On the equivariant Tamagawa number Conjecture for Tate motives, Part II.

Dedicated to John Coates

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Abstract. Let K be any finite abelian extension of $\mathbb{Q}$, k any subfield of K and r any integer. We complete the proof of the equivariant Tamagawa Number Conjecture for the pair $(h^0(\mathrm{Spec}(K))(r), \mathbb{Z}[\mathrm{Gal}(K/k)])$.

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1. Introduction

Let K/k be a Galois extension of number fields with group G. For each complex character χ of G denote by $L(\chi, s)$ the Artin L-function of χ and let $\hat{G}$ be the set of irreducible characters. We call

$$\zeta_{K/k}(s) = (L(\chi, s))_{\chi\in\hat{G}}$$

the equivariant Dedekind Zeta function of K/k. It is a meromorphic function with values in the center $\prod_{\chi\in\hat{G}}\mathbb{C}$ of $\mathbb{C}[G]$. The 'equivariant Tamagawa number conjecture' that is formulated in [9, Conj. 4], when specialized to the motive $M := \mathbb{Q}(r)_K := h^0(\mathrm{Spec}(K))(r)$ and the order $\mathfrak{A} := \mathbb{Z}[G]$, gives a cohomological interpretation of the leading Taylor coefficient of $\zeta_{K/k}(s)$ at any integer argument $s = r$. We recall that this conjecture is a natural refinement of the seminal 'Tamagawa number conjecture' that was first formulated by Bloch and Kato in [5] and then both extended and refined by Fontaine and Perrin-Riou [18] and Kato [27]. If $K = k$ and $r \in \{0, 1\}$ then the conjecture specializes to the analytic class number formula and is therefore already a theorem.

The most succinct formulation of the equivariant Tamagawa number conjecture asserts the vanishing of a certain element $T\Omega(M,\mathfrak{A}) = T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ in the relative algebraic K-group $K_0(\mathbb{Z}[G],\mathbb{R})$. Further, the functional equation of the Artin L-functions is reflected by an equality

$$(1) \quad T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G]) + \psi^*(T\Omega(\mathbb{Q}(1-r)_K,\mathbb{Z}[G]^{\mathrm{op}})) = T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$$

where ψ^* is a natural isomorphism $K_0(\mathbb{Z}[G]^{\mathrm{op}},\mathbb{R}) \cong K_0(\mathbb{Z}[G],\mathbb{R})$ and $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ is an element of $K_0(\mathbb{Z}[G],\mathbb{R})$ of the form

$$(2) \ T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G]) = L^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G]) + \delta_{K/k}(r) + R\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$$

where $L^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ is an 'analytic' element constructed from the archimedean Euler factors and epsilon factors associated to both $\mathbb{Q}(r)_K$ and $\mathbb{Q}(1-r)_K$, the element $\delta_{K/k}(r)$ reflects sign differences between the regulator maps used in defining $T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ and $T\Omega(\mathbb{Q}(1-r)_K,\mathbb{Z}[G]^{\mathrm{op}})$ (cf. §2.1) and $R\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ is an 'algebraic' element constructed from the various realisations of $\mathbb{Q}(r)_K$. (We caution the reader that some of the notation we use in (1) and (2) is slightly different from that which is used in [9]. See §2.1 for details of these changes.)

In this article we shall further specialize to the case where K is an *abelian* extension of $\mathbb{Q}$ and prove that $T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G]) = 0$ for all integers r and all subgroups G of $\mathrm{Gal}(K/\mathbb{Q})$. In fact, taking advantage of previous work in this area, the main new result which we prove here is the following refinement of the results of Benois and Nguyen Quang Do in [1].

Theorem 1.1. *If K is any finite abelian extension of $\mathbb{Q}$, G any subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ and r any strictly positive integer, then $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G]) = 0$.*

We now discuss some interesting consequences of Theorem 1.1. The first consequence we record is the promised verification of the equivariant Tamgawa number conjecture for Tate motives over absolutely abelian fields. This result therefore completes the proof of Theorem 5.1 in [17] and also refines the main result of Huber and Kings in [25] (for more details of the relationship between our approach and that of [25] see the Introduction to [11]).

Corollary 1.2. *If K is any finite abelian extension of $\mathbb{Q}$, G any subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ and r any integer, then $T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G]) = 0$.*

Proof. If $r \leq 0$, then the vanishing of $T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ is proved modulo powers of 2 by Greither and the first named author in [11, Cor. 8.1] and the argument necessary to cover the 2-primary part is provided by the second named author in [17]. For any $r > 0$, the vanishing of $T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ then follows by combining Theorem 1.1 with the equality (1). $\square$

Corollary 1.3. *The conjecture of Bloch and Kato* [5, Conj. (5.15)] *is valid for the Riemann-Zeta function at each integer strictly bigger than* 1.

Proof. If r is any integer strictly bigger than 1, then [5, Th. (6.1)] proves the validity of [5, Conj. (5.15)] for the leading term of the Riemann Zeta function at $s = r$, modulo powers of 2 and a certain compatibility assumption [5, Conj. (6.2)] concerning the 'cyclotomic elements' of Deligne and Soulé in algebraic K-theory. But the latter assumption was verified by Huber and Wildeshaus in [26] and Corollary 1.2 for $K = k = \mathbb{Q}$ now resolves the ambiguity at 2. $\square$

For any group G the image of the homomorphism $\delta_G : K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}[G])$ that occurs in the long exact sequence of relative K-theory is equal to the locally-free class group $\mathrm{Cl}(\mathbb{Z}[G])$ of the ring $\mathbb{Z}[G]$. In the following result we use the elements $\Omega(K/k, 1), \Omega(K/k, 2), \Omega(K/k, 3)$ and $w(K/k)$ of $\mathrm{Cl}(\mathbb{Z}[\mathrm{Gal}(K/k)])$ that are defined by Chinburg in [13].

Corollary 1.4. *If K is any finite abelian extension of $\mathbb{Q}$ and $k \subset K$ is any subfield, then one has $\Omega(K/k, 1) = \Omega(K/k, 2) = \Omega(K/k, 3) = w(K/k) = 0$. In particular, the Chinburg conjectures are all valid for K/k.*

Proof. In this first paragraph we do not assume that K is Galois over $\mathbb{Q}$ or that $G = \mathrm{Gal}(K/k)$ is abelian. We recall that from [10, (31)] one has

$$\delta_G(\psi^*(T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G]^{\mathrm{op}}))) = \Omega(K/k, 3) - w(K/k).$$

Further, [4, Prop. 3.1] implies δ_G sends $L^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G]) + \delta_{K/k}(1)$ to $-w(K/k)$ whilst the same type of argument as used in [4, §4.3] shows that $R\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ is equal to the element $R\Omega^{\mathrm{loc}}(K/k, 1)$ defined in [7, §5.1.1]. Hence, from [7, Rem. 5.5], we may deduce that

$$\delta_G(T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G])) = -w(K/k) + \Omega(K/k, 2). \tag{3}$$

We now assume that G is abelian. Then G has no irreducible complex symplectic characters and so the very definition of $w(K/k)$ ensures that $w(K/k) = 0$. Hence by combining the above displayed equalities with Theorem 1.1 (with $r = 1$) and Corollary 1.2 (with $r = 0$) we may deduce that $\Omega(K/k, 2) = \Omega(K/k, 3) = 0$. But from [13, (3.2)] one has $\Omega(K/k, 1) = \Omega(K/k, 2) - \Omega(K/k, 3)$, and so this also implies that $\Omega(K/k, 1) = 0$. $\square$

For extensions $K/\mathbb{Q}$ which are unramified at all places above 2, an alternative proof of the equality $\Omega(K/k, 2) = 0$ of Corollary 1.4 was first obtained by Greither in [21].

Before stating our next result we recall that, ever since the seminal results of Fröhlich in [19], the study of Quaternion extensions has been very important to the development of leading term conjectures in non-commutative settings. In the following result we provide a natural refinement of the main result of Hooper, Snaith and Tran in [24] (and hence extend the main result of Snaith in [35]).

Corollary 1.5. *Let K be any Galois extension of $\mathbb{Q}$ for which* $\mathrm{Gal}(K/\mathbb{Q})$ *is isomorphic to the Quaternion group of order* 8 *and k any subfield of K. Then one has $T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[\mathrm{Gal}(K/k)]) = 0$.*

Proof. We set $G := \mathrm{Gal}(K/\mathbb{Q})$ and let Γ denote the maximal abelian quotient of G with E the subfield of K such that $\Gamma = \mathrm{Gal}(E/\mathbb{Q})$ (so $E/\mathbb{Q}$ is biquadratic). We set $T\Omega^{\mathrm{loc}} := T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ and $T\Omega^{\mathrm{loc}}_E := T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_E, \mathbb{Z}[\Gamma])$.

Then from [9, Th. 5.1 and Prop. 4.1] we know that $T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[\mathrm{Gal}(K/k)])$ and $T\Omega^{\mathrm{loc}}_E$ are equal to the images of $T\Omega^{\mathrm{loc}}$ under the natural homomorphisms $K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}[\mathrm{Gal}(K/k)], \mathbb{R})$ and $K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}[\Gamma], \mathbb{R})$ respectively. In particular, it suffices to prove that $T\Omega^{\mathrm{loc}} = 0$.
Now [4, Cor. 6.3(i)] implies that $T\Omega^{\mathrm{loc}}$ is an element of finite order in the subgroup $K_0(\mathbb{Z}[G], \mathbb{Q})$ of $K_0(\mathbb{Z}[G], \mathbb{R})$ and so [10, Lem. 4] implies that $T\Omega^{\mathrm{loc}} = 0$ if and only if both $T\Omega^{\mathrm{loc}}_E = 0$ and $\delta_G(T\Omega^{\mathrm{loc}}) = 0$. But Theorem 1.1 implies $T\Omega^{\mathrm{loc}}_E = 0$ and, since $\delta_G(T\Omega^{\mathrm{loc}}) = -w(K/\mathbb{Q}) + \Omega(K/\mathbb{Q}, 2)$ (by (3)), the main result of [24] implies that $\delta_G(T\Omega^{\mathrm{loc}}) = 0$. □

The following result provides the first generalization to wildly ramified extensions of the algebraic characterization of tame symplectic Artin root numbers that was obtained by Cassou-Noguès and Taylor in [12].

COROLLARY 1.6. *Let K be any Galois extension of $\mathbb{Q}$ for which $G = \mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to the Quaternion group of order 8. Then the Artin root number of the irreducible 2-dimensional character of G is uniquely determined by the algebraic invariant $R\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ in $K_0(\mathbb{Z}[G], \mathbb{R})$.*

Proof. This is a direct consequence of combining Corollary 1.5 with a result of Breuning and the first named author [7, Th. 5.8] and the following facts: $L^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G]) + \delta_{K/\mathbb{Q}}(1)$ is equal to -1 times the element $\hat{\partial}^1_G(\epsilon_{K/\mathbb{Q}}(0))$ used in [7, §5.1.1] and $R\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ is equal to the element $R\Omega^{\mathrm{loc}}(K/\mathbb{Q}, 1)$ defined in loc. cit. □

To prove Theorem 1.1 we combine some classical and rather explicit computations of Hasse (concerning Gauss sums) and Leopoldt (concerning integer rings in cyclotomic fields) with a refinement of some general results proved in [9, §5] and a systematic use of the Iwasawa theory of complexes in the spirit of Kato [27, 3.1.2] and Nekovář [32] and of the generalization of the fundamental exact sequence of Coleman theory obtained by Perrin-Riou in [34].
We would like to point out that, in addition to the connections discussed above, there are also links between our approach and aspects of the work of Kato [28], Fukaya and Kato [20] and Benois and Berger [2]. In particular, the main technical result that we prove (the validity of equality (13)) is closely related to [28, Th. 4.1] and hence also to the material of [20, §3]. Indeed, Theorem 1.1 (in the case $r = 1$) provides a natural generalization of the results discussed in [20, §3.6]. However, the arguments of both loc. cit. and [28] do not cover the prime 2 and also leave open certain sign ambiguities, and much effort is required in the present article to deal with such subtleties.
Both authors were introduced to the subject of Tamagawa number conjectures by John Coates. It is therefore a particular pleasure for us to dedicate this paper to him on the occasion of his sixtieth birthday.
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2. Normalizations and notation.

2.1. Normalizations. In this section we fix an arbitrary Galois extension K/k, set $G := \mathrm{Gal}(K/k)$ and for each integer t write $T\Omega(t)$ and $T\Omega^{\mathrm{loc}}(t)$ for the elements $T\Omega(\mathbb{Q}(t)_K, \mathbb{Z}[G])$ and $T\Omega^{\mathrm{loc}}(\mathbb{Q}(t)_K, \mathbb{Z}[G])$ defined in [9].
Let r be a strictly positive integer. Then the computations of [10, 17] show that [9, Conj. 4(iv)] requires that the morphism $\vartheta_\infty : \mathbb{R}\otimes_{\mathbb{Q}}\Xi(\mathbb{Q}(1-r)_K) \to \mathbf{1}_{V(\mathbb{R}[G])}$ constructed in [9, §3.4] should be normalized by using -1 times the Dirichlet (resp. Beilinson) regulator map (rather than the Dirichlet (resp. Beilinson) regulator map itself as we speculated in [9]). To incorporate this observation we set

$$T\Omega(\mathbb{Q}(1-r)_K, \mathbb{Z}[G]) := T\Omega(1-r) + \delta_{K/k}(r) \tag{4}$$

where $\delta_{K/k}(r)$ is the image under the canonical map $K_1(\mathbb{R}[G]) \to K_0(\mathbb{Z}[G], \mathbb{R})$ of the element associated to multiplication by -1 on $K_{2r-1}(\mathcal{O}_K)^* \otimes_{\mathbb{Z}} \mathbb{R}$. To maintain the validity of (1) we are thus forced to make changes to the definitions of $T\Omega^{\mathrm{loc}}(r)$ and/or $T\Omega(r)$. Our proof of Theorem 1.1 now shows that the correct normalization is to set

$$T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G]) := T\Omega^{\mathrm{loc}}(r) + \delta_{K/k}(r) \tag{5}$$

and hence also

$$T\Omega(\mathbb{Q}(r)_K, \mathbb{Z}[G]) := T\Omega(r). \tag{6}$$

The elements defined in (4), (5) and (6) satisfy all of the functorial properties of $T\Omega(1-r), T\Omega^{\mathrm{loc}}(r)$ and $T\Omega(r)$ proved in [9, Th. 5.1, Prop. 4.1]. Further, the equality (2) is clear because $T\Omega^{\mathrm{loc}}(r) = L^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G]) + R\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])$ (cf. [9, p. 554]), and it can be shown that the conjectural equality $T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G]) = 0$ is compatible with the conjectures discussed in both [4] and [7].
Thus, in this article we always use the notation $T\Omega(\mathbb{Q}(1-r)_K, \mathbb{Z}[G])$, $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])$ and $T\Omega(\mathbb{Q}(r)_K, \mathbb{Z}[G])$ as defined in (4), (5) and (6).

2.2. The abelian case. Until explicitly stated otherwise, in the sequel we consider only abelian groups. Thus, following [9, §2.5], we use the graded determinant functor of [29] in place of virtual objects (for a convenient review of all relevant properties of the determinant functor see [11, §2]).
We note that, when proving Theorem 1.1, the functorial properties of the elements $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[\mathrm{Gal}(K/k)])$ allow us to assume that $k = \mathbb{Q}$ and also that K is generated by a primitive N-th root of unity ζ_N for some natural number $N \not\equiv 2 \mod 4$. Therefore, until explicitly stated otherwise, we henceforth fix the following notation:

$$K := \mathbb{Q}(\zeta_N); \quad G := \mathrm{Gal}(K/\mathbb{Q}); \quad M := \mathbb{Q}(r)_K, \quad r \geq 1; \quad A := \mathbb{Q}[G].$$

More generally, for any natural number n we shall set $\zeta_n = e^{2\pi i/n}$ and denote by σ_n the resulting complex embedding of the field $\mathbb{Q}(\zeta_n)$.

For any $\mathbb{Q}$-vector space W we set $W_{\mathbb{R}} = W \otimes_{\mathbb{Q}} \mathbb{R}$, $W_{\mathbb{C}} = W \otimes_{\mathbb{Q}} \mathbb{C}$ and $W_p = W \otimes_{\mathbb{Q}} \mathbb{Q}_p$ for each prime p.
For each complex character η of G we denote by $e_\eta = \frac{1}{|G|}\sum_{g\in G}\eta(g^{-1})g \in A_{\mathbb{C}}$ the associated idempotent. For each $\mathbb{Q}$-rational character (i.e. $\mathrm{Aut}(\mathbb{C})$-conjugacy class of $\mathbb{C}$-rational characters) χ of G we set $e_\chi = \sum_{\eta\in\chi} e_\eta \in A$ and denote by $\mathbb{Q}(\chi) = e_\chi A$ the field of values of χ. There is a ring decomposition $A = \prod_\chi \mathbb{Q}(\chi)$ and a corresponding decomposition $Y = \prod_\chi e_\chi Y$ for any A-module Y. We make similar conventions for $\mathbb{Q}_p$-rational characters of G.

3. An explicit analysis of $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])$

In this section we reduce the proof of Theorem 1.1 to the verification of an explicit local equality (cf. Proposition 3.4).

3.1. The archimedean component of $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])$.

The de Rham realization $H_{dR}(M)$ of M identifies with K, considered as a free rank one A-module (by means of the normal basis theorem). The Betti realisation $H_B(M)$ of M identifies with the $\mathbb{Q}$-vector space Y_Σ with basis equal to the set $\Sigma = \mathrm{Hom}(K, \mathbb{C})$ of field embeddings and is therefore a free A-module of rank one. Following [9, §5.1] we define

$$\Xi^{\mathrm{loc}}(M) := \mathrm{Det}_A(H_{dR}(M)) \otimes_A \mathrm{Det}_A^{-1}(H_B(M)) = K \otimes_A Y_\Sigma^{-1}.$$

We recall the isomorphism of graded invertible $A_{\mathbb{R}}$-modules

$$\vartheta_\infty^{loc} : \Xi^{\mathrm{loc}}(M)_{\mathbb{R}} \cong (A_{\mathbb{R}}, 0)$$

defined in [9, (57)] and the element $\mathcal{E}({}_AM) \in A_{\mathbb{R}}^\times$ defined in [9, §5.2]. By [9, Th. 5.2] we have $(\vartheta_\infty^{loc})^{-1}(\mathcal{E}({}_AM)^{-1}) \in \Xi^{\mathrm{loc}}(M) = K \otimes_A Y_\Sigma^{-1}$ and our first task will be to describe this element explicitly.

PROPOSITION 3.1. *We define an element $\epsilon_\infty = \sum_\chi \epsilon_{\infty,\chi} e_\chi$ of $A^\times$ by setting*

$$\epsilon_{\infty,\chi} := \begin{cases} -2 & \text{if } \chi(-1) = (-1)^r \\ -\frac{1}{2} & \text{if } \chi(-1) = -(-1)^r \text{ and } (\chi \neq 1 \text{ or } r > 1) \\ \frac{1}{2} & \text{if } \chi = 1 \text{ and } r = 1. \end{cases}$$

Then

$$(\vartheta_\infty^{loc})^{-1}(\mathcal{E}({}_AM)^{-1}) = \epsilon_\infty\, \beta_N \otimes \sigma_N^{-1} \in K \otimes_A Y_\Sigma^{-1}$$

where β_N is the element of $K = \prod_\chi e_\chi K$ which satisfies

$$e_\chi \beta_N := [K : \mathbb{Q}(\zeta_{f_\chi})]^{-1}(r-1)!f_\chi^{r-1} \cdot e_\chi \zeta_{f_\chi}$$

for all $\mathbb{Q}$-rational characters χ of G.

Proof. The functional equation of $L(\eta, s)$ reads

$$L(\eta,s)=\frac{\tau(\eta)}{2i^\delta}\left(\frac{2\pi}{f_\eta}\right)^s\frac{1}{\Gamma(s)\cos(\frac{\pi(s-\delta)}{2})}L(\bar\eta,1-s)$$

where f_η is the conductor of η and

$$\tau(\eta)=\sum_{a=1}^{f_\eta}\eta(a)e^{2\pi ia/f_\eta};\quad \eta(-1)=(-1)^\delta,\ \delta\in\{0,1\} \tag{7}$$

(cf. [36, Ch. 4]). Now by its very definition [9, §5.2], the η-component of the element $\mathcal{E}({}_AM)^{-1}$ of $A_{\mathbb{C}}=\prod_\eta\mathbb{C}$ is the leading Taylor coefficient at $s=r$ of the meromorphic function

$$(-1)^{\rho_\eta}\frac{2i^\delta}{\tau(\eta)}\left(\frac{f_\eta}{2\pi}\right)^s\Gamma(s)\cos(\frac{\pi(s-\delta)}{2});\qquad \rho_\eta=\begin{cases}1 & r=1,\eta=1\\ 0 & \text{else.}\end{cases}$$

Hence we have

$$\mathcal{E}({}_AM)_\eta^{-1}=\begin{cases}\frac{2i^\delta}{\tau(\eta)}\left(\frac{f_\eta}{2\pi}\right)^r(r-1)!(-1)^{\frac{r-\delta}{2}} & r-\delta\text{ even}\\ (-1)^{\rho_\eta}\frac{2i^\delta}{\tau(\eta)}\left(\frac{f_\eta}{2\pi}\right)^r(r-1)!(-1)^{\frac{r-\delta+1}{2}}\frac{\pi}{2} & r-\delta\text{ odd}\end{cases}$$

which, after collecting powers of i and using the relation $\tau(\eta)\tau(\bar\eta)=\eta(-1)f_\eta$, can be written as

$$\mathcal{E}({}_AM)_\eta^{-1}=\begin{cases}2\,\tau(\bar\eta)(2\pi i)^{-r}f_\eta^{r-1}(r-1)! & r-\delta\text{ even}\\ (-1)^{\rho_\eta+1}\frac{1}{2}\,\tau(\bar\eta)(2\pi i)^{-(r-1)}f_\eta^{r-1}(r-1)! & r-\delta\text{ odd.}\end{cases}$$

LEMMA 3.2. *The isomorphism $Y^+_{\Sigma,\mathbb{C}}=(H_B(M)_{\mathbb{C}})^+\cong H_B(M)_{\mathbb{R}}=Y_{\Sigma,\mathbb{R}}$ from* [9, (54)] *is given by*

$$\sum_{g\in G}\alpha_g g^{-1}\sigma_N\mapsto\sum_{g\in G/<c>}\big(\Re(\alpha_g)(1+(-1)^rc)-2\pi\Im(\alpha_g)(1-(-1)^rc)\big)g^{-1}\sigma_N$$

where $c\in G$ is complex conjugation, G acts on Σ via $(g\sigma)(x)=\sigma(g(x))$ and $\Re(\alpha)$, resp. $\Im(\alpha)$, denotes the real, resp. imaginary, part of $\alpha\in\mathbb{C}$.

Proof. An element $x:=\sum_{g\in G}\alpha_g g^{-1}\sigma_N$ of $Y_{\Sigma,\mathbb{C}}$ belongs to the subspace $Y^+_{\Sigma,\mathbb{C}}$ if and only if one has $\alpha_{gc}=(-1)^r\bar\alpha_g$ for all $g\in G$. Writing

$$\alpha_g=\Re(\alpha_g)-(2\pi i)^{-1}(2\pi)\Im(\alpha_g),\quad \bar\alpha_g=\Re(\alpha_g)+(2\pi i)^{-1}(2\pi)\Im(\alpha_g)$$

we find that

$$x=\sum_{g\in G/<c>}\big(\Re(\alpha_g)(1+(-1)^rc)-(2\pi i)^{-1}2\pi\Im(\alpha_g)(1-(-1)^rc)\big)g^{-1}\sigma_N.$$

But $\sum_{g\in G/<c>}(2\pi i)^{-1}2\pi\Im(\alpha_g)(1-(-1)^rc)g^{-1}\sigma_N\in H_B(M)^-\otimes_{\mathbb{Q}}\mathbb{R}\cdot i$ and the isomorphism [9, (56)] which occurs in [9, (54)] sends $(2\pi i)^{-1}$ to 1. This implies the Lemma. $\square$

The canonical comparison isomorphism $K_{\mathbb{C}} = H_{dR}(M)_{\mathbb{C}} \cong H_B(M)_{\mathbb{C}} = Y_{\Sigma,\mathbb{C}}$ sends any element β of K to

$$\sum_{g\in G} \sigma_N(g\beta)(2\pi i)^{-r} g^{-1}\sigma_N = \sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \sigma_N\tau_a(\beta)(2\pi i)^{-r}\tau_a^{-1}\sigma_N$$

where $\tau_a(\zeta) = \zeta^a$ for each N-th root of unity ζ. In particular, after composing the comparison isomorphism with the isomorphism of Lemma 3.2 we find that ζ_f is sent to the following element of $Y_{\Sigma,\mathbb{R}}$

$$\sum_a \big(\Re(e^{2\pi ia/f}(2\pi i)^{-r})(1+(-1)^r c) - 2\pi\Im(e^{2\pi ia/f}(2\pi i)^{-r})(1-(-1)^r c)\big)\tau_a^{-1}\sigma_N$$

where the summation runs over all elements a of $(\mathbb{Z}/N\mathbb{Z})^\times/\pm 1$. For a Dirichlet character η the η-component of the last displayed element is equal to $e_\eta\sigma_N$ multiplied by

$$\begin{aligned} &\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times/\pm 1} (2\pi)^{-r}\Re(e^{2\pi ia/f}i^{-r})\overline{\eta(a)}\cdot 2 \\ =&\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times/\pm 1} (2\pi i)^{-r}(e^{2\pi ia/f}+(-1)^r e^{-2\pi ia/f})\overline{\eta(a)} \\ =&(2\pi i)^{-r}\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} e^{2\pi ia/f}\overline{\eta(a)} \end{aligned}$$

if $\eta(-1) = (-1)^r$ (so $\delta - r$ is even), resp. by

$$\begin{aligned} &-2\pi\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times/\pm 1} (2\pi)^{-r}\Im(e^{2\pi ia/f}i^{-r})\overline{\eta(a)}\cdot 2 \\ =&-2\pi\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times/\pm 1} (2\pi i)^{-r}\frac{e^{2\pi ia/f}-(-1)^r e^{-2\pi ia/f}}{i}\overline{\eta(a)} \\ =&(2\pi i)^{-(r-1)}\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} e^{2\pi ia/f}\overline{\eta(a)} \end{aligned}$$

if $\eta(-1) = -(-1)^r$ (so $\delta - r$ is odd). Taking $f = f_\eta$ we find that the (η-part of the) comparison isomorphism sends

$$e_\eta\zeta_{f_\eta}\otimes(e_\eta\sigma_N)^{-1} \mapsto \begin{cases} (2\pi i)^{-r}[K:\mathbb{Q}(f_\eta)]\tau(\bar\eta) & \text{if } \eta(-1)=(-1)^r \\ (2\pi i)^{-(r-1)}[K:\mathbb{Q}(f_\eta)]\tau(\bar\eta) & \text{if } \eta(-1)=-(-1)^r. \end{cases} \tag{8}$$

We now note that the isomorphism ϑ_∞^{loc} involves [9, (54)] as well as the sign factors ϵ_{dR} and ϵ_B defined before [9, (57)]. But one immediately verifies that $e_{dR} = 1$, that $(\epsilon_B)_\chi = -1$ for $\chi(-1) = (-1)^r$ and that $(\epsilon_B)_\chi = 1$ otherwise. Upon comparing (8) with the description of $\mathcal{E}({}_AM)_\eta^{-1}$ before Lemma 3.2 we therefore deduce the statement of Proposition 3.1. $\square$

3.2. Reduction to the p-primary component. By [9, Th. 5.2] we know that $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])$ belongs to the subgroup $K_0(\mathbb{Z}[G], \mathbb{Q})$ of $K_0(\mathbb{Z}[G], \mathbb{R})$. Recalling the direct sum decomposition $K_0(\mathbb{Z}[G], \mathbb{Q}) \cong \bigoplus_\ell K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell)$ over all primes ℓ of [9, (13)], we may therefore prove Theorem 1.1 by showing that, for each prime ℓ, the projection $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])_\ell$ of $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])$ to $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell)$ vanishes. Henceforth we therefore fix a prime number p and shall analyze $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])_p$.

We denote by

$$T_p := \mathrm{Ind}_K^{\mathbb{Q}} \mathbb{Z}_p(r) \subset V_p := \mathrm{Ind}_K^{\mathbb{Q}} \mathbb{Q}_p(r) = H_p(M)$$

the natural lattice in the p-adic realisation V_p of M. Then the $\mathbb{Z}_p[G]$-module T_p is free of rank one and we denote its $\mathbb{Z}_p[G]$-dual by $T_p^{-1} := \mathrm{Hom}_{\mathbb{Z}_p[G]}(T_p, \mathbb{Z}_p[G])$. Similarly we define $V_p^{-1} := \mathrm{Hom}_{A_p}(V_p, A_p)$. In [9, (67)] we defined an isomorphism of graded invertible A_p-modules

$$\theta_p : \Xi^{\mathrm{loc}}(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \left(\bigotimes_{\ell | Np} \mathrm{Det}_{A_p}^{-1} R\Gamma(\mathbb{Q}_\ell, V_p) \right) \otimes_{A_p} V_p^{-1}. \tag{9}$$

Then by combining the definition of $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])$ via (5) and the material in [9, §5.2] together with the explicit description of Proposition 3.1 one finds that $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])_p = 0$ if and only if one has

$$\mathbb{Z}_p[G] \cdot \epsilon(r)\epsilon(S,p) \cdot \theta_p' \circ \theta_p(\epsilon_\infty \beta_N \otimes \sigma_N^{-1}) = \Lambda_p(S, T_p)$$

where $\Lambda_p(S, T_p)$, θ_p' and $\epsilon(S,p) \in A_p^\times$ are defined in [9, (71), (78) and (82)] and $\epsilon(r) := \mathrm{Det}_A(-1 \mid K_{2r-1}(\mathcal{O}_K)^* \otimes_{\mathbb{Z}} \mathbb{Q})$.

Lemma 3.3. *Let ϵ_{id} be the element of $A_p^\times$ that is defined in Proposition 6.3 (with $T_p = \mathrm{Ind}_K^{\mathbb{Q}} \mathbb{Z}_p(r)$ and $\alpha = \mathrm{id}_{T_p}$). Then, with ϵ_∞ as defined in Proposition 3.1, there exists an element $u(r)$ of $\mathbb{Z}_p[G]^\times$ such that $\epsilon(r)\epsilon(S,p)\epsilon_\infty = u(r)\epsilon_{\mathrm{id}}$.*

Proof. We recall that $\epsilon(S,p)$ is a product of factors $\mathrm{Det}_{A_p}(-1|R\Gamma_{/f}(\mathbb{Q}_\ell, V_p))$. Further, each complex $R\Gamma_{/f}(\mathbb{Q}_\ell, V_p) \cong R\Gamma_f(\mathbb{Q}_\ell, V_p^*(1))^*[-2]$ is quasi-isomorphic to a complex of the form $W \to W$ (indeed, this is clear if $\ell \neq p$ and is true in the case $\ell = p$ because the tangent space of the motive $\mathbb{Q}(1-r)_K$ vanishes for $r \geq 1$) and so one has $\epsilon(S,p) = 1$.

We next note that if $\epsilon(r) := \sum_\chi \epsilon(r)_\chi e_\chi$ with $\epsilon(r)_\chi \in \{\pm 1\}$, then the explicit structure of the $\mathbb{R}[G]$-module $K_{2r-1}(\mathcal{O}_K)^* \otimes_{\mathbb{Z}} \mathbb{R}$ (cf. [17, p. 86, p. 105]) implies that $\epsilon(r)_\chi = 1$ if either $r = 1$ and χ is trivial or if $\chi(-1) = (-1)^r$, and that $\epsilon(r)_\chi = -1$ otherwise.

Thus, after recalling the explicit definitions of ϵ_∞ and ϵ_{id}, it is straightforward to check that the claimed equality is valid with $u(r) = -(-1)^r c$ where $c \in G$ is complex conjugation. $\square$

Upon combining the last lemma with the discussion which precedes it and the result of Proposition 6.3 we may therefore deduce that $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])_p = 0$

if and only if one has

$$\mathbb{Z}_p[G]\cdot\theta_p(\beta_N\otimes\sigma_N^{-1}) = \left(\bigotimes_{\ell|Np}\mathrm{Det}^{-1}_{\mathbb{Z}_p[G]} R\Gamma(\mathbb{Q}_\ell, T_p)\right)\otimes_{\mathbb{Z}_p[G]} T_p^{-1}. \tag{10}$$

Now Shapiro's Lemma allows us to identify the complexes $R\Gamma(\mathbb{Q}_\ell, T_p)$ and $R\Gamma(\mathbb{Q}_\ell, V_p)$ with $R\Gamma(K_\ell, \mathbb{Z}_p(r))$ and $R\Gamma(K_\ell, \mathbb{Q}_p(r))$ respectively. Further, the complex $R\Gamma(K_p, \mathbb{Q}_p(r))$ is acyclic outside degree 1 for $r > 1$, and for $r = 1$ one has a natural exact sequence of $\mathbb{Q}_p[G]$-modules

$$0\to\hat{\mathcal{O}}^\times_{K_p}\to\hat{K}_p^\times\cong H^1(K_p,\mathbb{Q}_p(1))\xrightarrow{\mathrm{val}}\prod_{v|p}\mathbb{Q}_p\cong H^2(K_p,\mathbb{Q}_p(1))\to 0 \tag{11}$$

where the isomorphisms are induced by Kummer theory and the invariant map on the Brauer group. Our notation here is that $\hat{M} := \varprojlim_n M/p^nM\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ for any abelian group M. We let

$$K_p = D_{dR}(V_p)\xrightarrow{\exp} H^1_f(K_p,\mathbb{Q}_p(r))$$

denote the exponential map of Bloch and Kato for the $G_{\mathbb{Q}_p}$-representation V_p. This map is bijective (since $r > 0$) and $H^1_f(K_p, \mathbb{Q}_p(r))$ coincides with $\hat{\mathcal{O}}^\times_{K_p}$ for $r = 1$ and with $H^1(K_p, \mathbb{Q}_p(r))$ for $r > 1$ (cf. [5]). Both source and target for the map exp are free, rank one A_p-modules. By using the sequence (11) for $r = 1$ we therefore find that for each $r \geq 1$ there exists an isomorphism of invertible A_p-modules of the form

$$\widetilde{\exp}: K_p\xrightarrow{\exp} H^1_f(K_p,\mathbb{Q}_p(r))\cong\mathrm{Det}^{-1}_{A_p} R\Gamma(K_p,\mathbb{Q}_p(r)). \tag{12}$$

For any subgroup $H \subseteq G$ we set

$$e_H = \sum_{\chi(H)=1} e_\chi = \frac{1}{|H|}\sum_{g\in H} g,$$

and for any prime ℓ we denote by $I_\ell \subseteq D_\ell \subset G$ the inertia and decomposition group at ℓ. For $x \in A_p$ we set $e_\ell(x) = 1 + (x-1)e_{I_\ell} \in A_p$ (so $x \mapsto e_\ell(x)$ is a multiplicative map that preserves the maximal order of A_p) and we denote by $\mathrm{Fr}_\ell \in G \subset A$ any choice of a Frobenius element.

PROPOSITION 3.4. *Define an element $e_p^*(1 - p^{r-1}\,\mathrm{Fr}_p^{-1})$ of $A_p^\times$ by setting*

$$e_\chi e_p^*(1-p^{r-1}\,\mathrm{Fr}_p^{-1}) = \begin{cases} e_\chi e_p(1-p^{r-1}\,\mathrm{Fr}_p^{-1}) & \text{if } r>1 \text{ or } \chi(D_p)\neq 1\\ |D_p/I_p|^{-1}e_\chi & \text{otherwise.}\end{cases}$$

Then one has $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])_p = 0$ if and only if

$$\mathbb{Z}_p[G]\cdot\prod_{\substack{\ell|N\\ \ell\neq p}} e_\ell(-\mathrm{Fr}_\ell^{-1})\, e_p(1-\frac{\mathrm{Fr}_p}{p^r})^{-1} e_p^*(1-p^{r-1}\,\mathrm{Fr}_p^{-1})\,\widetilde{\exp}(\beta_N) = \mathrm{Det}^{-1}_{\mathbb{Z}_p[G]} R\Gamma(K_p,\mathbb{Z}_p(r)). \tag{13}$$

Proof. It suffices to prove that (13) is equivalent to (10). Now, by its very definition in [9, (67)], the isomorphism θ_p in (9) is induced by isomorphisms

$$\theta_p^{p\text{-part}} : \Xi^{\mathrm{loc}}(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{Det}_{A_p}^{-1} R\Gamma(K_p, \mathbb{Q}_p(r)) \otimes_{A_p} V_p^{-1}$$

and

$$\theta_p^{\ell\text{-part}} : A_p \cong \mathrm{Det}_{A_p}^{-1} R\Gamma(K_\ell, \mathbb{Q}_p(r))$$

for $\ell \neq p$. Concerning $\theta_p^{p\text{-part}}$, we have an exact triangle

$$R\Gamma_f(\mathbb{Q}_p, V_p) \to R\Gamma(\mathbb{Q}_p, V_p) \to R\Gamma_f(\mathbb{Q}_p, V_p^*(1))^*[-2] \tag{14}$$

where the complexes $R\Gamma_f$ are given by an exact triangle

$$K_p[-1] \to R\Gamma_f(\mathbb{Q}_p, V_p) \to \left(D_{cris}(V_p) \xrightarrow{1-p^{-r}\,\mathrm{Fr}_p} D_{cris}(V_p) \right)$$

(with this last complex in degrees 0 and 1) and an isomorphism

$$R\Gamma_f(\mathbb{Q}_p, V_p^*(1))^*[-2] \cong \left(D_{cris}(V_p) \xrightarrow{1-p^{r-1}\,\mathrm{Fr}_p^{-1}} D_{cris}(V_p) \right),$$

this latter complex being concentrated in degrees 1 and 2. The identity map on $D_{cris}(V_p)$ induces isomorphisms

$$K_p \cong \mathrm{Det}_{A_p}^{-1} R\Gamma_f(\mathbb{Q}_p, V_p); \quad A_p \cong \mathrm{Det}_{A_p} R\Gamma_f(\mathbb{Q}_p, V_p^*(1))^*[-2]. \tag{15}$$

The isomorphism $\theta_p^{p\text{-part}}$ is then induced by (15) and (14) together with the (elementary) comparison isomorphism

$$\gamma : Y_\Sigma \otimes_{\mathbb{Q}} \mathbb{Q}_p = H_B(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong H_p(M) = V_p$$

between the Betti and étale realization of M. The isomorphism $\tilde{\exp}$, on the other hand, arises by first passing to the cohomology sequence of (14) and using the identifications in (11) if $r = 1$. From [8, Lem. 1, Lem. 2] one has

$$\theta_p^{p\text{-part}} = e_p(1 - p^{-r}\,\mathrm{Fr}_p)^{-1} e_p^*(1 - p^{r-1}\,\mathrm{Fr}_p^{-1}) \tilde{\exp} \otimes \gamma^{-1}. \tag{16}$$

Now if ℓ is any prime different from p, and $\sigma_\ell \in \mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{ur}}/\mathbb{Q}_\ell)$ is the Frobenius automorphism, then Proposition 6.1 below implies that

$$\mathrm{Det}_{A_p}(-\sigma_\ell \ell^{-1} | V_p^{I_\ell})^{-1} \theta_p^{\ell\text{-part}}(\mathbb{Z}_p[G]) = \mathrm{Det}_{\mathbb{Z}_p[G]}^{-1} R\Gamma(K_\ell, \mathbb{Z}_p(r)).$$

Since $\gamma(\sigma_N)$ is a $\mathbb{Z}_p[G]$-basis of T_p we find that (10) holds if and only if the element

$$\prod_{\substack{\ell | N \\ \ell \neq p}} \mathrm{Det}_{A_p}(-\sigma_\ell \ell^{-1} | V_p^{I_\ell})\, e_p(1 - p^{-r}\,\mathrm{Fr}_p)^{-1} e_p^*(1 - p^{r-1}\,\mathrm{Fr}_p^{-1})\, \tilde{\exp}(\beta_N)$$

is a $\mathbb{Z}_p[G]$-basis of $\mathrm{Det}_{\mathbb{Z}_p[G]}^{-1} R\Gamma(K_p, \mathbb{Z}_p(r))$. But

$$\mathrm{Det}_{A_p}(-\sigma_\ell \ell^{-1} | V_p^{I_\ell}) = \mathrm{Det}_{A_p}(-\mathrm{Fr}_\ell^{-1} \ell^{r-1} | A_p \cdot e_{I_\ell}) = e_\ell(-\mathrm{Fr}_\ell^{-1}) e_\ell(\ell^{r-1})$$

and so Proposition 3.4 is implied by Lemma 3.5 below with u equal to the function which sends 0 to ℓ^{r-1} and all non-zero integers to 1. □

LEMMA 3.5. *Fix a prime ℓ that is not equal to p. If $u : \mathbb{Z} \to \mathbb{Z}_p[G]^\times$ is any function such that $\ell - 1$ divides $u(0) - u(1)$ in $\mathbb{Z}_p[G]$, then $\sum_\chi u(\mathrm{ord}_\ell(f_\chi))e_\chi$ is a unit of $\mathbb{Z}_p[G]$.*

Proof. If $\ell-1$ divides $u(0)-u(1)$, then $\ell-1$ divides $(u(1)-u(0))/u(1)u(0) = u(0)^{-1}-u(1)^{-1}$. It follows that the function u^{-1} also satisfies the hypothesis of the lemma and so it suffices to prove that the element $x_u := \sum_\chi u(\mathrm{ord}_\ell(f_\chi))e_\chi$ belongs to $\mathbb{Z}_p[G]$.
To this end we let $I_\ell = I_{\ell,0} \subseteq G$ denote the inertia subgroup at ℓ and $I_{\ell,k} \subseteq I_{\ell,k-1} \subseteq \cdots \subseteq I_{\ell,1} \subseteq I_{\ell,0}$ its canonical filtration, so that a character χ satisfies $\mathrm{ord}_\ell(f_\chi) = k$ if and only if $\chi(I_{\ell,k}) = 1$ (and $\chi(I_{\ell,k-1}) \neq 1$ if $k > 0$). Then

$$x_u = \sum_{k=0}^{K} u(k)(e_{I_{\ell,k}} - e_{I_{\ell,k-1}}) = \sum_{k=0}^{K-1} (u(k) - u(k+1))e_{I_{\ell,k}} + u(K)e_{I_{\ell,K}}$$

where $K = \mathrm{ord}_\ell(N)$ and we have set $e_{I_{\ell,-1}} := 0$. For $k \geq 1$ one has $e_{I_{\ell,k}} \in \mathbb{Z}_p[G]$ since $I_{\ell,k}$ is an ℓ-group and $\ell \neq p$. If $K = 0$, then $e_{I_{\ell,0}} = e_{I_{\ell,K}} = 1$ also lies in $\mathbb{Z}_p[G]$. Otherwise the assumptions that $\ell - 1$ divides $u(0) - u(1)$ and that $\ell \neq p$ combine to imply that

$$(u(0) - u(1))e_{I_{\ell,0}} = \frac{u(0) - u(1)}{(\ell - 1)\ell^{K-1}} \sum_{g \in I_{\ell,0}} g \in \mathbb{Z}_p[G].$$

$\square$

4. Local Iwasawa theory

As preparation for our proof of (13) we now prove a result in Iwasawa theory. For any natural number n we set $G_n = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$. We write

$$N = N_0 p^\nu; \qquad \nu \geq 0,\ p \nmid N_0$$

and set

$$\Lambda := \mathbb{Z}_p[[G_{Np^\infty}]] = \varprojlim_n \mathbb{Z}_p[G_{Np^n}] \cong \mathbb{Z}_p[G_{N_0\tilde{p}}][[T]].$$

Here we have set $\tilde{p} = p$ for odd p and $\tilde{p} = 4$ for $p = 2$, and the last isomorphism depends on a choice of generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q}(\zeta_{N_0\tilde{p}})) \cong \mathbb{Z}_p$. We set

$$T_p^\infty = \varprojlim_n \mathrm{Ind}_{\mathbb{Q}(\zeta_{Np^n})}^{\mathbb{Q}} \mathbb{Z}_p(r)$$

which is a free rank one Λ-module upon which $G_\mathbb{Q}$ acts by the character $(\chi_{\mathrm{cyclo}})^r\tau^{-1}$ where $\chi_{\mathrm{cyclo}} : G_\mathbb{Q} \to \mathbb{Z}_p^\times$ is the cyclotomic character and $\tau : G_\mathbb{Q} \to G_{Np^\infty} \subseteq \Lambda^\times$ is the tautological character. In this section we shall describe (in Proposition 4.2) a basis of the invertible Λ-module $\mathrm{Det}_\Lambda^{-1}(R\Gamma(\mathbb{Q}_p, T_p^\infty))$.
We note first that the cohomology of $R\Gamma(\mathbb{Q}_p, T_p^\infty)$ is naturally isomorphic to

$$H^i(\mathbb{Q}_p, T_p^\infty) \cong \begin{cases} (\varprojlim_n \mathbb{Q}(\zeta_{Np^n})_p^\times/p^n) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r-1) & i = 1 \\ \prod_{v|p} \mathbb{Z}_p(r-1) & i = 2 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

where the limit is taken with respect to the norm maps (and $\mathbb{Q}(\zeta_{Np^n})_p = \mathbb{Q}(\zeta_{Np^n}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a finite product of local fields). The valuation map induces a natural short exact sequence

$$0 \to \tilde{Z} := \varprojlim_n \mathcal{O}^\times_{\mathbb{Q}(\zeta_{Np^n})_p}/p^n \to \varprojlim_n \mathbb{Q}(\zeta_{Np^n})^\times_p/p^n \xrightarrow{\text{val}} \prod_{v|p} \mathbb{Z}_p \to 0 \tag{18}$$

and in addition Perrin-Riou has constructed an exact sequence [34, Prop. 4.1.3]

$$0 \to \prod_{v|p} \mathbb{Z}_p(r) \to \tilde{Z}(r-1) \xrightarrow{\theta_r^{PR}} R \to \prod_{v|p} \mathbb{Z}_p(r) \to 0 \tag{19}$$

where

$$R := \{f \in \mathbb{Z}[\zeta_{N_0}]_p[[X]] \mid \psi(f) := \sum_{\zeta^p=1} f(\zeta(1+X)-1) = 0\}$$

and $\mathbb{Z}[\zeta_{N_0}]_p$ denotes the finite étale $\mathbb{Z}_p$-algebra $\mathbb{Z}[\zeta_{N_0}] \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We remark that, whilst p is assumed to be odd in [34] one can check that the same arguments show that the sequence (19) exists and is exact also in the case $p = 2$. The $\mathbb{Z}_p$-module R carries a natural continuous G_{Np^∞}-action [34, 1.1.4], and with respect to this action all maps in (17), (18) and (19) are Λ-equivariant. In addition, if $r = 1$, then the exact sequence (19) is due to Coleman and the map θ_1^{PR} is given by

$$\theta_1^{PR}(u) = \left(1 - \frac{\phi}{p}\right) \log(f) \tag{20}$$

where f is the (unique) Coleman power series of the norm compatible system of units u with respect to $(\zeta_{p^n})_{n\geq 1}$ and one has $\phi(f)(X) = f^{\mathrm{Fr}_p}((1+X)^p - 1)$.

LEMMA 4.1. *The Λ-module R is free of rank one with basis*

$$\beta^\infty_{N_0} := \xi_{N_0}(1+X); \qquad \xi_{N_0} := \sum_{N_1|d|N_0} \zeta_d$$

where $N_1 = \prod_{\ell|N_0} \ell$.

Proof. The element ξ_{N_0} is a $\mathbb{Z}_p[G_{N_0}]$-basis of $\mathbb{Z}[\zeta_{N_0}]_p$. Indeed, this observation (which is due originally to Leopoldt [30]) can be explicitly deduced from [31, Th. 2] after observing that the idempotents ε_d of loc. cit. belong to $\mathbb{Z}_p[G_{N_0}]$. On the other hand, Perrin-Riou shows in [33, Lem. 1.5] that if W is the ring of integers in any finite unramified extension of $\mathbb{Z}_p$, then $W[[X]]^{\psi=0}$ is a free rank one $W[[G_{p^\infty}]]$-module with basis $1+X$ (her proof applies for all primes p, including $p=2$). Since $\mathbb{Z}[\zeta_{N_0}]_p$ is a finite product of such rings W and since $G_{Np^\infty} \cong G_{N_0} \times G_{p^\infty}$, the result follows. □

PROPOSITION 4.2. *Let Q be the total ring of fractions of Λ (so Q is a finite product of fields). Viewing $\beta^\infty_{N_0}$ as a Q-basis of*

$$R \otimes_\Lambda Q \cong \tilde{Z}(r-1) \otimes_\Lambda Q \cong H^1(\mathbb{Q}_p, T_p^\infty) \otimes_\Lambda Q \cong (\mathrm{Det}_\Lambda^{-1} R\Gamma(\mathbb{Q}_p, T_p^\infty)) \otimes_\Lambda Q,$$

one has $\Lambda \cdot \beta^\infty_{N_0} = \mathrm{Det}_\Lambda^{-1} R\Gamma(\mathbb{Q}_p, T_p^\infty) \subset (\mathrm{Det}_\Lambda^{-1} R\Gamma(\mathbb{Q}_p, T_p^\infty)) \otimes_\Lambda Q$.

Proof. We note first that, since Λ is noetherian, Cohen-Macauley and semilocal, it is enough to prove that $\beta_{N_0}^{\infty}$ is a $\Lambda_{\mathfrak{q}}$-basis of $\mathrm{Det}_{\Lambda_{\mathfrak{q}}}^{-1} R\Gamma(\mathbb{Q}_p, T_p^{\infty})_{\mathfrak{q}}$ for all height one prime ideals $\mathfrak{q}$ of Λ (see, for example, [17, Lem. 5.7]). In view of (17), (18) and (19) this claim is immediate for prime ideals $\mathfrak{q}$ which are not in the support of the (torsion) Λ-modules $\prod_{v|p} \mathbb{Z}_p(r-1)$ and $\prod_{v|p} \mathbb{Z}_p(r)$. On the other hand, since these modules are each p-torsion free, any prime $\mathfrak{q}$ which does lie in their support is *regular* in the sense that $p \notin \mathfrak{q}$ (see, for example, [17, p. 90]). In particular, in any such case $\Lambda_{\mathfrak{q}}$ is a discrete valuation ring and so it suffices to check cancellation of the Fitting ideals of the occurring torsion modules. But the Fitting ideal of $H^2(\mathbb{Q}_p, T_p^{\infty})_{\mathfrak{q}}$ cancels against that of the module $(\prod_{v|p} \mathbb{Z}_p(r-1))_{\mathfrak{q}}$ which occurs in the $(r-1)$-fold twist of (18), whilst the Fitting ideals of the kernel and cokernel of θ_r^{PR} obviously cancel against each other. □

5. Descent calculations

In this section we deduce equality (13) as a consequence of Proposition 4.2 and thereby finish the proof of Theorem 1.1.

The natural ring homomorphism

$$\Lambda \to \mathbb{Z}_p[G] \subseteq \mathbb{Q}_p[G] = \prod_{\chi} \mathbb{Q}_p(\chi) \tag{21}$$

induces an isomorphism of perfect complexes of $\mathbb{Z}_p[G]$-modules

$$R\Gamma(\mathbb{Q}_p, T_p^{\infty}) \otimes_{\Lambda}^{\mathbb{L}} \mathbb{Z}_p[G] \cong R\Gamma(\mathbb{Q}_p, T_p)$$

and hence an isomorphism of determinants

$$\mathrm{Det}_{\Lambda}^{-1} R\Gamma(\mathbb{Q}_p, T_p^{\infty}) \otimes_{\Lambda} \mathbb{Z}_p[G] \cong \mathrm{Det}_{\mathbb{Z}_p[G]}^{-1} R\Gamma(\mathbb{Q}_p, T_p).$$

Taken in conjunction with Proposition 4.2, this shows that $\beta_{N_0}^{\infty} \otimes 1$ is a $\mathbb{Z}_p[G]$-basis of $\mathrm{Det}_{\mathbb{Z}_p[G]}^{-1} R\Gamma(\mathbb{Q}_p, T_p)$. If we define an element u of $\mathbb{Q}_p[G]^{\times}$ by means of the equality

$$\prod_{\ell | N_0} e_{\ell}(-\mathrm{Fr}_{\ell}^{-1})\, e_p(1 - \frac{\mathrm{Fr}_p}{p^r})^{-1} e_p^*(1 - p^{r-1}\,\mathrm{Fr}_p^{-1})\, \tilde{\exp}(\beta_N) = u \cdot \beta_{N_0}^{\infty} \otimes 1, \tag{22}$$

it is thus clear that the equality (13) is valid if and only if $u \in \mathbb{Z}_p[G]^{\times}$.

5.1. The unit u'. To prove that the element u defined above belongs to $\mathbb{Z}_p[G]^{\times}$ we will compare it to the unit described by the following result.

Lemma 5.1. *There exists a unit $u' \in \mathbb{Z}_p[G]^{\times}$ such that for any integer k with $0 \le k \le \nu$ and any $\mathbb{Q}_p$-rational character χ of G the element $e_{\chi}(\zeta_{p^k} \xi_{N_0}^{\mathrm{Fr}_p^{-k}})$ is given by*

$$\begin{cases} \chi(u') \prod_{\ell | N_0, \ell \nmid f_{\chi}} \frac{1}{\ell - 1} \prod_{\ell | N_0} e_{\ell}(-\mathrm{Fr}_{\ell}^{-1}) e_{\chi} \zeta_{f_{\chi}} & \text{if } k = \mathrm{ord}_p(f_{\chi}) \\ \chi(u')(-\mathrm{Fr}_p^{-1}) \prod_{\ell | N, \ell \nmid f_{\chi}} \frac{1}{\ell - 1} \prod_{\ell | N_0} e_{\ell}(-\mathrm{Fr}_{\ell}^{-1}) e_{\chi} \zeta_{f_{\chi}} & \text{if } k = 1,\ \mathrm{ord}_p(f_{\chi}) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $d \mid N_0$ and $k \geq 0$ we set $d_k := p^k d$ and

$$a(d) := (d,1) \in (\mathbb{Z}/p^\nu\mathbb{Z})^\times \times (\mathbb{Z}/N_0\mathbb{Z})^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times \cong G$$

so that $\zeta_{p^k}\zeta_d^{\mathrm{Fr}_p^{-k}} = \zeta_{d_k}^{a(d)}$. Since $\xi_{N_0} = \sum_{N_1|d|N_0}\zeta_d$ Lemma 5.2 below implies

$$e_\chi(\zeta_{p^k}\xi_{N_0}^{\mathrm{Fr}_p^{-k}}) = \sum_{N_1|d|N_0, f_\chi|d_k} \frac{\phi(f_\chi)}{\phi(d_k)}\mu(\frac{d_k}{f_\chi})\chi^{-1}(\frac{d_k}{f_\chi})\chi(a(d))e_\chi\zeta_{f_\chi}. \tag{23}$$

The only non-vanishing summands in (23) are those for which d_k/f_χ is both square-free and prime to f_χ. Given the nature of the summation condition there is a unique such summand corresponding to

$$d_k/f_\chi = \begin{cases} \prod_{\ell|N_0,\ell\nmid f_\chi}\ell & \text{if } k = \mathrm{ord}_p(f_\chi) \\ \prod_{\ell|N,\ell\nmid f_\chi}\ell & \text{if } k=1 \text{ and } \mathrm{ord}_p(f_\chi)=0. \end{cases}$$

If neither of these conditions on k and $\mathrm{ord}_p(f_\chi)$ is satisfied, then $e_\chi(\zeta_{p^k}\xi_{N_0}^{\mathrm{Fr}_p^{-k}}) = 0$. By using the multiplicativity of μ, ϕ and χ and the equalities $\mu(\ell) = -1$ and $\phi(\ell) = \ell - 1$ we then compute that (23) equals

$$\begin{cases} \chi(a(d_\chi))\prod_{\ell|N_0,\ell\nmid f_\chi}\left(\frac{1}{\ell-1}\big(-\chi^{-1}(\ell)\big)\right)e_\chi\zeta_{f_\chi} & \text{if } k = \mathrm{ord}_p(f_\chi) \\ \chi(a(d_\chi))\prod_{\ell|N,\ell\nmid f_\chi}\left(\frac{1}{\ell-1}\big(-\chi^{-1}(\ell)\big)\right)e_\chi\zeta_{f_\chi} & \text{if } k=1,\ \mathrm{ord}_p(f_\chi)=0 \\ 0 & \text{otherwise} \end{cases}$$

where d_χ is the index of the unique nonvanishing summand in (23), i.e. $d_\chi = f_{\chi,0}\prod_{\ell|N_0,\ell\nmid f_\chi}\ell$ with $f_{\chi,0}$ the prime to p-part of f_χ. Now the element

$$u' := \sum_\chi \chi\big(a(d_\chi)\big)e_\chi \in \mathbb{Q}_p[G]^\times$$

belongs to $\mathbb{Z}_p[G]^\times$ by Lemma 3.5 (indeed the function $d \mapsto a(d)$ is multiplicative, $d_\chi = d(\mathrm{ord}_\ell(f_\chi))$ is a function of $\mathrm{ord}_\ell(f_\chi)$ only and satisfies $d(0) = d(1)$ as such). From here the explicit description of Lemma 5.1 follows because the definition of e_ℓ ensures that $\prod_{\ell|N_0,\ell\nmid f_\chi}\big(-\chi^{-1}(\ell)\big) = \prod_{\ell|N_0} e_\ell(-\mathrm{Fr}_\ell^{-1})e_\chi$. □

Lemma 5.2. *For any $\mathbb{Q}$-rational (resp. $\mathbb{Q}_p$-rational) character χ of $G \cong (\mathbb{Z}/N\mathbb{Z})^\times$, any $d \mid N$ and any* primitive *d-th root of unity ζ_d^a we have*

$$e_\chi\zeta_d^a = \begin{cases} 0 & \text{if } f_\chi \nmid d \\ \dfrac{\phi(f_\chi)}{\phi(d)}\mu(\dfrac{d}{f_\chi})\chi^{-1}(\dfrac{d}{f_\chi})\chi(a)e_\chi\zeta_{f_\chi} & \text{if } f_\chi \mid d \end{cases} \tag{24}$$

in K (resp. K_p). Here $\phi(m)$ is Euler's ϕ-function, $\mu(m)$ is the Möbius function and $\chi(m) = 0$ if $(m, f_\chi) > 1$.

Proof. Recall that we view a $\mathbb{Q}$-rational character χ as the tautological homomorphism $G \to A^\times \to \mathbb{Q}(\chi)^\times$ to the field $\mathbb{Q}(\chi) := e_\chi A$ which is a direct ring factor of A. Any complex embedding $j : \mathbb{Q}(\chi) \to \mathbb{C}$ induces a complex character $j\chi = \eta : G \to \mathbb{C}^\times$. Put $b := N/d$. Under the $\mathbb{C}$-linear map $\sigma_N : K_\mathbb{C} \to \mathbb{C}$ the element

$$(j\, e_\chi)\zeta_d^a = e_\eta \zeta_N^{ab} = \frac{1}{|G|}\sum_{g\in G} \eta(g^{-1}) g\, \zeta_N^{ab} = \frac{1}{\phi(N)} \sum_{x \bmod N} \bar\eta(x)\zeta_N^{xab} \in K_\mathbb{C}$$

is sent to the general Gaussian sum $\phi(N)^{-1}\tau(\bar\eta_N|\zeta_N^{ab})$ in the notation of Hasse's book [22, §20.1]. By [22, §20.2.IV] we have

$$\tau(\bar\eta_N|\zeta_N^{ab}) = \begin{cases} 0 & f_\eta \nmid d \\ \frac{\phi(N)}{\phi(d)}\mu(\frac{d}{f_\chi})\bar\eta(\frac{d}{f_\chi})\eta(a)\tau(\bar\eta) & f_\eta \mid d \end{cases}$$

where the Gaussian sum $\tau(\eta)$ attached to the character η was defined in (7). For $d = f_\chi$ and $\zeta_d^a = \zeta_{f_\chi}$ we find $\tau(\bar\eta) = \phi(f_\chi)\sigma_N((j\, e_\chi)\zeta_{f_\chi})$. This yields the image of (24) under σ_N. Note that $K_\mathbb{C} \cong \prod_{g\in G}\mathbb{C}$ via $x \mapsto (\sigma_N gx)_{g\in G}$ and both sides of (24) are multiplied by $\chi(g)$ after applying g. Since $j\chi(g) = \eta(g)$ is a scalar and σ_N is $\mathbb{C}$-linear we find that (24) holds in $K_\mathbb{C}$, hence in K, hence also in K_p for all p. $\square$

To prove Theorem 1.1 it is enough for us to prove that $uu' \in \mathbb{Z}_p[G]^\times$. Recalling Lemma 3.5 it thus suffices to prove that for each $\mathbb{Q}_p$-rational character χ one has

$$\chi(uu') = \frac{f_{\chi,0}^{r-1}\prod_{\ell|N_0,\ell\nmid f_\chi}(\ell-1)}{[\mathbb{Q}(\zeta_{N_0}) : \mathbb{Q}(\zeta_{f_{\chi,0}})]} \tag{25}$$

where $f_{\chi,0}$ denotes for the prime to p-part of f_χ. (In this regard note that the expression on the right hand side of (25) belongs to $\mathbb{Z}_p^\times$.)
We shall use explicit descent computations to prove that (25) is a consequence of the definition of u in (22). To this end, for each $\mathbb{Q}_p$-character χ of G we let $\mathfrak{q}_\chi$ be the kernel of the homomorphism $\Lambda \to \mathbb{Q}_p(\chi)$ in (21). Then $\mathfrak{q}_\chi$ is a regular prime ideal of Λ and $\Lambda_{\mathfrak{q}_\chi}$ is a discrete valuation ring with residue field $\mathbb{Q}_p(\chi)$. To apply [17, Lem. 5.7] we need to describe a $\Lambda_{\mathfrak{q}_\chi}$-basis of $H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}$ and for this purpose we find it convenient to split the argument into several different cases.

5.2. THE CASE $r > 1$ OR $\chi(D_p) \neq 1$. In this section we shall prove (25) for all characters χ except those which are trivial on D_p in the case that $r = 1$. In particular, the material of this section completes the proof of Theorem 1.1 in the case $r > 1$.
We note first that if either $r > 1$ or $\chi(D_p) \neq 1$, then $\mathfrak{q}_\chi$ does not lie in the support of either $\prod_{v|p}\mathbb{Z}_p(r-1)$ or $\prod_{v|p}\mathbb{Z}_p(r)$. From (17), (18) and (19) we find that $\beta_{N_0}^\infty$ is a $\Lambda_{\mathfrak{q}_\chi}$-basis of $H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi} = (\mathrm{Det}_\Lambda^{-1} R\Gamma(\mathbb{Q}_p, T_p^\infty))_{\mathfrak{q}_\chi}$ and that $\beta_{N_0}^\infty \otimes 1$ is equal to the image of $\beta_{N_0}^\infty$ under the composite map

$$H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi} \to H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi} \otimes_{\Lambda_{\mathfrak{q}_\chi}} \mathbb{Q}_p(\chi) \cong H^1(K_p, \mathbb{Q}_p(r)) \otimes_{A_p} \mathbb{Q}_p(\chi)$$

where the isomorphism is induced by the vanishing of $H^2(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}$ (cf. [17, Lem. 5.7]).

5.2.1. *The descent diagram.* By an obvious semi-local generalization of the argument of [1, §2.3.2] there exists a commutative diagram of Λ-modules

$$\begin{tikzcd} \tilde{Z}(r-1) \arrow[r, "\theta_r^{PR}"] \arrow[d] & R \arrow[d, "\Xi_{r,\nu}"] \\ H^1(K_p, \mathbb{Q}_p(r)) & K_p \arrow[l, "(r-1)!\exp"'] \end{tikzcd} \tag{26}$$

where $\nu = \mathrm{ord}_p(N)$ is as defined at the beginning of §4,

$$\Xi_{r,\nu}(f) = \begin{cases} \sum_{k=1}^{\nu} p^{rk-\nu} f^{\mathrm{Fr}_p^{-k}}(\zeta_{p^k} - 1) + p^{-\nu}(1 - \frac{\mathrm{Fr}_p}{p^r})^{-1} f(0) & \nu \geq 1 \\ \mathrm{Tr}_{K(\zeta_p)/K}(\Xi_{r,1}(f)) & \nu = 0 \end{cases}$$

is the map of [1, Lem. 2.2.2] and the choice of Frobenius element $\mathrm{Fr}_p \in G \cong G_{N_0} \times G_{p^\nu}$ is that which acts trivially on p-power roots of unity. (We are grateful to Laurent Berger for pointing out that the methods of [3] show that the diagram (26) commutes even in the case $p = 2$.)
Now for $f = \beta_{N_0}^\infty = \xi_{N_0}(1+X)$ this formula gives

$$\Xi_{r,\nu}(\beta_{N_0}^\infty) = \begin{cases} \displaystyle\sum_{k=1}^{\nu} p^{rk-\nu} \zeta_{p^k} \mathrm{Fr}_p^{-k} \xi_{N_0} + p^{-\nu}\big(1 - \frac{\mathrm{Fr}_p}{p^r}\big)^{-1} \xi_{N_0} & \nu \geq 1 \\ \mathrm{Tr}_{K(\zeta_p)/K}\left(p^{r-1}\zeta_p \mathrm{Fr}_p^{-1} + \frac{1}{p}(1 - \frac{\mathrm{Fr}_p}{p^r})^{-1}\right)\xi_{N_0} = & \\ \left(-p^{r-1}\mathrm{Fr}_p^{-1} + \frac{p-1}{p}(1 - \frac{\mathrm{Fr}_p}{p^r})^{-1}\right)\xi_{N_0} = & \\ (1 - \frac{\mathrm{Fr}_p}{p^r})^{-1}(1 - p^{r-1}\mathrm{Fr}_p^{-1})\xi_{N_0} & \nu = 0. \end{cases}$$

In addition, since either $r > 1$ or $\chi(D_p) \neq 1$, one has $e_\chi \exp = e_\chi \widetilde{\exp}$ and so the commutativity of (26) implies that the e_χ-projection of (22) is equivalent to an equality in $e_\chi K_p$ of the form

$$\begin{aligned} &\prod_{\ell \mid N_0} e_\ell(-\mathrm{Fr}_\ell^{-1})\, e_p(1 - \frac{\mathrm{Fr}_p}{p^r})^{-1} e_p(1 - p^{r-1}\mathrm{Fr}_p^{-1}) e_\chi \beta_N \\ =&\chi(u)(r-1)!\left(\sum_{k=1}^{\nu} p^{rk-\nu} e_\chi(\zeta_{p^k}\xi_{N_0}^{\mathrm{Fr}_p^{-k}}) + p^{-\nu}\left(1 - \frac{\mathrm{Fr}_p}{p^r}\right)^{-1} e_\chi \xi_{N_0}\right) \end{aligned} \tag{27}$$

if $\nu \geq 1$, resp.

$$\prod_{\ell \mid N_0} e_\ell(-\mathrm{Fr}_\ell^{-1}) e_\chi \beta_N = \chi(u)(r-1)! e_\chi \xi_{N_0} \tag{28}$$

if $\nu = 0$.

5.2.2. *The case* $\mathrm{ord}_p(f_\chi) > 0$. In this case $\nu > 0$ and $e_p(x)e_\chi = e_\chi$ for all $x \in A_p^\times$ and so we may leave out all factors of the form $e_p(-)$ on the left hand side of (27). In addition, Lemma 5.1 implies that the only non-vanishing term in the summation on the right hand side of (27) is that corresponding to $k = \mathrm{ord}_p(f_\chi)$ and moreover that (27) is equivalent to an equality

$$e_\chi\beta_N = \chi(uu')(r-1)!p^{rk-\nu}\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1}e_\chi\zeta_{f_\chi}.$$

Now since $k = \mathrm{ord}_p(f_\chi)$ and $\nu = \mathrm{ord}_p(N)$ we have

$$p^{rk-\nu}\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1} = \frac{[\mathbb{Q}(\zeta_{N_0}):\mathbb{Q}(\zeta_{f_{\chi,0}})]}{f_{\chi,0}^{r-1}\prod_{\ell|N_0,\ell\nmid f_\chi}(\ell-1)}\frac{f_\chi^{r-1}}{[K:\mathbb{Q}(\zeta_{f_\chi})]}.$$

To deduce the required equality (25) from the last two displayed formulas one need only substitute the expression for $e_\chi\beta_N$ given in Proposition 3.1.

5.2.3. *The case* $\mathrm{ord}_p(f_\chi) = 0$ *and* $\nu > 0$. In this case Lemma 5.1 shows that the only nonzero terms in the summation on the right hand side of (27) are those which correspond to $k = 0$ and $k = 1$. Moreover, one has $e_p(x)e_\chi = xe_\chi$ for $x \in A_p^\times$. By Lemma 5.1 equation (27) is thus equivalent to

$$\begin{aligned}(29)\quad &\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)^{-1}(1-p^{r-1}\mathrm{Fr}_p^{-1})e_\chi\beta_N = \\ &\chi(uu')(r-1)!\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1}\left(\frac{p^{r-\nu}}{p-1}(-\mathrm{Fr}_p^{-1})+p^{-\nu}\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)^{-1}\right)e_\chi\zeta_{f_\chi}.\end{aligned}$$

But

$$\begin{aligned}&\frac{p^{r-\nu}}{p-1}(-\mathrm{Fr}_p^{-1})+p^{-\nu}\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)^{-1}\\ =&\frac{p^{-\nu+1}}{p-1}\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)^{-1}\left(p^{r-1}(-\mathrm{Fr}_p^{-1})\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)+\frac{p-1}{p}\right)\\ =&\frac{1}{\phi(p^\nu)}\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)^{-1}(1-p^{r-1}\mathrm{Fr}_p^{-1})\end{aligned}$$

and so (29) implies that

$$e_\chi\beta_N = \chi(uu')(r-1)!\frac{1}{\phi(p^\nu)}\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1}e_\chi\zeta_{f_\chi}.$$

The required equality (25) follows from this in conjunction with the equality

$$\frac{1}{\phi(p^\nu)}\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1} = \frac{[\mathbb{Q}(\zeta_{N_0}):\mathbb{Q}(\zeta_{f_{\chi,0}})]}{f_{\chi,0}^{r-1}\prod_{\ell|N_0,\ell\nmid f_\chi}(\ell-1)}\frac{f_\chi^{r-1}}{[K:\mathbb{Q}(\zeta_{f_\chi})]}$$

and the expression for $e_\chi\beta_N$ given in Proposition 3.1.

5.2.4. *The case $\nu = \mathrm{ord}_p(N) = 0$.* In this case (25) results directly upon substituting the formulas of Proposition 3.1 and Lemma 5.1 (with $k = 0$) into (28).

5.3. The case $r = 1$ and $\chi(D_p) = 1$. In this case $\mathfrak{q}_\chi$ lies in the support of $\prod_{v|p} \mathbb{Z}_p(r-1)$ (but not of $\prod_{v|p} \mathbb{Z}_p(r)$) and $\beta^\infty_{N_0}$ is not a $\Lambda_{\mathfrak{q}_\chi}$-basis of $H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}$. We fix a generator γ of $\mathbb{Z}_p \cong \mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/K(\zeta_p)) \subseteq G_{Np^\infty}$ and then define a uniformizer of $\Lambda_{\mathfrak{q}_\chi}$ by setting

$$\varpi := 1 - \gamma.$$

The p-adic places of the fields $K = \mathbb{Q}(\zeta_{N_0p^\nu})$ and $\mathbb{Q}(\zeta_{N_0p^\infty})$ are in natural bijection. We fix one such place v_0 and set

$$\begin{aligned}\eta^\infty := (1-\zeta_{p^n})_{n\geq 1} \in \; & \varprojlim_n (\mathbb{Q}(\zeta_{N_0p^n})^\times_{v_0})/p^n \\ & \subseteq \varprojlim_n \prod_{v|p} (\mathbb{Q}(\zeta_{N_0p^n})^\times_v)/p^n = H^1(\mathbb{Q}_p, T_p^\infty).\end{aligned}$$

Then the image $\bar\eta^\infty$ of η^∞ in

$$H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}/\varpi \subseteq H^1(K_p, \mathbb{Q}_p(1)) \otimes_{A_p} \mathbb{Q}_p(\chi) =: H^1(K_p, \mathbb{Q}_p(1))_\chi$$

coincides with that of $p \in \mathbb{Q}(\zeta_{N_0})^\times_{v_0}$ and so is nonzero. In particular η^∞ is a $\Lambda_{\mathfrak{q}_\chi}$-basis of $H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}$. By [17, Lem. 5.7] there is an exact sequence

$$0 \to H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}/\varpi \to H^1(K_p, \mathbb{Q}_p(1))_\chi \xrightarrow{\beta} H^2(K_p, \mathbb{Q}_p(1))_\chi \to 0$$

where the Bockstein map β is the χ-projection of the map

$$H^1(K_p, \mathbb{Q}_p(1)) \cong \hat{K}_p^\times \to \prod_{v|p} \mathbb{Q}_p \cong H^2(K_p, \mathbb{Q}_p(1)); \qquad u_v \to \frac{Tr_{K_v/\mathbb{Q}_p} \log_p(u_v)}{\log_p(\chi_{\mathrm{cyclo}}(\gamma))}$$

(see [17, Lem. 5.8] and its proof). This exact sequence induces an isomorphism

$$\phi_\varpi : \mathrm{Det}^{-1}_{\mathbb{Q}_p(\chi)} R\Gamma(\mathbb{Q}_p, V_p)_\chi \cong H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}/\varpi$$

and [17, Lem. 5.7] implies that

$$\bar\lambda \cdot \beta^\infty_{N_0} \otimes 1 = \phi_\varpi^{-1}(\bar\eta^\infty) \tag{30}$$

where the elements $\lambda \in \Lambda^\times_{\mathfrak{q}_\chi}$ and $e \in \mathbb{Z}$ are defined by the equality

$$\lambda \cdot \beta^\infty_{N_0} = \varpi^e \eta^\infty \in H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi} \tag{31}$$

and $\bar\lambda$ denotes the image of λ in $\Lambda_{\mathfrak{q}_\chi}/\varpi$. This description of $\bar\eta^\infty$ implies that $\mathrm{val}(\bar\eta^\infty) = 1 \cdot v_0 = \beta(\exp(b))$ where val is the normalized valuation map for the field K appearing in (11) and

$$b := |D_p|^{-1} \log_p(\chi_{\mathrm{cyclo}}(\gamma)) \in \mathbb{Q}_p \subseteq K_{v_0} \subseteq \prod_{v|p} K_v = K_p.$$

LEMMA 5.3. *The element λ defined above belongs to Λ and in $\mathbb{Q}(\zeta_{N_0})_p = \prod_{v|p}\mathbb{Q}(\zeta_{N_0})_v$ one has*

$$b = -|D_p|^{-1}\left(1-\frac{1}{p}\right)^{-1}\lambda\cdot\xi_{N_0}.$$

This formula uniquely determines the image $\bar{\lambda}$ of λ in $\mathbb{Q}_p(\chi)$.

Proof. With respect to the system $(\zeta_{p^n})_{n\geq 1}$ the Coleman power series associated to the norm compatible system of units $(\eta_\infty)^\varpi = \eta_\infty^{(1-\gamma)}$ is

$$f(X) := \frac{X}{(1+X)^{\chi_{\mathrm{cyclo}}(\gamma)}-1} \equiv \chi_{\mathrm{cyclo}}(\gamma)^{-1} \mod (X).$$

Since R is a free rank one Λ-module we have $\lambda\cdot\beta_{N_0} = \theta^{PR}((\eta_\infty)^\varpi)$ for some $\lambda\in\Lambda$ (which then also satisfies (31)). Computing constant terms in this power series identity and using (20) we have

$$\begin{aligned}\lambda\cdot\xi_{N_0} = (1-\frac{\phi}{p})\log(f(X))\Big|_{X=0}\\ =&(1-\frac{1}{p})\log_p(\chi_{\mathrm{cyclo}}(\gamma)^{-1})\\ =&-(1-\frac{1}{p})|D_p|b,\end{aligned}$$

as required to prove the first assertion of the lemma. On the other hand, the second assertion of the lemma is clear because ξ_{N_0} is a $\mathbb{Q}_p[G_{N_0}]$-basis of $\mathbb{Q}(\zeta_{N_0})_p$ and $\mathbb{Q}_p(\chi) = \Lambda_{\mathfrak{q}_\chi}/\varpi$ is a quotient of $\mathbb{Q}_p[G_{N_0}]$. $\square$

With $\tilde{\exp}$ denoting the map in (12), the last lemma implies that

$$\begin{aligned}\phi_\varpi^{-1}(\bar{\eta}^\infty) =& \bar{\eta}^\infty\wedge\exp(b)\otimes\beta(\exp(b))^{-1}\\ =&-\exp(b)\wedge\bar{\eta}^\infty\otimes\mathrm{val}(\bar{\eta}^\infty)^{-1}\\ =&\tilde{\exp}(-b)\\ =&\tilde{\exp}\left(|D_p|^{-1}\left(1-\frac{1}{p}\right)^{-1}\bar{\lambda}\cdot e_\chi\xi_{N_0}\right),\end{aligned}$$

and hence, using (30), that

$$
\begin{aligned}
&\tilde{\exp}^{-1}(\beta^{\infty}_{N_0}\otimes 1)\\
=&|D_p|^{-1}(1-\frac{1}{p})^{-1}e_\chi(\xi_{N_0})\\
=&|D_p|^{-1}(1-\frac{1}{p})^{-1}\chi(u')\prod_{\ell\mid N_0,\ell\nmid f_\chi}\frac{1}{\ell-1}\prod_{\ell\mid N_0}e_\ell(-\mathrm{Fr}_\ell^{-1})e_\chi\zeta_{f_\chi}\\
=&|D_p|^{-1}(1-\frac{1}{p})^{-1}\chi(u')\prod_{\ell\mid N_0,\ell\nmid f_\chi}\frac{1}{\ell-1}\prod_{\ell\mid N_0}e_\ell(-\mathrm{Fr}_\ell^{-1})[K:\mathbb{Q}(\zeta_{f_\chi})]e_\chi\beta_N\\
=&|D_p/I_p|^{-1}(1-\frac{1}{p})^{-1}\chi(u')\frac{[\mathbb{Q}(\zeta_{N_0}):\mathbb{Q}(\zeta_{f_{\chi,0}})]}{f_{\chi,0}^{r-1}\prod_{\ell\mid N_0,\ell\nmid f_\chi}(\ell-1)}\prod_{\ell\mid N_0}e_\ell(-\mathrm{Fr}_\ell^{-1})e_\chi\beta_N
\end{aligned}
$$

where the second equality follows from Lemma 5.1, the third from Proposition 3.1 and the fourth from the fact that $r=1$, $f_\chi=f_{\chi,0}$ and

$$\frac{[K:\mathbb{Q}(\zeta_{f_\chi})]}{|I_p|\prod_{\ell\mid N_0,\ell\nmid f_\chi}(\ell-1)}=\frac{[\mathbb{Q}(\zeta_{N_0}):\mathbb{Q}(\zeta_{f_{\chi,0}})]}{f_{\chi,0}^{r-1}\prod_{\ell\mid N_0,\ell\nmid f_\chi}(\ell-1)}.$$

The required equality (25) is now obtained by comparing the above formula for $\tilde{\exp}^{-1}(\beta^{\infty}_{N_0}\otimes 1)$ to the definition of u in (22).
This completes our proof of Theorem 1.1.

6. Some remarks concerning $T\Omega^{\mathrm{loc}}(M,\mathfrak{A})$

In this section we prove two results that were used in the proof of Theorem 1.1 but which are most naturally formulated in a much more general setting. These results extend the computations of [9, §5].

We fix a motive M that is defined over $\mathbb{Q}$ (if M is defined over a general number field K as in [9], then we use induction to reduce to the base field $\mathbb{Q}$) and we assume that M is endowed with an action of a finite dimensional semisimple $\mathbb{Q}$-algebra A. (Thus, we stress, in this section M is not assumed to be a Tate motive and A is not assumed to be either commutative or a group ring). For any prime number p we denote by $V_p=H_p(M)$ the p-adic realisation of M. We fix a $\mathbb{Z}_p$-order $\mathfrak{A}_p$ in A_p for which there exists a full projective Galois stable $\mathfrak{A}_p$-sublattice T_p of V_p. We denote by S a finite set of places of $\mathbb{Q}$ containing ∞ and all primes of bad reduction for M and set $S_p:=S\cup\{p\}$ and $S_{p,f}:=S_p\setminus\{\infty\}$. For $R=A_p$ or $R=\mathfrak{A}_p$ we write $D^{\mathrm{perf}}(R)$ for the derived catgeory of perfect complexes of R-modules. We also let $\mathrm{Det}_R:D^{\mathrm{perf}}(R)\to V(R)$ denote the universal determinant functor to the Picard category of virtual objects of R (which is denoted by $P\mapsto[P]$ in [9]) and $\otimes$ the product functor in $V(R)$ (denoted by $\boxtimes$ in [9]). In particular, if R is commutative, then Det_R is naturally isomorphic to the Knudsen-Mumford functor to graded invertible R-modules. We denote by $\mathbf{1}_R$ a unit object of $V(R)$ and we recall that the group $K_1(R)$ naturally identifies with $\mathrm{Aut}_{V(R)}(L)$ for any object L of $V(R)$. For each automorphism

$\alpha : W \to W$ of a finitely generated A_p-module W we denote by $\mathrm{Det}_{A_p}(\alpha|W)$ the element of $K_1(A_p)$ that is represented by α.

6.1. THE CONTRIBUTION FROM PRIMES $\ell \neq p$. An endomorphism μ of an A_p-module H will be said to be *semisimple* if the natural composite map

$$\ker(\mu) \subseteq H \to \mathrm{coker}(\mu) \tag{32}$$

is bijective. In any such case we set

$$\mathrm{Det}^*_{A_p}(\mu) := \mathrm{Det}_{A_p}(\mu|D)$$

where D is any $A_p[\mu]$-invariant complement of $\ker(\mu)$ in H (this element of $K_1(A_p)$ is independent of the choice of D).

PROPOSITION 6.1. *For a prime number $\ell \neq p$ we let σ_ℓ denote the Frobenius automorphism in $\mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{ur}}/\mathbb{Q}_\ell)$. We assume that the endomorphisms $1-\sigma_\ell^{-1}$ and $1-\ell^{-1}\sigma_\ell$ of the A_p-module $V_p^{I_\ell}$ are both semisimple. If*

$$\theta_p^{\ell\text{-}part} : \mathbf{1}_{A_p} \cong \mathrm{Det}^{-1}_{A_p} R\Gamma(\mathbb{Q}_\ell, V_p)$$

denotes the isomorphism in $V(A_p)$ which occurs in [9, (67)], *then*

$$\mathrm{Det}_{A_p}(-\sigma_\ell\ell^{-1}|V_p^{I_\ell})^{-1}\ \theta_p^{\ell\text{-}part}$$

is induced by an isomorphism $\mathbf{1}_{\mathfrak{A}_p} \cong \mathrm{Det}^{-1}_{A_p} R\Gamma(\mathbb{Q}_\ell, T_p)$ in $V(\mathfrak{A}_p)$.

Proof. We shall make use of three natural trivializations of $\mathrm{Det}^{-1}_{A_p} R\Gamma(\mathbb{Q}_\ell, V_p)$. The first two are induced by the exact triangle of complexes of A_p-modules

$$R\Gamma_f(\mathbb{Q}_\ell, V_p) \to R\Gamma(\mathbb{Q}_\ell, V_p) \to R\Gamma_f(\mathbb{Q}_\ell, V_p^*(1))^*[-2] \tag{33}$$

where

$$R\Gamma_f(\mathbb{Q}_\ell, V_p) := [V_p^{I_\ell} \xrightarrow{1-\sigma_\ell^{-1}} V_p^{I_\ell}]$$

is concentrated in degrees 0 and 1 and

$$R\Gamma_f(\mathbb{Q}_\ell, V_p^*(1))^*[-2] := [V_p^{I_\ell} \xrightarrow{1-\ell^{-1}\sigma_\ell} V_p^{I_\ell}]$$

is concentrated in degrees 1 and 2. By definition the map $\theta_p^{\ell\text{-part}}$ is induced by (33) and the maps $\mathrm{id}_{V_p^{I_\ell},\mathrm{triv}}$ on both displayed complexes. The second trivialization $\theta_p^{\ell\text{-part},2}$ is induced by the long exact cohomology sequence

$$\ker(1-\sigma_\ell^{-1}) \cong H^0(\mathbb{Q}_\ell, V_p) \to 0 \to \mathrm{coker}(1-\sigma_\ell^{-1}) \to H^1(\mathbb{Q}_\ell, V_p) \to$$
$$\ker(1-\ell^{-1}\sigma_\ell) \to 0 \to H^2(\mathbb{Q}_\ell, V_p) \cong \mathrm{coker}(1-\ell^{-1}\sigma_\ell)$$

of (33) (where kernels and cokernels are taken on $V_p^{I_\ell}$) together with the isomorphisms (32) for $H = V_p^{I_\ell}$ and $\mu = 1-\sigma_\ell^{-1}$, resp. $\mu = 1-\ell^{-1}\sigma_\ell$, which arise from the assumed semisimplicity of $1-\sigma_\ell^{-1}$, resp. $1-\ell^{-1}\sigma_\ell$. By applying Lemma 6.2 below with $C = V_p^{I_\ell}[0]$ and $\mu = 1-\sigma_\ell^{-1}$, resp. $C = V_p^{I_\ell}[-1]$ and $\mu = 1-\ell^{-1}\sigma_\ell$, we may therefore deduce that

$$\theta_p^{\ell\text{-part}} = \mathrm{Det}^*_{A_p}(1-\sigma_\ell^{-1}|V_p^{I_\ell})^{-1}\,\mathrm{Det}^*_{A_p}(1-\ell^{-1}\sigma_\ell|V_p^{I_\ell})\,\theta_p^{\ell\text{-part},2}. \tag{34}$$

The third trivialization of $\mathrm{Det}_{A_p}^{-1} R\Gamma(\mathbb{Q}_\ell, V_p)$ is the scalar extension of the isomorphism

$$\theta_p^{\ell\text{-part},3} : \mathbf{1}_{\mathfrak{A}_p} \cong \mathrm{Det}_{\mathfrak{A}_p}^{-1} R\Gamma(\mathbb{Q}_\ell, T_p)$$

which is induced by the short exact sequence of complexes of $\mathfrak{A}_p$-modules

$$0 \to R\Gamma(\mathbb{Q}_\ell, T_p) \to R\Gamma(\mathbb{Q}_\ell^{ur}, T_p) \xrightarrow{1-\sigma_\ell^{-1}} R\Gamma(\mathbb{Q}_\ell^{ur}, T_p) \to 0 \tag{35}$$

and the identity map on $R\Gamma(\mathbb{Q}_\ell^{ur}, T_p)$ where $\mathbb{Q}_\ell^{ur}$ is the maximal unramified extension of $\mathbb{Q}_\ell$. Now $H^i(\mathbb{Q}_\ell^{ur}, V_p)$ vanishes for $i \geq 2$ and there are natural isomorphisms

$$H^0(\mathbb{Q}_\ell^{ur}, V_p) = V_p^{I_\ell}; \quad H^1(\mathbb{Q}_\ell^{ur}, V_p) \cong H^0(\mathbb{Q}_\ell^{ur}, V_p^*(1))^* = V_p(-1)^{I_\ell}$$

and so the long exact cohomology sequence of (35) induces isomorphisms

$$H^0(\mathbb{Q}_\ell, V_p) \cong \ker(1 - \sigma_\ell^{-1}); \quad \mathrm{coker}(1 - \ell\sigma_\ell^{-1}) \cong H^2(\mathbb{Q}_\ell, V_p)$$

and a short exact sequence

$$0 \to \mathrm{coker}(1 - \sigma_\ell^{-1}) \to H^1(\mathbb{Q}_\ell, V_p) \to \ker(1 - \ell\sigma_\ell^{-1}) \to 0$$

where kernels and cokernels are taken on $V_p^{I_\ell}$. By applying Lemma 6.2 below with $C = R\Gamma(\mathbb{Q}_\ell^{ur}, V_p)$ and $\phi = 1 - \sigma_\ell^{-1}$ one can therefore deduce that

$$\theta_p^{\ell\text{-part},2} = \mathrm{Det}^*_{A_p}(1 - \sigma_\ell^{-1} | V_p^{I_\ell})\, \mathrm{Det}^*_{A_p}(1 - \ell\sigma_\ell^{-1} | V_p^{I_\ell})^{-1} (\theta_p^{\ell\text{-part},3} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

By combining this with (34) we find that

$$\begin{aligned}\theta_p^{\ell\text{-part}} &= \mathrm{Det}^*_{A_p}(1 - \ell^{-1}\sigma_\ell | V_p^{I_\ell})\, \mathrm{Det}^*_{A_p}(1 - \ell\sigma_\ell^{-1} | V_p^{I_\ell})^{-1} (\theta_p^{\ell\text{-part},3} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \\ &= \mathrm{Det}_{A_p}(-\sigma_\ell \ell^{-1} | V_p^{I_\ell}) (\theta_p^{\ell\text{-part},3} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)\end{aligned}$$

and so the Proposition is clear. □

In the next result we say that a morphism of complexes of A_p-modules $\phi : C \to C$ is 'cohomologically semisimple' if in each degree i the endomorphism $H^i(\phi)$ of the A_p-module $H^i(C)$ is semisimple. We use the same conventions for mapping cones and exact triangles as described in [7, §2.2.1], and we write C_ϕ for the (-1)-shift of the mapping cone of ϕ. We also write $q : C[-1] \to C_\phi$, resp. $p : C_\phi \to C$, for the morphism defined by $q^i(x) = (x, 0)$, resp. $p^i((x, y)) = y$, in each degree i.

LEMMA 6.2. *Let C be any perfect complex of A_p-modules and $\phi : C \to C$ any morphism of complexes of A_p-modules that is cohomologically semisimple. Then there is a commutative diagram in $V(A_p)$*

$$\begin{tikzcd} \mathrm{Det}_{A_p} C_\phi \arrow[r, "\alpha"] \arrow[d, "h"'] & \mathbf{1}_{A_p} \arrow[d, "\lambda"'] \\ \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_{A_p}^{(-1)^i}(H^i(C_\phi)) \arrow[r, "\tilde{\alpha}"] & \mathbf{1}_{A_p}, \end{tikzcd}$$

where h is the canonical isomorphism to cohomology,

$$\lambda := \prod_{i\in\mathbb{Z}} \mathrm{Det}^*_{A_p}(H^i(\phi))^{(-1)^i},$$

α is induced by the exact triangle

$$C_\phi \xrightarrow{p} C \xrightarrow{\phi} C \xrightarrow{q[1]} C_\phi[1] \tag{36}$$

and the identity map on C, and $\tilde{\alpha}$ is induced by combining the short exact sequences

$$0 \to \mathrm{coker}(H^{i-1}(\phi)) \to H^i(C_\phi) \to \ker(H^i(\phi)) \to 0$$

which result from the long exact cohomology sequence of (36) together with the isomorphism (32) with $\mu = H^i(\phi)$ (in each degree i).

Proof. It can be shown that α coincides with the identification that is induced by the short exact sequence of complexes of A_p-modules

$$0 \to C[-1] \xrightarrow{q} C_\phi \xrightarrow{-p} C \to 0 \tag{37}$$

and the identity map on C, and that $\tilde{\alpha}$ coincides with the identification induced by combining the short exact sequences

$$0 \to \mathrm{coker}(H^{i-1}(\phi)) \to H^i(C_\phi) \to \ker(H^i(\phi)) \to 0 \tag{38}$$

which result from the long exact cohomology sequence of (37) together with the isomorphism (32) with $\mu = H^i(\phi)$ (in each degree i). Hence in the sequel we shall use (37) in place of (36).

We let n_C denote the number of integers i for which $H^i(C) \neq 0$. If $n_C = 0$, then C and C_ϕ are both acyclic, $\lambda = 1$ and the claimed result is clear. Also, if $n_C = 1$, then the claimed result can be deduced from [8, Lem. 1] (indeed, whilst the latter result is stated only for commutative algebras, [9, Lem. 2] implies that the commutativity of the given diagram can be proved by reduction to the commutative case - note also that, when comparing results, the functor $\det_{A_p}$ used in [8, Lem. 1] uses a different sign convention from that we use here to define Det_{A_p}). To prove the general case we assume that $n_C > 1$ and argue by induction on n_C.

We let n denote the largest integer i for which $H^i(C) \neq 0$. After replacing C by its truncation in degrees less than or equal to n we may assume that $C^i = 0$ for all $i > n$. Thus, if we set $C_2 := H^n(C)[-n]$, then the tautological surjection $C^n \to H^n(C)$ induces a morphism of complexes $\rho : C \to C_2$ and we let C_1 denote the 'kernel' of this morphism. Thus one has $C_1^i = C^i$ for each $i < n$, C_1^n is the submodule of boundaries of C in degree n, and there exists a canonical short exact sequence of (perfect) complexes of A_p-modules

$$0 \to C_1 \xrightarrow{\subseteq} C \xrightarrow{\rho} C_2 \to 0.$$

For $j \in \{1,2\}$ we write ϕ_j for the morphism $C_j \to C_j$ induced by ϕ, let $h_j : \mathrm{Det}_{A_p} C_{\phi_j} \cong \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_{A_p}^{(-1)^i}(H^i(C_{\phi_j}))$ denote the canonical isomorphism to cohomology and $\alpha_j : \mathrm{Det}_{A_p} C_{\phi_j} \cong \mathbf{1}_{A_p}$ the isomorphism induced by the analogue of (37) (with C and ϕ replaced by C_j and ϕ_j) and the identity map on C_j. Then ϕ_j is cohomologically semisimple and so we obtain an isomorphism $\tilde{\alpha}_j : \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_{A_p}^{(-1)^i}(H^i(C_{\phi_j})) \cong \mathbf{1}_{A_p}$ by combining (38) with ϕ replaced by ϕ_j and the isomorphisms (32) with $\mu = H^i(\phi_j)$ in each degree i. For the same reason we can define elements $\lambda_1 := \prod_{i\in\mathbb{Z}} \mathrm{Det}^*_{A_p}(H^i(\phi_1))^{(-1)^i} = \prod_{i\le n-1} \mathrm{Det}^*_{A_p}(H^i(\phi))^{(-1)^i}$ and $\lambda_2 := \prod_{i\in\mathbb{Z}} \mathrm{Det}^*_{A_p}(H^i(\phi_2))^{(-1)^i} = \mathrm{Det}^*_{A_p}(H^n(\phi))^{(-1)^n}$.
To prove the required inductive step we consider the following diagram in $V(A_p)$

(39)

$$\begin{tikzcd}
& \mathbf{1}_{A_p} \arrow[rr, equal] \arrow[dd, "\lambda"] & & \mathbf{1}_{A_p}\otimes\mathbf{1}_{A_p} \arrow[dd, "\lambda_1\otimes\lambda_2"] \\
\mathrm{Det}_{A_p} C_\phi \arrow[ur, "\alpha"] \arrow[rr] \arrow[dd, "h"] & & \mathrm{Det}_{A_p} C_{\phi_1} \otimes \mathrm{Det}_{A_p} C_{\phi_2} \arrow[ur, "\alpha_1\otimes\alpha_2"] \arrow[dd, "h_1\otimes h_2"] & \\
& \mathbf{1}_{A_p} \arrow[rr, equal] & & \mathbf{1}_{A_p}\otimes\mathbf{1}_{A_p} \\
\bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_{A_p}^{(-1)^i} H^i(C_\phi) \arrow[ur, "\tilde{\alpha}"] \arrow[rr] & & \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_{A_p}^{(-1)^i} H^i(C_{\phi_1}) \otimes \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_{A_p}^{(-1)^i} H^i(C_{\phi_2}). \arrow[ur, "\tilde{\alpha}_1\otimes\tilde{\alpha}_2"] &
\end{tikzcd}$$

The rear face of this diagram commutes because $\lambda = \lambda_1\lambda_2$ and the right hand face commutes by the inductive hypothesis applied to the morphism $\phi_j : C_j \to C_j$ for both $j = 1$ and $j = 2$. To investigate further we use the following diagram of short exact sequences of complexes

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_1[-1] \arrow[r] \arrow[d, "\subseteq"'] & C_{\phi_1} \arrow[r] \arrow[d, "\subseteq"'] & C_1 \arrow[r] \arrow[d, "\subseteq"'] & 0 \\
0 \arrow[r] & C[-1] \arrow[r, "q"] \arrow[d, "\rho[-1]"'] & C_\phi \arrow[r, "-p"] \arrow[d] & C \arrow[r] \arrow[d, "\rho"'] & 0 \\
0 \arrow[r] & C_2[-1] \arrow[r] \arrow[d] & C_{\phi_2} \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{40}
$$

where the second row is (37), the first and third rows are the analogues of (37) for ϕ_1 and ϕ_2 respectively and the third arrow in the central column is induced by ρ and $\rho[-1]$ in the obvious way. Indeed, the upper, resp. lower, front arrow in (39) denotes the isomorphism induced by the central column of (40), resp. by the long exact cohomology sequence of this column, and so the front face of (39) commutes by virtue of [6, Th. 3.3] and the upper face commutes because the diagram (40) is commutative. In addition, the lower face of (39) commutes because the long exact cohomology sequence of the central column of (40) decomposes to give equalities $H^i(C_{\phi_1}) = H^i(C_\phi)$ in each degree $i < n$, an exact sequence $0 \to H^n(C_{\phi_1}) \cong \mathrm{coker}(H^{n-1}(\phi)) \to H^n(C_\phi) \to \ker(H^n(\phi)) \cong H^n(C_{\phi_2}) \to 0$ which coincides with (38) for $i = n$, and an identification $H^{n+1}(C_\phi) \cong \mathrm{coker}(H^n(\phi)) = H^{n+1}(C_{\phi_2})$. At this stage we may deduce that the left hand face of (39) is commutative and this proves the required inductive step. □

6.2. Artin-Verdier Duality. In this section we extend [9, Lem. 14] to include the case $p = 2$ (and hence resolve the issue raised in [9, Rem. 16]). Before stating the main result we recall that [9, (78)] defines an isomorphism in $V(A_p)$ of the form

$$\theta'_p : \left(\bigotimes_{\ell\in S_{p,f}} \mathrm{Det}^{-1}_{A_p} R\Gamma(\mathbb{Q}_\ell, V_p)\right) \otimes_{A_p} \mathrm{Det}^{-1}_{A_p}(V_p) \cong A_p \otimes_{\mathfrak{A}_p} \Lambda_p(S, T_p)$$

where $\Lambda_p(S, T_p) := \mathrm{Det}_{\mathfrak{A}_p} C(\mathbb{Q}, T_p)$ with $C(\mathbb{Q}, T_p)$ a certain canonical perfect complex of $\mathfrak{A}_p$-modules (as occurs in the diagram (42) below with $E = \mathbb{Q}$).

Proposition 6.3. *There exists an element α of $\mathrm{Aut}_{\mathfrak{A}_p}(T_p)$ such that, setting*

$$\epsilon_\alpha := \mathrm{Det}_{A_p}(2|H^0(\mathbb{R}, V_p))\,\mathrm{Det}_{A_p}(2\alpha|H^0(\mathbb{R}, V_p(1)))^{-1} \in K_1(A_p),$$

the isomorphism $\epsilon_\alpha \cdot \theta'_p$ *is induced by an isomorphism in* $V(\mathfrak{A}_p)$ *of the form*

$$\left(\bigotimes_{\ell \in S_{p,f}} \mathrm{Det}^{-1}_{\mathfrak{A}_p} R\Gamma(\mathbb{Q}_\ell, T_p)\right) \otimes_{\mathfrak{A}_p} \mathrm{Det}^{-1}_{\mathfrak{A}_p}(T_p) \cong \Lambda_p(S, T_p). \tag{41}$$

Furthermore, if either

i) p *is odd, or*
ii) K *is a finite imaginary Galois extension of* $\mathbb{Q}$, $T_p = \mathrm{Ind}^{\mathbb{Q}}_K \mathbb{Z}_p(r)$ *for any prime* p *and any integer* r *and* $\mathfrak{A}_p = \mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})]$,

then one can take $\alpha = \mathrm{id}_{T_p}$.

The proof of this result occupies the remainder of this section. We note first that if p is odd, then $\epsilon_\alpha \in K_1(\mathfrak{A}_p)$ for any $\alpha \in \mathrm{Aut}_{\mathfrak{A}_p}(T_p)$ and so the above claim simply asserts that θ'_p induces an isomorphism in $V(\mathfrak{A}_p)$ of the form (41). Since this is precisely the statement of [9, Lem. 14] we shall assume henceforth that $p = 2$.

If E is any number field, then [9, (81)] gives a true nine term diagram

$$\begin{tikzcd}
\bigoplus_{v\in S_\infty} R\Gamma_\Delta(E_v, T_p^*(1))^*[-4] \arrow[r, equal] \arrow[d] & \bigoplus_{v\in S_\infty} R\Gamma_\Delta(E_v, T_p^*(1))^*[-4] \arrow[d, "\alpha(E)"'] & \\
\tilde{R\Gamma}_c(\mathcal{O}_{E,S_p}, T_p^*(1))^*[-4] \arrow[r] \arrow[d] & {}_3L(S_p, T_p)[-1] \arrow[r] \arrow[d] & R\Gamma_c(\mathcal{O}_{E,S_p}, T_p) \arrow[d, equal] \\
R\Gamma_c(\mathcal{O}_{E,S_p}, T_p^*(1))^*[-4] \arrow[r] & C(K, T_p) \arrow[r] & R\Gamma_c(\mathcal{O}_{E,S_p}, T_p)
\end{tikzcd} \tag{42}$$

where the complex ${}_3L(S_p, T_p)$ is endowed with a natural quasi-isomorphism

$$\beta(E) : {}_3L(S_p, T_p) \cong \bigoplus_{v \in S_p} R\Gamma(E_v, T_p).$$

To prove the Proposition we make an explicit study of the composite morphism $\beta(\mathbb{Q}) \circ \alpha(\mathbb{Q})$. To do this we observe that if E is any Galois extension of $\mathbb{Q}$ with group Γ, then (42), resp. $\beta(E)$, is a true nine-term diagram, resp. quasi-isomorphism, of complexes of $\mathfrak{A}_p[\Gamma]$-modules and the same arguments as used in [8, Lem. 11] show that application of $R\,\mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(\mathbb{Z}_p, -)$ to (42), resp. $\beta(E)$, renders a diagram which is naturally isomorphic to the corresponding diagram for $E = \mathbb{Q}$, resp. a quasi-isomorphism which identifies naturally with $\beta(\mathbb{Q})$.

We now fix E to be an imaginary quadratic field and set $\Gamma = \mathrm{Gal}(E/\mathbb{Q})$ and $R\Gamma_{\mathrm{Tate}}(E_v, -) = R\Gamma(E_v, -)$ for each non-archimedean place v. Then for each $v_0 \in S$ one has a natural morphism $R\Gamma_{\mathrm{Tate}}(E_{v_0}, -) \to R\Gamma(E_{v_0}, -)$ and we let $\gamma_{v_0}(E)$ denote the following composite morphism in $D(\mathfrak{A}_p[\Gamma])$

$$R\Gamma_\Delta(E_\infty, T_p^*(1))^*[-4] \xrightarrow{\beta(E)\circ\alpha(E)} \bigoplus_{v\in S_p} R\Gamma(E_v, T_p) \to R\Gamma(E_{v_0}, T_p).$$

Now if v_0 is non-archimedean, then $\gamma_{v_0}(E)$ is equal to the composite

$$R\Gamma_\Delta(E_\infty, T_p^*(1))^*[-4] \to \bigoplus_{v\in S_p} R\Gamma_{\mathrm{Tate}}(E_v, T_p) \to R\Gamma(E_{v_0}, T_p) \tag{43}$$

where the first arrow is the diagonal morphism in the following commutative diagram in $D(\mathfrak{A}_p[\Gamma])$

$$\begin{tikzcd}
R\Gamma_\Delta(E_\infty, T_p^*(1))^*[-3] \arrow[r] \arrow[d, equal] & \tilde{R\Gamma}_c(\mathcal{O}_{E,S_p}, T_p^*(1))^*[-3] \arrow[d] & R\Gamma(\mathcal{O}_{E,S_p}, T_p) \arrow[l, "\mathrm{AV}"'] \arrow[d] \\
R\Gamma_\Delta(E_\infty, T_p^*(1))^*[-3] \arrow[r] & \bigoplus_{v\in S_p} R\Gamma_{\mathrm{Tate}}(E_v, T_p^*(1))^*[-2] & \bigoplus_{v\in S_p} R\Gamma_{\mathrm{Tate}}(E_v, T_p). \arrow[l, "\oplus \mathrm{AV}_v"']
\end{tikzcd}$$

in which the left, resp. right, hand square follows directly from the definition of $R\Gamma_\Delta(E_\infty, T_p^*(1))$ in [9, (80)], resp. from the compatibility of local and global Artin-Verdier duality as in [9, Lem. 12]. Since the image of the lower left hand arrow in this diagram is contained in the summand $R\Gamma_{\mathrm{Tate}}(E_\infty, T_p^*(1))^*[-2]$ it is therefore clear that (43) is the zero morphism. Hence, there exists a natural isomorphism in $D(\mathfrak{A}_p[\Gamma])$ of the form

$$C(E, T_p) \cong C_\infty(E, T_p)[-1] \oplus \bigoplus_{v\in S_{p,f}} R\Gamma(E_v, T_p)[-1]$$

where $C_\infty(E, T_p)$ is a complex which lies in a distinguished triangle

$$R\Gamma_\Delta(E_\infty, T_p^*(1))^*[-3] \xrightarrow{\gamma_\infty(E)} R\Gamma(E_\infty, T_p) \to C_\infty(E, T_p) \to . \tag{44}$$

Now $R\Gamma_\Delta(E_\infty, T_p^*(1))^*[-3] = T_p(-1)[-3]$ and $R\Gamma(E_\infty, T_p) = T_p[0]$ and so we may identify $\gamma_\infty(E)$ with an element of

$$\mathrm{Hom}_{D(\mathfrak{A}_p[\Gamma])}(T_p(-1)[-3], T_p[0]) = \mathrm{Ext}^3_{\mathfrak{A}_p[\Gamma]}(T_p(-1), T_p).$$

With respect to this identification, $C_\infty(E, T_p)$ represents $\gamma_\infty(E)$ viewed as a Yoneda extension and so can be obtained as a push-out

$$\begin{tikzcd}
0 \arrow[r] & T_p \arrow[r] \arrow[d, "\mu"'] & T_p[\Gamma] \arrow[r, "1-c"] \arrow[d] & T_p[\Gamma] \arrow[r, "1+c"] \arrow[d, equal] & T_p[\Gamma] \arrow[r] \arrow[d, equal] & T_p(-1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_p \arrow[r] & B_\mu \arrow[r] & T_p[\Gamma] \arrow[r, "1+c"] & T_p[\Gamma] \arrow[r] & T_p(-1) \arrow[r] & 0
\end{tikzcd} \tag{45}$$

for some $\mathfrak{A}_p[\Gamma]$-homomorphism μ. Here (and in the sequel) we write c for the diagonal action of complex conjugation that is induced via the inclusion $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \subset G_\mathbb{Q}$ corresponding to a fixed embedding $\mathbb{Q}^c \subset \mathbb{C}$ (together with the natural action of $G_\mathbb{Q}$ on T_p and the action of $G_\mathbb{Q}$ on Γ via the projection $G_\mathbb{Q} \to \Gamma$ that corresponds to a fixed embedding $K \subset \mathbb{Q}^c$). Now for any $\mathfrak{A}_p[\Gamma]$-module X the above diagram induces a commutative diagram of the form

$$\begin{tikzcd}
\mathrm{Ext}^i_{\mathfrak{A}_p[\Gamma]}(T_p, X) \arrow[r] & \mathrm{Ext}^{i+3}_{\mathfrak{A}_p[\Gamma]}(T_p(-1), X) \\
\mathrm{Ext}^i_{\mathfrak{A}_p[\Gamma]}(T_p, X) \arrow[r] \arrow[u, "\mu^*"] & \mathrm{Ext}^{i+3}_{\mathfrak{A}_p[\Gamma]}(T_p(-1), X). \arrow[u, equal]
\end{tikzcd}$$

But $C_\infty(E,T_p)$ belongs to $D^{\mathrm{perf}}(\mathfrak{A}_p[\Gamma])$ (since the lower row of (42) belongs to $D^{\mathrm{perf}}(\mathfrak{A}_p[\Gamma]))$ and so the projective dimension of the $\mathfrak{A}_p[\Gamma]$-module B_μ is finite and therefore at most 1. This implies that the upper (resp. lower) horizontal map in the last diagram is bijective for $i \geq 2$ and surjective for $i = 1$ (resp. bijective for $i \geq 1$). The map μ_* is therefore bijective for each $i \geq 2$ and surjective for $i = 1$ and so [23, Th. 3.1] implies that there exists $\alpha \in \mathrm{Aut}_{\mathfrak{A}_p}(T_p)$ and a projective $\mathfrak{A}_p[\Gamma]$-module P such that $\mu - \alpha$ is a composite of the form $T_p \to P \to T_p$. Now the Γ-module

$$\mathrm{Hom}_{\mathfrak{A}_p}(T_p(-1), P) \cong \mathrm{Hom}_{\mathfrak{A}_p}(T_p(-1), \mathfrak{A}_p) \otimes_{\mathfrak{A}_p} P =: T^* \otimes_{\mathfrak{A}_p} P$$

is cohomologically trivial (indeed, it suffices to check this for $P = \mathfrak{A}_p[\Gamma]$ in which case $T^* \otimes_{\mathfrak{A}_p} \mathfrak{A}_p[\Gamma] = T^* \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma] \cong \mathbb{Z}_p[\Gamma]^d$ with $d = \mathrm{rank}_{\mathbb{Z}_p} T^*$) and so $\mathrm{Ext}^3_{\mathfrak{A}_p[\Gamma]}(T_p(-1), P) = H^3(\Gamma, \mathrm{Hom}_{\mathfrak{A}_p}(T_p(-1), P)) = 0$. In the diagram (45) we may thus assume that $\mu = \alpha \in \mathrm{Aut}_{\mathfrak{A}_p}(T_p)$ and so there exists an isomorphism in $D^{\mathrm{perf}}(\mathfrak{A}_p[\Gamma])$ between $C_\infty(E,T_p)$ and the complex

$$T_p[\Gamma] \xrightarrow{\alpha(1-c)} T_p[\Gamma] \xrightarrow{1+c} T_p[\Gamma]$$

(where the first term $T_p[\Gamma]$ occurs in degree 0) which induces the identity map in all degrees of cohomology. Writing $C_\infty(T_p)$ for the complex

$$T_p \xrightarrow{\alpha(1-c)} T_p \xrightarrow{1+c} T_p$$

(where the first term is placed in degree 0), we may thus deduce the existence of a composite isomorphism in $D(\mathfrak{A}_p)$ of the form

$$\begin{aligned} & C_\infty(T_p)[-1] \oplus \bigoplus_{v \in S_{p,f}} R\Gamma(\mathbb{Q}_v, T_p)[-1] \\ \cong\, & R\,\mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(\mathbb{Z}_p, C_\infty(E,T_p)[-1] \oplus \bigoplus_{v \in S_{p,f}} R\Gamma(E_v, T_p)[-1]) \\ \cong\, & R\,\mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(\mathbb{Z}_p, C(K,T_p)) \\ \cong\, & C(\mathbb{Q}, T_p). \end{aligned}$$

When taken in conjunction with the isomorphism

$$j(T_p) : \mathrm{Det}_{\mathfrak{A}_p} C_\infty(T_p)[-1] = \mathrm{Det}^{-1}_{\mathfrak{A}_p}(T_p) \otimes \mathrm{Det}_{\mathfrak{A}_p}(T_p) \otimes \mathrm{Det}^{-1}_{\mathfrak{A}_p}(T_p) \cong \mathrm{Det}^{-1}_{\mathfrak{A}_p}(T_p)$$

that is induced by the isomorphism which cancels the last two factors in the central expression, the above composite isomorphism induces an isomorphism in $V(\mathfrak{A}_p)$ of the form

$$\theta''_p : \left(\bigotimes_{\ell \in S_{p,f}} \mathrm{Det}^{-1}_{\mathfrak{A}_p} R\Gamma(\mathbb{Q}_\ell, T_p) \right) \otimes \mathrm{Det}^{-1}_{\mathfrak{A}_p}(T_p) \cong \mathrm{Det}_{\mathfrak{A}_p} C(\mathbb{Q}, T_p) =: \Lambda_p(S, T_p).$$

Now $A_p \otimes_{\mathfrak{A}_p} \theta''_p$ differs from θ'_p only in the following sense: in place of the map $A_p \otimes_{\mathfrak{A}_p} j(T_p)$ the isomorphism θ'_p involves the composite isomorphism

$$j(V_p) : A_p \otimes_{\mathfrak{A}_p} \mathrm{Det}_{\mathfrak{A}_p} C_\infty(T_p)[-1] \cong \mathrm{Det}^{-1}_{A_p}(A_p \otimes_{\mathfrak{A}_p} H^0(C_\infty(T_p))) \otimes \mathrm{Det}^{-1}_{A_p}(A_p \otimes_{\mathfrak{A}_p} H^2(C_\infty(T_p))) \cong \mathrm{Det}^{-1}_{A_p} V_p$$

where the first isomorphism is the canonical 'passage to cohomology' map and the second is induced by the natural identifications $A_p \otimes_{\mathfrak{A}_p} H^0(C_\infty(T_p)) = H^0(\mathbb{R}, V_p), A_p \otimes_{\mathfrak{A}_p} H^2(C_\infty(T_p)) = H^0(\mathbb{R}, V_p(-1))$ and $H^0(\mathbb{R}, V_p) \oplus H^0(\mathbb{R}, V_p(-1)) = V_p$. But $A_p \otimes_{\mathfrak{A}_p} H^0(C_\infty(T_p))$ identifies with

$$V_p^+ \oplus V_p^- \xrightarrow{(0,2\alpha)} V_p^+ \oplus V_p^- \xrightarrow{(2,0)} V_p^+ \oplus V_p^-$$

where $V_p^+ := H^0(\mathbb{R}, V_p)$ and $V_p^- := H^0(\mathbb{R}, V_p(-1))$ and so, by an explicit computation, one has $A_p \otimes_{\mathfrak{A}_p} j(T_p) = \epsilon_\alpha \cdot j(V_p)$ where ϵ_α is the element of $K_1(\mathfrak{A}_p)$ defined in the statement of Proposition 6.3. From the induced equality

$$A_p \otimes_{\mathfrak{A}_p} \theta''_p = \epsilon_\alpha \cdot \theta'_p$$

we deduce the first claim of the Proposition.

At this stage it only remains for us to prove that under the conditions of case ii) of the Proposition one can take $\alpha = \mathrm{id}_{T_p}$. To do this we fix a (complex) archimedean place v of K and write G_v for its decomposition group. Then there are natural identifications in $D(\mathbb{Z}_p[G])$ of the form

$$R\Gamma_\Delta(\mathbb{Q}_\infty, T_p^*(1))^* \cong \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p[G_v]} R\Gamma_\Delta(K_v, \mathbb{Z}_p(1-r))^*$$

and

$$R\Gamma(\mathbb{Q}_\infty, T_p) \cong \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p[G_v]} R\Gamma(K_v, \mathbb{Z}_p(r))$$

and, with respect to these identifications, the same argument as used to prove [8, Prop. 3.4] allows one to show that

$$R\,\mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(\mathbb{Z}_p, \gamma_\infty(E)) = \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p[G_v]} \gamma_v \tag{46}$$

where $\gamma_\infty(E)$ is as defined just prior to (43) and γ_v is a morphism in $D(\mathbb{Z}_p[G_v])$ of the form

$$\gamma_v : \mathbb{Z}_p(r-1)[-3] \cong R\Gamma_\Delta(K_v, \mathbb{Z}_p(1-r))^*[-4] \to R\Gamma(K_v, \mathbb{Z}_p(r)) \cong \mathbb{Z}_p(r)[0].$$

Now since $C_\infty(E, T_p)$ belongs to $D^{\mathrm{perf}}(\mathbb{Z}_p[G][\Gamma])$ the triangle (44) combines with the decomposition (46) to imply that γ_v corresponds to a non-zero element of $\mathrm{Ext}^3_{G_v}(\mathbb{Z}_p(r), \mathbb{Z}_p(r-1)) \cong \mathbb{Z}_p/(2)$. The claimed result now follows because the unique non-zero element of this group is also represented by the Yoneda extension

$$0 \to \mathbb{Z}_p(r) \xrightarrow{\delta} \mathbb{Z}_p(r)[G_v] \xrightarrow{1-c} \mathbb{Z}_p(r)[G_v] \xrightarrow{1+c} \mathbb{Z}_p(r)[G_v] \xrightarrow{\pi} \mathbb{Z}_p(r-1) \to 0$$

where, if c_v denotes the generator of G_v, then $\delta(a) = a + (-1)^r a \cdot c_v$ and π is induced by the natural isomorphism between $\mathbb{Z}_p(r-1)$ and the submodule $\mathbb{Z}_p(r) \cdot (1 - c_v)$ of $\mathbb{Z}_p(r)[G_v]$.

This completes the proof of Proposition 6.3.

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