that there exists a canonical exact sequence

$$0\to \mathrm{Cl}_\Pi^{\Pi'}(E)^\vee \to \mathrm{Sel}_\Pi^{\Pi'}(E)\to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{E,\Pi,\Pi'}^\times,\mathbb{Z})\to 0, \tag{5.1.1}$$

and a canonical transpose $\mathrm{Sel}_\Pi^{\Pi'}(E)^{\mathrm{tr}}$ to $\mathrm{Sel}_\Pi^{\Pi'}(E)$ (in the sense of Jannsen's homotopy theory of modules [46]) that lies in a canonical exact sequence

$$0\longrightarrow \mathrm{Cl}_\Pi^{\Pi'}(E)\longrightarrow \mathrm{Sel}_\Pi^{\Pi'}(E)^{\mathrm{tr}} \xrightarrow{\varrho_{E,\Pi}} X_{E,\Pi}\longrightarrow 0. \tag{5.1.2}$$

5.2. **Modified étale cohomology complexes.** We set $G := G_{L/K}$ and write $\mathsf{D}(\mathbb{Z}[G])$ for the derived category of G-modules. We also write $\mathsf{D}^{\mathrm{lf},0}(\mathbb{Z}[G])$ for its full triangulated subcategory comprising complexes isomorphic to a bounded complex of finitely generated locally-free G-modules C with the property that the Euler characteristic of $\mathbb{Q}\otimes_{\mathbb{Z}} C$ in $\mathrm{K}_0(\mathbb{Q}[G])$ vanishes.

The complexes that are used in the next result are described in terms of the complexes $\mathsf{R}\Gamma_{c,\Pi'}((\mathcal{O}_{L,\Pi})_{\mathcal{W}},\mathbb{Z})$ introduced by Kurihara and the current authors in [18, Prop. 2.4]. We recall, in particular, that the latter complexes can be naturally interpreted in terms of the Weil-étale cohomology theory that Lichtenbaum has constructed for global function fields [59] and conjectured to exist for number fields [60] (see [18, Rem. 2.5] for more details).

Lemma 5.1. *Let Π be a finite non-empty set of places of K containing $S_K^\infty\cup S_{\mathrm{ram}}(L/K)$, and let Π' be a finite set of places of K that is disjoint from Π. Then the complex*

$$C_{L,\Pi,\Pi'} := \mathsf{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}\Gamma_{c,\Pi'}((\mathcal{O}_{L,\Pi})_{\mathcal{W}},\mathbb{Z}),\mathbb{Z})[-2]$$

belongs to $\mathsf{D}^{\mathrm{lf},0}(\mathbb{Z}[G])$ and has the following properties.

(i) $C_{L,\Pi,\Pi'}$ *is acyclic outside degrees zero and one and there are canonical identifications* $H^0(C_{L,\Pi,\Pi'})$ *and* $H^1(C_{L,\Pi,\Pi'})$ *with* $\mathcal{O}^\times_{L,\Pi,\Pi'}$ *and* $\mathrm{Sel}_\Pi^{\Pi'}(L)^{\mathrm{tr}}$ *respectively.*

(ii) *If* Π_1 *is any finite set of places of* K *that contains* Π *and is disjoint from* Π'*, then there is a canonical exact triangle in* $\mathsf{D}^{\mathrm{lf},0}(\mathbb{Z}[G])$ *of the form*

$$C_{L,\Pi,\Pi'} \to C_{L,\Pi_1,\Pi'} \to \big(\bigoplus\nolimits_{w\in(\Pi_1\setminus\Pi)_L} \mathsf{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}\Gamma((\kappa_w)_{\mathcal{W}},\mathbb{Z}),\mathbb{Z})\big)[-1] \to C_{L,\Pi,\Pi'}[1],$$

where each complex $\mathsf{R}\Gamma((\kappa_w)_{\mathcal{W}},\mathbb{Z})$ *is as defined in* [18, Prop. 2.4(ii)].

(iii) *If* Π_1' *is any finite set of places of* K *that contains* Π' *and is disjoint from* Π*, then there is a canonical exact triangle in* $\mathsf{D}^{\mathrm{lf},0}(\mathbb{Z}[G])$ *of the form*

$$C_{L,\Pi,\Pi_1'} \to C_{L,\Pi,\Pi'} \to \big(\bigoplus\nolimits_{w\in(\Pi_1'\setminus\Pi')_L}\kappa_w^\times\big)[0] \to C_{L,\Pi,\Pi_1'}[1].$$

(iv) *For any normal subgroup* H *of* G *there is a canonical 'projection formula' isomorphism in* $\mathsf{D}^{\mathrm{lf},0}(\mathbb{Z}[G/H])$

$$\mathbb{Z}[G/H]\otimes^{\mathsf{L}}_{\mathbb{Z}[G]} C_{L,\Pi,\Pi'} \cong C_{L^H,\Pi,\Pi'},$$

and hence also a canonical isomorphism of $\mathbb{Z}[G/H]$*-modules*

$$\mathbb{Z}[G/H]\otimes_{\mathbb{Z}[G]}\mathrm{Sel}_\Pi^{\Pi'}(L)^{\mathrm{tr}} \cong \mathrm{Sel}_\Pi^{\Pi'}(L^H)^{\mathrm{tr}}.$$

(v) *If* Π *contains every* p*-adic place of* K*, then there exists a canonical exact triangle in* $\mathsf{D}(\mathbb{Z}_p[G])$ *of the form*

$$C_{L,\Pi,\Pi',p} \to \mathsf{R}\,\mathrm{Hom}_{\mathbb{Z}_p}(\mathsf{R}\Gamma_c(\mathcal{O}_{L,\Pi},\mathbb{Z}_p),\mathbb{Z}_p)[-2] \to \big(\mathbb{Z}_p\otimes_{\mathbb{Z}}\bigoplus\nolimits_{w\in\Pi_L'}\kappa_w^\times\big)[0] \to C_{L,\Pi,\Pi',p}[1]$$

in which $\mathsf{R}\Gamma_c(\mathcal{O}_{L,\Pi},\mathbb{Z}_p)$ *denotes the compactly-supported* p*-adic cohomology of* $\mathbb{Z}_p$ *over the scheme* $\mathrm{Spec}(\mathcal{O}_{L,\Pi})$.

Proof. The descriptions in claim (i) follow directly from [18, Def. 2.6 and Rem. 2.7]. In addition, since Π is assumed to contain all places which ramify in L/K, the fact that $C_{L,\Pi,\Pi'}$ belongs to $\mathsf{D}^{\mathrm{lf},0}(\mathbb{Z}[G])$ follows from the argument used to prove [18, Lem. 2.8].

The canonical exact triangle in claim (ii), resp. (iii), results directly from applying the functor $C\mapsto \mathsf{R}\,\mathrm{Hom}_{\mathbb{Z}}(C,\mathbb{Z})[-2]$ to the triangle given by the right-hand column of the diagram in claim (i), resp. the exact triangle in claim (ii), of Proposition 2.4 in loc. cit.

The first displayed isomorphism in claim (iv) follows by combining the construction of $C_{L,\Pi,\Pi'}$ in [18] with the canonical projection formula isomorphism in étale cohomology

$$\mathbb{Z}[G/H]\otimes^{\mathsf{L}}_{\mathbb{Z}[G]}\mathsf{R}\Gamma_c((\mathcal{O}_{L,\Pi})_{\text{ét}},\mathbb{Z}) \cong \mathsf{R}\Gamma_c((\mathcal{O}_{L^H,\Pi})_{\text{ét}},\mathbb{Z}).$$

The claimed isomorphism of $\mathbb{Z}[G/H]$-modules then follows directly from this isomorphism and the explicit description of cohomology groups given in claim (i).

Lastly we note that the existence of a canonical triangle as in claim (v) follows from the discussion in [19, §2.2]. □

Remark 5.2. If, in the setting of Lemma 5.1(ii), v is any place in $\Pi_1\setminus\Pi$, then the direct sum of $\mathsf{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}\Gamma((\kappa_w)_{\mathcal{W}},\mathbb{Z}),\mathbb{Z})[-1]$ over places w of L above v is a complex of (left) G-modules that identifies with

$$\mathbb{Z}[G] \xrightarrow{x\mapsto x(1-\mathrm{Fr}_v^{-1})} \mathbb{Z}[G],$$

where the first term is placed in degree zero and Fr_v is the Frobenius automorphism in G of some fixed place of L above v.

Remark 5.3. If the group $\mathcal{O}^\times_{L,\Pi,\Pi'}$ is torsion-free, then Lemma 5.1(i) implies that the complex $C^*_{L,\Pi,\Pi'} := \mathsf{R}\Gamma_{c,\Pi'}((\mathcal{O}_{L,\Pi})_{\mathcal{W}}, \mathbb{Z})$ is acyclic outside degrees one and two. Since $H^2(C^*_{L,\Pi,\Pi'})$ identifies with $\mathrm{Sel}_\Pi^{\Pi'}(L)$ (by [18, Prop. 2.4(iii)]), a similar argument to that in Lemma 5.1(iv) implies the existence in this case, for any normal subgroup H of G, of a canonical isomorphism of $\mathbb{Z}[G/H]$-modules $\mathbb{Z}[G/H] \otimes_{\mathbb{Z}[G]} \mathrm{Sel}_\Pi^{\Pi'}(L) \cong \mathrm{Sel}_\Pi^{\Pi'}(L^H)$.

6. Non-commutative Euler systems for $\mathbb{G}_m$

6.1. **Hypotheses and definitions.** In this section we fix a number field K, with algebraic closure K^c, and set $G_K := \mathrm{Gal}(K^c/K)$.

We write $\mathrm{Ir}(K)$ for the set of irreducible $\mathbb{Q}^c$-valued characters of G_K that have open kernel.

For each character χ in $\mathrm{Ir}(K)$ we fix an associated (finite-dimensional) representation V_χ of G_K over $\mathbb{Q}^c$ and we assume that all reduced exterior powers occurring in the sequel are defined relative to these fixed representations (cf. Remark 2.1).

6.1.1. We fix a Galois extension $\mathcal{K}$ of K in K^c and a finite set S of places of K with

$$S_K^\infty \subseteq S.$$

We write $\Sigma_S(\mathcal{K})$ for the subset of S comprising places that split completely in $\mathcal{K}$ and set

$$r_S = r_{S,\mathcal{K}} := |\Sigma_S(\mathcal{K})|.$$

We assume that there exists a prime number p and a (possibly empty) finite set of places T of K that is disjoint from S and such that the following condition is satisfied.

Hypothesis 6.1.

(i) $\mathcal{K}$ is unramified at all places of T, and
(ii) no element of $\mathcal{K}^\times$ of order p is congruent to 1 modulo all places in $T_{\mathcal{K}}$.

Remark 6.2. Hypothesis 6.1 is widely satisfied: for example, if $\mathcal{K}^\times$ contains no element of order p, then one can take T to be empty.

We write $\Omega(\mathcal{K}) = \Omega(\mathcal{K}/K)$ for the set of finite *ramified* Galois extensions of K in $\mathcal{K}$. For each F in $\Omega(\mathcal{K})$ we set

$$S(F) := S \cup S_{\mathrm{ram}}(F/K) \quad \text{and} \quad \mathcal{G}_F := G_{F/K}$$

and we identify $\mathrm{Ir}(\mathcal{G}_F)$ with the subset of $\mathrm{Ir}(K)$ comprising characters that factor through the restriction map $G_K \to \mathcal{G}_F$.

We write S_K^{all} for the set of all places of K and fix an ordering

$$S_K^{\mathrm{all}} = \{v(i)\}_{i\in\mathbb{N}} \tag{6.1.1}$$

in such a way that

$$\Sigma_S(\mathcal{K}) = \{v(i)\}_{i\in[r_S]}. \tag{6.1.2}$$

In the sequel we use, for each set of places of K, the ordering that is induced by (6.1.1). In particular, in all of the (exterior product) constructions that are made in the sequel, we regard the sets $S(F)$ to be ordered in this way.

For each place v of K we fix a place w_v of $\mathcal{K}$ and, by abuse of notation, also write $w_v = w_{F,v}$ for the restriction of w_v to any field F in $\Omega(\mathcal{K})$. If v is not in $S(F)$ for a given F in $\Omega(\mathcal{K})$ then we denote by $\mathcal{G}_{F,v}$ the decomposition subgroup of $\mathcal{G}_F$ relative to w_v and write

$$\mathrm{Fr}_v = \mathrm{Fr}_{F,v} \in \mathcal{G}_{F,v}$$

for the Frobenius automorphism relative to w_v.

6.1.2. The functoriality of reduced exterior powers implies that for F and F' in $\Omega(\mathcal{K})$ with $F \subseteq F'$, and any non-negative integer r, the norm map $\mathrm{N}_{F'/F} : (F')^\times \to F^\times$ induces a homomorphism of $\zeta(\mathbb{Q}[\mathcal{G}_{F'}])$-modules

$$\mathrm{N}^r_{F'/F} : {\bigwedge}^r_{\mathbb{Q}[\mathcal{G}_{F'}]}(\mathbb{Q}\cdot\mathcal{O}^\times_{F',S(F')}) \to {\bigwedge}^r_{\mathbb{Q}[\mathcal{G}_F]}(\mathbb{Q}\cdot\mathcal{O}^\times_{F,S(F')}).$$

Since $\mathcal{O}^\times_{F',S(F'),T,p}$ is torsion-free (as a consequence of Hypothesis 6.1(ii)), the general result of [24, Lem. 6.10(ii)] implies $\mathrm{N}^r_{F'/F}$ restricts to give a homomorphism of $\xi(\mathbb{Z}_p[\mathcal{G}_{F'}])$-modules

$$\bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}_{F'}]}\mathcal{O}^\times_{F',S(F'),T,p} \to \bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}_F]}\mathcal{O}^\times_{F,S(F'),T,p}. \tag{6.1.3}$$

This fact helps motivate the following definition.

Definition 6.3. Let r be a non-negative integer. Then a 'pre-Euler system of rank r' for $\mathbb{G}_m$ with respect to the data $\mathcal{K}/K, S$ and p is a family of elements

$$(c_F)_F \in \prod\nolimits_{F\in\Omega(\mathcal{K})}\mathbb{C}_p\otimes_{\mathbb{Q}}{\bigwedge}^r_{\mathbb{Q}[\mathcal{G}_F]}(\mathbb{Q}\cdot\mathcal{O}^\times_{F,S(F)})$$

with the property that for every F and F' in $\Omega(\mathcal{K})$ with $F \subset F'$ one has

$$\mathrm{N}^r_{F'/F}(c_{F'}) = \left(\prod\nolimits_{v\in S(F')\setminus S(F)}\mathrm{Nrd}_{\mathbb{Q}[\mathcal{G}_F]}(1-\mathrm{Fr}^{-1}_{F,v})\right)(c_F) \tag{6.1.4}$$

in $\mathbb{C}_p\otimes_{\mathbb{Q}}{\bigwedge}^r_{\mathbb{Q}[\mathcal{G}_F]}(\mathbb{Q}\cdot\mathcal{O}^\times_{F,S(F')})$.

An 'Euler system of rank r' for $\mathbb{G}_m$ with respect to the data $\mathcal{K}/K, S, T$ and p is a pre-Euler system $(c_F)_F$ for $\mathcal{K}/K, S$ and p with the additional property that

$$c_F \in \bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}_F]}\mathcal{O}^\times_{F,S(F),T,p}$$

for every F in $\Omega(\mathcal{K})$.

We write $\mathrm{pES}_r(\mathcal{K}/K,S,p)$ and $\mathrm{ES}_r(\mathcal{K}/K,S,T,p)$ for the respective collections of all such pre-Euler systems and Euler systems.

It is clear that $\mathrm{ES}_r(\mathcal{K}/K,S,T,p)$ is an abelian group that is endowed with a natural action of the algebra

$$\xi_p(\mathcal{K}/K) := \varprojlim_{F\in\Omega(\mathcal{K})}\xi(\mathbb{Z}_p[\mathcal{G}_F]),$$

where the transition morphisms are induced by the projection maps $\mathbb{Z}_p[\mathcal{G}_{F'}] \to \mathbb{Z}_p[\mathcal{G}_F]$ for $F \subseteq F'$ (and are surjective by [24, Lem. 2.7(iv)]).

Remark 6.4. There are useful relations between different modules of (pre-)Euler systems of any given rank.
(i) If $\mathcal{K}'$ is a Galois extension of K in $\mathcal{K}$, then the restriction of a system to the subset $\Omega(\mathcal{K}')$ of $\Omega(\mathcal{K})$ defines a homomorphism of $\xi_p(\mathcal{K}/K)$-modules

$$\mathrm{ES}_r(\mathcal{K}/K, S, T, p) \to \mathrm{ES}_r(\mathcal{K}'/K, S, T, p).$$

We refer to the image of ε under this homomorphism as the 'restriction of ε to $\mathcal{K}'$'.
(ii) Let v be a (non-archimedean) place of K outside $S \cup T$ and σ an element of G_K that acts as the inverse of the Frobenius automorphism of a place above v on every F in $\Omega(\mathcal{K})$ in which v is unramified. Then there exists a homomorphism of $\xi_p(\mathcal{K}/K)$-modules

$$\mathrm{ES}_r(\mathcal{K}/K, S, T, p) \to \mathrm{ES}_r(\mathcal{K}/K, S \cup \{v\}, T, p)$$

that sends each ε to the system $\varepsilon_\sigma = (\varepsilon_{\sigma,F})_F$ specified at each F in $\Omega(\mathcal{K})$ by

$$\varepsilon_{\sigma,F} := \begin{cases} (\mathrm{Nrd}_{\mathbb{Q}[\mathcal{G}_F]}(1-\sigma))(\varepsilon_F), & \text{if } v \text{ is unramified in } F, \\ \varepsilon_F, & \text{otherwise.} \end{cases}$$

In such a case we say that the system ε is a 'refinement' of the system ε_σ.

6.2. **Euler systems and L-series.** In this section we define a canonical family of pre-Euler systems in terms of the leading terms at zero of Artin L-series and discuss conditions under which this family comprises Euler systems.

6.2.1. We must first discuss some necessary preliminaries in the general setting of §5.

For this we fix a finite Galois extension L/K of global fields of group G, a finite non-empty set of places Π of K containing $S_K^\infty \cup S_{\mathrm{ram}}(L/K)$ and a finite set Π' of places of K that is disjoint from Π. For each intermediate field F of L/K we then set

$$\Sigma(F) = \Sigma_\Pi(F) := \{v \in \Pi : v \text{ splits completely in } F\}.$$

We write

$$R_{L,\Pi} : \mathbb{R} \cdot \mathcal{O}_{L,\Pi}^\times \to \mathbb{R} \cdot X_{L,\Pi} \tag{6.2.1}$$

for the isomorphism of $\mathbb{R}[G]$-modules that, for every u in $\mathcal{O}_{L,\Pi}^\times$, satisfies

$$R_{L,\Pi}(u) = -\sum\nolimits_{w \in \Pi_L} \log|u|_w \cdot w,$$

where $|\cdot|_w$ denotes the absolute value at w (normalized as in [82, Chap. 0, 0.2]).

Then, for each non-negative integer a, the map $R_{L,\Pi}$ induces an isomorphism of $\zeta(\mathbb{R}[G])$-modules

$$\lambda_{L,\Pi}^a : \bigwedge\nolimits_{\mathbb{R}[G]}^a (\mathbb{R} \cdot \mathcal{O}_{L,\Pi}^\times) \xrightarrow{\sim} \bigwedge\nolimits_{\mathbb{R}[G]}^a (\mathbb{R} \cdot X_{L,\Pi}).$$

For each such a, the 'a-th derived Stickelberger function' of the data $L/K, \Pi$ and Π' is defined to be the $\zeta(\mathbb{C}[G])$-valued meromorphic function of a complex variable z

$$\theta_{L/K,\Pi,\Pi'}^a(z) := \sum\nolimits_{\chi \in \mathrm{Ir}(G)} (z^{-a\chi(1)} L_{\Pi,\Pi'}(\check{\chi}, z)) \cdot e_\chi,$$

where $L_{\Pi,\Pi'}(\check{\chi}, z)$ is the Π-truncated Π'-modified Artin L-function for the contragredient $\check{\chi}$ of χ and we use the primitive central idempotent

$$e_\chi := \chi(1)|G|^{-1}\sum_{g\in G}\chi(g)g^{-1}$$

of $\mathbb{C}[G]$. In the case $a = 0$, we set

$$\theta_{L/K,\Pi,\Pi'}(z) := \theta^0_{L/K,\Pi,\Pi'}(z) \tag{6.2.2}$$

and refer to this function as the 'Stickelberger function' of $L/K, \Pi$ and Π'.

An explicit analysis of the functional equation of Artin L-functions (as in [82, Chap. I, Prop. 3.4]) shows that for each χ in $\mathrm{Ir}(G)$ one has

$$\mathrm{ord}_{z=0}L_\Pi(\chi, z) = \chi(1)^{-1}\cdot \dim_{\mathbb{C}}\big(e_\chi(\mathbb{C}\cdot X_{L,\Pi})\big). \tag{6.2.3}$$

This formula implies, in particular, that if Π contains a proper subset of a elements that split completely in L, then the function $\theta^a_{L/K,\Pi,\Pi'}(z)$ is holomorphic at $z = 0$ and it is then easily checked that its value $\theta^a_{L/K,\Pi,\Pi'}(0)$ at $z = 0$ belongs to the subring $\zeta(\mathbb{R}[G])$ of $\zeta(\mathbb{C}[G])$.

This observation allows us to make the following definition.

Definition 6.5. Let Σ be a subset of $\Sigma(L)$ with $\Sigma \neq \Pi$ and fix $v' \in \Pi\setminus\Sigma$. Then the '(non-commutative) Rubin-Stark element' associated to $L/K, \Pi, \Pi'$ and Σ is the unique element $\varepsilon^\Sigma_{L/K,\Pi,\Pi'}$ of ${\bigwedge}^{|\Sigma|}_{\mathbb{R}[G]}(\mathbb{R}\cdot\mathcal{O}^\times_{L,\Pi})$ that satisfies

$$\lambda^{|\Sigma|}_{L,\Pi}(\varepsilon^\Sigma_{L/K,\Pi,\Pi'}) = \theta^{|\Sigma|}_{L/K,\Pi,\Pi'}(0)\cdot\wedge_{v\in\Sigma}(w_v - w_{v'}),$$

where the exterior product is defined with respect to the ordering of Σ induced by (6.1.1).

Remark 6.6.

(i) The condition $\Sigma \neq \Pi$ is automatically satisfied if $\Sigma(L) \neq \Pi$ and hence, for example, if L/K is ramified.

(ii) Since every place v in Σ splits completely in L, the elements $\{w_v - w_{v'}\}_{v\in\Sigma}$ span a free G-module of rank $|\Sigma|$. Given this, the explicit definition (via (2.1.1)) of reduced exterior powers implies that the element $e_\chi(\wedge_{v\in\Sigma}(w_v - w_{v'}))$ is non-zero for every χ in $\mathrm{Ir}(G)$. In particular, since $\lambda^{|\Sigma|}_{L,\Pi}$ is injective, the equality defining $\varepsilon^\Sigma_{L/K,\Pi,\Pi'}$ implies, for each χ, that

$$e_\chi(\varepsilon^\Sigma_{L/K,\Pi,\Pi'}) \neq 0 \Longleftrightarrow e_\chi\cdot\theta^{|\Sigma|}_{L/K,\Pi,\Pi'}(0) \neq 0.$$

(iii) If $\Sigma \neq \Sigma(L)$, then $|\Sigma| < |\Sigma(L)|$ and, in this case, (6.2.3) combines with the observation in (ii) to imply that either $\varepsilon^\Sigma_{L/K,\Pi,\Pi'}$ vanishes or both $L = K$ (so that $\Sigma(L) = \Pi$) and $|\Sigma| = |\Sigma(L)| - 1$. By a similar argument one checks that, if $\Sigma(L) \neq \Pi$ (so that $L \neq K$), then $\theta^{|\Sigma(L)|}_{L/K,\Pi,\Pi'}(0)\cdot(w_{v'_1} - w_{v'}) = 0$ for every v'_1 in $\Pi\setminus\Sigma(L)$ and so the element

$$\varepsilon_{L/K,\Pi,\Pi'} := \varepsilon^{\Sigma(L)}_{L/K,\Pi,\Pi'} \tag{6.2.4}$$

depends only on the data $L/K, \Pi$ and Π'.

(iv) If Σ is empty, then the map $\lambda^{|\Sigma|}_{L,\Pi} = \lambda^0_{L,\Pi}$ identifies with the identity automorphism of the space

$${\bigwedge}^0_{\mathbb{C}[G]}(\mathbb{C}\cdot\mathcal{O}^\times_{L,\Pi}) = \zeta(\mathbb{C}[G]) = {\bigwedge}^0_{\mathbb{C}[G]}(\mathbb{C}\cdot X_{L,\Pi})$$

and $\wedge_{v\in\Sigma}(w_v - w_{v'})$ with the element 1 of $\zeta(\mathbb{C}[G])$. In this case, therefore, $\varepsilon^{\Sigma}_{L/K,\Pi,\Pi'}$ coincides with the (non-commutative) 'Stickelberger element'

$$\theta_{L/K,\Pi,\Pi'}(0) := \sum\nolimits_{\chi\in\mathrm{Ir}(G)} L_{\Pi,\Pi'}(\check{\chi},0)\cdot e_\chi \in \zeta(\mathbb{Q}[G])$$

that was first studied by Hayes in [44]. (We note here that, whilst $\theta_{L/K,\Pi,\Pi'}(0)$ belongs, a priori, only to $\zeta(\mathbb{C}[G])$, a classical result of Siegel [78] combines with Brauer induction to imply, via [82, Th. 1.2, p. 70], that it belongs to $\zeta(\mathbb{Q}[G])$, as indicated above.)

Remark 6.7. We now have all of the ingredients that are required to justify the observations in Example 4.7(ii) and Remark 4.9(ii). To do this, we fix data $L/K,\Pi,\Pi'$ as above and use the complex $C = C_{L,\Pi,\Pi'}$ defined in Lemma 5.1. Then C belongs to $\mathsf{D}^{\mathrm{lf},0}(\mathbb{Z}[G])$ and the isomorphism $R_{L,\Pi}$ from (6.2.1) combines with the explicit descriptions of cohomology in Lemma 5.1(i) to induce a canonical isomorphism $\lambda : \mathrm{d}_{\mathbb{R}[G]}(C_{\mathbb{R}}) \xrightarrow{\sim} (\zeta(\mathbb{R}[G]),0)$ in $\mathcal{P}(\zeta(\mathbb{R}[G]))$. In this setting, the zeta element $z_{\lambda,x}$ associated by Definition 4.6 to the leading term $x = \theta^*_{L/K,\Pi,\Pi'}(0)$ at $z=0$ of $\theta_{L/K,\Pi,\Pi'}(z)$ generalizes to non-abelian extensions the 'zeta elements' $z_{L/K,\Pi,\Pi'}$ defined in [18, Def. 3.5]. Further, an argument similar to Example 4.7(i) shows Theorem 4.8(iii) implies that the 'lifted root number conjecture' for L/K of Gruenberg, Ritter and Weiss [43] is equivalent to asserting that the element $z_{\lambda,x}$ is a locally-primitive basis of $\mathrm{d}_{\mathbb{Z}[G]}(C)$. This latter condition directly extends the leading term conjecture 'LTC(L/K)' formulated for abelian L/K in [18, Conj. 3.6]. In addition, if $z_{\lambda,x}$ is a locally-primitive basis of $\mathrm{d}_{\mathbb{Z}[G]}(C)$, then Theorem 4.8(v) combines with the Hasse-Schilling-Maass Norm Theorem to imply that $z_{\lambda,x}$ is a primitive basis of $\mathrm{d}_{\mathbb{Z}[G]}(C)$ if and only if, for every irreducible complex symplectic character χ of G, the leading term at $z=0$ of $L_\Pi(\chi,z)$ is a strictly positive real number.

6.2.2. We next state a precise conjecture concerning the 'integral' properties of the Rubin-Stark elements from Definition 6.5.

In particular, in this conjecture we fix data $L/K,\Pi$ and Π' as in §6.2.1. We also fix a prime p and an isomorphism of fields $\mathbb{C}\cong\mathbb{C}_p$ and identify each element of the form $\varepsilon^{\Sigma}_{L/K,\Pi,\Pi'}$ with its image under the induced embedding of $\zeta(\mathbb{R}[G])$-modules

$$\bigwedge\nolimits^{|\Sigma|}_{\mathbb{R}[G]}(\mathbb{R}\cdot\mathcal{O}^\times_{L,\Pi}) \to \bigwedge\nolimits^{|\Sigma|}_{\mathbb{C}_p[G]}(\mathbb{C}_p\cdot\mathcal{O}^\times_{L,\Pi,p}).$$

Conjecture 6.8. (Non-commutative Rubin-Stark Conjecture) *Let Σ be a subset of $\Sigma(L)$ with $\Sigma\neq\Pi$. Then, for every prime p for which the group $\mathcal{O}^\times_{L,\Pi,\Pi',p}$ is torsion-free, one has*

$$\varepsilon^{\Sigma}_{L/K,\Pi,\Pi'} \in \bigcap\nolimits^{|\Sigma|}_{\mathbb{Z}_p[G]}\mathcal{O}^\times_{L,\Pi,\Pi',p}.$$

Remark 6.9.
(i) If Tate's formulation [82, Chap. I, Conj. 5.1] of Stark's principal conjecture is valid for L/K (as is automatically the case if K has positive characteristic), then the validity of Conjecture 6.8 can be shown to be independent of the choice of isomorphism $j : \mathbb{C}\cong\mathbb{C}_p$. For this reason, we do not explicitly indicate the choice of j either in the statement of Conjecture 6.8 or in the arguments that follow.

(ii) If $\Sigma = \Sigma(L)$ is empty, then ($\Sigma \neq \Pi$ and) $\varepsilon^{\Sigma}_{L/K,\Pi,\Pi'} = \theta_{L/K,\Pi,\Pi'}(0)$ (cf. Remark 6.6(iv)) and also $|\Sigma| = 0$ so $\bigcap^{|\Sigma|}_{\mathbb{Z}_p[G]}\mathcal{O}^{\times}_{L,\Pi,\Pi',p} = \xi(\mathbb{Z}_p[G])$ (by [24, Th. 4.19(i)]). In this case, therefore, Conjecture 6.8 asserts that $\theta_{L/K,\Pi,\Pi'}(0)$ belongs to $\xi(\mathbb{Z}_p[G])$ for every prime p for which $\mathcal{O}^{\times}_{L,\Pi,\Pi',p}$ is torsion-free. Recent results of Ellerbrock and Nickel [34, Th. 1, Th. 2] provide evidence in support of this prediction.
(iii) If $\mathcal{O}^{\times}_{L,\Pi,\Pi'}$ is itself torsion-free, then Conjecture 6.8 (for all p) combines with [24, Th. 4.19(iii)] to predict that the element $\varepsilon_{L/K,\Pi,\Pi'}$ (from (6.2.4)) belongs to $\bigcap^{|\Sigma(L)|}_{\mathbb{Z}[G]}\mathcal{O}^{\times}_{L,\Pi,\Pi'}$. This prediction is a natural generalisation to arbitrary Galois extensions of the Rubin-Stark Conjecture that is formulated for abelian extensions in [74]. For this reason, we shall in the sequel refer to Conjecture 6.8 as the 'Rubin-Stark Conjecture' for the data $(L/K, \Pi, \Pi', p)$.

6.2.3. As a final preliminary step, we establish some useful properties of a family of central idempotents of $\mathbb{Q}[G]$ that will play an important role in the sequel.

To do this we note that, for any proper subset Π_1 of the set of places Π of K fixed above, one obtains an idempotent of $\zeta(\mathbb{Q}[G])$ by setting

$$e_{L/K,\Pi,\Pi_1} := \sum\nolimits_{\chi} e_{\chi} \tag{6.2.5}$$

where in the sum χ runs over all characters in $\mathrm{Ir}(G)$ for which $e_\chi(\mathbb{Q}^c \otimes_{\mathbb{Z}} X_{L,\Pi\setminus\Pi_1})$ vanishes.

In the following result, we fix, for each χ in $\mathrm{Ir}(G)$, an associated $\mathbb{Q}^c$-representation V_χ (cf. the beginning of §6.1).

Lemma 6.10. *Fix $L/K, \Pi$ and Π_1 as above. Then, for every character χ in $\mathrm{Ir}(G)$, the following conditions are equivalent.*

(i) $e_\chi \cdot e_{L/K,\Pi,\Pi_1} \neq 0$.
(ii) *The projection map $e_\chi(\mathbb{Q}^c \otimes_{\mathbb{Z}} X_{L,\Pi}) \to e_\chi(\mathbb{Q}^c \otimes_{\mathbb{Z}} Y_{L,\Pi_1})$ is bijective.*
(iii) *The multiplicity of V_χ in the decompositions of both $\mathbb{Q}^c \otimes_{\mathbb{Z}} \mathcal{O}^{\times}_{L,\Pi}$ and $\mathbb{Q}^c \otimes_{\mathbb{Z}} X_{L,\Pi}$ is equal to $\sum_{v\in\Pi_1}\dim_{\mathbb{Q}^c}(V_\chi^{G_v})$, where G_v is the decomposition subgroup in G of any fixed place w of L above v.*
(iv) *The order of vanishing of $L_\Pi(\chi, z)$ at $z = 0$ is equal to $\sum_{v\in\Pi_1}\dim_{\mathbb{Q}^c}(V_\chi^{G_v})$.*
(v) *If χ is non-trivial, then $V_\chi^{G_v}$ vanishes for each v in $\Pi \setminus \Pi_1$. If χ is trivial, then $|\Pi \setminus \Pi_1| = 1$.*

Proof. The definition (6.2.5) of $e_{L/K,\Pi,\Pi_1}$ ensures that $e_\chi \cdot e_{L/K,\Pi,\Pi_1} \neq 0$ if and only if the space $e_\chi(\mathbb{Q}^c \otimes_{\mathbb{Z}} X_{L,\Pi\setminus\Pi_1})$ vanishes. The equivalence of conditions (i) and (ii) is therefore a consequence of the natural exact sequence

$$0 \to X_{L,\Pi\setminus\Pi_1} \to X_{L,\Pi} \to Y_{L,\Pi_1} \to 0.$$

Conditions (ii) and (iii) are equivalent since $X_{L,\Pi}$ and $\mathcal{O}^{\times}_{L,\Pi}$ span isomorphic $\mathbb{Q}^c[G]$-modules and for each place v in Π_1 the space $e_\chi(\mathbb{Q}^c \otimes_{\mathbb{Z}} Y_{L,\{v\}})$ is isomorphic to the G_v-invariants of V_χ.

The latter fact also combines with the explicit formula (6.2.3) to imply equivalence of the conditions (ii) and (iv) and with the exact sequence

$$0 \to X_{L,\Pi\setminus\Pi_1} \to Y_{L,\Pi\setminus\Pi_1} \to \mathbb{Z} \to 0$$

to imply equivalence of the conditions (ii) and (v). □

Remark 6.11. If the set Π_1 in Lemma 6.10 is the subset Σ of $\Sigma(L)$ that occurs in Definition 6.5, then G_v is the trivial group for each v in $\Pi_1 = \Sigma$ and so, for every χ in $\mathrm{Ir}(G)$, one has

$$\sum\nolimits_{v\in\Pi_1}\dim_{\mathbb{Q}^c}(V_\chi^{G_v}) = \sum\nolimits_{v\in\Sigma}\dim_{\mathbb{Q}^c}(V_\chi) = |\Sigma|\cdot\chi(1).$$

In this case, therefore, the equivalence of conditions (i) and (iv) in the above result implies, for any auxiliary set of places Π' as in §6.2.1, that

$$\begin{aligned} e_\chi\cdot e_{L/K,\Pi,\Sigma}\neq 0 &\Longleftrightarrow e_\chi\cdot\theta^{|\Sigma|}_{L/K,\Pi,\Pi'}(0)\neq 0\\ &\Longleftrightarrow e_\chi(\varepsilon^{\Sigma}_{L/K,\Pi,\Pi'}),\end{aligned}$$

where the second equivalence follows from Remark 6.6(ii). In the space $\bigwedge^{|\Sigma|}_{\mathbb{R}[G]}(\mathbb{R}\cdot\mathcal{O}^\times_{L,\Pi})$, one therefore has an equality

$$\varepsilon^{\Sigma}_{L/K,\Pi,\Pi'} = e_{L/K,\Pi,\Sigma}(\varepsilon^{\Sigma}_{L/K,\Pi,\Pi'}).$$

6.2.4. We now return to the setting of §6.1.1.

For every F in $\Omega(\mathcal{K})$ the set $\Sigma_S(\mathcal{K})$ is a proper subset of $S(F)$ (since $S_{\mathrm{ram}}(F/K)\neq\emptyset$) and so Remark 6.6(iii) implies that the non-commutative Rubin-Stark element

$$\varepsilon_{F,S,T} := \varepsilon^{\Sigma_S(\mathcal{K})}_{F/K,S(F),T} \tag{6.2.6}$$

depends only on the data $F/K, S, T$ and $\mathcal{K}$.

Lemma 6.12. *The collection*

$$\varepsilon^{\mathrm{RS}}_{\mathcal{K},S,T} := (\varepsilon_{F,S,T})_{F\in\Omega(\mathcal{K})}$$

has the following properties.

(i) $\varepsilon^{\mathrm{RS}}_{\mathcal{K},S,T}$ *belongs to* $\mathrm{pES}_{r_S}(\mathcal{K}/K,S,p)$.

(ii) *If, for every F in $\Omega(\mathcal{K})$, Conjecture 6.8 is valid for the data F/K, $S(F)$, T and p, then $\varepsilon^{\mathrm{RS}}_{\mathcal{K},S,T}$ belongs to* $\mathrm{ES}_{r_S}(\mathcal{K}/K,S,T,p)$.

Proof. Set $r := r_S = |\Sigma_S(\mathcal{K})|$. Then, if Conjecture 6.8 is valid for the data F/K, $S(F)$, T and p for every F in $\Omega(\mathcal{K})$, each element

$$(\varepsilon^{\mathrm{RS}}_{\mathcal{K},S,T})_F = \varepsilon_{F,S,T} = \varepsilon^{\Sigma_S(\mathcal{K})}_{F/K,S(F),T}$$

belongs to $\bigcap^r_{\mathbb{Z}_p[\mathcal{G}_F]}\mathcal{O}^\times_{F,S(F),T,p}$ and so claim (ii) is a consequence of claim (i).

To prove claim (i) it suffices to show that the distribution relation (6.1.4) is valid for all pairs F and F' with $F\subset F'$ if we set $c_F := \varepsilon^{\Sigma_S(\mathcal{K})}_{F/K,S(F),T}$.

For each finite set of places S' of K that contains $S(F)$, and each χ in $\mathrm{Ir}(\mathcal{G}_F)$, there is an equality of functions

$$L_{S'}(\check\chi,s) = \Big(\prod\nolimits_{v\in S'\setminus S(F)}\det\big(1-\mathrm{Fr}^{-1}_{F,v}\cdot(\mathrm{N}v)^{-s}\mid V_\chi\big)\Big)\cdot L_{S(F)}(\check\chi,s). \tag{6.2.7}$$

Taken together with the inflation invariance of Artin L-series this implies that

$$\pi_{F'/F}\big(\theta^r_{F'/K,S(F'),T}(0)\big) = \Big(\prod\nolimits_{v\in S(F')\setminus S(F)}\mathrm{Nrd}_{\mathbb{Q}[\mathcal{G}_F]}\big(1-\mathrm{Fr}^{-1}_{F,v}\big)\Big)\cdot\theta^r_{F/K,S(F),T}(0),$$

where $\pi_{F'/F}$ is the natural projection map $\zeta(\mathbb{C}[\mathcal{G}_{F'}]) \to \zeta(\mathbb{C}[\mathcal{G}_F])$. Given the explicit definition (in Definition 6.5) of each element $c_F = \varepsilon^{\Sigma_S(\mathcal{K})}_{F/K,S(F),T}$, the validity of (6.1.4) in this case will therefore follow if there exists a commutative diagram of $\zeta(\mathbb{R}[\mathcal{G}_{F'}])$-modules

$$\begin{array}{ccc} {\bigwedge}^r_{\mathbb{R}[\mathcal{G}_{F'}]}(\mathbb{R}\cdot\mathcal{O}^\times_{F',S(F')}) & \xrightarrow{\lambda^r_{F',S(F')}} & {\bigwedge}^r_{\mathbb{R}[\mathcal{G}_{F'}]}(\mathbb{R}\cdot X_{F',S(F')}) \\ {\scriptstyle \mathrm{N}^r_{F'/F}}\downarrow & & \downarrow{\scriptstyle\theta} \\ {\bigwedge}^r_{\mathbb{R}[\mathcal{G}_{F}]}(\mathbb{R}\cdot\mathcal{O}^\times_{F,S(F')}) & \xrightarrow{\lambda^r_{F,S(F')}} & {\bigwedge}^r_{\mathbb{R}[\mathcal{G}_{F}]}(\mathbb{R}\cdot X_{F,S(F')}), \end{array}$$

in which one has

$$\theta\big(\wedge_{v\in\Sigma_S(\mathcal{K})}(w_{v,F'} - w_{v',F'})\big) = \wedge_{v\in\Sigma_S(\mathcal{K})}(w_{v,F} - w_{v',F})$$

for any choice of place v' in $S(F)\setminus\Sigma_S(\mathcal{K})$. The existence of such a diagram is in turn an easy consequence of the fact that the following diagram commutes

$$\begin{array}{ccc} \mathbb{R}\cdot\mathcal{O}^\times_{F',S(F')} & \xrightarrow{R_{F',S(F')}} & \mathbb{R}\cdot X_{F',S(F')} \\ \uparrow & & \uparrow \\ \mathbb{R}\cdot\mathcal{O}^\times_{F,S(F')} & \xrightarrow{R_{F,S(F')}} & \mathbb{R}\cdot X_{F,S(F')}, \end{array}$$

where the left hand vertical map is the natural inclusion and the right hand vertical map is induced by sending each place $w_{v,F}$ to $\sum_{g\in\mathrm{Gal}(F'/F)} g(w_{v,F'})$ (cf. [82, bottom of p. 29]). □

Definition 6.13. The 'Rubin-Stark (non-commutative) Euler system' relative to the data $\mathcal{K}/K, S, T$ and p is the element $\varepsilon^{\mathrm{RS}}_{\mathcal{K},S,T}$ of $\mathrm{pES}_{r_S}(\mathcal{K}/K,S,p)$ described in Lemma 6.12.

Remark 6.14.
(i) If $\mathcal{K}/K$ is abelian, then $\varepsilon^{\mathrm{RS}}_{\mathcal{K},S,T}$ coincides with the pre-Euler system considered by Rubin in [74, §6].
(ii) If $\Sigma_S(\mathcal{K})$ is empty, then, for every F in $\Omega(\mathcal{K})$, Remark 6.9(ii) implies that

$$(\varepsilon^{\mathrm{RS}}_{\mathcal{K},S,T})_F = \theta_{F/K,S(F),T}(0).$$

In addition, Conjecture 6.8 predicts that $\theta_{F/K,S(F),T}(0)$ belongs to $\xi(\mathbb{Z}_p[\mathcal{G}_F])$ for every p for which $\mathcal{O}^\times_{F,S(F),T,p}$ is torsion-free (see Remark 6.9(ii)).

6.3. **Euler systems and Galois cohomology.** The 'reduced determinant' functor constructed in [24, §5] can be combined with the complexes constructed in Lemma 5.1 to give an unconditional construction of non-commutative Euler systems.

In this section we shall describe this construction and then use it to strengthen one of the main results of [24].

To do so, we introduce the following convenient notation: in the sequel, for each natural number d, we consider the ordered set

$$[d] := \{i\in\mathbb{Z} : 1\le i\le d\}.$$

For each finite group Γ, we then write $\{b_{\Gamma,i}\}_{i\in[d]}$ for the standard (ordered) $\mathbb{Z}[\Gamma]$-basis of $\mathbb{Z}[\Gamma]^d$ and $\{b^*_{\Gamma,i}\}_{i\in[d]}$ for the corresponding dual basis of $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}[\Gamma]^d,\mathbb{Z}[\Gamma])$.

In the case that $\Gamma = \mathcal{G}_L$ for some L in $\Omega(\mathcal{K}_\infty)$ we shall also use the abbreviations

$$b_{L,i} := b_{\mathcal{G}_L,i} \quad \text{and} \quad b^*_{L,i} := b^*_{\mathcal{G}_L,i}. \tag{6.3.1}$$

6.3.1. We start by proving a version of the construction made in [24, Lem. 8.8] that is relevant to our setting.

As usual, we endow each set of places of K with the ordering induced by (6.1.1).

Lemma 6.15. *Fix data $L/K, G, \Pi$ and Π' as in §6.2.1.*

Fix a place v' in Π, set $n := |\Pi\setminus\{v'\}| = |\Pi|-1$ and let p be a prime for which $\mathcal{O}^\times_{L,\Pi,\Pi',p}$ is torsion-free. Then there exists a natural number d with $d\geq n$ and a canonical family of complexes of $\mathbb{Z}_p[G]$-modules $C(\phi)$ of the form

$$\mathbb{Z}_p[G]^d \xrightarrow{\phi} \mathbb{Z}_p[G]^d$$

in which the first term is placed in degree zero and the following claims are valid.

- (i) *There exists an isomorphism $\kappa : C(\phi)\to C_{L,\Pi,\Pi',p}$ in $\mathsf{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ with the following property: for $i\in[d]$, the image of $b_{G,i}$ under the composite map*

$$\mathbb{Z}_p[G]^d \to \mathrm{cok}(\phi) \xrightarrow{H^1(\kappa)} \mathrm{Sel}^{\Pi'}_{\Pi}(L)^{\mathrm{tr}}_p \xrightarrow{\varrho_{L,\Pi,p}} X_{L,\Pi,p},$$

  where $\varrho_{L,\Pi}$ is as in (5.1.2), is equal to $w_{v_i}-w_{v'}$, with v_i the i-th element of $\Pi\setminus\{v'\}$, if $i\in[n]$ and is equal to 0 otherwise.
- (ii) *If $C(\phi')$ is any complex in the family, then $\phi' = \eta\circ\phi\circ(\eta')^{-1}$ where η and η' are automorphisms of $\mathbb{Z}_p[G]^d$ and η is represented, with respect to $\{b_{G,i}\}_{i\in[d]}$, by a block matrix*

$$\left(\begin{array}{c|c} I_n & * \\ \hline 0 & M_\eta \end{array}\right), \tag{6.3.2}$$

  where I_n is the $n\times n$ identity matrix and M_η belongs to $\mathrm{GL}_{d-n}(\mathbb{Z}_p[G])$.

Proof. We first fix a projective cover of $\mathbb{Z}_p[G]$-modules $\varpi' : P\to \ker(\varrho_{L,\Pi,p}) \cong \mathrm{Cl}^{\Pi'}_{\Pi}(L)_p$ and a module P' of minimal rank such that $P\oplus P'$ is a free $\mathbb{Z}_p[G]$-module. With d_0 denoting the rank of $P\oplus P'$, we fix an identification $P\oplus P' = \mathbb{Z}_p[G]^{d_0}$ and write $\varpi_1 = (\varpi', 0_{P'})$ for the induced surjective map $\mathbb{Z}_p[G]^{d_0}\to\ker(\varrho_{L,\Pi,p})$. We finally set $d := n+d_0$ and $\mathfrak{S}_L := \mathrm{Sel}^{\Pi'}_{\Pi}(L)^{\mathrm{tr}}_p$ and write $\varpi : \mathbb{Z}_p[G]^d\to\mathfrak{S}_L$ for the map of $\mathbb{Z}_p[G]$-modules that sends $b_{G,i}$ to a choice of pre-image of $w_{v_i}-w_{v'}$ under $\varrho_{L,\Pi,p}$ if $i\in[n]$ and to $\varpi_1(b_{G,i-n})$ if $i\in[d]\setminus[n]$. Then ϖ is surjective and such that

$$\varrho_{L,\Pi,p}(\varpi(b_{G,i})) = \begin{cases} w_{v_i}-w_{v'}, & \text{if } i\in[n] \\ 0, & \text{if } i\in[d]\setminus[n]. \end{cases} \tag{6.3.3}$$

Now, since $C = C_{L,\Pi,\Pi'}$ belongs to $\mathsf{D}^{\mathrm{lf},0}(\mathbb{Z}[G])$, and the module $U_L = H^0(C_p)$ is torsion-free, a standard argument (as in [21, Prop. 3.2]) shows the existence of an isomorphism $\kappa : C(\phi)\to C_p$ in $\mathsf{D}(\mathbb{Z}_p[G])$, where $C(\phi)$ has the required form $\mathbb{Z}_p[G]^d \xrightarrow{\phi} \mathbb{Z}_p[G]^d$ and $H^1(\kappa)$

is induced by the map ϖ constructed above. In view of (6.3.3) this construction has the properties described in claim (i).

To verify claim (ii), it is also enough to note that if $\kappa' : C(\phi') \to C_p$ is any alternative set of data constructed as above, then the argument of [21, Prop. 3.2(iv)] shows the existence of automorphisms η' and η of $\mathbb{Z}_p[G]^d$ that lie in an exact commutative diagram of the form

(6.3.4)
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{O}^\times_{L,\Pi,\Pi',p} & \xrightarrow{H^0(\kappa)^{-1}} & \mathbb{Z}_p[G]^d & \xrightarrow{\phi} & \mathbb{Z}_p[G]^d & \xrightarrow{H^1(\kappa)^\dagger} & \mathfrak{S}_L & \longrightarrow & 0 \\ & & \| & & \eta' \downarrow & & \downarrow \eta & & \| & & \\ 0 & \longrightarrow & \mathcal{O}^\times_{L,\Pi,\Pi',p} & \xrightarrow{H^0(\kappa')^{-1}} & \mathbb{Z}_p[G]^d & \xrightarrow{\phi'} & \mathbb{Z}_p[G]^d & \xrightarrow{H^1(\kappa')^\dagger} & \mathfrak{S}_L & \longrightarrow & 0. \end{array}$$

Here $H^1(\kappa)^\dagger$ and $H^1(\kappa')^\dagger$ are the composites of the respective tautological maps $\mathbb{Z}_p[G]^d \to \mathrm{cok}(\phi)$ and $\mathbb{Z}_p[G]^d \to \mathrm{cok}(\phi')$ with $H^1(\kappa)$ and $H^1(\kappa')$, and η is represented with respect to the basis $\{b_{G,i}\}_{i\in[d]}$ by a block matrix of the required sort (6.3.2). □

6.3.2. We next derive a useful consequence of Lemma 6.15 in the setting of §6.1.1.

To state the result we use the following notation: if Γ is a finite group and X a finitely generated $\xi(\mathbb{Z}_p[\Gamma])$-lattice, then we regard X as a subset of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} X$ in the natural way and, for each element a of $\zeta(\mathbb{Q}_p[\Gamma])$, we define a $\xi(\mathbb{Z}_p[\Gamma])$-submodule of X by setting

$$X[a] := \{x \in X : a \cdot x = 0\} = \{x \in X : x = (1-a)\cdot x\}.$$

We also note that, for every L in $\Omega(\mathcal{K})$, the set $S(L) \setminus \Sigma_S(L)$ is non-empty (since L/K is assumed to be ramified).

Proposition 6.16. *Fix L in $\Omega(\mathcal{K})$, a place v' in $S(L) \setminus \Sigma_S(L)$ and a subset Σ of $\Sigma_S(L)$ of cardinality a (so that $a \le |\Sigma_S(L)| < |S(L)|$). Set $G := \mathcal{G}_L$ and write e for the idempotent $e_{L/K,S(L),\Sigma}$ of $\zeta(\mathbb{Q}[G])$ defined in (6.2.5).*

Fix a prime p and a finite set of places T of K with $T \cap S(L) = \emptyset$ and such that $U_{L,p} := \mathcal{O}^\times_{L,S(L),T,p}$ is torsion-free. Let $\mathbb{Z}_p[G]^d \xrightarrow{\phi} \mathbb{Z}_p[G]^d$ be a representative of $C_{L,S(L),T,p}$ of the form constructed in Lemma 6.15 (with respect to the place v') and write

$$\iota_* : \bigcap\nolimits^a_{\mathbb{Z}_p[G]} U_{L,p} \to \bigcap\nolimits^a_{\mathbb{Z}_p[G]} \mathbb{Z}_p[G]^d$$

for the injective homomorphism of $\xi(\mathbb{Z}_p[G])$-modules that is induced by the isomorphism $U_{L,p} \cong H^0(C_{L,S(L),T,p}) \cong \ker(\phi)$. Then, with $\{b_i\}_{i\in[d]}$ denoting the standard basis of $\mathbb{Z}_p[G]^d$, there exists an (ordered) subset $I = I_\Sigma$ of $[d]$ of cardinality $d-a$ such that the element

(6.3.5)
$$x_L := (\wedge_{i\in I}(b_i^* \circ \phi))(\wedge_{j\in[d]} b_j)$$

of $\bigcap^a_{\mathbb{Z}_p[G]} \mathbb{Z}_p[G]^d$ has all of the following properties.

- (i) *There exists a unique element ε_L of $\bigcap^a_{\mathbb{Z}_p[G]} U_{L,p}$ that is independent of the choice of v' and such that $\iota_*(\varepsilon_L) = x_L$.*
- (ii) $\zeta(\mathbb{Q}_p[G]) \cdot \varepsilon_L = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \big(\bigcap^a_{\mathbb{Z}_p[G]} U_{L,p}\big)[1-e]$.
- (iii) *For each λ in $\zeta(\mathbb{Q}_p[G])$ the following conditions are equivalent.*
  - (a) $\lambda \cdot \varepsilon_L \in \bigcap^a_{\mathbb{Z}_p[G]} U_{L,p}$.
  - (b) $\lambda \cdot x_L \in \bigcap^a_{\mathbb{Z}_p[G]} \mathbb{Z}_p[G]^d$.

(c) $\lambda\cdot\mathrm{Nrd}_{\mathbb{Q}_p[G]}(M)\in\xi(\mathbb{Z}_p[G])$ *for all matrices* $M=(M_{ij})$ *in* $\mathrm{M}_d(\mathbb{Z}_p[G])$ *with the property that* $M_{ij}=(b_i^*\circ\phi)(b_j)$ *for* $i\in I$ *and* $j\in[d]$.

(iv) *Let* ε'_L *be an element of* $\bigcap^a_{\mathbb{Z}_p[G]}U_{L,p}$ *obtained by making the above constructions with respect to another choice of representative of* $C_{L,S(L),T,p}$ *of the form constructed in Lemma 6.15. Then there exists an element* μ *of the subgroup* $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\mathrm{K}_1(\mathbb{Z}_p[G]))$ *of* $\xi(\mathbb{Z}_p[G])^\times$ *such that* $\varepsilon'_L=\mu\cdot\varepsilon_L$.

Proof. Set $R:=\mathbb{Z}_p[G]$ and $A:=\mathbb{Q}_p[G]$.

Write $I'=I'_\Sigma$ for the subset of $[n]$ comprising integers i for which the i-th place v_i in $S(L)\setminus\{v'\}$ belongs to Σ and set $I:=[d]\setminus I'$. By relabelling if necessary, we can (and will) assume in the rest of the argument that $I'=[a]$, and hence $I=[d]\setminus[a]$.

Then, since every place in Σ splits completely in L, the construction of ϕ combines with the properties of the surjective map fixed in (6.3.3) to imply $b_i^*\circ\phi=0$ for each $i\in[a]$. In addition, the group $\mathrm{Ext}^1_R(\mathrm{im}(\phi),R)$ vanishes since $\mathrm{im}(\phi)$ is torsion-free.

Given these observations, and the fact that the idempotent e is defined via the condition (iii) in Lemma 6.10, claims (i) and (ii) are obtained directly upon applying the general result of [24, Prop. 4.18(i)], with the matrix M in the latter result taken to be the submatrix

$$\big((b_i^*\circ\phi)(b_j)\big)_{a<i\le d,1\le j\le d}$$

of the matrix of ϕ with respect to the basis $\{b_i\}_{1\le i\le d}$. We note, in particular, that the independence assertion in claim (i) is true since claim (ii) implies $\varepsilon_L=e(\varepsilon_L)$, whilst Lemma 6.10(iii) implies $e(v')=0$.

Next we note that the vanishing of $\mathrm{Ext}^1_R(\mathrm{im}(\phi),R)$ also combines with [24, Th. 4.19(iv)] to imply that the conditions (a) and (b) in claim (iii) are equivalent. Finally, we note that condition (b) is equivalent to condition (c) since for every subset $\{\theta_i\}_{1\le i\le a}$ of $\mathrm{Hom}_R(R^d,R)$ one has

$$\begin{aligned}(\wedge_{i=1}^{i=a}\theta_i)(\lambda\cdot x_L)&=\lambda\cdot(\wedge_{i=1}^{i=a}\theta_i)\big((\wedge_{i=a+1}^{i=d}(b_i^*\circ\phi))(\wedge_{j=1}^{j=d}b_j)\big)\\&=\lambda\cdot\mathrm{Nrd}_A(-1)^{a(d-a)}\cdot\big((\wedge_{i=1}^{i=a}\theta_i)\wedge(\wedge_{i=a+1}^{i=d}(b_i^*\circ\phi))\big)(\wedge_{j=1}^{j=d}b_j)\\&=\lambda\cdot\mathrm{Nrd}_A(-1)^{a(d-a)}\cdot\mathrm{Nrd}_A(M).\end{aligned}$$

Here M is the matrix in $\mathrm{M}_d(R)$ that satisfies

$$M_{ij}=\begin{cases}\theta_i(b_j), & \text{if } 1\le i\le a,\ 1\le j\le d\\ (b_i^*\circ\phi)(b_j), & \text{if } a<i\le d,\ 1\le j\le d,\end{cases}$$

and so the third equality is valid by [24, Lem. 4.10].

To prove claim (iv) we assume to be given a commutative diagram (6.3.4) and for any map of $\xi(R)$-modules θ we set $\theta_{\mathbb{Q}_p}:=\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\theta$.

Then, using the automorphisms η' and η from (6.3.4), we compute

$$\begin{aligned}(\wedge_A^a \eta'_{\mathbb{Q}_p})(x_L) &= (\wedge_A^a \eta'_{\mathbb{Q}_p})\big((\wedge_{j=a+1}^{j=d}(b_j^*\circ\phi))(\wedge_{i=1}^{i=d}b_i)\big)\\ &= (\wedge_{j=a+1}^{j=d}(b_j^*\circ\phi\circ(\eta')^{-1}))\big((\wedge_A^d \eta'_{\mathbb{Q}_p})(\wedge_{i=1}^{i=d}b_i)\big)\\ &= \mathrm{Nrd}_A(\eta'_{\mathbb{Q}_p})\cdot(\wedge_{j=a+1}^{j=d}(b_j^*\circ\eta^{-1}\circ\phi'))(\wedge_{i=1}^{i=d}b_i)\\ &= \mathrm{Nrd}_A(\eta'_{\mathbb{Q}_p})\cdot(\mathrm{Nrd}_A(\eta_{\mathbb{Q}_p})^{\#})^{-1}\cdot(\wedge_{j=a+1}^{j=d}(b_j^*\circ\phi'))(\wedge_{i=1}^{i=d}b_i),\end{aligned}$$

where we write $x\mapsto x^{\#}$ for the $\mathbb{Q}_p$-linear involution of $\zeta(A)$ that is induced by inverting elements of $\mathcal{G}_L$. Before justifying this computation, we note that, if true, it implies the validity of claim (iv) (with $\mu = \mathrm{Nrd}_A(\eta'_{\mathbb{Q}_p})\cdot(\mathrm{Nrd}_A(\eta_{\mathbb{Q}_p})^{\#})^{-1}$) since the commutativity of the first square in (6.3.4) implies that the composite map $(\wedge_A^a(\eta'_{\mathbb{Q}_p}))\circ\iota_*$ coincides with the embedding $\bigcap_{\mathbb{Z}_p[G]}^a U_{L,p}\to\bigcap_{\mathbb{Z}_p[G]}^a\mathbb{Z}_p[G]^d$ induced by $\hat{\iota}'$.

Now the first and second equalities in the above computation are clear and the third is true since

$$(\wedge_A^d\eta'_{\mathbb{Q}_p})(\wedge_{i\in[d]}b_i) = \mathrm{Nrd}_A(\eta'_{\mathbb{Q}_p})\cdot\wedge_{i\in[d]}b_i$$

and because the commutativity of the second square in (6.3.4) implies that

$$b_j^*\circ\phi\circ(\eta')^{-1} = b_j^*\circ\eta^{-1}\circ\phi'.$$

To verify the fourth equality it suffices (by virtue of the injectivity of the map $\mathrm{ev}^a_{\mathbb{Z}_p[G]^d}$ in (2.2.1)) to show that

$$(\wedge_{j=1}^{j=a}\theta_j)\wedge(\wedge_{j=a+1}^{j=d}(b_j^*\circ\eta^{-1}\circ\phi')) = (\mathrm{Nrd}_A(\eta_{\mathbb{Q}_p})^{\#})^{-1}\cdot\big((\wedge_{j=1}^{j=a}\theta_j)\wedge(\wedge_{j=a+1}^{j=d}(b_j^*\circ\phi'))\big)$$

for every subset $\{\theta_j\}_{1\le j\le a}$ of $\mathrm{Hom}_R(R^d,R)$.

To prove this we write $\tilde{\eta}$ for the automorphism of R^d that is represented with respect to the basis $\{b_i\}_{1\le i\le d}$ by the matrix

$$\left(\begin{array}{c|c} I_a & 0\\ \hline 0 & M'_\eta\end{array}\right),$$

where M'_η agrees with the corresponding $(d-a)\times(d-a)$ minor of the matrix (6.3.2) that represents η. Then it is clear that

$$\mathrm{Nrd}_A(\eta_{\mathbb{Q}_p}) = \mathrm{Nrd}_A(M_\eta) = \mathrm{Nrd}_A(M'_\eta) = \mathrm{Nrd}_A(\tilde{\eta}_{\mathbb{Q}_p})$$

and also that

$$\eta^{-1}\circ\phi' = \tilde{\eta}^{-1}\circ\phi'$$

since $\mathrm{im}(\phi')\subseteq R\cdot\{b_i\}_{a<i\le d}$. Thus, if for any given subset $\{\theta_i\}_{1\le i\le a}$ of $\mathrm{Hom}_R(R^d,R)$ we write φ for the (unique) map in $\mathrm{Hom}_R(R^d,R^d)$ with

$$b_j^*\circ\varphi = \begin{cases}\theta_j, & \text{if } 1\le j\le a\\ b_j^*\circ\phi', & \text{if } a<j\le d,\end{cases}$$

then for each j with $a<j\le d$ one has

$$b_j^*\circ\eta^{-1}\circ\phi' = (b_j^*\circ\tilde{\eta}^{-1})\circ\phi' = (b_j^*\circ\tilde{\eta}^{-1})\circ\varphi$$

and hence

$$\begin{aligned}(\wedge_{j=1}^{j=a}\theta_i) \wedge (\wedge_{j=a+1}^{j=d}(b_j^* \circ \eta^{-1} \circ \phi')) &= (\wedge_{j=1}^{j=a}(b_j^* \circ \tilde{\eta}^{-1} \circ \varphi)) \wedge (\wedge_{j=a+1}^{j=d}(b_j^* \circ \tilde{\eta}^{-1} \circ \varphi)) \\ &= \big((\wedge_{j=1}^{j=a}(b_j^* \circ \tilde{\eta}^{-1})) \wedge (\wedge_{j=a+1}^{j=d}(b_j^* \circ \tilde{\eta}^{-1}))\big) \circ \wedge_A^d(\varphi_{\mathbb{Q}_p}) \\ &= (\mathrm{Nrd}_A(\tilde{\eta}_{\mathbb{Q}_p})^{\#})^{-1} \cdot \big((\wedge_{j=1}^{j=a}b_j^*) \wedge (\wedge_{j=a+1}^{j=d}b_j^*)\big) \circ \wedge_A^d(\varphi_{\mathbb{Q}_p}) \\ &= (\mathrm{Nrd}_A(\eta_{\mathbb{Q}_p})^{\#})^{-1} \cdot \big((\wedge_{j=1}^{j=a}(b_j^* \circ \varphi)) \wedge (\wedge_{j=a+1}^{j=d}(b_j^* \circ \varphi))\big) \\ &= (\mathrm{Nrd}_A(\eta_{\mathbb{Q}_p})^{\#})^{-1} \cdot \big((\wedge_{j=1}^{j=a}\theta_j) \wedge (\wedge_{j=a+1}^{j=d}(b_j^* \circ \phi'))\big),\end{aligned}$$

as required. Here the third equality follows from [24, Lem. 4.13] and the fact that the reduced norm of the automorphism of $\mathrm{Hom}_R(R^d, R)$ that sends each b_i^* to $b_i^* \circ \tilde{\eta}^{-1}$ is the inverse of $\mathrm{Nrd}_A(\tilde{\eta}_{\mathbb{Q}_p})^{\#}$, and all other equalities are clear. □

Remark 6.17. Claim (ii) of Proposition 6.16 implies that there is an equality $\varepsilon_L = e(\varepsilon_L)$ in $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \big(\bigcap^a_{\mathbb{Z}_p[G]} U_{L,p}\big)$. Taken in conjunction with Remark 6.11, this shows that the elements ε_L have the same invariance properties as do the relevant Rubin-Stark elements.

6.3.3. For each F in $\Omega(\mathcal{K})$ we now set

$$\Xi(F) := \mathrm{d}_{\mathbb{Z}[\mathcal{G}_F]}(C_{F,S(F),T})$$

and for F' in $\Omega(\mathcal{K})$ with $F \subseteq F'$ we consider the composite surjective homomorphism of (graded) $\xi(\mathbb{Z}[\mathcal{G}_{F'}])$-modules

$$\begin{aligned}\nu_{F'/F} : \Xi(F') &\to \mathrm{d}_{\mathbb{Z}[\mathcal{G}_F]}(\mathbb{Z}[\mathcal{G}_F] \otimes^{\mathsf{L}}_{\mathbb{Z}[\mathcal{G}_{F'}]} C_{F',S(F'),T}) \\ &\to \mathrm{d}_{\mathbb{Z}[\mathcal{G}_F]}(C_{F,S(F),T}) \otimes \bigotimes\nolimits_{v \in S(F')\setminus S(F)} \mathrm{d}_{\mathbb{Z}[\mathcal{G}_F]}(C_v) \\ &\to \Xi(F),\end{aligned} \tag{6.3.6}$$

where we set

$$C_v := \bigoplus\nolimits_w \mathsf{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}\Gamma((\kappa_w)_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})[-1]$$

with w running over all places of F above v. Here the first map in (6.3.6) is induced by the standard base-change isomorphism (from [24, Th. 5.2(ii)]), the second is the isomorphism obtained by combining Lemma 5.1(iv) with an application of [24, Th. 5.2(i)] to the exact triangle in Lemma 5.1(ii) and the final map uses the canonical isomorphisms

$$\mathrm{d}_{\mathbb{Z}[\mathcal{G}_F]}(C_v) \cong (\xi(\mathbb{Z}[\mathcal{G}_F]), 0) \tag{6.3.7}$$

that are induced by the descriptions in Remark 5.2 (as per [24, (8.1.1)]).

We can therefore define a graded $\xi_p(\mathcal{K}/K)$-module of 'vertical systems' for the data $\mathcal{K}/K, S, T$ and p by means of the inverse limit

$$\mathrm{VS}(\mathcal{K}/K, S, T, p) := \varprojlim_{F \in \Omega(\mathcal{K})} \Xi(F)_p,$$

where the transition morphism for $F \subseteq F'$ is $\nu_{F'/F,p}$. The Hermite-Minkowski theorem implies that the ungraded part of $\mathrm{VS}(\mathcal{K}/K, S, T, p)$ is a free $\xi_p(\mathcal{K}/K)$-module of rank one but we make no use of this fact (and so do not give a proof).

Theorem 6.18. *There exists a canonical homomorphism of $\xi_p(\mathrm{K}/K)$-modules*

$$\Theta_{\mathcal{K}/K,S,T,p} : \mathrm{VS}(\mathcal{K}/K,S,T,p) \to \mathrm{ES}_{r_S}(\mathcal{K}/K,S,T,p).$$

This homomorphism is non-zero if and only if $\theta^{r_S}_{F/K,S(F)}(0) \neq 0$ for some F in $\Omega(\mathcal{K})$.

Proof. Set $\Sigma := \Sigma_S(\mathcal{K})$, $r := r_S(= |\Sigma|)$ and $U_F := \mathcal{O}^\times_{F,S(F),T}$ and $C_F := C_{F,S(F),T}$ for every F in $\Omega(\mathcal{K})$. Then, since each transition morphism $\nu_{F'/F,p}$ is surjective, to construct a map of the claimed sort it is enough to construct for each F a canonical homomorphism of $\zeta(\mathbb{Q}_p[\mathcal{G}_F])$-modules

$$\Theta^\Sigma_{F,S,T,p} : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Xi(F)_p \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}_F]} U_{F,p}$$

with all of the following properties:-

(a) $\Theta^\Sigma_{F,S,T,p}(\Xi(F)_p) \subseteq \bigcap^r_{\mathbb{Z}_p[\mathcal{G}_F]} U_{F,p}$.
(b) $\Theta^\Sigma_{F,S,T,p}$ is the zero map if and only if $\theta^r_{F/K,S(F)}(0) = 0$.
(c) For all F' in $\Omega(\mathcal{K})$ with $F \subset F'$ and all x in $\Xi(F')_p$ the equality (6.1.4) is valid with

$$c_{F'} = \Theta^\Sigma_{F',S,T,p}(x) \quad \text{and} \quad c_F = \Theta^\Sigma_{F,S,T,p}(\nu_{F'/F,p}(x)).$$

We write e_F for the idempotent $e_{F/K,S(F),\Sigma}$ of $\zeta(\mathbb{Q}[\mathcal{G}_F])$ defined in (6.2.5) and note that the space $e_F(\mathbb{Q} \otimes_\mathbb{Z} \ker(\alpha))$ vanishes, where α is the composite surjective homomorphism

$$H^1(C_F) \to X_{F,S(F)} \to Y_{F,\Sigma} \cong \mathbb{Z}[\mathcal{G}_F]^r.$$

Here the first map is induced by Lemma 5.1(i) and the exact sequence (5.1.2), the second is the natural projection and the isomorphism is induced by sending the chosen set of places $\{w_{v,F}\}_{v\in\Sigma}$ to the standard basis of $\mathbb{Z}[\mathcal{G}_F]^r$.

We then define $\Theta^\Sigma_{F,S,T,p}$ to be the scalar extension of the following composite homomorphism of $\zeta(\mathbb{Q}[\mathcal{G}_F])$-modules

$$\begin{aligned}
(6.3.8) \qquad & \mathrm{d}_{\mathbb{Q}[\mathcal{G}_F]}(\mathbb{Q} \cdot C_F) \\
&\cong \mathrm{d}^\diamond_{\mathbb{Q}[\mathcal{G}_F]}(\mathbb{Q} \cdot U_F) \otimes \mathrm{d}^\diamond_{\mathbb{Q}[\mathcal{G}_F]}(\mathbb{Q} \cdot H^1(C_F))^{-1} \\
&\to e_F\big(\mathbb{Q} \cdot \bigcap\nolimits^r_{\mathbb{Z}[\mathcal{G}_F]} U_F\big) \otimes_{\zeta(\mathbb{Q}[\mathcal{G}_F])} e_F\big(\mathrm{Hom}_{\zeta(\mathbb{Q}[\mathcal{G}_F])}\big(\mathbb{Q} \cdot \bigcap\nolimits^r_{\mathbb{Z}[\mathcal{G}_F]} \mathbb{Z}[\mathcal{G}_F]^r, \zeta(\mathbb{Q}[\mathcal{G}_F])\big)\big) \\
&\cong e_F\big(\mathbb{Q} \cdot \bigcap\nolimits^r_{\mathbb{Z}[\mathcal{G}_F]} U_F\big).
\end{aligned}$$

Here the first map is induced by the standard 'passage to cohomology' isomorphism (cf. [24, Prop. 5.14(i)]) and the descriptions in Lemma 5.1(i), the second is induced by multiplication by e_F, the isomorphism $e_F(\mathbb{Q} \otimes_\mathbb{Z} \alpha)$ and the argument of [24, Lem. 8.7(ii)] and the last map uses the isomorphism of $\xi(\mathbb{Z}[\mathcal{G}_F])$-modules $\bigcap^r_{\mathbb{Z}[\mathcal{G}_F]} \mathbb{Z}[\mathcal{G}_F]^r \cong \xi(\mathbb{Z}[\mathcal{G}_F])$ induced by the standard basis of $\mathbb{Z}[\mathcal{G}_F]^r$ (and [24, Prop. 5.6(i)]).

To verify that this definition of $\Theta^\Sigma_{F,S,T,p}$ has the required properties, we fix a place v' in $S(F) \setminus \Sigma_S(F)$ (so $v' \notin \Sigma$) and note that condition (6.1.2) ensures that Σ corresponds to the first r elements of the (ordered) set $S(F) \setminus \{v'\}$.

We use Hypothesis 6.1 to fix a representative of $C_{F,p}$ of the form

$$\mathbb{Z}_p[\mathcal{G}_F]^d \xrightarrow{\phi_F} \mathbb{Z}_p[\mathcal{G}_F]^d$$

used in Proposition 6.16. We write ε_F for the element of $\bigcap^r_{\mathbb{Z}_p[\mathcal{G}_F]} U_{F,p}$ that is obtained in this case via the formula (6.3.5) (noting that the present hypotheses imply the set I in the latter formula is equal to $[d] \setminus [r]$). We write $\{b_{F,i}\}_{i\in[d]}$ for the standard basis of $\mathbb{Z}_p[\mathcal{G}_F]^d$. Then, setting

$$\beta_F := ((\wedge_{i\in[d]} b_{F,i}) \otimes (\wedge_{i\in[d]} b^*_{F,i}), 0),$$

the argument establishing [24, (8.2.6)] shows that the restriction of $\Theta^{\Sigma}_{F,S,T,p}$ to $\Xi(F)_p$ coincides with the composite

$$\Xi(F)_p = \mathrm{d}_{\mathbb{Z}_p[\mathcal{G}_F]}(C_{F,p}) \xrightarrow{\sim} \xi(\mathbb{Z}_p[\mathcal{G}_F])\cdot \beta_F \to \bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}_F]} U_{F,p} \tag{6.3.9}$$

where the first map is the isomorphism induced by the given representative of $C_{F,p}$ and the second map sends β_F to ε_F.

Given this explicit description of $\Theta^{\Sigma}_{F,S,T,p}$, property (a) follows from Proposition 6.16(i) and property (b) upon combining the results of Proposition 6.16(ii) and Lemma 6.10.

Finally, the fact $\Theta^{\Sigma}_{F,S,T,p}$ has property (c) follows directly from the argument proving [24, (8.2.4)], after making the following changes: the map $\mathrm{Cor}^r_{F'/F}$ and element P_v that are used in loc. cit. are replaced by $\mathrm{N}^r_{F'/F}$ and $\mathrm{Nrd}_{\mathbb{Q}[\mathcal{G}_F]}(1 - \mathrm{Fr}^{-1}_{F,v})$; the exact triangle in [24, Lem. 8.1(iv)] and the isomorphisms occuring in [24, (8.1.2)] are replaced by the exact triangle in Lemma 5.1(ii) and the isomorphisms (6.3.7) that occur in the composite homomorphism (6.3.6). □

6.3.4. We now use Theorem 6.18 to strengthen the result of [24, Th. 1.1], thereby fulfilling a promise made in the Introduction to loc. cit.

To state the result we fix an embedding $\sigma : \mathbb{Q}^c \to \mathbb{C}$ and for each subfield F of $\mathbb{Q}^c$ write $w_{F,\sigma}$ for the archimedean place of F corresponding to σ. For each natural number n we write ζ_n for the unique primitive n-th root of unity in $\mathbb{Q}^c$ that satisfies $\sigma(\zeta_n) = e^{2\pi i/n}$. We write $\mathbb{Q}^{c,+}$ for the maximal totally real extension of $\mathbb{Q}$ in $\mathbb{Q}^c$.

For a finite group Γ we use the ideal $\delta(\mathbb{Z}[\Gamma])$ of $\zeta(\mathbb{Z}[\Gamma])$ introduced in [24, Def. 3.4]. Following [24, Def. 3.15], we then define the 'central pre-annihilator' of a Γ-module M by setting

$$\mathrm{pAnn}_{\mathbb{Z}[\Gamma]}(M) := \{x \in \zeta(\mathbb{Z}[\Gamma]) : x \cdot \delta(\mathbb{Z}[\Gamma]) \subseteq \mathrm{Ann}_{\mathbb{Z}[\Gamma]}(M)\}.$$

We note, in particular, that this lattice is a $\xi(\mathbb{Z}[\Gamma])$-submodule of $\zeta(\mathbb{Q}[\Gamma])$ and is equal to the annihilator of M in $\mathbb{Z}[\Gamma]$ if Γ is abelian.

Theorem 6.19. *For each odd prime p, there exists an Euler system*

$$\varepsilon^{\mathrm{cyc}} = (\varepsilon^{\mathrm{cyc}}_F)_F$$

in $\mathrm{ES}_1(\mathbb{Q}^{c,+}/\mathbb{Q}, \{\infty\}, \emptyset, p)$ that has the following properties at every F in $\Omega(\mathbb{Q}^{c,+}/\mathbb{Q})$.

(i) *If $F/\mathbb{Q}$ is abelian, and of conductor $f(F)$, then*

$$\varepsilon^{\mathrm{cyc}}_F = \mathrm{Norm}_{\mathbb{Q}(\zeta_{f(F)})/F}(1 - \zeta_{f(F)}).$$

(ii) *For φ in $\mathrm{Hom}_{\mathcal{G}_F}(\mathcal{O}^{\times}_{F,S(F)}, \mathbb{Z}[\mathcal{G}_F])$, and every prime ℓ that ramifies in F, one has*

$$\Big(\bigwedge\nolimits^1_{\mathbb{Q}[\mathcal{G}_F]} \varphi\Big)(\varepsilon^{\mathrm{cyc}}_F) \in \mathrm{pAnn}_{\mathbb{Z}[\mathcal{G}_F]}(\mathrm{Cl}(\mathcal{O}_F[1/\ell]))_p.$$

(iii) *For every* χ *in* $\mathrm{Ir}(\mathcal{G}_F)$ *there exists a non-zero element* $u_{F,\chi}$ *of* $\mathbb{C}_p$ *that satisfies both*

$$(\bigwedge\nolimits_{\mathbb{C}_p}^{\chi(1)} V_\chi^* \otimes_{\mathbb{R}[\mathcal{G}_F]} \mathrm{Reg}_{F,S(F)})(e_\chi \cdot \varepsilon_F^{\mathrm{cyc}}) = u_{F,\chi} \cdot L_{S(F)}^{\chi(1)}(\check{\chi},0) \cdot e_\chi(\bigwedge\nolimits_{\mathbb{C}_p[\mathcal{G}_F]}^{1}(w_{F,\sigma} - w_{F,p})),$$

and

$$\prod\nolimits_{\omega \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} u_{F,\chi^\omega} \in \mathbb{Z}_p^\times.$$

Proof. We set $\mathcal{K} := \mathbb{Q}^{c,+}$ and write K^{ab} for the maximal absolutely abelian subfield of $\mathcal{K}$. Then, since p is odd, Hypothesis 6.1 is satisfied for $\mathcal{K}$ with $S = \{\infty\}$ and $T = \emptyset$ (cf. Remark 6.2) and so Theorem 6.18 constructs a canonical map $\Theta := \Theta_{\mathcal{K}/\mathbb{Q},\{\infty\},\emptyset,p}$.

In addition, the known validity of the equivariant Tamagawa Number Conjecture in the relevant case implies (via the argument of [24, Lem. 9.4]) that the (free, rank one) $\xi_p(\mathcal{K}^{\mathrm{ab}}/\mathbb{Q})$-module $\mathrm{VS}(\mathcal{K}^{\mathrm{ab}}/\mathbb{Q},\{\infty\},\emptyset,p)$ has a basis element η^{ab} with the following property: for each F in $\Omega(\mathcal{K}^{\mathrm{ab}})$ the image η_F^{ab} of η^{ab} in $\Xi(F)_p$ is sent by the homomorphism $\Theta_F := \Theta_{F,\{\infty\},\emptyset,p}^{\{\infty\}}$ constructed in the proof of Theorem 6.18 to $\mathrm{Norm}_{\mathbb{Q}(\zeta_{f(F)})/F}(1-\zeta_{f(F)})$.

Since the projection map $\xi_p(\mathcal{K}/\mathbb{Q})^\times \to \xi_p(\mathcal{K}^{\mathrm{ab}}/\mathbb{Q})^\times$ is surjective (by [24, Lem. 9.5]), we can then fix a basis element η of the $\xi_p(\mathcal{K}/\mathbb{Q})$-module $\mathrm{VS}(\mathcal{K}^{\mathrm{ab}}/\mathbb{Q},\{\infty\},\emptyset,p)$ that the projection map

$$\mathrm{VS}(\mathcal{K}/\mathbb{Q},\{\infty\},\emptyset,p) \to \mathrm{VS}(\mathcal{K}^{\mathrm{ab}}/\mathbb{Q},\{\infty\},\emptyset,p)$$

sends to η^{ab}. We then obtain a system in $\mathrm{ES}_1(\mathcal{K}/\mathbb{Q},\{\infty\},\emptyset,p)$ by setting

$$\varepsilon^{\mathrm{cyc}} := \Theta(\eta).$$

This construction ensures directly that for every F in $\Omega(\mathcal{K}^{\mathrm{ab}})$ one has

$$\varepsilon_F^{\mathrm{cyc}} = \Theta_F(\eta_F^{\mathrm{ab}}) = \mathrm{Norm}_{\mathbb{Q}(\zeta_{f(F)})/F}(1-\zeta_{f(F)}),$$

as stated in claim (i).

On the other hand, the properties in claims (ii) and (iii) are verified by mimicking the arguments in [24, §9.2]. □

Remark 6.20. This result strengthens that of [24, Th. 1.1] since $\varepsilon^{\mathrm{cyc}}$ is a refinement (in the sense of Remark 6.4(ii)) of the system in $\mathrm{ES}_1(\mathcal{K}/\mathbb{Q},\{\infty\}\cup\{p\},\emptyset,p)$ that is constructed in loc. cit.

Remark 6.21. The displayed containment in Theorem 6.19(ii) can be interpreted as a special case of the annihilation results relating to Selmer modules of p-adic representations that are obtained by Macias Castillo and Tsoi in [61].

7. Higher rank non-commutative Iwasawa theory

In this section we use the constructions made in §6 to formulate an explicit main conjecture of non-commutative Iwasawa theory for $\mathbb{G}_m$ over arbitrary number fields.

We then show that this conjecture simultaneously extends both the higher rank main conjecture of (commutative) Iwasawa theory formulated by Burns, Kurihara and Sano in [19] and the general formalism of main conjectures in non-commutative Iwasawa theory following the approaches of Ritter and Weiss in [72] and of Coates et al in [27], and thereby deduce its validity in important special cases.

In the sequel we shall regard the prime p as fixed, write $L^{\rm cyc}$ for the cyclotomic $\mathbb{Z}_p$-extension of each number field L and set $\Gamma_L := {\rm Gal}(L^{\rm cyc}/L)$.

We also fix a rank one p-adic Lie extension $\mathcal{K}_\infty$ of K, set $\mathcal{G}_\infty := {\rm Gal}(\mathcal{K}_\infty/K)$ and for any infinite subquotient $\mathcal{G}$ of $\mathcal{G}_\infty$ we write $\Lambda(\mathcal{G})$ for the Iwasawa algebra $\mathbb{Z}_p[[\mathcal{G}]]$.

We recall that the total quotient ring $Q(\mathcal{G}_\infty)$ of $\Lambda(\mathcal{G}_\infty)$ is a semisimple algebra and hence that there exists a reduced norm homomorphism

$$\mathrm{Nrd}_{Q(\mathcal{G}_\infty)} : \mathrm{K}_1(Q(\mathcal{G}_\infty)) \to \zeta(Q(\mathcal{G}_\infty))^\times.$$

Finally we set

$$\mathbb{Q}_p[[\mathcal{G}_\infty]] := \varprojlim_{L\in\Omega(\mathcal{K}_\infty)} \mathbb{Q}_p[\mathcal{G}_L],$$

where the transition morphisms are the natural projection maps.

7.1. **Whitehead orders in Iwasawa theory.** To help set the context for our conjecture, we first clarify the link between the subrings $\xi_p(\mathcal{K}_\infty/K)$ and $\zeta(\Lambda(\mathcal{G}_\infty))$ of $\zeta(\mathbb{Q}_p[[\mathcal{G}_\infty]])$.

Lemma 7.1. *Set $\mathcal{R} := \zeta(\Lambda(\mathcal{G}_\infty))$ and $\mathcal{R}' := \xi_p(\mathcal{K}_\infty/K)$. Fix a central open subgroup $\mathcal{Z}$ of $\mathcal{G}_\infty$ that is topologically isomorphic to $\mathbb{Z}_p$ and for each natural number n write $\mathcal{R}_n$ for the subring $\Lambda(\mathcal{Z}^{p^n})$ of $\mathcal{R}$. Then the following claims are valid for each natural number n.*

(i) *The $\mathbb{Z}_p$-module $\mathcal{R}'/\big(\mathcal{R}'\cap\mathcal{R}\big)$ has finite exponent.*

(ii) *There exists a natural number t such that the element $p^t\cdot \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(p)$ belongs to $\mathcal{R}\cap\mathcal{R}'$ and any sufficiently large power of it annihilates $\mathcal{R}/\big(\mathcal{R}\cap(\mathcal{R}'\cdot\mathcal{R}_n)\big)$.*

(iii) *There are inclusions*

$$\mathcal{R}' \subseteq \mathcal{R}\left[p^{-1}\right] \subseteq (\mathcal{R}'\cdot\mathcal{R}_n)\left[p^{-1}, \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(p)^{-1}\right].$$

Proof. It is clearly enough to prove claims (i) and (ii) and to do this we use the fact (proved in [24, Lem. 8.13]) that for any matrix B in $\mathrm{M}_d(\Lambda(\mathcal{G}_\infty))$ the reduced norm $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(B)$ belongs to $\mathcal{R}'$ and is equal to

$$\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(B) = \big(\mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_L]}(B_L)\big)_{L\in\Omega(\mathcal{K}_\infty)} \tag{7.1.1}$$

where B_L is the image of B in $\mathrm{M}_d(\mathbb{Z}_p[\mathcal{G}_L])$.

To prove claim (i) we fix an element $x = (x_L)_L$ of $\mathcal{R}'$. Then, for each L in $\Omega(\mathcal{K}_\infty)$, there exists a finite index set I_L and, for each $i \in I_L$, an element a_i of $\mathbb{Z}_p$, a natural number d_i and a matrix $M_{L,i}$ in $\mathrm{M}_{d_i}(\mathbb{Z}_p[\mathcal{G}_L])$ such that in $\xi(\mathbb{Z}_p[\mathcal{G}_L])$ one has

$$x_L = \sum\nolimits_{i\in I_L} a_i\cdot \mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_L]}(M_{L,i}).$$

For each index i in I_L we fix a pre-image $M^\infty_{L,i}$ of $M_{L,i}$ under the natural (surjective) projection map $\mathrm{M}_{d_i}(\Lambda(\mathcal{G}_\infty)) \to \mathrm{M}_{d_i}(\mathbb{Z}_p[\mathcal{G}_L])$. Then $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(M^\infty_{L,i})$ belongs to the integral closure of $\Lambda(\mathcal{Z})$ in $\zeta(Q(\mathcal{G}_\infty))$ (this follows, for example, from the observation of Ritter and Weiss in [72, §5, Rem. (H)]) and therefore also to any choice of a maximal $\Lambda(\mathcal{Z})$-order in $Q(\mathcal{G}_\infty)$. Hence, by the central conductor formula of Nickel [68, Th. 3.5], there exists a natural number N (that is independent of both L and $M_{L,i}$) such that $p^N\cdot\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(M^\infty_{L,i})$ belongs to $\Lambda(\mathcal{G}_\infty)$.

The latter containment combines with (7.1.1) to imply $p^N \cdot \mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_L]}(M_{L,i}) \in \mathbb{Z}_p[\mathcal{G}_L]$ for each i and hence that $p^N \cdot x_L \in \mathbb{Z}_p[\mathcal{G}_L]$. It follows that $p^N \cdot x = (p^N \cdot x_L)_L$ belongs to

$$\zeta(\mathbb{Q}_p[[\mathcal{G}_\infty]]) \cap \prod\nolimits_L \mathbb{Z}_p[\mathcal{G}_L] = \mathcal{R},$$

and hence that $\mathcal{R}'/(\mathcal{R}' \cap \mathcal{R})$ is annihilated by a fixed power p^N of p, as required.

To prove claim (ii) we fix a maximal $\mathcal{R}_n$-order $\mathfrak{M}$ in $Q(\mathcal{G}_\infty)$ that contains $\Lambda(\mathcal{G}_\infty)$. We write $\mathcal{M}_n$ for the maximal $\mathcal{R}_n$-order in $\zeta(Q(\mathcal{G}_\infty))$ and claim that the $\mathcal{R}_n$-order generated by the elements $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(M)$ as M runs over $\bigcup_{n\ge 1} \mathrm{M}_n(\mathfrak{M})$ has finite index in $\mathcal{M}_n$. To show this we note $\mathfrak{M}$ is a finitely generated $\mathcal{R}_n$-module and hence that it is enough (by the structure theory of $\mathcal{R}_n$-modules) to show that the localization of $\mathcal{M}_n$ at every height one prime ideal $\wp$ of $\mathcal{R}_n$ is generated over $\mathcal{R}_{n,\wp}$ by $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(M)$ as M runs over $\bigcup_{n\ge 1} \mathrm{M}_n(\mathfrak{M}_\wp)$. In addition, since each such ring $\mathcal{R}_{n,\wp}$ is a discrete valuation ring and $\mathfrak{M}_\wp$ is a maximal $\mathcal{R}_{n,\wp}$-order in $\zeta(Q(\mathcal{G}_\infty))$, this follows in a straightforward fashion from the arithmetic of local division algebras (as in the proof of [28, Prop. (45.8)]).

We can therefore fix a natural number t_1 such that for each $x = (x_L)_L$ in $\mathcal{R}$, there exists a finite index set I_x, and for each i in I_x an element y_i of $\mathcal{R}_n$, a natural number n_i and a matrix $M_{x,i}$ in $\mathrm{M}_{n_i}(\mathfrak{M})$ such that

$$p^{t_1} \cdot x = \sum\nolimits_{i\in I_x} \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(M_{x,i}) \cdot y_i.$$

In addition, by another application of the central conductor formula [68, Th. 3.5], there exists a natural number t that is greater than or equal to the integer N fixed above and is such that for every i in I_x the matrix $p^t \cdot M_{x,i}$ belongs to $\mathrm{M}_{n_i}(\Lambda(\mathcal{G}_\infty))$. These facts combine to imply that for every x in $\mathcal{R}$ one has

$$p^{t_1} \cdot \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(p)^t \cdot x = \sum\nolimits_{i\in I_x} \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(p^t \cdot M_{x,i}) \cdot y_i \in \mathcal{R}' \cdot \mathcal{R}_n.$$

In particular, since $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(p) \in \mathcal{R}'$, this containment implies $(p^t \cdot \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(p))^{t'} \cdot x$ belongs to $\mathcal{R}' \cdot \mathcal{R}_n$ for any integer t' that is greater than both t_1/t and t. To complete the proof of claim (ii) it is therefore enough to note that, since $t \ge N$, the proof of claim (i) above implies that $p^t \cdot \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(p)$ belongs to $\mathcal{R}' \cap \mathcal{R}$. □

7.2. **A main conjecture of higher rank non-commutative Iwasawa theory for $\mathbb{G}_m$.**

7.2.1. For each object C of $\mathsf{D}^{\mathrm{perf}}(\Lambda(\mathcal{G}_\infty))$ we define a $\xi_p(\mathcal{K}_\infty/K)$-module by setting

$$\mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(C) := \varprojlim_{L\in\Omega(\mathcal{K}_\infty)} \mathrm{d}_{\mathbb{Z}_p[\mathcal{G}_L]}(\mathbb{Z}_p[\mathcal{G}_L] \otimes^{\mathsf{L}}_{\Lambda(\mathcal{G}_\infty)} C)$$

where the transition morphism are induced by [24, Th. 5.4(ii)]. This $\xi_p(\mathcal{K}_\infty/K)$-module is free of rank one and, following the approach of §3, we now introduce a canonical set of basis elements.

To do this we fix an isomorphism in $\mathsf{D}^{\mathrm{perf}}(\Lambda(\mathcal{G}_\infty))$ of the form

$$P^\bullet \cong C, \tag{7.2.1}$$

in which $P^\bullet$ is a bounded complex of finitely generated free $\Lambda(\mathcal{G}_\infty)$-modules.

In each degree a we write r_a for the rank of P^a and fix an ordered $\Lambda(\mathcal{G}_\infty)$-basis $\underline{b}_a = \{b_{a,j}\}_{1\le j\le r_a}$ of P^a. Then in every degree a and for every field L in $\Omega(\mathcal{K}_\infty)$ the image

$\underline{b}_{L,a} = \{b_{L,a,j}\}_{1\le j\le r_a}$ of $\underline{b}_a$ under the projection map $P^a \to P^a_L := H_0(\mathrm{Gal}(\mathcal{K}_\infty/L), P^a)$ is an ordered $\mathbb{Z}_p[\mathcal{G}_L]$-basis of P^a_L. The element

$$x(\underline{b}_\bullet)_L := (\bigotimes\nolimits_{a\in\mathbb{Z}}(\wedge_{j\in[r_a]} b_{L,a,j})^{(-1)^a}, 0)$$

is then a basis of the (graded) $\xi(\mathbb{Z}_p[\mathcal{G}_L])$-module $\mathrm{d}_{\mathbb{Z}_p[\mathcal{G}_L]}(\mathbb{Z}_p[\mathcal{G}_L]\otimes_{\Lambda(\mathcal{G}_\infty)} P^\bullet)$ that is compatible with the natural transition morphisms as L varies and so the tuple

$$x(\underline{b}_\bullet) = \big(x(\underline{b}_\bullet)_L\big)_{L\in\Omega(\mathcal{K}_\infty)}$$

is a $\xi_p(\mathcal{K}_\infty/K)$-basis of $\mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(P^\bullet)$.

We use this construction to define an Iwasawa-theoretic analogue of the notion of primitive-basis.

Definition 7.2. Fix C in $\mathsf{D}^{\mathrm{perf}}(\Lambda(\mathcal{G}_\infty))$. Then an element x of $\mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(C)$ is said to be a 'primitive basis element' if, for every resolution of C of the form (7.2.1), there exists a collection $\underline{b}_\bullet$ of ordered bases of the modules P^a such that the induced isomorphism $\mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(P^\bullet) \cong \mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(C)$ sends $x(\underline{b}_\bullet)$ to x. We write $\mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(C)^{\mathrm{pb}}$ for the subset of $\mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(C)$ comprising all primitive basis elements.

Remark 7.3. The fact that the elements $x(\underline{b}_\bullet)$ are defined as inverse limits of the corresponding elements $x(\underline{b}_\bullet)_L$ over finite extensions L of K in $\mathcal{K}_\infty$ is important. Specifically, this property combines with the argument of Proposition 3.2 to imply that, in order to show x belongs to $\mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(C)^{\mathrm{pb}}$ it is sufficient to check, for any *fixed* resolution (7.2.1) of C, that there exists a collection $\underline{b}_\bullet$ of ordered bases such that the induced isomorphism $\mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(P^\bullet) \cong \mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(C)$ sends $x(\underline{b}_\bullet)$ to x.

7.2.2. The set $S_{\mathrm{ram}}(\mathcal{K}_\infty/K)$ of places of K that ramify in $\mathcal{K}_\infty$ is finite and we define

$$S_{\mathcal{K}_\infty/K} := S_K^\infty \cup S_K^p \cup S_{\mathrm{ram}}(\mathcal{K}_\infty/K), \tag{7.2.2}$$

where S_K^p denotes the set of all p-adic places of K. We then fix a finite set S of places of K with the property that

$$S_{\mathcal{K}_\infty/K} \subseteq S.$$

We note that, since $S = S(L)$ for every L in $\Omega(\mathcal{K}_\infty)$, the construction in Lemma 5.1 gives rise to an object

$$C_{\mathcal{K}_\infty,S,T} := \varprojlim_{L\in\Omega(\mathcal{K}_\infty)} C_{L,S,T,p}$$

of $\mathsf{D}^{\mathrm{perf}}(\Lambda(\mathcal{G}_\infty))$, where the transition morphisms for $L\subset L'$ are induced by the morphisms in Lemma 5.1(iv). We further note that, since S contains S_K^p, Lemma 5.1(v) implies that this object can be naturally interpreted in terms of the compactly-supported p-adic cohomology of $\mathbb{Z}_p$.

We also set

$$\mathcal{O}^\times_{\mathcal{K}_\infty,S,T} := \varprojlim_{L\in\Omega(\mathcal{K}_\infty)} \mathcal{O}^\times_{L,S,T,p}$$

where the transition morphisms for $L \subseteq L'$ are induced by the field-theoretic norm maps $(L')^\times \to L^\times$. For each non-negative integer a we then define a $\xi_p(\mathcal{K}/K)$-module by setting

$$\bigwedge\nolimits^a_{\mathbb{C}_p\cdot\Lambda(\mathcal{G}_\infty)}(\mathbb{C}_p\cdot\mathcal{O}^\times_{\mathcal{K}_\infty,S,T}) := \varprojlim_{L\in\Omega(\mathcal{K}_\infty)} \bigwedge\nolimits^a_{\mathbb{C}_p[\mathcal{G}_L]}(\mathbb{C}_p\cdot\mathcal{O}^\times_{L,S,T,p})$$

and a submodule

$$\bigcap\nolimits^a_{\Lambda(\mathcal{G}_\infty)}\mathcal{O}^\times_{\mathcal{K}_\infty,S,T} := \varprojlim_{L\in\Omega(\mathcal{K}_\infty)} \bigcap\nolimits^a_{\mathbb{Z}_p[\mathcal{G}_L]}\mathcal{O}^\times_{L,S,T,p} \tag{7.2.3}$$

where, in both cases, the limits are taken with respect to (the scalar extensions of) the transition morphisms (6.1.3).

We set $\Sigma := \Sigma_S(\mathcal{K}_\infty)$ and $r := r_{S,\mathcal{K}_\infty}(= |\Sigma|)$. Then, since $S = S(L)$ for every L in $\Omega(\mathcal{K}_\infty)$, there are identifications

$$\mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(C_{\mathcal{K}_\infty,S,T}) = \mathrm{VS}(\mathcal{K}_\infty/K,S,T,p)$$

and

$$\mathrm{ES}_r(\mathcal{K}_\infty/K,S,T,p) = \bigcap\nolimits^r_{\Lambda(\mathcal{G}_\infty)}\mathcal{O}^\times_{\mathcal{K}_\infty,S,T},$$

and so the constructions in §6.3 gives a canonical homomorphism of $\xi_p(\mathcal{K}_\infty/K)$-modules

$$\Theta^\Sigma_{\mathcal{K}_\infty,S,T} : \mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(C_{\mathcal{K}_\infty,S,T}) \to \bigcap\nolimits^r_{\Lambda(\mathcal{G}_\infty)}\mathcal{O}^\times_{\mathcal{K}_\infty,S,T}.$$

For the same reason, the distribution relation (6.1.4) gives rise to an element

$$\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T} := (\varepsilon^\Sigma_{F/K,S,T})_{F\in\Omega(\mathcal{K}_\infty)} \tag{7.2.4}$$

of $\bigwedge^r_{\mathbb{C}_p\cdot\Lambda(\mathcal{G}_\infty)}(\mathbb{C}_p\cdot\mathcal{O}^\times_{\mathcal{K}_\infty,S,T})$.

We can now formulate an explicit main conjecture of non-commutative p-adic Iwasawa theory for $\mathbb{G}_m$ relative to $\mathcal{K}_\infty/K$.

Conjecture 7.4. (Higher Rank Non-commutative Main Conjecture for $\mathbb{G}_m$) *Fix $\mathcal{K}_\infty/K$ and S as above and set $\Sigma := \Sigma_S(\mathcal{K}_\infty)$ and $r := |\Sigma|$. Then one has*

$$\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(\mathrm{K}_1(\Lambda(\mathcal{G}_\infty)))\cdot\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T} = \Theta^\Sigma_{\mathcal{K}_\infty,S,T}(\mathrm{d}_{\Lambda(\mathcal{G}_\infty)}(C_{\mathcal{K}_\infty,S,T})^{\mathrm{pb}})$$

in $\bigcap^r_{\Lambda(\mathcal{G}_\infty)}\mathcal{O}^\times_{\mathcal{K}_\infty,S,T}$.

Remark 7.5. The validity of Conjecture 7.4 combines with the definition (7.2.3) (with $a = r$) to imply that for every F in $\Omega(\mathcal{K}_\infty)$ the element $\varepsilon^\Sigma_{F/K,S,T}$ belongs to $\bigcap^r_{\mathbb{Z}_p[\mathcal{G}_F]}\mathcal{O}^\times_{F,S,T,p}$ and hence implies the validity of Conjecture 6.8 for $F/K, S$ and T. (Here we recall, from Remark 6.6(iii), that for any field F in $\Omega(\mathcal{K}_\infty)$ for which $\Sigma \neq \Sigma_S(F)$ the element $\varepsilon^\Sigma_{F/K,S,T}$ vanishes.)

Remark 7.6. If $\mathcal{K}_\infty/K$ is abelian, then [19, Th. 3.5] implies Conjecture 7.4 is equivalent to the 'higher rank main conjecture' of Iwasawa theory formulated by Burns, Kurihara and Sano in [19, Conj. 3.1]. In particular, the argument of [19, Cor. 5.6] shows that if $K = \mathbb{Q}$ and $\mathcal{K}_\infty/K$ is abelian, then the validity of Conjecture 7.4 follows as a consequence of the classical Iwasawa main conjecture in this setting, as proved by Mazur and Wiles.

Remark 7.7. Following the general approach of Coates et al in [27], we let $\mathcal{K}'_\infty$ be any compact p-adic Lie extension of K in $\mathcal{K}$ that is ramified at only finitely many places and also contains an intermediate field K_∞ that is Galois over K and such $\mathrm{Gal}(K_\infty/K)$ is topologically isomorphic to $\mathbb{Z}_p$. Then $\mathcal{K}'_\infty$ is equal to the union of all compact p-adic Lie extensions $\mathcal{K}_\infty$ of rank one of K in $\mathcal{K}'_\infty$ and hence, by taking the limit of Conjecture 7.4 over all such extensions $\mathcal{K}_\infty/K$, one can formulate a 'main conjecture' for $\mathbb{G}_m$ relative to the extension $\mathcal{K}'_\infty/K$.

7.3. **Evidence for the rank zero case.** In this section we show that Conjecture 7.4 generalizes to arbitrary rank (of Euler systems) the standard formulation of main conjectures in non-commutative Iwasawa theory, and thereby deduce the validity of an appropriate component of Conjecture 7.4 for an important class of extensions.

7.3.1. Before stating the next result we recall that any object C of the category $\mathsf{D}^{\mathrm{perf}}(\Lambda(\mathcal{G}_\infty))$ for which $Q(\mathcal{G}_\infty)\otimes_{\Lambda(\mathcal{G}_\infty)} C$ is acyclic gives rise to a canonical Euler characteristic element $\chi^{\mathrm{ref}}_{\Lambda(\mathcal{G}_\infty)}(C)$ in $\mathrm{K}_0(\Lambda(\mathcal{G}_\infty), Q(\mathcal{G}_\infty))$.

Proposition 7.8. *Let ϵ be a non-zero idempotent of $\zeta(\Lambda(\mathcal{G}_\infty))$ such that the cohomology groups of $\epsilon\cdot C_{\mathcal{K}_\infty,S,T}$ are torsion $\Lambda(\mathcal{G}_\infty)$-modules.*

Then $\Sigma_S(\mathcal{K}_\infty)$ is empty (so that $r=0$) and the ϵ-component of Conjecture 7.4 is valid if and only if there exists an element λ of $\mathrm{K}_1(Q(\mathcal{G}_\infty)\epsilon)$ with the following two properties:

(i) *λ is sent by the canonical connecting homomorphism $\mathrm{K}_1(Q(\mathcal{G}_\infty)\epsilon)\to \mathrm{K}_0(\Lambda(\mathcal{G}_\infty)\epsilon, Q(\mathcal{G}_\infty)\epsilon)$ to the element $\chi^{\mathrm{ref}}_{\Lambda(\mathcal{G}_\infty)\epsilon}(\epsilon\cdot C_{\mathcal{K}_\infty,S,T})$;*

(ii) *λ is sent by the reduced norm map of the semisimple algebra $Q(\mathcal{G}_\infty)\epsilon$ to $\epsilon\cdot\theta_{\mathcal{K}_\infty,S,T}(0)$.*

Proof. Lemma 5.1 implies, under Hypothesis 6.1, the complex $C_{\mathcal{K}_\infty,S,T}$ is isomorphic in $\mathsf{D}^{\mathrm{perf}}(\Lambda(\mathcal{G}_\infty))$ to a complex $P^\bullet$ of the form

$$\Lambda(\mathcal{G}_\infty)^d \xrightarrow{\phi} \Lambda(\mathcal{G}_\infty)^d,$$

where the first term is placed in degree zero (for a detailed construction of such a complex see, for example, the proof of Proposition 8.2 below). We write

$$\underline{b}_\bullet := \{b_i\}_{i\in[d]}$$

for the standard basis of $\Lambda(\mathcal{G}_\infty)^d$.

Then, since the cohomology groups of $C := \epsilon\cdot C_{\mathcal{K}_\infty,S,T}$ are assumed to be torsion $\Lambda(\mathcal{G}_\infty)$-modules, the matrix

$$M := \big(\epsilon\cdot(b_i^*\circ\phi)(b_j)\big)_{i,j\in[d]}$$

belongs to $\mathrm{M}_d(\Lambda(\mathcal{G}_\infty))\cap \mathrm{GL}_d(Q(\mathcal{G}_\infty)\epsilon)$ and its class $\langle M\rangle$ in $\mathrm{K}_1(Q(\mathcal{G}_\infty)\epsilon)$ is a pre-image of $\chi^{\mathrm{ref}}_{\Lambda(\mathcal{G}_\infty)\varepsilon}(C)$ under the connecting homomorphism in claim (i).

The long exact sequence of relative K-theory therefore implies that the stated conditions (i) and (ii) are equivalent to asserting that

$$\mathrm{Nrd}_{Q(\mathcal{G}_\infty)\epsilon}(\mathrm{K}_1(\Lambda(\mathcal{G}_\infty)\epsilon))\cdot\theta_{\mathcal{K}_\infty,S,T}(0) = \{\mathrm{Nrd}_{Q(\mathcal{G}_\infty)\epsilon}(u\cdot\langle M\rangle) : u\in \mathrm{K}_1(\Lambda(\mathcal{G}_\infty)\epsilon)\}.$$

Remark 6.14(ii) implies that the left hand side of this equality is equal to the ϵ-component of the left hand side of the equality in Conjecture 7.4. To complete the proof it is thus enough to show that

$$\{\mathrm{Nrd}_{Q(\mathcal{G}_\infty)\epsilon}(u\cdot\langle M\rangle) : u\in \mathrm{K}_1(\Lambda(\mathcal{G}_\infty)\epsilon)\} = \Theta(\mathrm{d}_{\Lambda(\mathcal{G}_\infty)\epsilon}(C)^{\mathrm{pb}}),$$

where we write Θ in place of $\Theta^{\Sigma}_{\mathcal{K}_\infty,S,T}$. Since the description (7.1.1) of the reduced norm of $Q(\mathcal{G}_\infty)$ combines with the argument of §3.1 to imply

$$\Theta(\mathrm{d}_{\Lambda(\mathcal{G}_\infty)\epsilon}(C)^{\mathrm{pb}}) = \mathrm{Nrd}_{Q(\mathcal{G}_\infty)\epsilon}(\mathrm{K}_1(\Lambda(\mathcal{G}_\infty)\epsilon))\cdot\Theta(x(\underline{b}_\bullet)),$$

where the tuple $x(\underline{b}_\bullet)$ is constructed using $\underline{b}$ as the ordered basis of both non-zero terms of $P^\bullet$, it is therefore enough to show $\Theta(x(\underline{b}_\bullet))$ is equal to $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)\epsilon}(\langle M\rangle)$.

To prove this we first apply (7.1.1) to the matrix M to deduce that

$$\mathrm{Nrd}_{Q(\mathcal{G}_\infty)\epsilon}(\langle M\rangle) = \big(\mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_L]\epsilon_L}(M_L)\big)_{L\in\Omega(\mathcal{K}_\infty)}$$

with

$$M_L := \big(\epsilon_L\cdot(b^*_{L,i}\circ\phi_L)(b_{L,j})\big)_{i,j\in[d]}\in \mathrm{M}_d(\mathbb{Z}_p[\mathcal{G}_L]),$$

where the standard basis elements $b^*_{L,i}$ and $b_{L,j}$ are as specified in (6.3.1), ϵ_L is the image of ϵ in $\zeta(\mathbb{Z}_p[\mathcal{G}_L])$ and ϕ_L is the endomorphism of $\mathbb{Z}_p[\mathcal{G}_L]^d$ induced by ϕ. In addition, for L in $\Omega(\mathcal{K}_\infty)$ the equality (2.1.7) implies

$$\epsilon_L\cdot\big(\wedge_{i\in[d]}(b^*_{L,i}\circ\phi_L)\big)\big(\wedge_{j\in[d]}b_{L,j}\big) = \mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_L]\epsilon_L}(M_L).$$

It is thus enough to show the left hand side of this expression is equal to the image y_L of $\Theta(\epsilon\cdot x(\underline{b}_\bullet))$ under the projection map

$$\bigcap\nolimits^0_{\Lambda(\mathcal{G}_\infty)}\mathcal{O}^\times_{\mathcal{K}_\infty,S,T}\to\bigcap\nolimits^0_{\mathbb{Z}_p[\mathcal{G}_L]}\mathcal{O}^\times_{L,S,T,p} = \xi(\mathbb{Z}_p[\mathcal{G}_L]).$$

To verify this we note $\epsilon_L\cdot C_{L,S,T,p}$ is isomorphic in $\mathsf{D}^{\mathrm{perf}}(\mathbb{Z}_p[\mathcal{G}_L])$ to the complex

$$P^\bullet_L := \epsilon_L(\mathbb{Z}_p[\mathcal{G}_L]\otimes_{\Lambda(\mathcal{G}_\infty)}P^\bullet)$$

and hence that $(y_L,0)$ is the image of $\epsilon_L\cdot x(\underline{b}_\bullet)_L$ under the canonical morphism

$$\mathrm{d}_{\mathbb{Z}_p[\mathcal{G}_L]\epsilon_L}(P^\bullet_L)\subset \mathrm{d}_{\mathbb{Q}_p[\mathcal{G}_L]\epsilon_L}(\mathbb{Q}_p\otimes_{\mathbb{Z}_p}P^\bullet_L)\cong \mathrm{d}_{\mathbb{Q}_p[\mathcal{G}_L]\epsilon_L}(0) = (\zeta(\mathbb{Q}_p[\mathcal{G}_L]),0)$$

induced by the acyclicity of $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}P^\bullet_L$. The claimed result is thus true because the latter morphism sends the element

$$\epsilon_L\cdot x(\underline{b}_\bullet)_L = \epsilon_L\cdot((\wedge_{j\in[d]}b_{L,j})\otimes(\wedge_{i\in[d]}b^*_{L,i}),0)$$

to $(\epsilon_L\cdot z_L,0)$ with

$$z_L := (\wedge_{i\in[d]}b^*_{L,i})(\wedge_{j\in[d]}(\phi_L(b_{L,j}))) = (\wedge_{i\in[d]}(b^*_{L,i}\circ\phi_L))(\wedge_{j\in[d]}b_{L,j}).$$

□

7.3.2. In the sequel we write L^+ for the maximal totally real subfield of a number field L and $\mu_p(L)$ for the p-adic cyclotomic μ-invariant of L.

Corollary 7.9. *Assume K is totally real, L is CM and $\mathcal{K}_\infty = L^{\rm cyc}$. Write ϵ for the idempotent $(1-\tau)/2$ of $\zeta(\Lambda(\mathcal{G}_\infty))$, where τ is the (unique) non-trivial element of $\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}_\infty^+)$.*

Then, if $\mu_p(L)$ vanishes, the ϵ-component of Conjecture 7.4 is valid for the data $\mathcal{K}_\infty/K$, $S = S_{\mathcal{K}_\infty/K}$ and any auxiliary set of places T.

Proof. Set $\mathcal{G} := \mathcal{G}_\infty$ and write $\Lambda(\mathcal{G})^\#(1)$ for the (left) $\Lambda(\mathcal{G})$-module $\Lambda(\mathcal{G})$ endowed with the action of G_K whereby each element σ acts as right multiplication by $\chi_K(\sigma)\cdot\bar\sigma^{-1}$ where $\chi_K : G_K \to \mathbb{Z}_p^\times$ is the cyclotomic character and $\bar\sigma$ the image of σ in $\mathcal{G}$. We then write C' and C for the respective complexes $\epsilon\cdot \mathsf{R}\Gamma_{\text{ét},T}(\mathcal{O}_{F,S}, \Lambda(\mathcal{G})^\#(1))$ and $\epsilon \cdot C_{\mathcal{K}_\infty,S,T}$.

Then, since $\epsilon(Y_{F,S_K^\infty,p})$ vanishes for all F in $\Omega(\mathcal{K}_\infty)$, the complex $Q(\mathcal{G})\otimes_{\Lambda(\mathcal{G})} C$ is acyclic and the Artin-Verdier Duality theorem implies the existence of a canonical isomorphism $C \cong C'[1]$ in $\mathsf{D}^{\rm perf}(\Lambda(\mathcal{G}))$ and hence of an equality in $\mathrm{K}_0(\Lambda(\mathcal{G})\epsilon, Q(\mathcal{G})\epsilon)$

$$\chi^{\rm ref}_{\Lambda(\mathcal{G})\epsilon}(C) = -\chi^{\rm ref}_{\Lambda(\mathcal{G})\epsilon}(C').$$

We next note that $K^{\rm cyc} \subseteq \mathcal{K}_\infty$ and use a fixed choice of topological generator γ_K of Γ_K to identify (via the association $\gamma_K - 1 \leftrightarrow t$) the Iwasawa algebra $\mathbb{Z}_p[[\Gamma_K]]$ with a power series ring in one variable $\mathbb{Z}_p[[t]]$. We also write $A(\mathcal{G})$ for the set of irreducible $\mathbb{Q}_p^c$-valued characters of $\mathcal{G}$ that have open kernel.

We recall that for each χ in $A(\mathcal{G})$ Ritter and Weiss have in [72, Prop. 6] constructed a canonical homomorphism

$$j_\chi : \zeta(Q(\mathcal{G}))^\times \to (\mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} Q(\mathbb{Z}_p[[t]]))^\times.$$

The Weierstrass Preparation Theorem combines with [72, Prop. 5(3)] to imply an element x of $\zeta(Q(\mathcal{G}))^\times$ is uniquely determined by the value $j_\chi(x)(0)$ of $j_\chi(x)$ at $t=0$ for every χ in $A(\mathcal{G})$. In addition, if V_χ is a $\mathbb{Q}_p^c[[\mathcal{G}]]$-module of character χ and x belongs to $\zeta(Q(\mathcal{G}))\cap\zeta(\mathbb{Q}_p[[\mathcal{G}]])$, then $j_\chi(x)(0)$ is equal to the χ-component of the image of x in $\zeta(\mathbb{Q}_p[\mathrm{Gal}(\mathcal{K}_\infty^{\ker(\chi)}/K)])$. (This can be verified by an explicit computation and relies on the fact that the elements γ_χ and e_χ occurring in [72, Prop. 6] act trivially on V_χ.) In particular, since $\theta_{\mathcal{K}_\infty,S,T}$ belongs to $\zeta(Q(\mathcal{G}))\cap\zeta(\mathbb{Q}_p[[\mathcal{G}]])$ (as a consequence, for example, of [72, Prop. 11 and the proof of Th. 8]), one finds that if $-\tau$ acts as the identity on V_χ, then

$$j_\chi(\epsilon\cdot\theta_{\mathcal{K}_\infty,S,T})(0) = L_{S,T}(\check\chi,0) = L_{p,S,T}(\check\chi\cdot\omega_K,0), \tag{7.3.1}$$

where ω_K is the Teichmüller character of K and $L_{p,S,T}(\check\chi\cdot\omega_K,z)$ the S-truncated T-modified Deligne-Ribet p-adic Artin L-series of $\check\chi\cdot\omega_K$, as discussed by Greenberg in [40].

Next we note that j_χ is related to the map

$$\Phi_{\mathcal{G},\chi} : \mathrm{K}_1(Q(\mathcal{G})) \to (\mathbb{Q}_p^c\otimes_{\mathbb{Q}_p} Q(\mathbb{Z}_p[[t]]))^\times$$

defined by Coates et al in [27] by virtue of the fact (proved in [15, Lem. 3.1]) that for every element λ of $\mathrm{K}_1(Q(\mathcal{G}))$ one has

$$\Phi_{\mathcal{G},\chi}(\lambda) = j_\chi(\mathrm{Nrd}_{Q(\mathcal{G})}(\lambda)).$$

Given this, the above observations combine to imply an element λ of $\mathrm{K}_1(Q(\mathcal{G}))$ satisfies the conditions (i) and (ii) in Proposition 7.8 if and only if the connecting homomorphism

$\mathrm{K}_1(Q(\mathcal{G})) \to \mathrm{K}_0(\Lambda(\mathcal{G}), Q(\mathcal{G}))$ sends λ to $-\chi^{\mathrm{ref}}_{\Lambda(\mathcal{G})\epsilon}(C')$ and, in addition, for every χ in $A(\mathcal{G})$ one has

$$\Phi_{\mathcal{G},\chi}(\lambda)(0) = L_{p,S,T}(\check{\chi} \cdot \omega_K, 0).$$

In view of Proposition 7.8, it is therefore enough to note that if $\mu_p(L)$ vanishes, then the existence of such an element λ is deduced in [16, Prop. 7.1] from the proof (under the given assumption on $\mu_p(L)$) of the main conjecture of non-commutative Iwasawa theory for totally real fields, due to Ritter and Weiss [73] and, independently, Kakde [50]. □

Remark 7.10. In [49] Johnston and Nickel identify families of (non-abelian) Galois extensions L/K for which one can prove the main conjecture of non-commutative p-adic Iwasawa theory for L^{cyc}/K without assuming $\mu_p(L)$ vanishes (or that p does not divide $[L:K]$). In all such cases the argument of Corollary 7.9 shows that the $(1-\tau)/2$-component of Conjecture 7.4 is valid for the data L^{cyc}/K, $S = S_{\mathcal{K}_\infty/K}$ and any auxiliary set of places T.

8. Canonical resolutions and semisimplicity in Iwasawa theory

8.1. **Canonical resolutions in Iwasawa theory.** In this section we fix a compact p-adic Lie extension $\mathcal{K}_\infty$ of K of strictly positive rank for which $S_{\mathrm{ram}}(\mathcal{K}_\infty/K)$ is finite and non-empty, and set $\mathcal{G}_\infty := \mathrm{Gal}(\mathcal{K}_\infty/K)$.

We also fix a finite set S of places of K that contains the set $S_{\mathcal{K}_\infty/K}$ specified in (7.2.2) and a finite set of places T of K that is disjoint from S and such that Hypothesis 6.1 is satisfied (with $\mathcal{K}$ taken to be $\mathcal{K}_\infty$).

Then, since $S_{\mathrm{ram}}(\mathcal{K}_\infty/K)$ is non-empty, one has $\Sigma_S(\mathcal{K}_\infty) \neq S$ and so we can fix a place v' in $S \setminus \Sigma_S(\mathcal{K}_\infty)$. We then set

$$S' := S \setminus \{v'\}.$$

8.1.1. We first describe an important aspect of the descent properties of transpose Selmer modules.

Lemma 8.1. *Fix L in $\Omega(\mathcal{K}_\infty)$, with $G := \mathcal{G}_L$, and a normal subgroup H of G with $E := L^H$. Consider the composite surjective homomorphism of G/H-modules*

$$\beta_{E,S} : \mathrm{Sel}_S^T(E)^{\mathrm{tr}} \xrightarrow{\varrho_{E,S}} X_{E,S} \xrightarrow{\alpha_{E,S,S'}} Y_{E,S'},$$

in which $\varrho_{E,S}$ comes from (5.1.2) and $\alpha_{E,S,S'}$ is the natural projection map.

Then the image of $\ker(\beta_{L,S})$ under the composite homomorphism

$$\mathrm{Sel}_S^T(L)^{\mathrm{tr}} \to \mathbb{Z}[G/H] \otimes_{\mathbb{Z}[G]} \mathrm{Sel}_S^T(L)^{\mathrm{tr}} \cong \mathrm{Sel}_S^T(E)^{\mathrm{tr}},$$

in which the isomorphism is as in Lemma 5.1(iv), is equal to $\ker(\beta_{E,S})$.

Proof. The claimed result is obtained directly by applying the Snake Lemma to the following exact commutative diagram

$$\begin{array}{ccccccccc} & & H_0(H, \ker(\beta_{L,S})) & \longrightarrow & H_0(H, \mathrm{Sel}_S^T(L)^{\mathrm{tr}}) & \xrightarrow{\beta_{L,S}} & H_0(H, Y_{L,S'}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker(\beta_{E,S}) & \longrightarrow & \mathrm{Sel}_S^T(E)^{\mathrm{tr}} & \xrightarrow{\beta_{E,S}} & Y_{E,S'} & \longrightarrow & 0. \end{array}$$

Here the lower row is the tautological exact sequence, the upper row is obtained by taking H-coinvariants of the analogous tautological exact sequence, the right-hand vertical arrow is the canonical isomorphism that maps the class of each place w in S'_L to the restriction of w to E, the middle vertical arrow is the canonical isomorphism coming from Lemma 5.1(iv) and the left hand vertical arrow is induced by the commutativity of the second square. □

8.1.2. In the sequel we will use the $\Lambda(\mathcal{G}_\infty)$-modules that are defined by the inverse limits

$$\mathcal{S}_S^T(\mathcal{K}_\infty)^{\rm tr} := \varprojlim_{F\in\Omega(\mathcal{K}_\infty)} \mathrm{Sel}_S^T(F)_p^{\rm tr}$$

and, if Hypothesis 6.1 is satisfied, also

$$\mathcal{S}_S^T(\mathcal{K}_\infty) := \varprojlim_{F\in\Omega(\mathcal{K}_\infty)} \mathrm{Sel}_S^T(F)_p.$$

Here the respective transition morphisms for $F \subset F'$ are the composite maps

$$\mathrm{Sel}_S^T(F')_p^{\rm tr} \to \mathbb{Z}_p[\mathcal{G}_F] \otimes_{\mathbb{Z}_p[\mathcal{G}_{F'}]} \mathrm{Sel}_S^T(F')_p^{\rm tr} \cong \mathrm{Sel}_S^T(F)_p^{\rm tr},$$

and

$$\mathrm{Sel}_S^T(F')_p \to \mathbb{Z}_p[\mathcal{G}_F] \otimes_{\mathbb{Z}_p[\mathcal{G}_{F'}]} \mathrm{Sel}_S^T(F')_p \cong \mathrm{Sel}_S^T(F)_p,$$

where the first map in both cases is the obvious projection and the second is the isomorphism given by Lemma 5.1(iv), respectively Remark 5.3 (if Hypothesis 6.1 is satisfied).

In the following result we also use the object $C_{\mathcal{K}_\infty,S,T}$ of $\mathsf{D}^{\rm perf}(\Lambda(\mathcal{G}_\infty))$ defined in §7.2.2.

Proposition 8.2. *Set $n := |S| - 1$. Then there exists a natural number d with $d \geq n$ and a canonical family of complexes of $\Lambda(\mathcal{G}_\infty)$-modules $C(\phi)$ of the form*

$$\Lambda(\mathcal{G}_\infty)^d \xrightarrow{\phi} \Lambda(\mathcal{G}_\infty)^d$$

in which the first term is placed in degree zero and the following properties are satisfied.

(i) *$C(\phi)$ is isomorphic in $\mathsf{D}^{\rm perf}(\Lambda(\mathcal{G}_\infty))$ to $C_{\mathcal{K}_\infty,S,T}$.*

(ii) *If $C(\tilde{\phi})$ is any other complex in the family, then $\tilde{\phi} = \eta\circ\phi\circ(\eta')^{-1}$ where η and η' are automorphisms of $\Lambda(\mathcal{G}_\infty)^d$ and η is represented, with respect to the standard basis of $\Lambda(\mathcal{G}_\infty)^d$, by a block matrix of the form (6.3.2).*

(iii) *For each L in $\Omega(\mathcal{K}_\infty)$ we set $\mathfrak{S}_L := \mathrm{Sel}_S^T(L)_p^{\rm tr}$ and write ϕ_L, ι_L and ϖ_L respectively for the endomorphism of $\mathbb{Z}_p[\mathcal{G}_L]^d$ induced by ϕ and the embedding $\mathcal{O}^\times_{L,S,T,p} \cong \ker(\phi_L) \subseteq \mathbb{Z}_p[\mathcal{G}_L]^d$ and surjection $\mathbb{Z}_p[\mathcal{G}_L]^d \to \mathrm{cok}(\phi_L) \cong \mathfrak{S}_L$ that are induced by the descent isomorphism*

$$C_{L,S,T,p} \cong \mathbb{Z}_p[\mathcal{G}_L] \otimes^{\mathsf{L}}_{\Lambda(\mathcal{G}_\infty)} C_{\mathcal{K}_\infty,S,T} \cong \mathbb{Z}_p[\mathcal{G}_L] \otimes_{\Lambda(\mathcal{G}_\infty)} C(\phi).$$

Then there exists a natural number d_L with $n \leq d_L \leq d$ and a commutative diagram of $\mathbb{Z}_p[\mathcal{G}_L]$-modules of the form

$$\begin{array}{ccccccccc} 0 \to \mathcal{O}^\times_{L,S,T,p} & \xrightarrow{\iota_L} & \mathbb{Z}_p[\mathcal{G}_L]^d & \xrightarrow{\phi_L} & \mathbb{Z}_p[\mathcal{G}_L]^d & \xrightarrow{\varpi_L} & \mathfrak{S}_L \to 0 \\ \Vert & & \downarrow{\scriptstyle \kappa'_L} & & \downarrow{\scriptstyle \kappa_L} & & \Vert \\ 0 \to \mathcal{O}^\times_{L,S,T,p} & \xrightarrow{(\hat{\iota}_L,0)} & \mathbb{Z}_p[\mathcal{G}_L]^{d_L+(d-d_L)} & \xrightarrow{(\hat{\phi}_L,\mathrm{id})} & \mathbb{Z}_p[\mathcal{G}_L]^{d_L+(d-d_L)} & \xrightarrow{(\hat{\pi}_L,0)} & \mathfrak{S}_L \to 0. \end{array} \tag{8.1.1}$$

Here κ'_L and κ_L are bijective, the matrix of κ_L with respect to the standard basis of $\mathbb{Z}_p[\mathcal{G}_L]^d = \mathbb{Z}_p[\mathcal{G}_L]^{d_L} \oplus \mathbb{Z}_p[\mathcal{G}_L]^{(d-d_L)}$ has the form $\left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & * \end{array}\right)$ and the exact sequence

$$0 \to \mathcal{O}^\times_{L,S,T,p} \xrightarrow{\hat{\iota}_L} \mathbb{Z}_p[\mathcal{G}_L]^{d_L} \xrightarrow{\hat{\phi}_L} \mathbb{Z}_p[\mathcal{G}_L]^{d_L} \xrightarrow{\hat{\pi}_L} \mathfrak{S}_L \to 0 \tag{8.1.2}$$

is constructed as in diagram (6.3.4).

Proof. We set $\Omega_\infty := \Omega(\mathcal{K}_\infty)$, $R_\infty := \Lambda(\mathcal{G}_\infty)$ and $\mathfrak{S}_\infty := \mathcal{S}^T_S(\mathcal{K}_\infty)^{\mathrm{tr}}$. For $v \in S$ we set $Y_{\infty,v} := \varprojlim_{F\in\Omega_\infty} Y_{F,\{v\},p}$, where the transition maps are the natural projection maps. We then set $Y_{\infty,S'} := \bigoplus_{v\in S'} Y_{\infty,v}$ and consider the homomorphism

$$\beta_\infty := \varprojlim_{F\in\Omega_\infty} \beta_{F,S,p} : \mathfrak{S}_\infty \to Y_{\infty,S'},$$

where $\beta_{F,S}$ is the map of $\mathcal{G}_F$-modules in Lemma 8.1. We note β_∞ is surjective since each map $\beta_{F,S,p}$ is surjective and each module $\ker(\beta_{F,S,p})$ is compact.

For $i \in [n]$ we write v_i for the i-th element of S' (with respect to the ordering induced by (6.1.1)). For each such i, we fix a place $w_{i,\infty}$ of $\mathcal{K}_\infty$ above v_i and write $\pi_{1,\infty} : R^n_\infty \to Y_{\infty,S'}$ for the surjective map of R_∞-modules that sends the i-th element in the standard basis of R^n_∞ to $w_{i,\infty}$. We choose a lift

$$\pi'_{1,\infty} : R^n_\infty \to \mathfrak{S}_\infty$$

of $\pi_{1,\infty}$ through β_∞.

The algebra R_∞ is semiperfect since it is both semilocal and complete with respect to its Jacobson radical (as $\mathcal{G}_\infty$ is a compact p-adic Lie group). In view of [28, Th. (6.23)], we may therefore fix a projective cover

$$\pi_{2,\infty} : P \to \ker(\beta_\infty)$$

of $\ker(\beta_\infty) = \varprojlim_{F\in\Omega_\infty} \ker(\beta_{F,S,p})$.

We next choose a R_∞-module P' such that $P \oplus P'$ is a free module of minimal rank, n' say, fix an isomorphism $j : P \oplus P' \cong R^{n'}_\infty$ and write $\pi'_{2,\infty}$ for the map $(\pi_{2,\infty}, 0)\circ j^{-1}$ on $R^{n'}_\infty$. We set $d := n + n'$ and consider the homomorphism of R_∞-modules

$$\pi_\infty : R^d_\infty = R^n_\infty \oplus R^{n'}_\infty \xrightarrow{(\pi'_{1,\infty},\pi'_{2,\infty})} \mathfrak{S}_\infty. \tag{8.1.3}$$

For L in Ω_∞ we set

$$R_L := \mathbb{Z}_p[\mathcal{G}_L],\ U_L := \mathcal{O}^\times_{L,S,T,p} \quad \text{and} \quad \mathcal{H}_L := \mathrm{Gal}(\mathcal{K}_\infty/L).$$

Then the $\mathcal{H}_L$-coinvariants of π_∞ gives a surjective homomorphism $\varpi_L : R^d_L \to \mathfrak{S}_L$ and, for each L and L' in Ω_∞ with $L \subset L'$, the argument of [21, Prop. 3.2] allows us to fix an exact commutative diagram of the form

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & U_{L'} & \xrightarrow{\hat{\iota}_{L'}} & R^d_{L'} & \xrightarrow{\phi_{L'}} & R^d_{L'} & \xrightarrow{\varpi_{L'}} & \mathfrak{S}_{L'} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \omega^0_{L'/L}} & & \downarrow{\scriptstyle \omega^1_{L'/L}} & & \downarrow{\scriptstyle \omega_{L'/L}} & & \\ 0 & \longrightarrow & U_L & \xrightarrow{\hat{\iota}_L} & R^d_L & \xrightarrow{\phi_L} & R^d_L & \xrightarrow{\varpi_L} & \mathfrak{S}_L & \longrightarrow & 0. \end{array} \tag{8.1.4}$$

Here $\omega^1_{L'/L}$ and $\omega_{L'/L}$ are the natural projection maps, $\omega^0_{L'/L}$ sends each element $b_{L',i}$ in the standard basis of $R^d_{L'}$ to any choice of element x_i with

$$\phi_L(x_i) = \omega^1_{L'/L}(\phi_{L'}(b_{L',i}))$$

and the following property is satisfied: if F denotes either L or L', then the complex $C(\phi_F)$ given by $R^d_F \xrightarrow{\phi_F} R^d_F$, where the first module is placed in degree zero and the cohomology groups are identified with U_F and $\mathfrak{S}_F$ by means of the maps in the respective row of the diagram, then there exists an isomorphism $C(\phi_F) \cong C_{F,S,T,p}$ in $\mathsf{D}^{\rm perf}(R_F)$ that induces the identity map on both cohomology groups. With these identifications the canonical descent isomorphism $R_L \otimes^{\mathsf{L}}_{R_{L'}} C_{L',S,T,p} \cong C_{L,S,T,p}$ implies that the morphism $R_L \otimes_{R_{L'}} C(\phi_{L'}) \cong C(\phi_L)$ induced by the maps $\omega^0_{L'/L}$ and $\omega^1_{L'/L}$ is a quasi-isomorphism. Since $\omega^1_{L'/L}$ is surjective, this in turn implies that the map $\omega^0_{L'/L}$ is surjective and hence that the limit $\varprojlim_{F\in\Omega_\infty} R^d_F$ with respect to the transition morphisms are $\omega^0_{F'/F}$ is isomorphic to R^d_∞.

The limit of the complexes $\{C(\phi_L)\}_{L\in\Omega(\mathcal{K}_\infty)}$ with respect to the morphisms in (8.1.4) is therefore a complex of the form $R^d_\infty \xrightarrow{\phi} R^d_\infty$ that is isomorphic in $\mathsf{D}^{\rm perf}(R_\infty)$ to $C_{\mathcal{K}_\infty,S,T,p}$. Thus, denoting this complex by $C(\phi)$, claim (i) is clear.

Turning to claim (ii), we note that if $\tilde\pi_\infty$ is any map defined in the same way as π_∞ but with respect to a different choices either of projective cover $\pi_{2,\infty}$, isomorphism j or lift $\pi'_{1,\infty}$ of $\pi_{1,\infty}$, then an easy exercise shows that $\pi_\infty = \tilde\pi_\infty \circ \eta$, where η is an automorphism of R^d_∞ that is represented with respect to the standard basis $\{b_i\}_{i\in[d]}$ of R^d_∞ by a block matrix of the form (6.3.2). (Here, with respect to the decomposition of R^d_∞ used in (8.1.3) we identify b_i for $i \in [d]\setminus[n]$ with the $(i-n)$-th element of the standard basis of $R^{n'}_\infty$.)

Then, if $C(\tilde\phi)$ is any complex obtained in the same way by using $\tilde\pi_\infty$ rather than π_∞, there exists a quasi-isomorphism $\xi : C(\tilde\phi) \cong C(\phi)$ that is represented by an exact commutative diagram of the form

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{O}^\times_{\mathcal{K}_\infty,S,T} & \xrightarrow{\iota_\infty} & R^d_\infty & \xrightarrow{\phi} & R^d_\infty & \xrightarrow{\pi_\infty} & \mathfrak{S}_\infty & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta} & & \| & & \\ 0 & \longrightarrow & \mathcal{O}^\times_{\mathcal{K}_\infty,S,T} & \longrightarrow & R^d_\infty & \xrightarrow{\tilde\phi} & R^d_\infty & \xrightarrow{\tilde\pi_\infty} & \mathfrak{S}_\infty & \longrightarrow & 0 \end{array} \tag{8.1.5}$$

To deduce claim (ii) it is thus sufficient to note that, since η is bijective, the Five Lemma implies that η' is also bijective.

Finally, to prove claim (iii), we fix L in Ω_∞ and set $\mathcal{H} := \mathcal{H}_L$. We note that the projection map $\ker(\beta_\infty)_\mathcal{H} \to \ker(\beta_{L,S,p})$ is surjective (by Lemma 8.1(ii)) and hence that there exists a direct sum decomposition

$$P_\mathcal{H} = P_L \oplus Q_L$$

of R_L-modules so that $(\pi_{2,\infty})_\mathcal{H}$ is zero on Q_L and restricts to P_L to give a projective cover of $\ker(\beta_{L,S,p})$. We fix a projective R_L-module P'_L of minimal rank so $P_L \oplus P'_L$ is a free R_L-module, write n'_L for the rank of the latter module and set $d_L := n + n'_L$ and $\delta_L := n' - n'_L$.

Then $d = d_L + \delta_L$ and the Krull-Schmidt Theorem implies $\delta_L \geq 0$ and that, for any choice of an isomorphism of R_L-modules $j' : R_L^{n'_L} \cong P_L \oplus P'_L$, there exists an isomorphism

$$\iota_L : Q_L \oplus P'_{\mathcal{H}} \to P'_L \oplus R_L^{\delta_L}$$

of R_L-modules and a commutative diagram of the form

$$\begin{array}{ccccc} R_L^n \oplus R_L^{n'} & \xrightarrow{(\mathrm{id}, j_{\mathcal{H}})} & R_L^n \oplus (P \oplus P')_{\mathcal{H}} & \xrightarrow{(\pi'_{1,\infty}, (\pi_{2,\infty}, 0))_{\mathcal{H}}} & (\mathfrak{S}_\infty)_{\mathcal{H}} \\ {\scriptstyle \kappa_L}\downarrow & & {\scriptstyle (\mathrm{id}, \iota'_L)}\downarrow & & \downarrow \\ R_L^n \oplus R_L^{n'_L} \oplus R_L^{\delta_L} & \xrightarrow{(\mathrm{id}, j', \mathrm{id})} & R_L^n \oplus (P_L \oplus P'_L) \oplus R_L^{\delta_L} & \xrightarrow{((\pi'_{1,\infty})_{\mathcal{H}}, ((\pi_{2,\infty})_{\mathcal{H}}, 0), 0)} & \mathfrak{S}_L. \end{array}$$

Here j is the isomorphism $R_\infty^{n'} \cong P \oplus P'$ fixed just before (8.1.3), ι'_L is the isomorphism

$$(P \oplus P')_{\mathcal{H}} = P_L \oplus (Q_L \oplus P'_{\mathcal{H}}) \xrightarrow{(\mathrm{id}, \iota_L)} P_L \oplus P'_L \oplus R_L^{\delta_L},$$

$\kappa_L = (\mathrm{id}, \tilde{\kappa}_L)$ with $\tilde{\kappa}_L$ the automorphism of $R_L^{n'} = R_L^{n'_L} \oplus R_L^{\delta_L}$ given by $(j', \mathrm{id})^{-1} \circ \iota'_L \circ j_{\mathcal{H}}$ and the right hand vertical arrow is the isomorphism induced by Lemma 8.1(i).

The upper and lower composite horizontal maps in this diagram are respectively equal to the map ϖ_L in diagram (8.1.4) and to $\hat{\pi} \oplus 0$, where $\hat{\pi}$ is a surjective map $R_L^{d_L} \to \mathfrak{S}_L$ constructed as in diagram (6.3.4). Hence, if we fix an embedding $\hat{\iota} : U_L \to R_L^{d_L}$ and an endomorphism $\hat{\phi}$ of $R_L^{d_L}$ as in the upper row of (6.3.4) (for this choice of $\hat{\pi}$), then there exists a commutative diagram of R_L-modules of the form

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & U_L & \xrightarrow{\hat{\iota}_L} & R_L^d & \xrightarrow{\phi_L} & R_L^d & \xrightarrow{\varpi_L} & \mathfrak{S}_L & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \kappa'_L} & & \downarrow{\scriptstyle \kappa_L} & & \| & & \\ 0 & \longrightarrow & U_L & \xrightarrow{(\hat{\iota}, 0)} & R_L^{d_L + \delta_L} & \xrightarrow{(\hat{\phi}, \mathrm{id})} & R_L^{d_L \oplus \delta_L} & \xrightarrow{(\hat{\pi}, 0)} & \mathfrak{S}_L & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & U_L & \xrightarrow{\hat{\iota}} & R_L^{d_L} & \xrightarrow{\hat{\phi}} & R_L^{d_L} & \xrightarrow{\hat{\pi}} & \mathfrak{S}_L & \longrightarrow & 0. \end{array}$$

Here the unlabelled vertical maps are the natural inclusions and so the construction of the sequence in (6.3.4) implies that the upper two rows of this diagram represent the same element of the Yoneda Ext-group $\mathrm{Ext}^2_{R_L}(\mathfrak{S}_L, U_L)$. Given this, the existence of a map κ'_L that makes the first and second upper squares commute (and hence is bijective) follows from the fact that κ_L is bijective and that the upper third square commutes. Finally, we note that the upper part of the above diagram satisfies all of the assertions in claim (iii). $\square$

Remark 8.3. If $\mathcal{G}_\infty$ has rank one, then the resolution $C(\phi)$ of $C_{\mathcal{K}_\infty,S,T}$ constructed in Proposition 8.2 leads to the following explicit interpretation of Conjecture 7.4. For L in $\Omega(\mathcal{K}_\infty)$ the complex $C_{L,S,T,p}$ is isomorphic in $\mathsf{D}^{\mathrm{perf}}(\mathbb{Z}_p[\mathcal{G}_L])$ to

$$\mathbb{Z}_p[\mathcal{G}_L]^d \xrightarrow{\phi_L} \mathbb{Z}_p[\mathcal{G}_L]^d,$$

where the first term is placed in degree zero, and the alternative description of $\Theta^{\Sigma}_{L,S,T,p}$ given in (6.3.9) implies that the image of $\Theta^{\Sigma}_{\mathcal{K}_\infty,S,T}(x(\underline{b}_\bullet))$ under the natural projection map $\bigcap^r_{\Lambda(\mathcal{G}_\infty)}\mathcal{O}^\times_{\mathcal{K}_\infty,S,T} \to \bigcap^r_{\mathbb{Z}_p[\mathcal{G}_L]}\mathcal{O}^\times_{L,S,T,p}$ is

$$\Theta^{\Sigma}_{L,S,T,p}(x(\underline{b}_\bullet)_L) = (\wedge^{i=d}_{i=r+1}(b^*_{L,i}\circ\phi_L))(\wedge_{j\in[d]}b_{L,j}).$$

The equality predicted in Conjecture 7.4 is therefore valid if and only if there exists an element u in $\mathrm{K}_1(\Lambda(\mathcal{G}_\infty))$ such that in $\bigcap^r_{\Lambda(\mathcal{G}_\infty)}\Lambda(\mathcal{G}_\infty)^d$ one has

$$\iota_{\infty,*}(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T}) = \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(u)\cdot\big((\wedge^{i=d}_{i=r+1}(b^*_{L,i}\circ\phi_L)(\wedge_{j\in[d]}b_{L,j})\big)_{L\in\Omega(\mathcal{K}_\infty)}, \tag{8.1.6}$$

where $\iota_{\infty,*}$ is the homomorphism of $\xi_p(\mathcal{K}_\infty/K)$-modules $\bigcap^r_{\Lambda(\mathcal{G}_\infty)}\mathcal{O}^\times_{\mathcal{K}_\infty,S,T}\to\bigcap^r_{\Lambda(\mathcal{G}_\infty)}\Lambda(\mathcal{G}_\infty)^d$ that is induced by the injective map of $\Lambda(\mathcal{G}_\infty)$-modules $\iota_\infty : \mathcal{O}^\times_{\mathcal{K}_\infty,S,T}\to\Lambda(\mathcal{G}_\infty)^d$.

Remark 8.4. The detailed descent properties established in Proposition 8.2(iii) are finer than is strictly necessary for the present article. However, we have included the detail since it is used in the associated article [22] of Burns, Puignau, Sano and Seo in order to discuss non-commutative generalizations of the 'refined class number formulas' conjectured independently by Mazur and Rubin in [63] and by Sano in [75].

8.2. **Semisimplicity for Selmer modules.** The hypothesis that a finitely generated torsion module over the classical Iwasawa algebra be 'semisimple at zero' allows one to make explicit descent computations even in the presence of trivial zeroes (see, for example, [16], though the notion arises in many earlier articles).

In this section we introduce a generalization of the notion of semisimplicity in the context of Selmer modules that will play an important role in later sections.

To do this we fix data $\mathcal{K}_\infty/K$, S and T as in §8.1. We assume throughout this section that $\mathcal{G}_\infty$ has rank one and fix a field E in $\Omega(\mathcal{K}_\infty)$ for which there is an isomorphism of topological groups

$$\mathcal{H}_\infty := \mathrm{Gal}(\mathcal{K}_\infty/E)\cong\mathbb{Z}_p.$$

We then also fix a subset Σ of $\Sigma_S(E)$ such that

$$\begin{cases}\Sigma=\Sigma_S(E), & \text{if } \Sigma_S(E)\neq S,\\ \Sigma_S(\mathcal{K}_\infty)\subseteq\Sigma \text{ and } |\Sigma|=|S|-1, & \text{if } \Sigma_S(E)=S\end{cases} \tag{8.2.1}$$

and set

$$r:=|\Sigma_S(\mathcal{K}_\infty)| \quad\text{and}\quad r':=|\Sigma|$$

(so that $r'\geq r$).

Remark 8.5. The restrictions on Σ given by (8.2.1) are motivated by the observations made in Remark 6.6(iii). If $\Sigma_S(E)=S$ (which occurs, for example, if $\mathcal{K}_\infty$ is a $\mathbb{Z}_p$-extension of $E=K$), then there exists a choice of Σ as above since $\Sigma_S(\mathcal{K}_\infty)\neq S$ and the definitions and results in the rest of this section are independent of this choice.

For any element γ of $\mathcal{G}_\infty$, we define an element of $\xi_p(\mathcal{K}_\infty/K)$ by setting

$$\lambda(\gamma):=\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(\gamma-1)\in\xi_p(\mathcal{K}_\infty/K).$$

We also fix a topological generator γ_E of $\mathcal{H}_\infty$, and define an ideal of $\xi_p(\mathcal{K}_\infty/K)$ by setting

$$I_E(\mathcal{G}_\infty) := \xi_p(\mathcal{K}_\infty/K) \cdot \lambda(\gamma_E).$$

Finally, we write $x \mapsto x^\#$ for the $\mathbb{Q}_p$-linear involution of $Q(\mathcal{G}_\infty)$ that is induced by inverting elements of $\mathcal{G}_\infty$.

Remark 8.6. We record two important properties of the ideal $I_E(\mathcal{G}_\infty)$.
(i) As the notation suggests, $I_E(\mathcal{G}_\infty)$ is independent of the choice of $\gamma := \gamma_E$ (and hence only depends on $\mathcal{G}_\infty$ and E). To see this note that any other topological generator of $\mathcal{H}_\infty$ is equal to γ^a for some $a \in \mathbb{Z}_p^\times$ and that the corresponding quotient $x_a := (\gamma^a - 1)/(\gamma - 1)$ belongs to $\Lambda(\mathcal{H}_\infty)^\times \subseteq \Lambda(\mathcal{G}_\infty)^\times$. From the explicit description of reduced norm given in (7.1.1), it then follows that $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(x_a)$ belongs to $\xi_p(\mathcal{K}_\infty/K)^\times$, and hence that $\lambda(\gamma^a) = \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(x_a) \cdot \lambda(\gamma)$ generates $I_E(\mathcal{G}_\infty)$ over $\xi_p(\mathcal{K}_\infty/K)$.
(ii) One has $I_E(\mathcal{G}_\infty) = I_E(\mathcal{G}_\infty)^\#$. To see this one can combine (7.1.1) with [24, (3.4.1)] to deduce that, for each matrix $M = (M_{ij})$ in $\mathrm{M}_d(\Lambda(\mathcal{G}_\infty))$ there is an equality

$$\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(M)^\# = \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(M^\#),$$

where $M^\#$ denotes the matrix $(M_{ij}^\#)$. These equalities imply that $\xi_p(\mathcal{K}_\infty/K) = \xi_p(\mathcal{K}_\infty/K)^\#$. Since $(\gamma_E - 1)^\# = (-\gamma_E^{-1})(\gamma_E - 1)$ and $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(-\gamma_E^{-1}) \in \xi_p(\mathcal{K}_\infty/K)^\times$, it then also follows that $I_E(\mathcal{G}_\infty) = I_E(\mathcal{G}_\infty)^\#$, as claimed.

8.2.1. We start by defining an analogue for Iwasawa-theoretic Selmer modules of the higher non-commutative Fitting invariants introduced in [24].

To do this, for each endomorphism ϕ of $\Lambda(\mathcal{G}_\infty)^d$ we write $\mathfrak{G}_r(\phi)$ for the subset of $\mathrm{M}_d(\Lambda(\mathcal{G}_\infty))$ comprising all matrices that are obtained by replacing the elements in any selection of r columns of the matrix of ϕ with respect to the standard basis of $\Lambda(\mathcal{G}_\infty)^d$ by arbitrary elements of $\Lambda(\mathcal{G}_\infty)$.

Definition 8.7. For each endomorphism ϕ of $\Lambda(\mathcal{G}_\infty)^d$ constructed as in Proposition 8.2, we define an ideal of $\xi_p(\mathcal{K}_\infty/K)$ by setting

$$\mathrm{Fit}^r_{\Lambda(\mathcal{G}_\infty)}(\mathcal{S}_S^T(\mathcal{K}_\infty)) := \xi_p(\mathcal{K}_\infty/K) \cdot \{\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(M) : M \in \mathfrak{G}_r(\phi)\}^\#.$$

The basic properties of this ideal are described in the following result.

Lemma 8.8. *For each ϕ as above, the following claims are valid.*

(i) $\mathrm{Fit}^r_{\Lambda(\mathcal{G}_\infty)}(\mathcal{S}_S^T(\mathcal{K}_\infty))$ *depends only on the $\Lambda(\mathcal{G}_\infty)$-module $\mathcal{S}_S^T(\mathcal{K}_\infty)$.*
(ii) $\mathrm{Fit}^r_{\Lambda(\mathcal{G}_\infty)}(\mathcal{S}_S^T(\mathcal{K}_\infty))$ *is contained in $I_E(\mathcal{G}_\infty)^{r'-r}$.*

Proof. The commutative diagram (8.1.5) shows that the collection of endomorphisms ϕ that are constructed via the approach in Proposition 8.2 constitutes a distinguished family of free resolutions of the $\Lambda(\mathcal{G}_\infty)$-module $\mathcal{S}_S^T(\mathcal{K}_\infty)^{\mathrm{tr}}$. These endomorphisms are uniquely determined by their linear duals $\mathrm{Hom}_{\Lambda(\mathcal{G}_\infty)}(\phi, \Lambda(\mathcal{G}_\infty))$ which (Remark 5.3 implies) in turn constitute a distinguished family of resolutions of $\Lambda(\mathcal{G}_\infty)$-module $\mathcal{S}_S^T(\mathcal{K}_\infty)$.

To prove claim (i) it is therefore enough to show that $\mathrm{Fit}^r_{\Lambda(\mathcal{G}_\infty)}(\mathcal{S}_S^T(\mathcal{K}_\infty))$ is unchanged if one replaces ϕ by any other endomorphism constructed via Proposition 8.2.

To do this we write $M(\phi)$ for the matrix of ϕ with respect to the standard basis $\{b_i\}_{i\in[d]}$ of $\Lambda(\mathcal{G}_\infty)^d$. Then, in view of the property (6.1.2) (with $\mathcal{K}$ taken to be $\mathcal{K}_\infty$), the construction of the map (8.1.3) implies that the i-th column of $M(\phi)$ is zero for every i in $[r]$. This means that all non-zero contributions to the ideal $\mathrm{Fit}^r_{\Lambda(\mathcal{G}_\infty)}(\mathcal{S}^T_S(\mathcal{K}_\infty))$ arise from the reduced norms of block matrices of the form $\big(N \,\big|\, M(\phi)^\dagger\big)$, where N is an arbitrary matrix in $\mathrm{M}_{d,r}(\Lambda(\mathcal{G}_\infty))$ and for any matrix M in $\mathrm{M}_d(\Lambda(\mathcal{G}_\infty))$ we write $M^\dagger$ for the matrix in $\mathrm{M}_{d,d-r}(\Lambda(\mathcal{G}_\infty))$ given by the last $d-r$ columns of M.

Now if $\tilde{\phi}$ is any other endomorphism constructed as in Proposition 8.2, then claim (ii) of that result implies $U\cdot M(\tilde{\phi}) = M(\phi)\cdot V$ where U and V are matrices in $\mathrm{GL}_d(\Lambda(\mathcal{G}_\infty))$ and V is a block matrix of the form (6.3.2). In particular, one has $U\cdot M(\tilde{\phi})^\dagger = (M(\phi)\cdot V)^\dagger$ and, for any N in $\mathrm{M}_{d,r}(\Lambda(\mathcal{G}_\infty))$, also

$$\begin{aligned} U\cdot\big(N \,\big|\, M(\tilde{\phi})^\dagger\big) &= \big(U\cdot N \,\big|\, U\cdot M(\tilde{\phi})^\dagger\big)\\ &= \big(U\cdot N \,\big|\, (M(\phi)\cdot V)^\dagger\big)\\ &= \big(U\cdot N \,\big|\, M(\phi)^\dagger\big)\cdot V, \end{aligned}$$

where the last equality follows from the nature of the block matrix V. Claim (i) is then a consequence of the resulting equalities

$$\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}\big(\big(U\cdot N \,\big|\, M(\phi)^\dagger\big)\big) = \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(U)\cdot \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}\big(\big(N \,\big|\, M(\tilde{\phi})^\dagger\big)\big)\cdot \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(V)^{-1}$$

and the fact $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(U)$ and $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(V)$ are both units of $\xi_p(\mathcal{K}_\infty/K) = \xi_p(\mathcal{K}_\infty/K)^\#$, where the last equality follows from Remark 8.6(ii).

In a similar way, Remark 8.6(ii) reduces the proof of claim (ii) to showing that for every N in $\mathrm{M}_{d,r}(\Lambda(\mathcal{G}_\infty))$ one has

$$\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}\big(\big(N \,\big|\, M(\phi)^\dagger\big)\big) \in \lambda(\gamma_E)^{r'-r}\cdot \xi_p(\mathcal{K}_\infty/K).$$

To show this we assume, as we may (under the hypothesis (8.2.1)), that the place v' of $S\setminus\Sigma_S(\mathcal{K}_\infty)$ used in the constructions of Proposition 8.2 does not belong to Σ. We then define a subset $J = J_{S,\Sigma,v'}$ of $[n]$ by setting

$$J := \{j\in[n]\setminus[r] : \text{the } j\text{-th element of } S\setminus\{v'\} \text{ belongs to } \Sigma\setminus\Sigma_S(\mathcal{K}_\infty)\}. \tag{8.2.2}$$

Then one has $|J| = r'-r$ and the nature of the map (8.1.3) implies that for each j in J, and every i in $[d]$ there exist a (unique) element c_{ij} of $\Lambda(\mathcal{G}_\infty)$ with

$$M(\phi)_{ij} = c_{ij}(\gamma_E - 1)$$

and hence

$$\big(N \,\big|\, M(\phi)^\dagger\big) = M'\cdot\Delta_E \tag{8.2.3}$$

where M' is the matrix in $\mathrm{M}_d(\Lambda(\mathcal{G}_\infty))$ defined by

$$M'_{ij} = \begin{cases} c_{ij}, & \text{if } j\in J \text{ and } i\in[d]\\ (N \,\big|\, M(\phi)^\dagger)_{ij}, & \text{if } j\in[d]\setminus J \text{ and } i\in[d], \end{cases} \tag{8.2.4}$$

and Δ_E is the diagonal matrix in $\mathrm{M}_d(\Lambda(\mathcal{G}_\infty))$ with

$$\Delta_{E,ij} := \begin{cases} \gamma_E - 1, & \text{if } i = j \in J \\ 1, & \text{if } i = j \in [d] \setminus J \\ 0, & \text{if } i \neq j. \end{cases}$$

The required result is therefore true since $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(\Delta_E) = \lambda(\gamma_E)^{r'-r}$, whilst (7.1.1) implies $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(M')$ belongs to $\xi_p(\mathcal{K}_\infty/K)$. □

8.2.2. With the result of Lemma 8.8 in mind, we now introduce a restriction on the structure of the $\Lambda(\mathcal{G}_\infty)$-module $\mathcal{S}_S^T(\mathcal{K}_\infty)$ that will play an important role in the sequel.

For a character χ in $\mathrm{Ir}_p(\mathcal{G}_E)$ we define a prime ideal of $\xi_p(\mathcal{K}_\infty/K)$ by setting

$$\wp_\chi(\mathcal{K}_\infty/K) := \ker\big(\xi_p(\mathcal{K}_\infty/K) \to \zeta(\mathbb{Q}_p^c[\mathcal{G}_E]) \xrightarrow{x \mapsto x_\chi} \mathbb{Q}_p^c\big), \tag{8.2.5}$$

where the unlabelled arrow is the natural projection.

Definition 8.9. Fix χ in $\mathrm{Ir}_p(\mathcal{G}_E)$. Then the data $\mathcal{K}_\infty/K, E$ and S is said to be 'semisimple at χ' if $\mathrm{Fit}^r_{\Lambda(\mathcal{G}_\infty)}(\mathcal{S}_S^T(\mathcal{K}_\infty))$ is not contained in $I_E(\mathcal{G}_\infty)^{r'-r} \cdot \wp_\chi(\mathcal{K}_\infty/K)$.

Before proceeding, we explain the motivation for our use of the word 'semisimple' in this context. In particular, we note that the stated property of the module Q in the following result implies that, after localizing at $\wp_\chi(\mathcal{K}_\infty/K)$, it is 'semisimple at zero' in the sense relevant to Iwasawa-theoretic descent computations (cf. [16]).

Lemma 8.10. *Fix χ in $\mathrm{Ir}_p(\mathcal{G}_E)$. Then, if the data $\mathcal{K}_\infty/K, E$ and S is semisimple at χ, there exists an exact sequence of $\Lambda(\mathcal{G}_\infty)$-modules*

$$\Lambda(\mathcal{G}_\infty)^r \to \mathcal{S}_S^T(\mathcal{K}_\infty) \to Q \to 0 \tag{8.2.6}$$

in which Q has the following property: the natural map

$$Q^{\gamma_E = 1} \oplus (\gamma_E - 1)Q \to Q$$

is bijective after localizing at $\wp_\chi(\mathcal{K}_\infty/K)$.

Proof. Set $\wp := \wp_\chi(\mathcal{K}_\infty/K)$. Then, under the stated hypothesis, the argument of Lemma 8.8 implies, via the product decomposition (8.2.3), that there exists a matrix N in $\mathrm{M}_{d,r}(\Lambda(\mathcal{G}_\infty))$ such that

$$\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(M') \in \xi_p(\mathcal{K}_\infty/K) \setminus \wp^\#, \tag{8.2.7}$$

where the matrix M' is defined in terms of N as in (8.2.4).

For each matrix M in $\mathrm{M}_d(\Lambda(\mathcal{G}_\infty))$ we set $M^* := M^{\mathrm{tr},\#}$. We also identify the matrices $M(\phi)^*$, $M(\phi, N) := \big(N \,\big|\, M(\phi)^\dagger\big)^*$, $(M')^*$ and Δ_E^* with endomorphisms of $\Lambda(\mathcal{G}_\infty)^d$ in the obvious way. Then the cokernel of $M(\phi)^*$ is isomorphic to $\mathcal{S}_S^T(\mathcal{K}_\infty)$ and so there exists an exact sequence (8.2.6) in which Q is the cokernel of $M(\phi, N)$. In addition, from the decomposition $M(\phi, N) = \Delta_E^* \cdot (M')^*$, one deduces that this choice of Q lies in an exact sequence of $\Lambda(\mathcal{G}_\infty)$-modules $Q' \to Q \to \mathrm{cok}\big(\Delta_E^*\big) \to 0$, with $Q' := \mathrm{cok}\big((M')^*\big)$.

Since $\mathrm{cok}\big(\Delta_E^*\big)$ is isomorphic to $\Lambda(\mathcal{G}_E)^{r'-r}$, to deduce that Q has the claimed property it is therefore enough to show that $Q'_\wp$ vanishes. Now (8.2.7) implies that the reduced norm $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}((M')^*) = \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}((M'))^\#$ does not belong to $\wp$, and this implies $\gamma_E - 1$ acts invertibly on $Q'_\wp$. Thus, since $\gamma_E - 1$ belongs to $\wp$, Nakayama's Lemma implies that $Q'_\wp$ vanishes, as required. □

8.2.3. In the sequel, for any subfield $\mathcal{K}$ of $\mathbb{Q}^c$ we set

$$A_S(\mathcal{K}) := \varprojlim_F \mathrm{Cl}_S(F)_p \quad \text{and} \quad A_S^T(\mathcal{K}) := \varprojlim_F \mathrm{Cl}_S^T(F)_p$$

where in both limits F runs over all finite extensions of $\mathbb{Q}$ in $\mathcal{K}$ and the transition morphisms for $F \subset F'$ are the natural norm maps.

In the next result we describe the link between semisimplicity in the sense of Definition 8.9 and the structural properties of modules of the form $A_S(\mathcal{K})$.

In condition (ii) of this result we use the idempotent $e_{E/K,S,\Sigma}$ defined in (6.2.5) (and we recall that, under the hypotheses of this section, one has $\Sigma \neq S$).

Proposition 8.11. *The data $\mathcal{K}_\infty/K, S$ and E is semisimple at every character χ in $\mathrm{Ir}_p(\mathcal{G}_E)$ that satisfies all of the following conditions:*

(i) *The space $e_\chi\big(\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} A_S(\mathcal{K}_\infty)_{\mathcal{H}_\infty}\big)$ vanishes.*
(ii) $e_\chi \cdot e_{E/K,S,\Sigma} \neq 0$.

Proof. The essential idea of this argument is that, under the stated hypotheses, one can 'reverse' the direction of the argument in Lemma 8.10.

To make this precise, we write $\xi'_p(\mathcal{K}_\infty/K)$ for the subring of $\zeta(Q(\mathcal{G}_\infty))$ generated over $\zeta(\Lambda(\mathcal{G}_\infty))$ by the set $\{\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(M) : M \in \bigcup_{m>0} \mathrm{M}_m(\Lambda(\mathcal{G}_\infty))\}$. We also fix an open normal subgroup $\mathcal{Z}$ of $\mathcal{H}_\infty$ that is central in $\mathcal{G}_\infty$ and contained in the decomposition subgroup in $\mathcal{G}_\infty$ of $w_{j,\infty}$ for each j belonging to the set J defined in (8.2.2) (so $v_j \in \Sigma \setminus \Sigma_S(\mathcal{K}_\infty)$). We then write E' for finite extension of E that is obtained as the fixed field of $\mathcal{Z}$ in $\mathcal{K}_\infty$.

We note $\xi'_p(\mathcal{K}_\infty/K)$ is a $\Lambda(\mathcal{Z})$-order in $\zeta(Q(\mathcal{G}_\infty))$ (cf. the proof of Lemma 7.1) and also that the explicit description of reduced norm given in (7.1.1) implies that the projection map $\zeta(\Lambda(\mathcal{G}_\infty)) \to \zeta(\mathbb{Z}_p[\mathcal{G}_E])$ extends to a well-defined ring homomorphism ϱ_E from $\xi'_p(\mathcal{K}_\infty/K)$ to $\zeta(\mathbb{Q}_p^c[\mathcal{G}_E])$. For any fixed character χ in $\mathrm{Ir}_p(\mathcal{G}_E)$ we can therefore define a prime ideal of $\xi'_p(\mathcal{K}_\infty/K)$ by setting

$$\wp' = \wp'_\chi := \ker\big(\xi'_p(\mathcal{K}_\infty/K) \xrightarrow{\varrho_E} \zeta(\mathbb{Q}_p^c[\mathcal{G}_E]) \xrightarrow{x \mapsto x_\chi} \mathbb{Q}_p^c\big).$$

We write $\wp$ for the prime ideal $\wp' \cap \zeta(\Lambda(\mathcal{G}_\infty))$ of $\zeta(\Lambda(\mathcal{G}_\infty))$ and $M_\wp$ for any $\Lambda(\mathcal{G}_\infty)$-module M for the $\Lambda(\mathcal{G}_\infty)$-module obtained by localizing M (as a $\zeta(\Lambda(\mathcal{G}_\infty))$-module) at $\wp$.

We now fix an endomorphism ϕ as constructed in Proposition 8.2 with respect to a place v' chosen in $S \setminus \Sigma$, and set

$$\mathcal{E}_E(\mathcal{K}_\infty) := \Lambda(\mathcal{G}_\infty)^r \oplus \mathbb{Z}_p[\mathcal{G}_E]^{r'-r}.$$

We then claim that it suffices to prove that the given hypotheses on χ imply the existence of a commutative diagram of $\Lambda(\mathcal{G}_\infty)_\wp$-modules of the form

$$\begin{array}{ccccccc}
\Lambda(\mathcal{G}_\infty)_\wp^d & \xrightarrow{\phi_\wp} & \Lambda(\mathcal{G}_\infty)_\wp^d & \xrightarrow{\pi_{\infty,\wp}} & \mathcal{S}_S^T(\mathcal{K}_\infty)_\wp^{\mathrm{tr}} & \longrightarrow & 0 \\
\nu\big\downarrow & & \big\| & & \nu'\big\downarrow & & \\
\Lambda(\mathcal{G}_\infty)_\wp^d & \xrightarrow{\varpi} & \Lambda(\mathcal{G}_\infty)_\wp^d & \xrightarrow{\varpi'} & \mathcal{E}_E(\mathcal{K}_\infty)_\wp & \longrightarrow & 0,
\end{array} \tag{8.2.8}$$

in which ν is bijective and the rows are obtained by respectively localizing the upper row of (8.1.5) and the standard resolution of $\mathcal{E}_E(\mathcal{K}_\infty)$. In particular, for each i in $[d]$ one has

$$\varpi(b_i) = \begin{cases} 0 \\ (\gamma_E - 1)(b_i) \\ b_i \end{cases} \quad \text{and} \quad \varpi'(b_i) = \begin{cases} b_i, & \text{if } i \in [r], \\ b_{E,a}, & \text{if } i \text{ is the } a\text{-th place in } J, \\ 0, & \text{otherwise.} \end{cases}$$

We assume for the moment such a diagram exists. We write U for the matrix of ν with respect to the standard basis of $\Lambda(\mathcal{G}_\infty)^d_\wp$ and N for the matrix in $\mathrm{M}_{d,r}(\Lambda(\mathcal{G}_\infty)_\wp)$ for which $U^{-1} \cdot N$ is the transpose of the block matrix $\big(I_r \mid 0\big)$.

Then the argument of Lemma 8.8 shows the commutativity of the first square in (8.2.8) implies $U^{-1} \cdot \big(N \mid M(\phi)^\dagger\big)$ is the $d \times d$ diagonal matrix Δ_E with ii-th entry equal to $\gamma_E - 1$ if $i \in J$ and equal to 1 otherwise. Thus, if we choose an element x of $\zeta(\Lambda(\mathcal{G}_\infty)) \setminus \wp$ for which xN belongs to $\mathrm{M}_{d,r}(\Lambda(\mathcal{G}_\infty))$, then one has

$$\begin{aligned} &\lambda(\gamma_E)^{r-r'} \cdot \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}\big(\big(xN \mid M(\phi)^\dagger\big)\big) \\ =\, &\lambda(\gamma_E)^{r-r'} \cdot \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(x)^r \cdot \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(\Delta_E) \cdot \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(U) \\ =\, &\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(x)^r \cdot \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(U) \end{aligned}$$

This implies the claimed semisimplicity at χ since $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}\big(\big(xN \mid M(\phi)^\dagger\big)\big)$ belongs to $\mathrm{Fit}^r_{\Lambda(\mathcal{G}_\infty)}(\mathcal{S}^T_S(\mathcal{K}_\infty))$ whilst neither $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(x)$ nor $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(U)$ belongs to $\wp'_\chi$.

It therefore suffices to prove the given assumptions imply the existence of a diagram of the form (8.2.8). To do this we note that the second square of (8.2.8) clearly commutes if we take ν' to be the composite homomorphism of $\Lambda(\mathcal{G}_\infty)_\wp$-modules

$$\nu' : \mathcal{S}^T_S(\mathcal{K}_\infty)^{\mathrm{tr}}_\wp \xrightarrow{\beta_\infty} (Y_{\infty,S'})_\wp \to \bigoplus\nolimits_{v\in\Sigma} Y_{\infty,v,\wp} \to \mathcal{E}_E(\mathcal{K}_\infty)_\wp,$$

where S' denotes $S \setminus \{v'\}$, the second map is the natural projection and the third is induced by sending $w_{i,\infty}$ for $i \in [r]$, respectively $i \in J$, to b_i, respectively to $b_{E,i-r}$.

The key claim we make now is that ν' is bijective. To show this we take the limit over L in $\Omega_E(\mathcal{K}_\infty)$ of the exact sequence (5.1.2) to obtain an exact sequence of $\Lambda(\mathcal{G}_\infty)$-modules

$$0 \to A^T_S(\mathcal{K}_\infty) \to \mathcal{S}^T_S(\mathcal{K}_\infty)^{\mathrm{tr}} \to \bigoplus\nolimits_{v\in S} Y_{\infty,v} \to \mathbb{Z}_p \to 0.$$

This sequence implies that ν' is bijective if all of the following conditions are satisfied: $A^T_S(\mathcal{K}_\infty)_\wp$ vanishes; $Y_{\infty,v,\wp}$ vanishes for each $v \in S' \setminus \Sigma$; for every $j \in J$ the $\wp$-localization of the morphism $Y_{\infty,v_j} \to \mathbb{Z}_p[\mathcal{G}_E]$ sending $w_{j,\infty}$ to $b_{E,j-r}$ is bijective; if $Y_{\infty,v',\wp}$ does not vanish, then the $\wp$-localization of the natural projection map $Y_{\infty,v'} \to \mathbb{Z}_p$ is bijective.

To verify these conditions we write $E(v)$ for each v in $S \setminus \Sigma_S(\mathcal{K}_\infty)$ for the maximal extension of K in $\mathcal{K}_\infty$ in which $w_{v,\infty}$ splits completely. We also write $E'(S)$ for the finite extension of K in $\mathcal{K}_\infty$ that is obtained as the compositum of E' and the fields $E(v)$ and set $G' := \mathcal{G}_{E'(S)}$.

We then note that [68, Th. 3.5] implies $p^m \cdot \xi'_p(\mathcal{K}_\infty/K)$ is contained in $\zeta(\Lambda(\mathcal{G}_\infty))$ for any large enough integer m. This fact implies the existence of an element t_χ of $\zeta(\Lambda(\mathcal{G}_\infty))$ whose projection to $\zeta(\mathbb{Z}_p[G'])$ is a p-power multiple of the idempotent $e_{(\chi)}$ of $\zeta(\mathbb{Q}_p[G'])$ that corresponds to the irreducible $\mathbb{Q}_p$-valued character of G' that contains χ as a component. It follows that $t_\chi \in \zeta(\Lambda(\mathcal{G}_\infty)) \setminus \wp$ and hence that for each place $v \in S \setminus \Sigma_S(\mathcal{K}_\infty)$ one has

$$Y_{\infty,v,\wp} = t_\chi\big(Y_{\infty,v,\wp}\big) = t_\chi\big(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} Y_{\infty,v}\big)_\wp = t_\chi\big(\mathbb{Q}_p \cdot Y_{E'(S),\{v\}}\big)_\wp = e_{(\chi)}(\mathbb{Q}_p \cdot Y_{E,\{v\}})_\wp. \tag{8.2.9}$$

In particular, since for each v in Σ, the map

$$e_{(\chi)}(\mathbb{Q}_p \cdot Y_{E,\{v\}})_\wp \xrightarrow{w_v \mapsto 1} t_\chi(\mathbb{Z}_p[\mathcal{G}_E]_\wp) = \mathbb{Z}_p[\mathcal{G}_E]_\wp$$

is bijective, we deduce that for every $j \in J$ the $\wp$-localization of the morphism $Y_{\infty,v_j} \to \mathbb{Z}_p[\mathcal{G}_E]$ sending $w_{j,\infty}$ to $b_{E,j-r}$ is bijective, as required.

By combining (8.2.9) with the result of Lemma 6.10 and the assumed validity of condition (ii) we also derive the following consequences: the localisation $Y_{\infty,v,\wp}$ vanishes for all $v \in S \setminus \Sigma$; if $Y_{\infty,v',\wp}$ does not vanish, then χ is trivial, $S \setminus \Sigma = \{v'\}$ and the natural map $Y_{\infty,v',\wp} = e_{(\chi)}(\mathbb{Q}_p \cdot Y_{E,\{v'\}})_\wp \to (\mathbb{Z}_p)_\wp$ is bijective.

To complete the proof that ν' is bijective, it now suffices to show $A_S^T(\mathcal{K}_\infty)_\wp$ vanishes. To do this we fix a topological generator $\gamma_{E'}$ of $\mathcal{Z}$ and note $\Lambda(\mathcal{G}_\infty)_\wp$ is an order over the discrete valuation ring $\Lambda(\mathcal{Z})_\mathfrak{p}$ obtained by localizing $\Lambda(\mathcal{Z})$ at the prime ideal $\mathfrak{p}$ generated by $\gamma_{E'} - 1$. Nakayama's Lemma therefore implies that $A_S^T(\mathcal{K}_\infty)_\wp$ vanishes if the space

$$t_\chi\big(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A_S^T(\mathcal{K}_\infty)_\mathcal{Z}\big) = e_{(\chi)}\big(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A_S^T(\mathcal{K}_\infty)_\mathcal{Z}\big) = e_{(\chi)}\big(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A_S^T(\mathcal{K}_\infty)_{\mathcal{H}_\infty}\big)$$

vanishes, and this follows directly from condition (i).

At this stage we have established that the right hand square in (8.2.8) commutes when ν' is the isomorphism specified above. From this it follows that $\mathrm{im}(\phi_\wp) = \ker(\pi_{\infty,\wp})$ is equal to the free $\Lambda(\mathcal{G}_\infty)_\wp$-submodule of $\Lambda(\mathcal{G}_\infty)_\wp^d$ that has basis

$$\{(\gamma_E - 1)(b_i)\}_{i \in J} \cup \{b_i\}_{i \in [d] \setminus J^\dagger},$$

where we set

$$J^\dagger := J \cup [r]. \tag{8.2.10}$$

We may therefore choose a section σ to $\phi_\wp$ and thereby obtain an isomorphism of $\Lambda(\mathcal{G}_\infty)_\wp$-modules

$$\Lambda(\mathcal{G}_\infty)_\wp^d = \ker(\phi_\wp) \oplus \sigma(\mathrm{im}(\phi_\wp)) \cong \ker(\phi_\wp) \oplus \Lambda(\mathcal{G}_\infty)_\wp^{d-r}.$$

Upon applying the Krull-Schmidt Theorem for the $\Lambda(\mathcal{Z})_\mathfrak{p}$-order $\Lambda(\mathcal{G}_\infty)_\wp$ to this isomorphism, we deduce that the $\Lambda(\mathcal{G}_\infty)_\wp$-module $\ker(\phi_\wp)$ is free of rank r, and hence isomorphic to the kernel $\bigoplus_{i=1}^{i=r} \Lambda(\mathcal{G}_\infty)_\wp \cdot b_i$ of ϖ.

In particular, if we fix an isomorphism of $\Lambda(\mathcal{G}_\infty)_\wp$-modules $\nu_1 : \ker(\phi_\wp) \cong \ker(\varpi)$, and write $\nu_2 : \sigma(\mathrm{im}(\phi_\wp)) \to \Lambda(\mathcal{G}_\infty)_\wp^d$ for the map of $\Lambda(\mathcal{G}_\infty)_\wp$-modules that sends the element

$$\begin{cases} \sigma((\gamma_E - 1)(b_i)), & \text{if } i \in J, \\ \sigma(b_i), & \text{if } i \in [d] \setminus J^\dagger \end{cases}$$

to b_i, then the homomorphism $\nu = (\nu_1, \nu_2)$ is an automorphism of $\Lambda(\mathcal{G}_\infty)_\wp^d$ that makes the first square in (8.2.8) commute, as required to complete the proof. □

Remark 8.12. For a number field E, write Γ_E for the Galois group over E of its cyclotomic $\mathbb{Z}_p$-extension $E^{\rm cyc}$. Then it is conjectured by Jaulent in [47] that, for every E, the Γ_E-coinvariants $A_S(E^{\rm cyc})_{\Gamma_E}$ of $A_S(E^{\rm cyc})$ should be finite. In addition, if E is a CM Galois extension of a totally real field K, then an observation of Kolster in [56, Th. 1.14] (where the result is attributed to Kuz'min [57]) implies that the finiteness of $A_S(E^{\rm cyc})^-_{\Gamma_E}$ is equivalent to the earlier conjecture [41, Conj. 1.15] of Gross and hence also, by [16, Th. 5.2(ii)], to the validity of Gross's 'Order of Vanishing Conjecture' [41, Conj. 2.12a)] for all totally odd characters of $\mathcal{G}_E$. In particular, if K contains at most one p-adic place that splits completely in E/E^+, then $A_S(E^{\rm cyc})^-_{\Gamma_E}$ is finite as a consequence of [41, Prop. 2.13] (which itself relies Brumer's p-adic version of Baker's theorem). In general, if $\mathcal{E}$ is any $\mathbb{Z}_p$-extension of a number field E, then $A_S(\mathcal{E})_{{\rm Gal}(\mathcal{E}/E)}$ is known to be finite in each of the following cases.

(i) E is abelian over $\mathbb{Q}$ (cf. Greenberg [39]).
(ii) E is an abelian extension of an imaginary quadratic field and $\mathcal{E} = E^{\rm cyc}$ (cf. Maksoud [62]).
(iii) $\mathcal{E} = E^{\rm cyc}$ and E has at most two p-adic places (cf. Kleine [53]).
(iv) $\mathcal{E}$ is totally real and the Leopoldt conjecture is valid for E at p (cf. Kolster [56, Cor. 1.3]).

9. A conjectural derivative formula for Rubin-Stark Euler systems

In this section we define a notion of 'the value of a higher derivative' of the Rubin-Stark non-commutative Euler system (from Definition 6.13) and formulate an explicit conjectural formula for such values.

We also show that this conjectural derivative formula specializes to recover the classical Gross-Stark Conjecture (from [41, Conj. 2.12b)]) and hence deduce its validity in an important family of examples.

Throughout the section we fix a rank one compact p-adic Lie extension $\mathcal{K}_\infty$ of K in $\mathcal{K}$ that is ramified at only finitely many places.

We fix sets of places S and T of K as specified at the beginning of §8.1, and use the abbreviations

$$\mathcal{G} := \mathcal{G}_\infty = G_{\mathcal{K}_\infty/K}, \;\; R_\infty := \Lambda(\mathcal{G}) \quad \text{and} \quad R_L := \mathbb{Z}_p[\mathcal{G}_L]$$

for each L in $\Omega(\mathcal{K}_\infty)$.

We also fix a normal subgroup $\mathcal{H}$ of $\mathcal{G}$ that is topologically isomorphic to $\mathbb{Z}_p$ and write E for the fixed field of $\mathcal{H}$ in $\mathcal{K}_\infty$. We then fix a subset Σ of $\Sigma_S(E)$ that satisfies the condition (8.2.1) and set

$$r := |\Sigma_S(\mathcal{K}_\infty)| \quad \text{and} \quad r' := |\Sigma|$$

(so that $r \le r' < |S|$ since $\Sigma_S(\mathcal{K}_\infty) \subseteq \Sigma \subsetneq S$).

9.1. Derivatives of Rubin-Stark non-commutative Euler systems.

For a natural number t and non-negative integer a we set

$$\bigcap\nolimits^a_{R_\infty} R^t_\infty := \varprojlim_{L'} \bigcap\nolimits^a_{R_{L'}} R^t_{L'}$$

and

$$\bigcap\nolimits^a_{\mathbb{C}_p\cdot R_\infty}(\mathbb{C}_p\cdot R_\infty)^t := \varprojlim_{L'}\big(\mathbb{C}_p\otimes_{\mathbb{Z}_p}\bigcap\nolimits^a_{R_{L'}}R^t_{L'}\big),$$

where in both limits L' runs over $\Omega(\mathcal{K}_\infty)$ and the transition morphisms are induced by the natural projection maps $R^t_{L'}\to R^t_L$ for $L\subseteq L'$. For each L in $\Omega(\mathcal{K}_\infty)$ we also use the natural projection map

$$\pi^a_L:\bigcap\nolimits^a_{\mathbb{C}_p\cdot R_\infty}(\mathbb{C}_p\cdot R_\infty)^t\to\mathbb{C}_p\otimes_{\mathbb{Z}_p}\bigcap\nolimits^a_{R_L}R^t_L.$$

We use the element $\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T}$ of $\bigwedge^r_{\mathbb{C}_p\cdot\Lambda(\mathcal{G}_\infty)}(\mathbb{C}_p\cdot\mathcal{O}^\times_{\mathcal{K}_\infty,S,T})$ defined in (7.2.4).

Proposition 9.1. *For each topological generator γ of $\mathcal{H}$, there exists an element*

$$\partial^{r'-r}_\gamma(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T})\in\mathbb{C}_p\cdot\bigcap\nolimits^r_{R_E}\mathcal{O}^\times_{E,S,T,p}$$

that depends only on the data $\mathcal{K}_\infty/K,\gamma,S$ and T and has the following property: for every pair of embeddings $\iota_\infty:\mathcal{O}^\times_{\mathcal{K}_\infty,S,T}\to R^d_\infty$ and $\iota_E:\mathcal{O}^\times_{E,S,T,p}\to R^d_E$ constructed as in Proposition 8.2(iii), there exists an element $y(\gamma)$ of $\bigcap^r_{\mathbb{C}_p\cdot R_\infty}(\mathbb{C}_p\cdot R_\infty)^d$ such that both

$$\iota_{\infty,*}(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T})=\mathrm{Nrd}_{Q(\mathcal{G})}(\gamma-1)^{r'-r}\cdot y(\gamma)$$

and

$$\iota_{E,*}\big(\partial^{r'-r}_\gamma(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T})\big)=\pi^r_E(y(\gamma)).$$

Proof. We fix a resolution $R^d_\infty\xrightarrow{\phi}R^d_\infty$ of $C_{\mathcal{K}_\infty,S,T}$ as constructed in Proposition 8.2 with respect to a place v' chosen in $S\setminus\Sigma$, and hence also an associated exact sequence as in the upper row of (8.1.5). This leads to fixed embeddings ι_∞ and ι_E of the stated form. In addition, for each L in $\Omega_E(\mathcal{K}_\infty)$ we obtain an induced resolution $R^d_L\xrightarrow{\phi_L}R^d_L$ of $C_{L,S,T,p}$.

We note, in particular, that, for each j belonging to the subset J of $[d]$ defined in (8.2.2), the composite map $b^*_{E,j}\circ\phi_E$ is zero and so there exists a (unique) homomorphism

$$\widehat{\phi}_j=(\widehat{\phi}_{j,F})_{F\in\Omega_E(\mathcal{K}_\infty)}\in\mathrm{Hom}_{R_\infty}(R^d_\infty,R_\infty)$$

with

$$b^*_j\circ\phi=(\gamma-1)(\widehat{\phi}_j). \tag{9.1.1}$$

We claim that this implies an equality

$$\begin{aligned}\big(\wedge_{j\in J}(b^*_{F,j}\circ\phi_F)\big)_F&=\big(\wedge_{j\in J}(\gamma-1)(\widehat{\phi}_{j,F})\big)_F\\&=\lambda(\gamma)^{r'-r}\cdot\big(\wedge_{j\in J}\widehat{\phi}_{j,F}\big)_F,\end{aligned} \tag{9.1.2}$$

where in each case F runs over $\Omega_E(\mathcal{K}_\infty)$. Here the first equality follows directly from the defining relations (9.1.1) and, after taking account of (7.1.1) (with N taken to be the 1×1 matrix $(\gamma-1)$), the second equality can be verified by showing that, for each F in $\Omega_E(\mathcal{K}_\infty)$, one has

$$\wedge_{j\in J}\theta_{j,F}=\mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_F]}(\gamma_{(F)}-1)^{r'-r}\cdot\big(\wedge_{j\in J}\theta'_{j,F}\big),$$

where $\gamma_{(F)}$ is the image of γ in $\mathcal{G}_F$ and we set $\theta'_{j,F}:=\widehat{\phi}_{j,F}$ and $\theta_{j,F}:=(\gamma_{(F)}-1)(\theta'_{j,F})$. It is in turn enough to verify this last displayed equality after applying the projection functor $\zeta(A)\otimes_{\zeta(\mathbb{Q}_p[\mathcal{G}_F])}-$ for each simple Wedderburn component A of $\mathbb{Q}_p[\mathcal{G}_F]$. Further, if we write

$\theta_{j,A}$ and $\theta'_{j,A}$ for the corresponding projections of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \theta_{j,F}$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \theta'_{j,F}$, then the explicit definition (2.1.1) of reduced exterior products implies that the elements $\wedge_{j\in J}\theta_{j,A}$ and $\wedge_{j\in J}\theta'_{j,A}$ vanish unless the A-module W that is generated by $\{\theta'_{j,A}\}_{j\in J}$ is free of rank $r'-r$. Then, in the latter case, the required equality follows by applying the general result of [24, Lem. 4.13] with φ taken to be the element of $\mathrm{End}_A(W)$ that sends each basis element $\theta'_{j,A}$ of W to $\theta_{j,A} = (\gamma_{(F)} - 1)(\theta'_{j,A})$.

For each L in $\Omega_E(\mathcal{K}_\infty)$ we consider the element

$$x_L := (\wedge_{j=r+1}^{j=d}(b^*_{L,j} \circ \phi_L))(\wedge_{j\in[d]} b_{L,j}) \in {\bigcap}^r_{R_L} R_L^d.$$

For each integer j in $[d] \setminus [r]$ we also set

$$\theta_{L,j} := \begin{cases} \widehat{\phi}_{j,L}, & \text{if } j \in J \\ b^*_{L,j} \circ \phi_L, & \text{otherwise.} \end{cases}$$

Then the family $x := (x_L)_L$ belongs to ${\bigcap}^r_{R_\infty} R^d_\infty$ and, setting $\lambda(\gamma) = \mathrm{Nrd}_{Q(\mathcal{G})}(\gamma - 1)$, the relations (9.1.2) (as F runs over $\Omega_E(\mathcal{K}_\infty)$) imply that

$$x = \lambda(\gamma)^{r'-r} \cdot x' \quad \text{with} \quad x' := \big((\wedge_{j=r+1}^{j=d}\theta_{L,j})(\wedge_{j\in[d]} b_{L,j})\big)_L \in {\bigcap}^r_{R_\infty} R^d_\infty. \tag{9.1.3}$$

This equality determines the element x' uniquely since multiplication by $\lambda(\gamma)$ on ${\bigcap}^r_{R_\infty} R^d_\infty$ is injective (see Remark 9.2 below).

For L in $\Omega(\mathcal{K}_\infty)$ we write $\iota_{L,*}$ for the injective homomorphism ${\bigcap}^{r'}_{R_L} \mathcal{O}^\times_{L,S,T,p} \to {\bigcap}^{r'}_{R_L} R^d_L$ that is induced by our fixed resolution of $C_{L,S,T,p}$. Then the result of Proposition 6.16(ii) implies the existence of a unique element z_L of $\mathbb{C}_p[\mathcal{G}_L]e_L$ with $\iota_{L,*}(\varepsilon^\Sigma_{L/K,S,T,p}) = z_L \cdot x_L$. In addition, the element $z := (z_L)_L$ belongs to $\varprojlim_L \mathbb{C}_p[\mathcal{G}_L]$ and one has

$$\iota_{\infty,*}(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T}) = (\iota_{L,*}(\varepsilon^\Sigma_{L/K,S,T,p}))_L = z \cdot x.$$

The equality (9.1.3) therefore implies that

$$\iota_{\infty,*}(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T}) = \lambda(\gamma)^{r'-r} \cdot y(\gamma) \quad \text{with} \quad y(\gamma) := z \cdot x' \in {\bigcap}^r_{\mathbb{C}_p\cdot R_\infty} (\mathbb{C}_p \cdot R_\infty)^d. \tag{9.1.4}$$

In addition, since $v_i \in \Sigma$ for all i in the set $J^\dagger = J \cup [r]$ introduced in (8.2.10), Proposition 6.16(i) implies that the element

$$w_E := (\wedge_{j\in[d]\setminus J^\dagger}(b^*_{E,j} \circ \phi_E))(\wedge_{j\in[d]} b_{E,j})$$

belongs to $\iota_{E,*}\big({\bigcap}^{r'}_{R_E} \mathcal{O}^\times_{E,S,T,p}\big)$. This implies that the element

$$\begin{aligned} \pi^r_E(y(\gamma)) &= z_E \cdot x'_E \\ &= \mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_E]}(-1)^{t_E} \cdot z_E \cdot (\wedge_{j\in J}\widehat{\phi}_{j,E})(w_E) \end{aligned}$$

belongs to $\iota_{E,*}\big(\mathbb{C}_p \otimes_{\mathbb{Z}_p} {\bigcap}^r_{R_E} \mathcal{O}^\times_{E,S,T,p}\big)$, where, following [24, Lem. 4.13], the integer t_E is fixed so that

$$\wedge_{j\in[d]\setminus[r]}\theta_{L,j} = \mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_E]}((-1)^{t_E}) \cdot (\wedge_{j\in[d]\setminus J^\dagger}(b^*_{E,j} \circ \phi_E)) \wedge (\wedge_{j\in J}\widehat{\phi}_{j,E}).$$

It is therefore enough to show that the unique element $\partial_\gamma^{r'-r}(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T})$ of $\mathbb{C}_p\otimes_{\mathbb{Z}_p}\bigcap^r_{R_E}\mathcal{O}^\times_{E,S,T,p}$ that satisfies

$$\iota_{E,*}(\partial_\gamma^{r'-r}(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T})) = \pi^r_E(y(\gamma)) \tag{9.1.5}$$

is independent of the resolution of $C_{\mathcal{K}_\infty,S,T}$ fixed above. To check this we can assume to be given a commutative diagram of the form (8.1.5) and we write $\tilde\iota_E$, $\widehat{\tilde\phi}_{j,E}$, $\tilde z$, $\tilde w_E$ and $\tilde y(\gamma)$ for the corresponding data that arises when making the above constructions with respect to the resolution given by the lower (rather than upper) row of this diagram. Then the commutativity of this diagram combines with the argument of Proposition 6.16(iv) to imply the existence of an element $\mu = (\mu_L)_L$ of $\xi(R_\infty)^\times$ such that both $\tilde z = \mu^{-1}\cdot z$ and

$$(\tilde\iota_{E,*})^{-1}\big((\wedge_{j\in J}\widehat{\tilde\phi}_{j,E})(\tilde w_E)\big) = (\iota_{E,*})^{-1}\big(\mu_E\cdot(\wedge_{j\in J}\widehat{\phi}_{j,E})(w_E)\big)$$

and hence also

$$\begin{aligned}(\tilde\iota_{E,*})^{-1}(\tilde y(\gamma)) &= (\tilde\iota_{E,*})^{-1}(\tilde z_E\cdot(\wedge_{j\in J}\widehat{\tilde\phi}_{j,E})(\tilde w_E))\\ &= (\iota_{E,*})^{-1}(z_E\cdot(\wedge_{j\in J}\widehat{\phi}_{j,E})(w_E)) = (\iota_{E,*})^{-1}(y(\gamma)),\end{aligned}$$

as required. □

Remark 9.2. To show that multiplication by $\lambda(\gamma)$ is injective on $\bigcap^r_{R_\infty}R^d_\infty$ it suffices to take $z=(z_L)_L\in\bigcap^r_{R_\infty}R^d_\infty$ with $\lambda(\gamma)\cdot z=0$ and show that this implies $(\wedge_{j\in[r]}\theta_j)(z_F)=0$ for each F in $\Omega_E(\mathcal{K}_\infty)$ and each subset $\{\theta_j\}_{j\in[r]}$ of $\mathrm{Hom}_{R_F}(R^d_F,R_F)$. However, if one fixes a pre-image $(\theta_{j,L})_L$ of each θ_j under the surjection $\mathrm{Hom}_{R_\infty}(R^d_\infty,R_\infty)\to\mathrm{Hom}_{R_F}(R^d_F,R_F)$, then one has

$$\lambda(\gamma)\cdot((\wedge_{j\in[r]}\theta_{j,L})(z_L))_L = ((\wedge_{j\in[r]}\theta_{j,L})_L)\big(\lambda(\gamma)\cdot z\big)=0$$

and so, since $((\wedge_{j\in[r]}\theta_{j,L})(z_L))_L$ belongs to $\xi(R_\infty)$, the argument in Proposition 6.16(i) implies that $((\wedge_{j\in[r]}\theta_{j,L})(z_L))_L=0$ and hence also $(\wedge_{j\in[r]}\theta_j)(z_F)=0$, as required.

Motivated by the result of Proposition 9.1, we now make the following key definition.

Definition 9.3. The '$(r'-r)$-th order derivative at γ' of the Rubin-Stark element $\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T}$ is the element

$$\partial_\gamma^{r'-r}(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T})$$

of $\mathbb{C}_p\cdot\bigcap^r_{\mathbb{Z}_p[G]}\mathcal{O}^\times_{E,S,T,p}$.

9.2. **$\mathscr{L}$-invariant maps and the Generalized Gross-Stark Conjecture.** Our aim is to formulate an explicit conjectural formula for the derivative element $\partial_\gamma^{r'-r}(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T})$ introduced above.

For this purpose we shall need an appropriate generalization of the notion of '$\mathscr{L}$-invariant' that occurs in the classical Gross-Stark Conjecture and we next prove a technical result that plays a key role in the construction of such a generalization.

9.2.1. We abbreviate $\mathcal{G}_E$ to G and fix an (ordered) set of places

$$\Sigma' \subseteq \Sigma \setminus S_\infty^K$$

of K. Then, since each place v in Σ' is non-archimedean, we fix a place w_v of E above v and write ϕ_v^{ord} for the map in $\mathrm{Hom}_{\mathbb{Z}_p[G]}(\mathcal{O}_{E,S,p}^\times, \mathbb{Z}_p[G])$ that satisfies

$$\phi_v^{\mathrm{ord}}(u) := \sum\nolimits_{g\in G} \mathrm{ord}_{w_v}(g^{-1}(u)) \cdot g \tag{9.2.1}$$

for each u in $\mathcal{O}_{E,S,p}^\times$, where ord_{w_v} is the normalized additive valuation at w_v.

Lemma 9.4. *Write e for the idempotent $e_{E/K,S,\Sigma}$ defined in (6.2.5). Then the reduced exterior product $\wedge_{v\in\Sigma'}\phi_v^{\mathrm{ord}}$ induces an isomorphism of $\zeta(\mathbb{Q}_p[G])$-modules*

$$\phi_{E,S,\Sigma'}^{\mathrm{ord}} : e\big(\mathbb{Q}_p \cdot {\bigcap}_{\mathbb{Z}_p[G]}^{|\Sigma|} \mathcal{O}_{E,S,p}^\times\big) \xrightarrow{\sim} e\big(\mathbb{Q}_p \cdot {\bigcap}_{\mathbb{Z}_p[G]}^{|\Sigma\setminus\Sigma'|} \mathcal{O}_{E,S\setminus\Sigma',p}^\times\big).$$

Proof. Set $U := \mathcal{O}_{E,S,p}^\times$, $U' := \mathcal{O}_{E,S\setminus\Sigma',p}^\times$, $r' := |\Sigma|$ and $r^* := |\Sigma \setminus \Sigma'| \le r'$.

Then, since each place in Σ' splits completely in E, the definition of e implies that the $\mathbb{Q}_p[G]e$-modules $e(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} U) \cong e(\mathbb{Q}_p \otimes_{\mathbb{Z}} Y_{E,S})$ and $e(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} U') \cong e(\mathbb{Q}_p \otimes_{\mathbb{Z}} Y_{E,S\setminus\Sigma'})$ are respectively free of ranks r' and r^* and so [24, Th. 4.19(v)] implies that the $\zeta(\mathbb{Q}_p[G])e$-modules $e\big(\mathbb{Q}_p \cdot \bigcap_{\mathbb{Z}_p[G]}^{r'} U\big)$ and $e\big(\mathbb{Q}_p \cdot \bigcap_{\mathbb{Z}_p[G]}^{r^*} U'\big)$ are each free of rank one.

We now fix a representative of $C_{E,S,T,p}$ as in the upper row of (6.3.4) (with $L = E$ and $\Pi = S$) and use the subsets J and $J^\dagger$ of $[d]$ defined in (8.2.2) and (8.2.10). Then Proposition 6.16(ii) implies that the element

$$\varepsilon := (\wedge_{j\in[d]\setminus J^\dagger}(b_{E,j}^* \circ \phi))(\wedge_{i\in[d]} b_{E,i})$$

generates $e\big(\mathbb{Q}_p \cdot \bigcap_{\mathbb{Z}_p[G]}^{r'} U\big)$ over $\zeta(\mathbb{Q}_p[G])$ and so it is enough to prove that $\phi_{E,S,\Sigma'}^{\mathrm{ord}}(\varepsilon)$ is a generator of the $\zeta(\mathbb{Q}_p[G])$-modules $e\big(\mathbb{Q}_p \cdot \bigcap_{\mathbb{Z}_p[G]}^{r^*} U'\big)$.

To do this we fix an isomorphism in $\mathsf{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ between $C_{E,S,T,p}$ and the complex $P^\bullet$ given by

$$\mathbb{Z}_p[G]^d \xrightarrow{\phi} \mathbb{Z}_p[G]^d,$$

where the first term is placed in degree zero and the cohomology groups are identified with those of $C_{E,S,T,p}$ by the maps in the upper row of (6.3.4). We also write $P_{\Sigma'}^\bullet$ for the complex

$$\mathbb{Z}_p[G]^{|\Sigma'|} \xrightarrow{0} \mathbb{Z}_p[G]^{|\Sigma'|},$$

where the first term is placed in degree zero, and we choose a morphism of complexes of $\mathbb{Z}_p[G]$-modules $\alpha : P^\bullet \to P_{\Sigma'}^\bullet$ that represents the composite morphism in $\mathsf{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$

$$P^\bullet \cong C_{E,S,T,p} \to \mathbb{Z}_p \otimes_{\mathbb{Z}} \big(\bigoplus\nolimits_{w\in\Sigma'_E} \mathsf{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}\Gamma((\kappa_w)_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})\big)[-1] \cong P_{\Sigma'}^\bullet,$$

where the first map is the fixed isomorphism, the second is induced by the exact triangle in Lemma 5.1(ii) and the third is the canonical isomorphism induced by Remark 5.2 (and the fact that each place in Σ' splits completely in E).

We write $J_{\Sigma'}$ for the subset of $J^\dagger$ comprising indices j for which the j-th place of $S\setminus\{v'\}$ belongs to Σ'. Then there is a short exact sequence of complexes of $\mathbb{Z}_p[G]$-modules (with horizontal differentials and the first term in the upper complex placed in degree one)

$$\begin{array}{ccccc} & & \mathbb{Z}_p[G]^{|\Sigma'|} & \xrightarrow{\mathrm{id}} & \mathbb{Z}_p[G]^{|\Sigma'|} \\ & & \big\downarrow{\scriptstyle(-\mathrm{id},\iota_1)} & & \big\downarrow{\scriptstyle\mathrm{id}} \\ \mathbb{Z}_p[G]^d & \xrightarrow{(\alpha^0,\phi)} & \mathbb{Z}_p[G]^{|\Sigma'|}\oplus\mathbb{Z}_p[G]^d & \xrightarrow{(0,\pi_1)} & \mathbb{Z}_p[G]^{|\Sigma'|} \\ {\scriptstyle\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle(\iota_1,\mathrm{id})} & & \\ \mathbb{Z}_p[G]^d & \xrightarrow{\phi_\alpha} & \mathbb{Z}_p[G]^d. & & \end{array} \tag{9.2.2}$$

Here ι_1 is the inclusion that sends the i-th element in the standard basis of $\mathbb{Z}_p[G]^{|\Sigma'|}$ to the $k(i)$-th element in the standard basis of $\mathbb{Z}_p[G]^d$ where $k(i)\in J_{\Sigma'}$ is such that $v_{k(i)}$ is the i-th place in Σ', and π_1 denotes the corresponding projection. In addition, the endomorphism ϕ_α is such that for each j in $[d]$ one has

$$b^*_{E,j}\circ\phi_\alpha = \begin{cases} b^*_{E,i}\circ\alpha^0, & \text{if } j=k(i)\in J_{\Sigma'} \text{ for } i\in[|\Sigma'|],\\ 0, & \text{if } J^\dagger\setminus J_{\Sigma'},\\ b^*_{E,j}\circ\phi, & \text{if } j\in[d]\setminus J^\dagger.\end{cases}$$

In particular, since $H^0(\alpha)$ coincides with the composite

$$\mathcal{O}^\times_{E,S,T,p} \xrightarrow{\epsilon\mapsto\sum_{w\in\Sigma'_E}\mathrm{ord}_w(\epsilon)\cdot w} Y_{E,\Sigma',p}\cong\mathbb{Z}_p[G]^{|\Sigma'|} \tag{9.2.3}$$

where the isomorphism sends the ordered basis $\{w_v\}_{v\in\Sigma'}$ to the standard basis of $\mathbb{Z}_p[G]^{|\Sigma'|}$, for every $j=k(i)\in J_{\Sigma'}$ one has $b^*_{E,i}\circ H^0(\alpha)=\phi^{\mathrm{ord}}_{v_j}$. Hence, for a suitable integer a, there are equalities

$$\begin{aligned}\phi^{\mathrm{ord}}_{E,S,\Sigma'}(\varepsilon) &= (\wedge_{v\in\Sigma'}\phi^{\mathrm{ord}}_v)(\varepsilon)\\ &= (\wedge_{j\in J_{\Sigma'}}(b^*_{E,j}\circ\phi_\alpha))(\varepsilon)\\ &= (\wedge_{j\in J_{\Sigma'}}(b^*_{E,j}\circ\phi_\alpha))((\wedge_{j\in[d]\setminus J^\dagger}(b^*_{E,j}\circ\phi))(\wedge_{i\in[d]}b_{E,i}))\\ &= \mathrm{Nrd}_{\mathbb{Q}_p[G]}((-1)^a)\cdot(\wedge_{j\in J_{\Sigma'}\cup([d]\setminus J^\dagger)}(b^*_{E,j}\circ\phi_\alpha))(\wedge_{i\in[d]}b_{E,i})\end{aligned}$$

Since [24, Prop. 4.18(i)] implies the latter element is a generator of $e\big(\mathbb{Q}_p\cdot\bigcap^{r^*}_{\mathbb{Z}_p[G]}\ker(\phi_\alpha)\big)$ over $\zeta(\mathbb{Q}_p[G])$, it is therefore enough to show $\ker(\phi_\alpha)$ is isomorphic to U'. This is in turn true since the first complex in the short exact sequence (9.2.2) is acyclic and the second is equal to $\mathrm{Cone}(\alpha)[-1]$ and so the exact triangle in Lemma 5.1(ii) induces an isomorphism in $\mathsf{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ between $C_{E,S\setminus\Sigma',T,p}$ and the third complex in (9.2.2) and therefore also an isomorphism of $\mathbb{Z}_p[G]$-modules between $U'=H^0(C_{E,S\setminus\Sigma',T,p})$ and $\ker(\phi_\alpha)$. □

9.2.2. We set

$$R_\infty := \Lambda(\mathcal{G}) \quad\text{and}\quad \mathcal{I}_E := R_\infty\cdot(\gamma-1).$$

We note that $\mathcal{I}_E$ is a (two-sided) ideal of R_∞, that the assignment $x \mapsto x(\gamma-1)$ induces an isomorphism $R_\infty \cong \mathcal{I}_E$ of (left) R_∞-modules and that there is a natural exact sequence of (left) R_∞-modules

$$0 \to \mathcal{I}_E \xrightarrow{\subseteq} R_\infty \to R_E \to 0,$$

where the third arrow is the natural projection. This exact sequence combines with (the limit over L in $\Omega_E(\mathcal{K}_\infty)$) of the isomorphism in Lemma 5.1(iv) to give a canonical exact triangle in $\mathsf{D}(R_\infty)$

$$\mathcal{I}_E \otimes^{\mathsf{L}}_{R_\infty} C_{\mathcal{K}_\infty,S,T} \to C_{\mathcal{K}_\infty,S,T} \xrightarrow{\theta} C_{E,S,T} \xrightarrow{\theta'} (\mathcal{I}_E \otimes^{\mathsf{L}}_{R_\infty} C_{\mathcal{K}_\infty,S,T})[1] \tag{9.2.4}$$

and thereby also a composite Bockstein homomorphism

$$\begin{aligned} H^0(C_{E,S,T}) \xrightarrow{H^0(\theta')} & H^1(\mathcal{I}_E \otimes^{\mathsf{L}}_{R_\infty} C_{\mathcal{K}_\infty,S,T}) \\ \cong & \mathcal{I}_E \otimes_{R_\infty} H^1(C_{\mathcal{K}_\infty,S,T}) \\ \xrightarrow{\mathrm{id}\otimes H^1(\theta)} & \mathcal{I}_E \otimes_{R_\infty} H^1(C_{E,S,T}) \\ \cong & \left(\mathcal{I}_E/\mathcal{I}_E^2\right) \otimes_{R_E} H^1(C_{E,S,T}) \\ \cong & H^1(C_{E,S,T}). \end{aligned} \tag{9.2.5}$$

Here the first isomorphism is induced by the fact $C_{\mathcal{K}_\infty,S,T}$ is acyclic in degrees greater than one, the second by the fact $\mathcal{I}_E$ acts trivially on $H^1(C_{E,S,T})$ and the third by the isomorphism $R_E \cong \mathcal{I}_E/\mathcal{I}_E^2$ that sends 1 to the class of $\gamma-1$. (For an alternative, and more direct, description of the composite (9.2.5) as a Bockstein homomorphism see the proof of Lemma 9.10 below.)

For each v in Σ the latter map induces a composite homomorphism of $\mathbb{Z}_p[G]$-modules

$$\phi^{\mathrm{Bock}}_{\gamma,v} : \mathcal{O}^\times_{E,S,T,p} = H^0(C_{E,S,T}) \to H^1(C_{E,S,T}) = \mathrm{Sel}^T_S(E)^{\mathrm{tr}}_p \xrightarrow{\varrho_v} \mathbb{Z}_p[G].$$

Here the equalities come from Lemma 5.1(ii) and ϱ_v denotes the composite of the canonical projection $\mathrm{Sel}^T_S(E)^{\mathrm{tr}}_p \to \mathbb{Z}_p[G]\cdot w_v$ induced by (5.1.2) and the homomorphism of $\mathbb{Z}_p[G]$-modules $\mathbb{Z}_p[G]\cdot w_v \to \mathbb{Z}_p[G]$ that sends w_v to 1 (and is well-defined since v splits completely in E).

For any finite (ordered) subset Σ' of Σ the reduced exterior product of the maps $\phi^{\mathrm{Bock}}_{\gamma,v}$ over v in Σ' induces a homomorphism of $\zeta(\mathbb{Q}_p[G])$-modules

$$\phi^{\mathrm{Bock}}_{\gamma,S,\Sigma} : e_{E/K,S,\Sigma}\big(\mathbb{Q}_p \cdot {\bigcap}^{|\Sigma|}_{\mathbb{Z}_p[G]} \mathcal{O}^\times_{E,S,p}\big) \to e_{E/K,S,\Sigma}\big(\mathbb{Q}_p \cdot {\bigcap}^{|\Sigma\setminus\Sigma'|}_{\mathbb{Z}_p[G]} \mathcal{O}^\times_{E,S,p}\big)$$

where $e_{E/K,S,\Sigma}$ is the idempotent of $\zeta(\mathbb{Q}[G])$ defined in (6.2.5).

Taking advantage of Lemma 9.4, we can now define a canonical '$\mathscr{L}$-invariant map'.

Definition 9.5. The $\mathscr{L}$-invariant map associated to γ, S and a subset Σ' of $\Sigma \setminus S_K^\infty$ is the homomorphism of $\zeta(\mathbb{Q}_p[G])$-modules

$$\mathscr{L}^{\Sigma'}_{\gamma,S} : e_{E/K,S,\Sigma}\big(\mathbb{Q}_p \cdot {\bigcap}^{|\Sigma\setminus\Sigma'|}_{\mathbb{Z}_p[G]} \mathcal{O}^\times_{E,S\setminus\Sigma',p}\big) \to e_{E/K,S,\Sigma}\big(\mathbb{Q}_p \cdot {\bigcap}^{|\Sigma\setminus\Sigma'|}_{\mathbb{Z}_p[G]} \mathcal{O}^\times_{E,S,p}\big)$$

that is induced by the composite $\phi^{\mathrm{Bock}}_{\gamma,S,\Sigma'} \circ (\phi^{\mathrm{ord}}_{E,S,\Sigma'})^{-1}$.

The following result explains the significance of semisimplicity (in the sense of Definition 8.9) in this setting.

Lemma 9.6. *Assume that the set $\Sigma' := \Sigma\setminus\Sigma_S(\mathcal{K}_\infty)$ contains no archimedean places. Then, for any character χ in $\mathrm{Ir}_p(\mathcal{G}_E)$ at which $\mathcal{K}_\infty/K, E$ and S is semisimple, the χ-component of the map $\mathscr{L}^{\Sigma'}_{\gamma,S}$ is injective.*

Proof. We use the notation of the proof of Lemma 8.8. In particular, one has $r = |\Sigma_S(\mathcal{K}_\infty)|$, $\Sigma_S(\mathcal{K}_\infty) = \{v_i : i \in [r]\}$, $r' = |\Sigma|$, $\Sigma' = \{v_j : j \in J\}$ and $J^\dagger := J \cup [r]$ (so $|\Sigma'| = r' - r$ and $|J^\dagger| = r'$). We have also fixed a resolution $C(\phi)$ of $C_{\mathcal{K}_\infty,S,T}$ as in Remark 8.3 and, for each $L \in \Omega(\mathcal{K}_\infty)$, we write $C(\phi_L)$ for the induced resolution of $C_{L,S,T,p}$. We abbreviate the notation γ_E from loc. cit. to γ.

Then, as Proposition 6.16(ii) implies that the element ε_E of $\bigcap^{r'}_{\mathbb{Z}_p[G]}\mathcal{O}^\times_{E,S,p}$ constructed by the argument of Proposition 6.16 (with respect to the complex $C(\phi_E)$) generates the $\zeta(\mathbb{Q}_p[G])$-module $e_{E/K,S,\Sigma}(\mathbb{Q}_p\cdot\bigcap^{r'}_{\mathbb{Z}_p[G]}\mathcal{O}^\times_{E,S,p})$, it is enough for us to show the given semisimplicity hypothesis on χ implies that

$$e_\chi\big(\phi^{\mathrm{Bock}}_{\gamma,S,\Sigma'}(\varepsilon_E)\big) \neq 0. \tag{9.2.6}$$

The key point in showing this is that, under the given hypothesis, the argument of Lemma 8.8 implies the existence of a matrix N in $\mathrm{M}_{d,r}(R_\infty)$ for which one has

$$\mathrm{Nrd}_{Q(\mathcal{G})}\big(\big(N \mid M(\phi)^\dagger\big)\big) = \lambda(\gamma)^{r'-r}\cdot\eta \quad\text{with}\quad \eta\in\xi_p(\mathcal{K}_\infty/K)\setminus\wp_\chi(\mathcal{K}_\infty/K). \tag{9.2.7}$$

To interpret this equality, we write $\theta_i = (\theta_{i,F})_F$ for each index $i \in [r]$ for the element of $\mathrm{Hom}_{R_\infty}(R^d_\infty, R_\infty)$ that corresponds to the i-th column of N and, for each $F \in \Omega_E(\mathcal{K}_\infty)$, we use the element

$$x_F := (\wedge_{i\in[d]\setminus J^\dagger}(b^*_{F,i}\circ\phi_F))(\wedge_{j\in[d]}b_{F,j}) \in \bigcap^{r'}_{\mathbb{Z}_p[\mathcal{G}_F]}\mathbb{Z}_p[\mathcal{G}_F]^d.$$

In particular, we note that (7.1.1) combines with [24, Lem. 4.10] to imply that, for a suitable integer t, one has

$$\begin{aligned}&\big(\mathrm{Nrd}_{Q(\mathcal{G})}\big(\big(N \mid M(\phi)^\dagger\big)\big)\\ &=\mathrm{Nrd}_{Q(\mathcal{G})}((-1)^t)\cdot\big((\wedge_{i\in[r]}\theta_{i,F})((\wedge_{j\in J}(b^*_{F,j}\circ\phi_F))(x_F))\big)_{F\in\Omega_E(\mathcal{K}_\infty)}\\ &=\lambda(\gamma)^{r'-r}\cdot\mathrm{Nrd}_{Q(\mathcal{G})}((-1)^t)\cdot\big((\wedge_{i\in[r]}\theta_{i,F})((\wedge_{j\in J}\widehat{\phi}_{j,F})(x_F))\big)_{F\in\Omega_E(\mathcal{K}_\infty)}\end{aligned}$$

where the second equality follows from (9.1.2). Upon comparing the latter equality with (9.2.7), we deduce that

$$\begin{aligned}\big((\wedge_{i\in[r]}\theta_{i,F})((\wedge_{j\in J}\widehat{\phi}_{j,F})(x_F))\big)_{F\in\Omega_E(\mathcal{K}_\infty)} &= \mathrm{Nrd}_{Q(\mathcal{G})}((-1)^t)\cdot\eta\\ &\in \xi_p(\mathcal{K}_\infty/K)\setminus\wp_\chi(\mathcal{K}_\infty/K).\end{aligned}$$

In view of the explicit definition (8.2.5) of the prime ideal $\wp_\chi(\mathcal{K}_\infty/K)$, it follows that

$$e_\chi\big((\wedge_{i\in[r]}\theta_{i,E}))((\wedge_{j\in J}\widehat{\phi}_{j,E})(x_E))\big) \neq 0.$$

To derive (9.2.6) from here, it is clearly enough to prove that $(\wedge_{j\in J}\widehat{\phi}_{j,E})(x_E)$ is equal to $\phi^{\rm Bock}_{\gamma,S,\Sigma'}(\varepsilon_E)$. To show this, we recall from Proposition 6.16(i) that

$$(\wedge_{j\in J}\widehat{\phi}_{j,E})(x_E) = (\wedge_{j\in J}(\widehat{\phi}_{j,E}\circ\iota_{E,*}))(\varepsilon_E),$$

where $\iota_{E,*}$ the injective map $\mathcal{O}^\times_{E,S,T,p}\to\mathbb{Z}_p[G]^d$ that is induced by the given resolution $C(\phi_E)$ of $C_{E,S,T,p}$. Given the explicit definition of $\phi^{\rm Bock}_{\gamma,S,\Sigma'}$, it is therefore enough for us to show that, for each $j\in J$ (so that $v_j\in\Sigma'$), one has

$$\widehat{\phi}_{j,E}\circ\iota_{E,*} = \phi^{\rm Bock}_{\gamma,v_j}. \tag{9.2.8}$$

This identity can then be verified by an explicit, and straightforward, computation that combines the defining equality $\phi_j = (\gamma^{-1}-1)(\widehat{\phi}_j)$ of the homomorphism $\widehat{\phi}_j$ with the fact $\phi^{\rm Bock}_{\gamma,v_j}$ can be computed as the composite $\varpi\circ\theta$, with θ the connecting homomorphism $H^0(C_{E,S,T,p}) = \ker(\phi_E)\to\operatorname{cok}(\phi)$ associated to the short exact sequence of complexes (with vertical differentials)

$$\begin{array}{ccccc}
(R_\infty(\gamma-1))^d & \overset{\subset}{\hookrightarrow} & R^d_\infty & \twoheadrightarrow & \mathbb{Z}_p[G]^d\\
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi_E}\\
(R_\infty(\gamma-1))^d & \overset{\subset}{\hookrightarrow} & R^d_\infty & \twoheadrightarrow & \mathbb{Z}_p[G]^d
\end{array}$$

and ϖ the composite map

$$\operatorname{cok}(\phi)\to\operatorname{cok}(\phi_E) = \operatorname{Sel}^T_S(E)^{\rm tr}_p \xrightarrow{\varrho_{v_j}} \mathbb{Z}_p[G]$$

in which the first arrow is the natural projection map. □

9.2.3. We can now use the $\mathscr{L}$-invariant map from Definition 9.5 to formulate an explicit conjectural formula for the $(r'-r)$-th order derivative at γ of $\varepsilon^{\rm RS}_{\mathcal{K}_\infty,S,T}$, as specified in Definition 9.3.

Conjecture 9.7. (Generalized Gross-Stark Conjecture) *Fix a subset Σ of $\Sigma_S(E)$ as in (8.2.1) and set $r := |\Sigma_S(\mathcal{K}_\infty)|$, $r' := |\Sigma|$ and $\Sigma' := \Sigma\setminus\Sigma_S(\mathcal{K}_\infty)$ (so that $r\le r'<|S|$ and $|\Sigma'| = r'-r$). Then, if no place in Σ' is archimedean, there is an equality*

$$\partial^{r'-r}_\gamma(\varepsilon^{\rm RS}_{\mathcal{K}_\infty,S,T}) = \mathscr{L}^{\Sigma'}_{\gamma,S}(\varepsilon^{\Sigma_S(\mathcal{K}_\infty)}_{E/K,S\setminus\Sigma',T})$$

in $\mathbb{C}_p\cdot\bigcap^r_{\mathbb{Z}_p[\mathcal{G}_E]}\mathcal{O}^\times_{E,S,p}$.

In the sequel we write κ_K for the homomorphism $\chi_K\cdot\omega_K^{-1}: G_K\to\mathbb{Z}_p^\times$.

Remark 9.8. Fix a in $\mathbb{Z}_p^\times$. Then the first displayed equality in Lemma 9.1 implies $y(\gamma) = x_a^{r'-r}\cdot y(\gamma^a)$ with $x_a := \mathrm{Nrd}_{Q(\mathcal{G})}\big((\gamma^a-1)/(\gamma-1)\big)$, and hence also

$$\pi^{r'}_E\big(y(\gamma)\big) = \pi^{r'}_E\big(x_a^{r'-r}\cdot y(\gamma)\big) = \pi^0_E(x_a)^{r-r'}\cdot\pi^{r'}_E\big(y(\gamma^a)\big).$$

By a straightforward computation, one can also show that $\mathscr{L}^{\Sigma'}_{\gamma,S} = \pi^0_E(x_a)^{r-r'}\cdot\mathscr{L}^{\Sigma'}_{\gamma^a,S}$ and hence that the validity of Conjecture 9.7 is independent of the choice of γ. In addition, one can explicitly compute $\pi^0_E(x_a)$ as follows. Fix a topological generator γ_0 of Γ_K and write

n for the element of $\mathbb{Z}_p$ for which the projection of γ to Γ_K is equal to γ_0^n. Then, since γ acts trivially on V_χ for each χ in $\mathrm{Ir}_p(G)$, an explicit computation of reduced norm (as in the proof of Corollary 7.9) shows that

$$j_\chi(\mathrm{Nrd}_{Q(\mathcal{G})}(\gamma^a-1)) = \Phi_{\mathcal{G},\chi}(\gamma^a-1) = ((1+t)^{na}-1)^{\chi(1)} \tag{9.2.9}$$

and hence that

$$\begin{aligned}\pi_E^0(x_a)_\chi &= j_\chi(x_a)(0)\\ &= \left(\frac{(1+t)^{an}-1}{(1+t)^n-1}\right)^{\chi(1)}(0)\\ &= a^{\chi(1)}\\ &= \left(\frac{\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\log_p(\kappa_K(\gamma^a)))}{\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\log_p(\kappa_K(\gamma)))}\right)_\chi.\end{aligned}$$

This equality implies, in particular, that each side of the equality in Conjecture 9.7 becomes independent of γ after multiplying by the normalization factor $\mathrm{Nrd}_{\mathbb{Q}_p[G]}\big(\log_p(\kappa_K(\gamma))^{r'-r}$.

Remark 9.9. The argument of Theorem 9.11 below shows Conjecture 9.7 recovers (in the relevant special case) the classical Gross-Stark Conjecture. For this reason, we refer to the derivative formula in Conjecture 9.7 as the 'Generalized Gross-Stark Conjecture' for the data $\mathcal{K}_\infty/K, E, S$ and T.

9.3. **Cyclotomic $\mathbb{Z}_p$-extensions.** In this section we investigate Conjecture 9.7 in the special case $\mathcal{K}_\infty = E^{\mathrm{cyc}}$.

9.3.1. In this case, for each v in $\Sigma\setminus S_K^\infty$, the map $\phi_{\gamma,v}^{\mathrm{Bock}}$ has a more explicit description. To give this description we recall that in [41, §1] Gross defines for each place w of E a local p-adic absolute value by means of the composite

$$||\cdot||_{w,p} : E_w^\times \xrightarrow{r_w} G_{E_w^{\mathrm{ab}}/E_w} \xrightarrow{\chi_w} \mathbb{Z}_p^\times \xrightarrow{x\mapsto x^{-1}} \mathbb{Z}_p^\times,$$

where E_w^{ab} is the maximal abelian extension of E_w in E_w^c, r_w is the local reciprocity map and $\chi_w = \chi_{E_w}$ is the cyclotomic character. We write ϕ_v^{Gross} for the homomorphism of $\mathbb{Z}_p[G]$-modules $\mathcal{O}_{E,S,T,p}^\times \to \mathbb{Z}_p[G]$ that sends each u to the element

$$\phi_v^{\mathrm{Gross}}(u) = \sum\nolimits_{g\in G}\log_p||g^{-1}(u)||_{w_v,p}\cdot g.$$

Lemma 9.10. *For each v in $\Sigma\setminus S_K^\infty$ one has*

$$\phi_{\gamma,v}^{\mathrm{Bock}} = \log_p(\kappa_K(\gamma))^{-1}\cdot\phi_v^{\mathrm{Gross}}.$$

Proof. We set $C_\infty := C_{\mathcal{K}_\infty,S,T}$ and $C := C_{E,S,T,p}$. We also fix a topological generator γ_K of Γ_K, write n for the element of $\mathbb{Z}_p$ such that the projection of γ to Γ_K is equal to γ_K^n and set $T_n := \sum_{i=0}^{i=n-1}\gamma_K^i \in \Lambda(\Gamma_K)$.

We consider the morphism of exact triangles in $\mathsf{D}(R_\infty)$

$$\begin{array}{ccccccc}
\mathcal{I}_E \otimes^{\mathsf{L}}_{R_\infty} C_\infty & \to & C_\infty & \to & C & \to & (\mathcal{I}_E \otimes^{\mathsf{L}}_{R_\infty} C_\infty)[1] \\
{\scriptstyle\theta_1}\downarrow & & {\scriptstyle\theta_2}\downarrow & & \| & & {\scriptstyle\theta_1[1]}\downarrow \\
\Lambda(\Gamma_K \times G) \otimes^{\mathsf{L}}_{R_\infty} C_\infty & \xrightarrow{\gamma_K - 1} & \Lambda(\Gamma_K \times G) \otimes^{\mathsf{L}}_{R_\infty} C_\infty & \to & C & \to & (\Lambda(\Gamma_K \times G) \otimes^{\mathsf{L}}_{R_\infty} C_\infty)[1].
\end{array}$$

In this diagram the upper triangle is (9.2.4); in the lower triangle each term $\Lambda(\Gamma_K \times G)$ is regarded as an R_∞-bimodule via the natural diagonal injection $\mathcal{G} \to \Gamma_K \times G$ and $\gamma_K - 1$ acts via left multiplication on the first factor in the tensor product and so the existence of the triangle is a consequence of the descent isomorphism $\mathbb{Z}_p[G] \otimes^{\mathsf{L}}_{R_\infty} C_\infty \cong C$; the morphism θ_1 sends each element $x(\gamma - 1) \otimes c^i$ to $(T_n \cdot \overline{x}) \otimes c^i$, where $\overline{x}$ is the image of x under the natural projection $R_\infty \to \Lambda(\Gamma_K) \subseteq \Lambda(\Gamma_K \times G)$, and θ_2 sends each element c^i to $1 \otimes c^i$.

We abbreviate the map (9.2.5) to β and write β' for the composite homomorphism

$$H^0(C) \to H^1(\Lambda(\Gamma_K \times G) \otimes^{\mathsf{L}}_{R_\infty} C_\infty) \to H^1(C),$$

where the arrows are the maps induced by the lower triangle in the above diagram. Then, since the image of T_n under the projection $\Lambda(\Gamma_K) \subseteq \Lambda(\Gamma_K \times G) \to \mathbb{Z}_p[G]$ is equal to n, the commutativity of the above diagram implies (without assuming $\mathcal{K}_\infty = E^{\rm cyc}$) that $\beta' = n \cdot \beta$.

We note next that, setting $c := \log_p(\kappa_K(\gamma_K)) = n^{-1} \cdot \log_p(\kappa_K(\gamma))$, the argument of [16, Th. 5.7(i)] proves $\phi_v^{\rm Gross} = c \cdot (\varrho_v \circ \beta')$. (The difference in sign between this formula and that in loc. cit. is accounted for by the fact that the proof of [16, Th. 5.7(i)] uses the shifted complexes $C_\infty[1]$ and $C[1]$ in place of C_∞ and C and the differentials of $C_\infty[1]$ and C_∞ differ by a sign.) This equality then implies that

$$\begin{aligned}
\phi_v^{\rm Gross} &= c \cdot (\varrho_v \circ \beta') \\
&= c \cdot (\varrho_v \circ (n \cdot \beta)) \\
&= (n \cdot c) \cdot (\varrho_v \circ \beta) \\
&= \log_p(\kappa_K(\gamma)) \cdot \phi^{\rm Bock}_{\gamma,v},
\end{aligned}$$

as required. □

9.3.2. In this section we explain the connection between Conjecture 9.7 and the classical Gross-Stark Conjecture and are thereby able to deduce its validity in an important family of examples.

To do this we further specialize to the case that K is totally real, E is CM and $\mathcal{K}_\infty = E^{\rm cyc}$. We then write τ for the (unique) non-trivial element of $\mathrm{Gal}(E^{\rm cyc}/(E^{\rm cyc})^+)$ and $e_\pm$ for the idempotent $(1 \pm \tau)/2$ of $\Lambda(\mathcal{G})$. We identify the elements τ and $e_\pm$ with their respective images in G and $\mathbb{Z}_p[G]$.

We write $\mathrm{Ir}_p^\pm(G)$ for the subsets of $\mathrm{Ir}_p(G)$ comprising characters for which $\chi(\tau) = \pm\chi(1)$. For any $\Lambda(\mathcal{G})$-module M we write $M^\pm$ for the $\Lambda(\mathcal{G})$-submodule $\{m \in M : \tau(m) = \pm m\}$. For an element m of M we also often abbreviate $e_\pm(m)$ to $m^\pm$ and use a similar convention for homomorphisms.

We consider the homomorphism of G-modules

$$\lambda^{\rm ord}_{E,S} : \mathcal{O}^{\times,-}_{E,S} \to Y^-_{E,S}$$

that sends each u to $\sum_w \mathrm{ord}_w(u)\cdot w$, where in the sum w runs over all places of E above those in $S\setminus S^K_\infty$, and also write

$$\lambda^{\mathrm{Gross}}_{E,S,p}:\mathcal{O}^{\times,-}_{E,S,p}\to Y^-_{E,S,p} \tag{9.3.1}$$

for the map of $\mathbb{Z}_p[G]$-modules that sends each u in $\mathcal{O}^{\times,-}_{E,S}$ to $\sum_{w\in S_E}\log_p||u||_{w,p}\cdot w$.

Then, since the scalar extension $\mathbb{Q}_p\otimes_{\mathbb{Z}}\lambda^{\mathrm{ord}}_{E,S}$ is bijective, for each χ in $\mathrm{Ir}^-_p(\mathcal{G}_E)$ we can define a $\mathbb{C}_p$-valued '$\mathscr{L}$-invariant' by setting

$$\mathscr{L}_S(\chi):=\det_{\mathbb{C}_p}((\mathbb{C}_p\otimes_{\mathbb{Z}_p}\lambda^{\mathrm{Gross}}_{E,S,p})\circ(\mathbb{C}_p\otimes_{\mathbb{Z}}\lambda^{\mathrm{ord}}_{E,S})^{-1}\mid \mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\chi}},\mathbb{C}_p\cdot Y^-_{E,S,p})).$$

In claim (ii) of the following result we refer to Gross's 'Order of Vanishing Conjecture' for p-adic Artin L-series, as discussed in Remark 8.12 (and originally formulated by Gross in [42, Conj. 2.12a)]).

Theorem 9.11. *Let E be a finite Galois CM extension of a totally real field K, set $G:=\mathcal{G}_E$ and $\mathcal{K}_\infty:=E^{\mathrm{cyc}}$ and fix a topological generator γ of $\mathrm{Gal}(E^{\mathrm{cyc}}/E)$. Then no archimedean place of K splits in E and so $\Sigma_S(E)\neq S$, $\Sigma_S(\mathcal{K}_\infty)=\emptyset$ and $|\Sigma_S(E)\setminus\Sigma_S(\mathcal{K}_\infty)|$ is equal to $r':=|\Sigma_S(E)|$. In addition, the following claims are valid.*

(i) *For every χ in $\mathrm{Ir}^-_p(\mathcal{G}_\infty)$ one has*

$$\partial^{r'}_\gamma(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T})_\chi=\log_p(\kappa_K(\gamma))^{-r'\chi(1)}\cdot L^{r'\chi(1)}_{p,S}(\check{\chi}\cdot\omega_K,0).$$

(ii) *For every χ in $\mathrm{Ir}^-_p(\mathcal{G}_\infty)$ one has*

$$\mathscr{L}^{\Sigma_S(E)}_{\gamma,S}(\varepsilon^{\Sigma_S(\mathcal{K}_\infty)}_{E/K,S\setminus\Sigma_S(E),T})_\chi=\log_p(\kappa_K(\gamma))^{-r'\chi(1)}\cdot\mathscr{L}_S(\chi)\cdot L_{S\setminus\Sigma_S(E)}(\check{\chi},0).$$

(iii) *The minus component of Conjecture 9.7 is valid if Gross's Order of Vanishing Conjecture is valid for every χ in $\mathrm{Ir}^-_p(G)$.*

Proof. The assertions concerning the sets $\Sigma_S(E)$ and $\Sigma_S(\mathcal{K}_\infty)$ are clear. In particular, in this case the integer $r:=r_{S,\mathcal{K}_\infty}$ is equal to 0 and no place in $\Sigma':=\Sigma_S(E)\setminus\Sigma_S(\mathcal{K}_\infty)=\Sigma_S(E)$ is archimedean.

Remark 6.14(ii) therefore implies that both

$$(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T})^-=\theta^-_{\mathcal{K}_\infty,S,T}\quad\text{and}\quad\varepsilon^{\Sigma_S(\mathcal{K}_\infty)}_{E/K,S\setminus\Sigma',T}=\theta_{E/K,S\setminus\Sigma_S(E),T}(0). \tag{9.3.2}$$

In particular, since $r=0$ and $\theta^-_{\mathcal{K}_\infty,S,T}\in\zeta(Q(\mathcal{G}))$ (see the proof of Corollary 7.9), the argument of Proposition 9.1 implies

$$\mathrm{Nrd}_{Q(\mathcal{G})}(\gamma-1)^{-r'}\cdot\theta^-_{\mathcal{K}_\infty,S,T}\in\zeta(Q(\mathcal{G}))\cap\varprojlim_L\zeta(\mathbb{Q}_p[\mathcal{G}_L])$$

and that

$$\partial^{r'}_\gamma(\varepsilon^{\mathrm{RS}}_{\mathcal{K}_\infty,S,T})^-=\pi_E(\mathrm{Nrd}_{Q(\mathcal{G})}(\gamma-1)^{-r'}\cdot\theta^-_{\mathcal{K}_\infty,S,T}).$$

To prove claim (i) it is thus enough to show that, for each χ in $\mathrm{Ir}^-_p(G)$ one has

$$\pi_E(\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(\gamma-1)^{-r'}\cdot\theta^-_{\mathcal{K}_\infty,S,T})_\chi=c(\gamma)^{-r'\chi(1)}\cdot L^{r'\chi(1)}_{p,S,T}(\check{\chi}\cdot\omega_K,0), \tag{9.3.3}$$

with $c(\gamma):=\log_p(\kappa_K(\gamma))$.

To prove this we note that $\kappa := \kappa_K$ factors through the projection $G_K \to \Gamma_K$ and recall that for each ψ in $A^-(\mathcal{G}_\infty)$ Deligne and Ribet have shown that there exists a unique element $f_{S,T,\psi}$ in $\mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} Q(\mathbb{Z}_p[[t]])$ for which

$$L_{p,S,T}(\psi \cdot \omega_K, 1-s) = f_{S,T,\psi}(\kappa(\gamma_K)^s - 1)$$

(cf. [40]).

For every ψ in $A(\Gamma_K)$ one has

$$\begin{aligned} j_\chi(\theta^-_{\mathcal{K}_\infty,S,T})(\psi(\gamma_K)-1) &= j_{\chi\cdot\psi}(\theta^-_{\mathcal{K}_\infty,S,T})(0) \\ &= L_{p,S,T}(\check{\chi}\cdot\check{\psi}\cdot\omega_K, 0) \\ &= f_{S,T,\check{\chi}\cdot\check{\psi}}(\kappa(\gamma_K)-1) \\ &= f_{S,T,\check{\chi}}(\check{\psi}(\gamma)\kappa(\gamma_K)-1) \\ &= \iota(f_{S,T,\check{\chi}})(\psi(\gamma_K)-1), \end{aligned}$$

where ι is the ring automorphism of $\mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} Q(\mathbb{Z}_p[[t]])$ that sends t to $\kappa(\gamma_K)(1+t)^{-1}-1$. Here the first and fourth equalities follow from a general property of the maps j_χ and functions $f_{S,T,\check{\chi}}$ that are respectively established in [72, Prop. 5 and (2), p. 563], the second follows from (7.3.1) and the other equalities are clear. Since the displayed equalities are true for every ψ in $A(\Gamma_K)$ it follows that $j_\chi(\theta_{\mathcal{K}_\infty,S,T}) = \iota(f_{S,T,\check{\chi}})$.

The left hand side of (9.3.3) is therefore equal to

$$\begin{aligned} &\big(j_\chi(\mathrm{Nrd}_{Q(\mathcal{G})}(\gamma-1)^{-r'})\cdot j_\chi(\theta^-_{\mathcal{K}_\infty,S,T})\big)(0) \\ =&\big(((1+t)^n-1)^{-r'\chi(1)}\cdot\iota(f_{S,T,\check{\chi}})\big)(0) \\ =&\, n^{-r'\chi(1)}\cdot\big(t^{-r'\chi(1)}\cdot\iota(f_{S,T,\check{\chi}})\big)(0) \\ =&\, n^{-r'\chi(1)}\cdot\log_p(\kappa(\gamma_K))^{-r'\chi(1)}\cdot L^{r'\chi(1)}_{p,S,T}(\tilde{\chi}\cdot\omega_K,0) \\ =&\, c(\gamma)^{-r'\chi(1)}\cdot L^{r'\chi(1)}_{p,S,T}(\tilde{\chi}\cdot\omega_K,0), \end{aligned}$$

as required to complete the proof of claim (i). Here the second equality uses (9.2.9), the third is established by the argument of [16, Lem. 5.9] and all other equalities are clear.

To prove claim (ii) we write ϱ for the natural projection map $Y^-_{E,S,p} \to Y^-_{E,\Sigma_S(E),p}$ and note that the definition of the idempotent $e = e_{E/K,S,\Sigma_S(E)}$ ensures that the map $e(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varrho)$ is bijective.

For each v in $\Sigma_S(E)$ we set

$$w_v^- := e_-(w_{v,E}) \in Y^-_{E,\Sigma_S(E),p}.$$

We write $\{(w_v^-)^*\}_{v\in\Sigma_S(E)}$ for the $\mathbb{Z}_p[G]^-$-basis of $\mathrm{Hom}_{\mathbb{Z}_p[G]}(Y^-_{E,\Sigma_S(E),p}, \mathbb{Z}_p[G]^-)$ that is dual to the basis $\{w_v^-\}_{v\in\Sigma_S(E)}$ of $Y^-_{E,\Sigma_S(E),p}$ and note that these maps gives rise to an isomorphism of $\xi(\mathbb{Z}_p[G])$-modules

$$\wedge_{v\in\Sigma_S(E)}(w_v^-)^* : {\bigcap}^{r'}_{\mathbb{Z}_p[G]} Y^-_{E,\Sigma_S(E),p} \to {\bigcap}^{0}_{\mathbb{Z}_p[G]} Y^-_{E,\Sigma_S(E),p} = \xi(\mathbb{Z}_p[G])^-$$

that sends $\wedge_{v\in\Sigma_S(E)} w_v^-$ to the identity element e_- of the ring $\xi(\mathbb{Z}_p[G])^-$.

In addition, for each v in $\Sigma' = \Sigma_S(E)$ one has

$$(\phi_v^{\rm ord})^- = (w_v^-)^* \circ \varrho \circ \lambda_{E,S,p}^{\rm ord}$$

and hence also

$$\phi_{E,S,\Sigma'}^{\rm ord,-} = e \cdot (\mathbb{Q}_p \cdot \wedge_{v\in\Sigma_S(E)}(w_v^-)^*) \circ \big(\bigwedge\nolimits_{\mathbb{Q}_p[G]}^{r'} \mathbb{Q}_p \cdot \varrho\big) \circ \bigwedge\nolimits_{\mathbb{Q}_p[G]}^{r'} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \lambda_{E,S,p}^{\rm ord}).$$

In a similar way, Lemma 9.10 implies, for each v in Σ', that

$$\begin{aligned}(\phi_{\gamma,v}^{\rm Bock})^- &= c(\gamma)^{-1} \cdot (\phi_v^{\rm Gross})^- \\ &= c(\gamma)^{-1} \cdot (w_v^-)^* \circ \varrho \circ \lambda_{E,S,p}^{\rm Gross}\end{aligned}$$

and hence also

$$\begin{aligned}&\phi_{\gamma,S,\Sigma'}^{\rm Bock,-} \\ &= {\rm Nrd}_{\mathbb{Q}_p[G]}(c(\gamma))^{-r'} e \cdot (\mathbb{Q}_p \cdot \wedge_{v\in\Sigma_S(E)}(w_v^-)^*) \circ \big(\bigwedge\nolimits_{\mathbb{Q}_p[G]}^{r'} \mathbb{Q}_p \cdot \varrho\big) \circ \bigwedge\nolimits_{\mathbb{Q}_p[G]}^{r'} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \lambda_{E,S,p}^{\rm Gross}).\end{aligned}$$

In this case therefore, the $\mathscr{L}$-invariant map $\mathscr{L}_{\gamma,S}^{\Sigma',-} = \phi_{\gamma,S,\Sigma'}^{\rm Bock,-} \circ (\phi_{E,S,\Sigma'}^{\rm ord,-})^{-1}$ is equal to the endomorphism of $\zeta(\mathbb{Q}_p[G])^- e$ that is given by multiplication by the element

$$\text{(9.3.4)} \qquad {\rm Nrd}_{\mathbb{Q}_p[G]}(c(\gamma)^{-r'})e \cdot {\rm Nrd}_{\mathbb{Q}_p[G]}\big((\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \lambda_{E,S,p}^{\rm Gross}) \circ (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \lambda_{E,S,p}^{\rm ord})^{-1}\big).$$

In particular, since

$${\rm Nrd}_{\mathbb{Q}_p[G]}(c(\gamma)^{-r'})_\chi = c(\gamma)^{-r'\chi(1)}$$

and $\mathscr{L}_S(\chi)$ is (by its very definition) equal to the χ-component of the second reduced norm in the product (9.3.4), the equality in claim (ii) now follows from the second equality in (9.3.2) and the fact that

$$\theta_{E/K,S\setminus\Sigma_S(E),T}(0)_\chi = L_{S\setminus\Sigma_S(E),T}(\check{\chi},0).$$

Given claims (i) and (ii), claim (iii) then follows directly from the fact that if Gross's Order of Vanishing Conjecture is valid for every χ in ${\rm Ir}_p^-(G)$, then the main result of Dasgupta, Kakde and Ventullo in [29] implies that, for every χ in ${\rm Ir}_p^-(G)$, the explicit quantities in claims (i) and (ii) are equal (for details of this deduction see [16, Prop. 2.6]). □

10. The equivariant Tamagawa Number Conjecture for $\mathbb{G}_m$

In this section we establish a precise connection between the Main Conjecture of Higher Rank Non-commutative Iwasawa Theory (Conjecture 7.4), the Generalized Gross-Stark Conjecture (Conjecture 9.7) and the equivariant Tamagawa Number Conjecture for $\mathbb{G}_m$ over general Galois extensions of number fields.

In this way we obtain a concrete strategy for obtaining new evidence in support of the latter conjecture, and thereby also extend (to general Galois extensions) the main result of the Burns, Kurihara and Sano in [19].

After reviewing the relevant case of the equivariant Tamagawa Number Conjecture in §10.1, the main result of this section will be stated and proved in §10.2.

10.1. **Review of the conjecture.** For the reader's convenience we shall first give an explicit statement of the equivariant Tamagawa Number Conjecture for $\mathbb{G}_m$ relative to an arbitrary finite Galois extension of number fields L/K and clarify what is currently known about this conjecture.

We set $G := \mathrm{Gal}(L/K)$ and fix a finite set of places S of K with

$$S_K^\infty \cup S_{\mathrm{ram}}(L/K) \subseteq S$$

and an auxiliary finite set of places T of K that is disjoint from S.

10.1.1. The leading term at $z = 0$ of the Stickelberger function $\theta_{L/K,S,T}(z)$ defined in (6.2.2) is

$$\theta^*_{L/K,S,T}(0) := \sum\nolimits_{\chi\in\widehat{G}} L^*_{S,T}(\check{\chi}, 0)e_\chi,$$

where $L^*_{S,T}(\chi, 0)$ is the leading term of $L_{S,T}(\chi, z)$ at $z = 0$, and belongs to $\zeta(\mathbb{R}[G])^\times$.

The equivariant Tamagawa Number Conjecture for $\mathbb{G}_m$ relative to L/K is then an equality in the relative algebraic K_0-group of the ring extension $\mathbb{Z}[G] \to \mathbb{R}[G]$ that relates $\theta^*_{L/K,S,T}(0)$ to Euler characteristic invariants of the complex

$$C := C_{L,S,T}$$

in $\mathsf{D}^{\mathrm{lf},0}(\mathbb{Z}[G])$ that is constructed in Lemma 5.1. In this section we interpret this conjectural equality in terms of the constructions made in §4.

To do so we note that, just as in Definition 7.2 (and Remark 4.10), for each prime p there exists a canonical subset $\mathrm{d}_{\mathbb{Z}_p[G]}(C_p)^{\mathrm{pb}}$ of the graded $\xi(\mathbb{Z}_p[G])$-module $\mathrm{d}_{\mathbb{Z}_p[G]}(C_p)$ comprising all primitive basis elements.

We next note that the Dirichlet regulator isomorphism $R_{L,S}$ combines with the explicit descriptions of cohomology groups in Lemma 5.1(i) (and the short exact sequence (5.1.2)) to induce a canonical isomorphism of graded $\zeta(\mathbb{R}[G])$-modules

$$\lambda_{L,S} : \mathrm{d}_{\mathbb{R}[G]}(\mathbb{R}\cdot C) \to (\zeta(\mathbb{R}[G]), 0).$$

From any fixed isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$ we obtain an induced ring embedding $j_* : \zeta(\mathbb{R}[G]) \to \zeta(\mathbb{C}_p[G])$ and hence also, via scalar extension, an isomorphism of graded $\zeta(\mathbb{C}_p[G])$-modules $\lambda^j_{L,S} := \mathbb{C}_p \otimes_{\mathbb{R},j} \lambda_{L,S}$. For each such j we can therefore write

$$z_p = z^j_{L/K,S,T} \tag{10.1.1}$$

for the zeta element associated (by Definition 4.6) to the isomorphism $\lambda^j_{L,S}$ and element $j_*(\theta^*_{L/K,S,T}(0))$ of $\zeta(\mathbb{C}_p[G])^\times$.

It then follows from Theorem 4.8 and Remarks 4.10 and 6.7 that the equivariant Tamagawa Number Conjecture for $\mathbb{G}_m$ relative to the extension L/K is valid if and only if the following conjecture is valid for all primes p.

Conjecture 10.1 ($\mathrm{TNC}_p(L/K)$)**.** *For each isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$ one has*

$$\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\mathrm{K}_1(\mathbb{Z}_p[G]))\cdot z_p = \mathrm{d}_{\mathbb{Z}_p[G]}(C_p)^{\mathrm{pb}}.$$

Remark 10.2. If Tate's formulation [82, Chap. I, Conj. 5.1] of Stark's principal conjecture is valid for L/K, then the same approach as in Remark 6.9(ii) shows that the zeta element z_p, and hence also each side of the above conjectural equality, is independent of the choice of isomorphism j. In addition, the independence of this equality from the choices of both S and T follows in a straightforward fashion from the properties of the complex C described in Lemma 5.1(ii) and (iii) respectively.

10.1.2. In [48, §4], Johnston and Nickel provide a clear and comprehensive overview of evidence in support of Conjecture $\mathrm{TNC}_p(L/K)$ circa 2015. In this section we assume that K is totally real and L is CM and recall several more recent developments concerning the conjecture in this case.

In the sequel we shall write '$\mathrm{TNC}_p(L/K)^{\pm}$ is valid' to denote that the displayed equality in Conjecture $\mathrm{TNC}_p(L/K)$ is valid after applying the exact scalar extension functor $\xi(\mathbb{Z}_p[G])e_{\pm} \otimes_{\xi(\mathbb{Z}_p[G])} -$, where the idempotent $e_{\pm}$ of $\zeta(\mathbb{Z}_p[G])$ is as defined in §9.3.2 (with E replaced by L).

We first recall what is known about Conjecture $\mathrm{TNC}_p(L/K)^+$. Before stating the result, we recall that the 'p-adic Stark Conjecture at $s = 1$' is discussed by Tate in [82, Chap. VI, §5], where it is attributed to Serre [77] (see also [14, Rem. 4.1.7]), and predicts an explicit formula for the leading term at $z = 1$ of $L_{p,S,T}(\psi, z)$ for each ψ in $\mathrm{Ir}_p^+(G)$.

Proposition 10.3. *$TNC_p(L/K)^+$ is valid if all of the following conditions are satisfied.*
- (i) *p is prime to $|G|$ or $\mu_p(L)$ vanishes.*
- (ii) *Breuning's Local Epsilon Constant Conjecture is valid for all extensions obtained by p-adically completing L/K.*
- (iii) *The p-adic Stark Conjecture at $s = 1$ is valid for all characters in $\mathrm{Ir}_p^+(G)$.*

Proof. It is shown in [14, §9.1] that $\mathrm{TNC}_p(L/K)^+$ is valid provided that all of the following conditions are satisfied: the p-adic Stark Conjecture at $s = 1$ is valid for all characters in $\mathrm{Ir}_p(G)$; if p divides $|G|$, then $\mu_p(L)$ vanishes; the p-component of a certain element $T\Omega^{\mathrm{loc}}(\mathbb{Q}(0)_L, \mathbb{Z}[G])$ of $\mathrm{K}_0(\mathbb{Z}[G], \mathbb{R}[G])$ vanishes.

The stated claim is therefore valid since the result [10, Th. 4.1] of Breuning simplies that the p-component of $T\Omega^{\mathrm{loc}}(\mathbb{Q}(0)_L, \mathbb{Z}[G])$ vanishes if the 'Local Epsilon Constant Conjecture' formulated in [10] is valid for all extensions obtained by p-adically completing L/K. □

Remark 10.4. Breuning's Local Epsilon Constant Conjecture has been shown to be valid for all tamely ramified extensions of local fields [10, Th. 3.6] and also for certain classes of wildly ramified extensions (cf. Bley and Cobbe [2] and the references contained therein). All such results lead to more explicit versions of Proposition 10.3.

Turning to Conjecture $\mathrm{TNC}_p(L/K)^-$, we first record an unconditional result. This result relies crucially on the recent verification by Dasgupta and Kakde [30] of the Strong Brumer-Stark Conjecture.

Proposition 10.5. *$TNC_p(L/K)^-$ is valid if the Sylow p-subgroups of G are abelian.*

Proof. We write M for the motive $h^0(\mathrm{Spec}(L))$, regarded as defined over K and with coefficients $\mathbb{Q}[G]$. Then, in terms of the notation of Remark 4.9(i), $\mathrm{TNC}_p(L/K)$ asserts the vanishing of the element of $\mathrm{K}_0(\mathbb{Z}_p[G], \mathbb{C}_p[G])$ given by

$$\delta_{\mathbb{Z}_p[G],\mathbb{C}_p}(L^*(M,0)) - \chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(C(M),t).$$

We must therefore show the stated conditions imply that the image $T\Omega(L/K)_p^-$ of this element under the natural projection $\mathrm{K}_0(\mathbb{Z}_p[G], \mathbb{C}_p[G]) \to K_0(\mathbb{Z}_p[G]e_-, \mathbb{C}_p[G]e_-)$ vanishes.

To do this, we first combine a result of Nickel [70, Th. 1] with [14, Rem. 6.1.1(iii)] to deduce that $T\Omega(L/K)_p^-$ has finite order. From [69, Prop. 6.2], we can then deduce that $T\Omega(L/K)_p^-$ vanishes if and only $T\Omega(L'/K')_p^-$ vanishes for every intermediate Galois CM extension L'/K' of L/K whose Galois group is either p-elementary or a direct product of a p-elementary group with $\{1, \tau\}$. In addition, by the argument of [70, §3], the assumption that the Sylow p-subgroups of G are abelian implies that every p-elementary subquotient of G is abelian.

To complete the proof, it is therefore enough to prove that $T\Omega(L'/K')_p^-$ vanishes for every intermediate Galois extension L'/K' of L/K in which L' is CM, K' is totally real and $\mathrm{Gal}(L'/K')$ is abelian. For any such extension L'/K', however, the vanishing of $T\Omega(L'/K')_p^-$ is derived from the seminal results of Dasgupta and Kakde in [30] by Bullach, Daoud, Seo and the first author in [13, Th. B (a)] (and see also the related works of Nickel [70], of Atsuta and Kataoka [1] and of Dasgupta, Kakde and Silliman [31]). □

In the general case, there is also the following result of Burns [16, Cor. 3.8(ii)].

Proposition 10.6. *$TNC_p(L/K)^-$ is valid if $\mu_p(L) = 0$ and Gross's Order of Vanishing Conjecture* [41, Conj. 2.12a)] *is valid for every totally odd character of G.*

Remark 10.7. If one sets $S := S_K^\infty \cup S_K^p$, then Remark 8.12 implies that the conditions in Proposition 10.6 are satisfied if and only if the $\mathbb{Z}_p$-module $A_S(L^{\mathrm{cyc}})$ is finitely generated and its quotient $A_S(L^{\mathrm{cyc}})^-_{\Gamma_L}$ is finite.

Remark 10.8. In addition, Nickel has proved in [69, Th. 6.8] that if $\mu_p(L) = 0$ and p is 'non-exceptional' in a certain technical sense (see [69, Def. 6.5]), then $\mathrm{TNC}_p(L/K)^-$ is valid. We recall, in particular, from loc. cit. that for any given extension L/K there can only be finitely many 'exceptional' primes p.

We next explain how Proposition 10.6 leads to unconditional evidence in support of Conjecture $\mathrm{TNC}_p(L/K)^-$ in the technically most difficult case that the Sylow p-subgroups of G are non-abelian (thereby complementing Proposition 10.5) and the relevant p-adic L-series possess trivial zeroes.

Corollary 10.9. *$TNC_p(L/K)^-$ is valid if all of the following conditions are satisfied.*

(i) *G is the semi-direct product of an abelian group A by a supersolvable group.*
(ii) *L^A is totally real and has at most one p-adic place that splits in L/L^+.*
(iii) *$\mu_p(L^P)$ vanishes where P is any given subgroup of G of p-power order.*

Proof. It suffices to show that the three given conditions imply the validity of the conditions stated in Proposition 10.6.

Condition (i) implies that for every ρ in $\mathrm{Ir}_p(G)$ there exists a subgroup A_ρ of G that contains A (and hence τ) and a linear $\mathbb{Q}_p^c$-valued character ρ' of A_ρ such that $\rho = \mathrm{Ind}_{A_\rho}^G(\rho')$

(for a proof of this fact see [76, II-22, Exercice] and the argument of [76, II-18]). It is also clear that if ρ belongs to $\mathrm{Ir}_p^-(G)$, then the field $L^{\ker(\rho')}$ is CM.

In particular, since the functorial properties of p-adic Artin L-series under induction and inflation imply that the conjecture [41, Conj. 2.12a)] is valid for ρ if and only if it is valid for ρ' we can assume (after replacing L/K by $L^{\ker(\rho')}/L^{A_\rho}$ and ρ by ρ') that ρ is linear. In view of condition (ii) we can also assume that K has at most one p-adic place that splits in L/L^+. We then recall that, if all of these hypotheses are satisfied, then [41, Conj. 2.12a)] is verified by Gross in [42, Prop. 2.13].

It is therefore enough to note that condition (iii) implies $\mu_p(F) = 0$. This is because if $\mu_p(E) = 0$ for some number field E, then Nakayama's Lemma implies $\mu_p(E') = 0$ for any p-power degree Galois extension E' of E. □

Example 10.10. If the field L^P in Corollary 10.9(iii) is abelian over $\mathbb{Q}$, then $\mu_p(L^P)$ vanishes by Ferrero-Washington [35]. Hence, if in such a case the field L^A has only one p-adic place, then Corollary 10.9 implies the unconditional validity of Conjecture $\mathrm{TNC}_p(L/K)^-$. It is straightforward to describe families of non-abelian extensions satisfying these hypotheses.
(i) Let F be a real quadratic field in which p does not split and assume that L is a CM abelian extension of F of exponent $2p^n$ for some natural number n. One can then set $K = \mathbb{Q}$ and $A = G_{L/F}$ and assume that L/K is Galois with (generalized) dihedral Galois group. Then $L^A = F$, P is normal in G and the quotient group G/P is abelian, as required.
(ii) Let E be a totally real A_4 extension of $\mathbb{Q}$ with the property that 3 does not split in its unique cubic subfield. Then for any imaginary quadratic field F the field $L := EF$ is a CM Galois extension of $\mathbb{Q}$ and $G_{L/\mathbb{Q}}$ is of the form $A \rtimes \mathbb{Z}/3$ with $A := \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ (where $\mathbb{Z}/3$ acts trivially on one copy of $\mathbb{Z}/2$ and cyclically permutes the non-trivial elements in the remaining factor $\mathbb{Z}/2 \times \mathbb{Z}/2$) and so is abelian-by-cyclic. The field L^A is then the unique cubic subfield of L and so is totally real with only one 3-adic place and the field $E_1 := L^A F$ is abelian over $\mathbb{Q}$ so $\mu_3(E_1)$ vanishes. One can also show that if $\mu_3(E_2)$ vanishes for any given quadratic extension E_2 of E_1 in L, then $\mu_3(L)$ also vanishes and so Corollary 10.9 applies to $L/\mathbb{Q}$.

Finally, to end this section, we take the opportunity to clarify an aspect of some results in [16]. To be specific, we give a concrete example to show that the above approach also allows one to describe situations in which the hypotheses of [16, Cor. 3.3] are satisfied by characters that are both faithful and of arbitrarily large degree. For all such examples one thus obtains a p-adic analytic construction of p-units that generate non-abelian Galois extensions of totally real fields and also encode explicit structural information about ideal class groups, thereby extending and refining the p-adic analytic approach to Hilbert's twelfth problem this is described for linear p-adic characters by Gross in [42, Prop. 3.14]. In the same way one deduces these examples verify a natural p-adic analogue of a question of Stark in [79] and a conjecture of Chinburg in [26] that were both formulated in the setting of characters of degree two.

Example 10.11. Fix a totally real field E and a cyclic CM extension E' of E in which precisely one p-adic place v of E splits completely and no other place of E that ramifies in $E/\mathbb{Q}$ splits completely. We let k be any subfield of E for which the restriction of v has absolute degree one and write F for the Galois closure of E' over k. Then F is a CM field and for any faithful linear character ψ' of $G_{E'/E}$ the character

$$\psi := \mathrm{Ind}_{G_{F/E}}^{G_{F/k}}(\mathrm{Inf}_{G_{E'/E}}^{G_{F/E}}(\psi'))$$

of $G_{F/k}$ is irreducible, totally odd, faithful and of degree $[E:k]$. Further, the functoriality of p-adic L-functions under induction and inflation combines with the result [42, Prop. 2.13] of Gross to imply ψ validates the hypotheses of [16, Cor. 3.3] with S taken to be $S_k^\infty \cup S_k^p \cup S_{\mathrm{ram}}(E/k)$ and v_1 the place of k below v.

10.2. **Main conjectures, derivative formulas and Tamagawa numbers.**

10.2.1. *Statement of the main result.* We now assume to be given a $\mathbb{Z}_p$-extension K^∞ of a number field K and for each extension L of K set

$$L^\infty := LK^\infty.$$

We also assume to be given a finite Galois extension E of K, a finite set S of places of K such that

$$S_K^\infty \cup S_{\mathrm{ram}}(E^\infty/K) \subseteq S \tag{10.2.1}$$

and an auxiliary finite set of places T of K that is disjoint from S and such that Hypothesis 6.1 is satisfied with $\mathcal{K}_\infty = E^\infty$. We then fix a place

$$v' \in S \setminus \Sigma_S(E^\infty) \tag{10.2.2}$$

(noting that this is always possible under the hypothesis (10.2.1) since $S_{\mathrm{ram}}(E^\infty/K)$ is both disjoint from $\Sigma_S(E^\infty)$ and non-empty).

We set

$$G := \mathcal{G}_E$$

and for each χ in $\mathrm{Ir}_p(G)$ we write E_χ for the fixed field of $\ker(\chi)$ in E and set

$$G_\chi := \mathcal{G}_{E_\chi} \quad \text{and} \quad \Sigma_\chi := \Sigma_{S\setminus\{v'\}}(E_\chi)$$

(so that $\Sigma_S(E^\infty) \subseteq \Sigma_\chi$, $\Sigma_S(E)\setminus\{v'\} \subseteq \Sigma_\chi$ and $\Sigma_\chi \neq S$).

We now introduce a convenient restriction on characters of G. We note that condition (ii) of this hypothesis is stated in terms of the idempotent $e_{E/K,S,\Sigma_\chi}$ of $\zeta(\mathbb{Q}[G])$ that is defined in (6.2.5) (so that a more explicit interpretation of the condition can be obtained via the equivalence of the properties (i) and (v) in Lemma 6.10).

Hypothesis 10.12. *χ is a character in $\mathrm{Ir}_p(G)$ that has all of the following properties.*

- (i) $\Sigma_S(E_\chi^\infty) = \Sigma_S(E^\infty)$ *and* $(\Sigma_\chi \setminus \Sigma_S(E^\infty)) \cap S_K^\infty = \emptyset$.
- (ii) $e_\chi \cdot e_{E/K,S,\Sigma_\chi} \neq 0$.
- (iii) *The space* $e_\chi\big(\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} A_S(E_\chi^\infty)_{\mathrm{Gal}(E_\chi^\infty/E_\chi)}\big)$ *vanishes.*
- (iv) *The Generalized Gross-Stark Conjecture (Conjecture 9.7) for the data* $E_\chi^\infty/K, E_\chi, S$ *and* T *is valid after multiplication by* e_χ.

Definition 10.13. We define idempotents of $\zeta(\mathbb{Q}_p[G])$ by setting

$$e_{E,1}^* := \sum\nolimits_\chi e_\chi \quad \text{and} \quad e_{E,2}^* := \sum\nolimits_\chi e_\chi,$$

where the sums are taken over all characters χ in $\mathrm{Ir}_p(G)$ that satisfy both of the conditions (i) and (ii), respectively (iii) and (iv), in Hypothesis 10.12. We then define an idempotent

$$e_E^* := e_{E,1}^* e_{E,2}^*$$

of $\zeta(\mathbb{Q}_p[G])$.

Remark 10.14. Under mild restrictions, the conditions (i) and (ii) in Hypothesis 10.12 are satisfied by every $\chi \in \mathrm{Ir}_p(G)$ so that one has

$$e_{E,1}^* = 1 \quad \text{and hence} \quad e_E^* = e_{E,2}^*.$$

For example, if K^∞ is the cyclotomic $\mathbb{Z}_p$-extension of K, then $\Sigma_S(E_\chi^\infty) \subseteq S_K^\infty$ and so $\Sigma_S(E^\infty) = \Sigma_S(E_\chi^\infty)$ if $S_K^\infty \cap S_{\mathrm{ram}}(L/K) = \emptyset$. Hence, if the latter condition is satisfied (as is the case, for example, if $|G|$ is odd), then Hypothesis 10.12(i) is satisfied by all $\chi \in \mathrm{Ir}_p(G)$. In addition, the equivalence of the conditions (i) and (ii) in Lemma 6.10 implies $\chi \in \mathrm{Ir}_p(G)$ validates Hypothesis 10.12(ii) if and only if one has $e_\chi(\mathbb{Q}^c \otimes_{\mathbb{Z}} X_{L,S\setminus\Sigma_\chi}) = 0$, or equivalently (by (6.2.3)) $\mathrm{ord}_{z=0} L_S(\chi, z) = \chi(1) \cdot |\Sigma_\chi|$. It is easily checked that this condition is satisfied by every linear χ and also, as a consequence of Frobenius reciprocity, by any non-linear (irreducible) χ whose restriction to G_v for each $v \in S \setminus S_K^\infty$ does not contain the trivial character of G_v. We note, in particular, that the latter condition is satisfied by any χ of the form $\mathrm{Ind}_H^G \phi$ with H a proper subgroup of G and ϕ a linear character of H for which one has $H \cap G_v \nsubseteq \ker(\phi)$ for all $v \in S \setminus S_K^\infty$.

These observations imply that (even if G is non-abelian) Hypothesis 10.12(ii) is satisfied by every χ in $\mathrm{Ir}_p(G)$ under a variety of explicit conditions. It can be checked, for example, that this situation arises in each of the following concrete cases:
(i) G is abelian;
(ii) $G_v = G$ for all $v \in S \setminus S_K^\infty$;
(iii) G is a Frobenius group whose Frobenius complement is abelian and Frobenius kernel is contained in G_v for all $v \in S \setminus S_K^\infty$;
(iv) G is non-abelian of order p^3 and its (non-trivial) centre is contained in G_v for all $v \in S \setminus S_K^\infty$.

In the next result we establish a concrete link between the Main Conjecture of Higher Rank Non-Commutative Iwasawa Theory (given by Conjecture 7.4), the Generalized Gross-Stark Conjecture (given by Conjecture 9.7) and the equivariant Tamagawa Number Conjecture for $\mathbb{G}_m$.

We observe in particular that, after taking account of Remark 10.14(i), this result generalizes to arbitrary finite Galois extensions the main result of Burns, Kurihara and Sano in [19] concerning abelian extensions (see Remark 10.17).

Theorem 10.15. *The validity of the Main Conjecture of Higher Rank Non-Commutative Iwasawa Theory for $E^\infty/K, S$ and T implies the validity of the equivariant Tamagawa Number Conjecture for the pair $(h^0(\mathrm{Spec}(E)), \mathbb{Z}_p[G]e_E^*)$.*

Our proof of this result will rely on the interpretation of the equivariant Tamagawa Number Conjecture in terms of the concept of locally-primitive bases, as given explicitly by Conjecture 10.1.

10.2.2. *The proof of Theorem 10.15.* At the outset, we set

$$r := |\Sigma_S(E^\infty)|,\ \mathcal{G} := \mathcal{G}_{E^\infty} \quad \text{and} \quad R_\infty := \Lambda(\mathcal{G}).$$

We also fix an endomorphism ϕ of R_∞^d that is constructed by the method of Proposition 8.2 with respect to the place v' fixed in (10.2.2).

Then, under the assumed validity of Conjecture 7.4, Remark 8.3 implies the existence of an element u of $\mathrm{K}_1(R_\infty)$ that validates the equality

$$\iota_{\infty,*}(\varepsilon^{\mathrm{RS}}_{E^\infty,S,T}) = \mathrm{Nrd}_{Q(\mathcal{G})}(u) \cdot \big((\wedge_{i=r+1}^{i=d}(b^*_{L,i} \circ \phi_L))(\varpi_L)\big)_{L\in\Omega(E^\infty)} \in {\bigcap}^r_{R_\infty} R^d_\infty \tag{10.2.3}$$

from (8.1.6) with $\mathcal{K}_\infty = E^\infty$, where for each $L \in \Omega(E^\infty)$ we set

$$R_L := \mathbb{Z}_p[\mathcal{G}_L] \quad \text{and} \quad \varpi_L := \wedge_{j\in[d]} b_{L,j} \in {\bigcap}^d_{R_L} R^d_L.$$

We next set

$$n = |S \setminus \{v'\}|$$

and use the ordering

$$S \setminus \{v'\} = \{v_j : j \in [n]\}$$

of $S \setminus \{v'\}$ that is induced by (6.1.1) and (6.1.2) for the field $\mathcal{K} = E^\infty$.

We then fix a character χ in $\mathrm{Ir_p(G)}$ that satisfies Hypothesis 10.12 and use the following convenient notation

$$\begin{aligned}
\Sigma_0 &:= \Sigma_S(E^\infty) = \Sigma_S(E^\infty_\chi) = \{v_j : j \in [r]\}\\
r_\chi &:= |\Sigma_\chi| \ \text{(so that } r_\chi \geq r \text{ since } \Sigma_0 \subseteq \Sigma_\chi)\\
\Sigma'_\chi &:= \Sigma_\chi \setminus \Sigma_0 \ \text{(so that } |\Sigma'_\chi| = r_\chi - r \ \text{ and } \ \Sigma'_\chi \cap S_K^\infty = \emptyset)\\
J_\chi &:= \{i \in [n] : v_i \in \Sigma'_\chi\}\\
J^\dagger_\chi &:= J_\chi \cup [r] = \{i \in [n] : v_i \in \Sigma_\chi\}\\
S'_\chi &:= S \setminus \Sigma'_\chi = (S \setminus \Sigma_\chi) \cup \Sigma_0\\
\mathcal{G}_\chi &:= \mathcal{G}_{E^\infty_\chi}\\
\mathcal{H}_\chi &:= \mathrm{Gal}(E^\infty_\chi/E_\chi)\\
G_\chi &:= \mathcal{G}_{E_\chi} \cong \mathcal{G}_\chi/\mathcal{H}_\chi\\
R_{\chi,\infty} &:= \Lambda(\mathcal{G}_\chi)\\
R_\chi &:= \mathbb{Z}_p[G_\chi].
\end{aligned}$$

We note that, with this notation, the idempotent

$$e_{(\chi)} := e_{E_\chi/K,S,\Sigma_\chi} \in \zeta(\mathbb{Q}_p[G_\chi])$$

defined in (6.2.5) coincides with $e_{E_\chi/K,S'_\chi,\Sigma_0}$.

We write Ω_χ for the subset $\Omega_{E_\chi}(E^\infty_\chi)$ of $\Omega(E^\infty_\chi)$ comprising fields that contain E_χ. For F in Ω_χ we write u_F for the projection of u to $\mathrm{K}_1(R_F)$ and set

$$\varepsilon_{F,\chi} := \mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_F]}((-1)^{m_\chi}) \cdot \mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_F]}(u_F) \cdot ((\wedge_{i\in[d]\setminus J^\dagger_\chi}(b^*_{F,i} \circ \phi_F))(\varpi_F)) \in {\bigcap}^{r_\chi}_{R_F} R^d_F,$$

with

$$m_\chi := |J_\chi| \cdot |[d] \setminus J_\chi^\dagger| = (r_\chi - r)(d - r_\chi). \tag{10.2.4}$$

In the case $F = E_\chi$ we further abbreviate u_F, $\varepsilon_{F,\chi}$ and ϖ_F to u_χ, ε_χ and ϖ_χ respectively.

We now fix a topological generator γ_χ of $\mathcal{H}_\chi$. Then, noting that $\Sigma_S(E^\infty) = \Sigma_S(E_\chi^\infty)$, we can compare the projection to $\bigcap^r_{R_{\chi,\infty}} R^d_{\chi,\infty}$ of the equality (10.2.3) to the explicit construction of $\partial^{r_\chi - r}_{\gamma_\chi}\big(\varepsilon^{\mathrm{RS}}_{E_\chi^\infty,S,T}\big)$ in Proposition 9.1 via the equalities (9.1.4) and (9.1.5) (with $\mathcal{K}_\infty/K$ and γ taken to be E_χ^∞/K and γ_χ).

In this way, we find that the element z in the latter formulas is, in the present context, equal to $\mathrm{Nrd}_{Q(\mathcal{G})}(u)$, and hence that there are equalities

$$\begin{aligned}
\iota_{E_\chi,*}(\partial^{r_\chi - r}_{\gamma_\chi}\big(\varepsilon^{\mathrm{RS}}_{E_\chi^\infty,S,T}\big)) &= \mathrm{Nrd}_{\mathbb{Q}_p[G_\chi]}((-1)^{m_\chi + t_\chi}) \cdot (\wedge_{j\in J_\chi}(\widehat{\phi}_{\chi,j,E_\chi}))(\varepsilon_\chi)\\
&= \mathrm{Nrd}_{\mathbb{Q}_p[G_\chi]}((-1)^{m_\chi + t_\chi}) \cdot (\wedge_{j\in J_\chi}(\widehat{\phi}_{\chi,j,E_\chi} \circ \iota'_{\chi,*}))(\hat{\varepsilon}_\chi)\\
&= \mathrm{Nrd}_{\mathbb{Q}_p[G_\chi]}((-1)^{m_\chi + t_\chi}) \cdot \phi^{\mathrm{Bock}}_{\gamma_\chi,S,\Sigma'_\chi}(\hat{\varepsilon}_\chi).
\end{aligned}$$

Here the integer t_χ is fixed so that

$$\mathrm{Nrd}_{\mathbb{Q}_p[G_\chi]}((-1)^{t_\chi}) \cdot \wedge_{j\in[d]\setminus[r]} b_{E_\chi,j} = \big(\wedge_{j\in[d]\setminus J_\chi^\dagger} b_{E_\chi,j}\big) \wedge \big(\wedge_{j\in J_\chi} b_{E_\chi,j}\big) \tag{10.2.5}$$

and each homomorphism $\widehat{\phi}_{\chi,i} = (\widehat{\phi}_{\chi,i,F})_F$ is as defined in (9.1.1) (with γ replaced by γ_χ). Further, setting $U_\chi := \mathcal{O}^\times_{E_\chi,S,T,p}$, we have written $\iota'_{\chi,*}$ for the embedding $\bigcap^{r_\chi}_{R_\chi} U_\chi \to \bigcap^{r_\chi}_{R_\chi} R^d_\chi$ induced by our fixed resolution of $C_{E_\chi,S,T,p}$ and, following Proposition 6.16, we write $\hat{\varepsilon}_\chi$ for the unique element of $e_{(\chi)}(\bigcap^{r_\chi}_{R_\chi} U_\chi)$ that is sent by $\wedge^{r_\chi}_{\mathbb{Q}_p[G_\chi]}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \iota'_{\chi,*})$ to ε_χ. Finally, we note that the last displayed equality follows from the corresponding case of the equality (9.2.8).

Now, under Hypothesis 10.12(iv), the above displayed formula combines with the validity of the χ-component of Conjecture 9.7 for the data $E_\chi^\infty/K, E_\chi, S$ and T to imply that

$$\begin{aligned}
\mathrm{Nrd}_{\mathbb{Q}_p[G_\chi]}((-1)^{m_\chi + t_\chi}) \cdot e_\chi\big(\phi^{\mathrm{Bock}}_{\gamma_\chi,S,\Sigma'_\chi}(\hat{\varepsilon}_\chi)\big) &= e_\chi\big(\mathscr{L}^{\Sigma'_\chi}_{\gamma_\chi,S}(\varepsilon^{\Sigma_0}_{E_\chi/K,S'_\chi,T})\big)\\
&= e_\chi\big(\phi^{\mathrm{Bock}}_{\gamma_\chi,S,\Sigma'_\chi}\big((\phi^{\mathrm{ord}}_{E_\chi,S,\Sigma'_\chi})^{-1}(\varepsilon^{\Sigma_0}_{E_\chi/K,S'_\chi,T})\big)\big).
\end{aligned}$$

In addition, since Hypothesis 10.12(ii) implies $e_\chi \cdot e_{(\chi)} \neq 0$, the conditions in Hypothesis 10.12(i), (ii) and (iii) imply that the hypotheses of Proposition 8.11 are satisfied by the data $E_\chi^\infty/K, E_\chi$ and S, and hence that the relevant case of Lemma 9.6 implies that the χ-component of the map $\phi^{\mathrm{Bock}}_{\gamma_\chi,S,\Sigma'_\chi}$ is injective.

From the last displayed equality, we can therefore deduce that

$$\mathrm{Nrd}_{\mathbb{Q}_p[G_\chi]}((-1)^{m_\chi + t_\chi}) \cdot e_\chi(\hat{\varepsilon}_\chi) = e_\chi\big((\phi^{\mathrm{ord}}_{E_\chi,S,\Sigma'_\chi})^{-1}(\varepsilon^{\Sigma_0}_{E_\chi/K,S'_\chi,T})\big). \tag{10.2.6}$$

In the next result, we describe the link between the elements $\varepsilon^{\Sigma_0}_{E_\chi/K,S'_\chi,T}$ and $\varepsilon^{\Sigma_\chi}_{E_\chi/K,S,T}$. In the case that E_χ/K is abelian, this result was first proved by Rubin in [74, Prop. 5.2].

Lemma 10.16. *In $\mathbb{C}_p \otimes_{\mathbb{Z}_p} \bigcap^r_{R_\chi} \mathcal{O}^\times_{E_\chi,S'_\chi,T,p}$ one has*

$$\varepsilon^{\Sigma_0}_{E_\chi/K,S'_\chi,T} = \phi^{\mathrm{ord}}_{E_\chi,S,\Sigma'_\chi}\big(\varepsilon^{\Sigma_\chi}_{E_\chi/K,S,T}\big).$$

Proof. We set $\tilde{w}_i := w_{v_i,E_\chi} - w_{v',E_\chi}$ for each index i in $J_\chi^\dagger$. We then consider the free R_χ-modules

$$Y := \bigoplus_{i\in[r]} R_\chi \cdot \tilde{w}_i \quad \text{and} \quad Y' := \bigoplus_{i\in J_\chi^\dagger} R_\chi \cdot \tilde{w}_i,$$

and use the isomorphism

$$\nu : \bigcap\nolimits_{R_\chi}^{r_\chi} Y' \cong \bigcap\nolimits_{R_\chi}^{r} Y$$

of (free, rank one) $\xi(R_\chi)$-modules that sends $\wedge_{j\in J_\chi^\dagger}\tilde{w}_j$ to $\wedge_{j\in[r]}\tilde{w}_j$.

We note that $\theta^r_{E_\chi/K,S'_\chi,T}(0)e_{(\chi)}$ belongs to $(\zeta(\mathbb{R}[G_\chi])e_{(\chi)})^\times$. Hence, after recalling the explicit definition (in Definition 6.5) of the elements $\varepsilon^{\Sigma_0}_{E/K,S'_\chi,T}$ and $\varepsilon^{\Sigma_\chi}_{E_\chi/K,S,T}$, and setting

$$\eta := \theta^{r_\chi}_{E_\chi/K,S,T}(0)\cdot(\theta^r_{E_\chi/K,S'_\chi,T}(0)e_{(\chi)})^{-1},$$

the verification of the claimed equality is reduced to proving commutativity of the following diagram

$$\begin{array}{ccc} e_{(\chi)}(\mathbb{C}_p\otimes_{\mathbb{Z}_p}\bigcap^{r_\chi}_{R_\chi}\mathcal{O}^\times_{E_\chi,S,T,p}) & \xrightarrow{\mathbb{C}_p\otimes_{\mathbb{R}}\lambda} & e_{(\chi)}(\mathbb{C}_p\otimes_{\mathbb{Z}_p}\bigcap^{r_\chi}_{R_\chi}Y') \\ {\scriptstyle \phi^{\rm ord}_{E_\chi,S,\Sigma'_\chi}}\downarrow & & \downarrow{\scriptstyle \eta\times(\mathbb{C}_p\otimes_{\mathbb{Z}_p}\nu)} \\ e_{(\chi)}(\mathbb{C}_p\otimes_{\mathbb{Z}_p}\bigcap^{r}_{R_\chi}\mathcal{O}^\times_{E_\chi,S'_\chi,T,p}) & \xrightarrow{\mathbb{C}_p\otimes_{\mathbb{R}}\lambda'} & e_{(\chi)}(\mathbb{C}_p\otimes_{\mathbb{Z}_p}\bigcap^{r}_{R_\chi}Y), \end{array}$$

where we abbreviate the maps $\lambda^{r_\chi}_{E_\chi,S}$ and $\lambda^r_{E_\chi,S'_\chi}$ to λ and λ' respectively.

Write $\mathrm{ord}_{S,\Sigma_\chi}$ for the map defined as in (9.2.3) (with E and Σ replaced by E_χ and Σ_χ) Then the commutativity of the above diagram is itself verified by noting that the identity (6.2.7) (with $F' = F = E_\chi$ and S' and $S(F)$ replaced by S and S'_χ respectively) implies an equality

$$\theta^{r_\chi}_{E_\chi/K,S,T}(0)e_{(\chi)} = \theta^r_{E_\chi/K,S'_\chi,T}(0)e_{(\chi)}\cdot\prod\nolimits_{j\in J_\chi}\log(\mathrm{N}w_{j,E_\chi}),$$

whilst the endomorphism

$$e_{(\chi)}((\mathbb{C}_p\otimes_{\mathbb{Z}_p}\mathrm{ord}_{S,\Sigma_\chi})\circ(\mathbb{C}_p\otimes_{\mathbb{R}}R_{E_\chi,S})^{-1})$$

of $e_{(\chi)}(\mathbb{C}_p\otimes_{\mathbb{Z}_p}Y')$ is represented, with respect to the ordered $\mathbb{C}_p[\mathcal{G}_\chi]e_{(\chi)}$-basis $\{e_{(\chi)}(\tilde{w}_j)\}_{j\in J_\chi^\dagger}$, by a block matrix of the form

$$\begin{pmatrix} M & 0 \\ * & \Delta \end{pmatrix}.$$

Here M is the matrix of the endomorphism

$$e_{(\chi)}((\mathbb{C}_p\otimes_{\mathbb{Z}_p}\mathrm{ord}_{S'_\chi,\Sigma})\circ(\mathbb{C}_p\otimes_{\mathbb{R}}R_{E_\chi,S'_\chi})^{-1})$$

of $e_{(\chi)}(\mathbb{C}_p\otimes_{\mathbb{Z}_p}Y)$ with respect to the ordered $\mathbb{C}_p[\mathcal{G}_\chi]e_{(\chi)}$-basis $\{e_{(\chi)}(\tilde{w}_j)\}_{j\in[r]}$, and Δ is the diagonal $(r_\chi - r)\times(r_\chi - r)$ matrix with j-th entry equal to $(\log(\mathrm{N}w_{v(j),E_\chi}))^{-1}$, where $v(j)$ here denotes the j-th place in the ordered set J_χ. □

Upon substituting the result of Lemma 10.16 into the equality (10.2.6) we deduce that, for every character χ that satisfies Hypothesis 10.12, there is an equality

$$\mathrm{Nrd}_{\mathbb{Q}_p[G_\chi]}((-1)^{m_\chi+t_\chi})\cdot e_\chi(\hat{\varepsilon}_\chi) = e_\chi(\varepsilon^{\Sigma_\chi}_{E_\chi/K,S,T}). \tag{10.2.7}$$

To interpret these equalities we write C_E for the complex $C_{E,S,T,p}$. Then, upon unwinding the definition of the isomorphism $\lambda^j_{E,S}$ that is discussed in §10.1.1, and setting $A := \mathbb{C}_p[G]$, one obtains a commutative diagram of $\zeta(A)e_{(\chi)}$-modules

$$\begin{array}{ccc} e_{(\chi)}\big(\mathbb{C}_p\cdot \mathrm{d}_{R_E}(C_E)\big) & \xrightarrow{\lambda^{j,\mathrm{u}}_{E,S}} & \zeta(A)e_{(\chi)} \\ {\scriptstyle \mathbb{C}_p\otimes_{\mathbb{Q}_p}\Theta^{\Sigma_\chi}_{E,S,T,p}}\big\downarrow & & \big\uparrow \\ e_{(\chi)}\big(\mathbb{C}_p\cdot \bigcap^{r_\chi}_{R_E} U_E\big) & \xrightarrow{\bigwedge^{r_\chi}_A(\mathbb{C}_p\otimes_{\mathbb{R},j} R_{E,S})} & e_{(\chi)}(\mathbb{C}_p\cdot \bigcap^{r_\chi}_{R_E} X_{E,S,p}). \end{array}$$

Here $\lambda^{j,\mathrm{u}}_{E,S}$ denotes the ungraded part of $\lambda^j_{E,S}$, the projection map $\Theta^{\Sigma_\chi}_{E,S,T,p}$ is as defined in (6.3.8) and the right hand side vertical arrow is induced by combining the natural identification

$$e_{(\chi)}\big(\mathbb{C}_p\cdot \bigcap\nolimits^{r_\chi}_{R_E} X_{E,S,p}\big) = e_{(\chi)}\big(\mathbb{C}_p\cdot \bigcap\nolimits^{r_\chi}_{R_\chi} Y_{E_\chi,\Sigma_\chi,p}\big)$$

together with the isomorphism of R_χ-modules $Y_{E_\chi,\Sigma_\chi,p}\cong R_\chi^{r_\chi}$ induced by the basis $\{w_{v,E_\chi} : v\in\Sigma_\chi\}$ and the canonical isomorphism of $\xi(R_\chi)$-modules $\bigcap^{r_\chi}_{R_\chi} R_\chi^{r_\chi}\cong \xi(R_\chi)$.

To proceed we now use the primitive basis

$$\beta_E := \big((\wedge_{i\in[d]} b_{E,i})\otimes(\wedge_{i\in[d]} b^*_{E,i}), 0\big)$$

of $\mathrm{d}_{R_E}(C_E)$ that is obtained when one represents C_E by the complex

$$R_E^d \xrightarrow{\phi_E} R_E^d,$$

with the first term placed in degree zero. We observe, in particular, that the explicit description of the map $\Theta^{\Sigma_\chi}_{E,S,T,p}$ given in (6.3.9) implies that

$$\mathrm{Nrd}_{\mathbb{Q}_p[G_\chi]}((-1)^{m_\chi+t_\chi})\cdot e_\chi(\hat{\varepsilon}_\chi) = e_\chi(\Theta^{\Sigma_\chi}_{E,S,T,p}(\beta_E)). \tag{10.2.8}$$

Note that the scalar multiplying factor $\mathrm{Nrd}_{\mathbb{Q}_p[G_\chi]}((-1)^{m_\chi+t_\chi})$ occurs here since the equality (10.2.5) and definition (10.2.4) combine to imply

$$\begin{aligned} &\mathrm{Nrd}_{\mathbb{Q}_p[G_\chi]}((-1)^{m_\chi+t_\chi})\cdot \wedge_{j\in[d]\setminus[r]} b_{E_\chi,j} \\ &= \mathrm{Nrd}_{\mathbb{Q}_p[G_\chi]}((-1)^{m_\chi})\cdot\big(\wedge_{j\in[d]\setminus J^\dagger_\chi} b_{E_\chi,j}\big)\wedge\big(\wedge_{j\in J_\chi} b_{E_\chi,j}\big) \\ &= \big(\wedge_{j\in J_\chi} b_{E_\chi,j}\big)\wedge\big(\wedge_{j\in[d]\setminus J^\dagger_\chi} b_{E_\chi,j}\big) \end{aligned}$$

and because the verification of the description (6.3.9) in the present setting assumes that $J_\chi = [r_\chi]\setminus[r]$.

Upon combining the equality (10.2.8) with (10.2.7), the definition of $\varepsilon^{\Sigma_\chi}_{E_\chi/K,S,T}$ and the commutativity of the above diagram, one finds that, for every χ validating Hypothesis 10.12, there are equalities

$$e_\chi(\lambda^{j,\mathrm{u}}_{E,S}(\beta_E)) = e_\chi(j_*(\theta^{r_\chi}_{E_\chi/K,S,T}(0))) = e_\chi(j_*(\theta^*_{E/K,S,T}(0)))$$

in $\zeta(\mathbb{C}_p[G])e_\chi$.

These equalities in turn imply that the zeta element z_p defined in (10.1.1) satisfies

$$e^*_E(z_p) = e^*_E\big((\lambda^j_{E,S})^{-1}\big((j_*(\theta^*_{E/K,S,T}(0)),0)\big)\big) = e^*_E(\beta_E).$$

In particular, since $e_E^*(\beta_E)$ is a primitive basis of the $\xi(R_E)e_E^*$-module that is generated by $\mathrm{d}_{R_E}(C_E)$, this equality implies the validity of the equivariant Tamagawa Number Conjecture for the pair $(h^0(\mathrm{Spec}(E)), R_E e_E^*)$.

This completes the proof of Theorem 10.15.

Remark 10.17. The result of Theorem 10.15 combines with the observations in Remark 10.14 to present a strategy for obtaining supporting evidence for Conjecture $\mathrm{TNC}_p(L/K)$ beyond the case of CM-extensions of totally real fields discussed in §10.1.2. In particular, Remark 10.14(i) implies that, upon specialisation to abelian extensions L/K, this strategy recovers that developed by Burns, Kurihara and Sano in [19, §5]. In addition, for non-abelian Galois extensions, it already gives a simpler proof of existing results such as Proposition 10.6. To explain the latter point, we assume that K is totally real, L is CM and K^∞ is the cyclotomic $\mathbb{Z}_p$-extension of K. Then, for each $\chi \in \mathrm{Ir}_p^-(G)$, one has $\Sigma_S(E_\chi^\infty) = \Sigma_S(E^\infty) = \emptyset$ and $\Sigma_\chi \cap S_K^\infty = \emptyset$ so Hypothesis 10.12(i) is satisfied, whilst (6.2.3) implies Hypothesis 10.12(ii) is satisfied (with the place v' in (10.2.2) taken to be archimedean) if $\mathrm{ord}_{z=0} L_S(\chi, z) = \chi(1)\cdot|\Sigma_\chi|$ and Remark 8.12 combines with Theorem 9.11(iii) to imply the conditions (iii) and (iv) in Hypothesis 10.12 are equivalent. Thus, if we write e_* for the idempotent of $\zeta(\mathbb{Q}_p[G])$ that is obtained by summing e_χ over all χ in $\mathrm{Ir}_p^-(G)$ for which one has both

$$e_\chi\big(\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} A_S(E_\chi^\infty)_{\mathrm{Gal}(E_\chi^\infty/E_\chi)}\big) = 0 \quad \text{ and } \quad \mathrm{ord}_{z=0} L_S(\chi, z) = \chi(1)\cdot|\Sigma_\chi|,$$

then Theorem 10.15 combines with Corollary 7.9 to imply the validity, modulo the assumed vanishing of $\mu_p(L)$, of the image of the equality in Conjecture $\mathrm{TNC}_p(L/K)$ under the functor $\mathbb{Z}_p[G]e_* \otimes_{\mathbb{Z}_p[G]} -$. We note, in particular, that this argument avoids the extensive descent computations that are required for the proof of Proposition 10.6 given in [16].

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