ON MAIN CONJECTURES OF GEOMETRIC IWASAWA THEORY AND RELATED CONJECTURES

DAVID BURNS

Abstract. We prove the main conjecture of non-commutative Iwasawa theory for flat, smooth sheaves on schemes that are separated and of finite type over a finite field. This result gives a natural non-commutative generalisation of the main conjectures that were proved by Crew and by Emerton and Kisin. By using the relevant descent formalism we then deduce a range of explicit consequences including an equivariant refinement of a leading term formula proved by Lichtenbaum, the proof of Chinburg's '$\Omega(3)$-Conjecture' for global functions fields and a non-abelian generalisation of Deligne's proof of the Brumer-Stark Conjecture.

1. Introduction and statement of main results

We fix a power q of a prime p, write $\mathbb{F}_q$ for the finite field of cardinality q and fix a separable closure $\mathbb{F}_q^c$ of $\mathbb{F}_q$. We also fix a separated variety X that is of finite type over $\mathbb{F}_q$ and a geometric point $\overline{x}$ of X. We then consider compact p-adic Lie groups G which lie in a commutative diagram of continuous homomorphisms of the form

$$\begin{tikzcd} \pi_1(X,\overline{x}) \arrow[rr, "\pi_{X,\overline{x}}"] \arrow[dr, "\pi'_G"'] & & \pi_1^p(\mathbb{F}_q,\mathbb{F}_q^c) \\ & G \arrow[ur, "\pi_G"'] & \end{tikzcd} \tag{1}$$

where $\pi_{X,\overline{x}}$ is the canonical homomorphism to the maximal pro-p quotient $\pi_1^p(\mathbb{F}_q,\mathbb{F}_q^c)$ of $\pi_1(\mathbb{F}_q,\mathbb{F}_q^c)$ and π'_G is surjective. In any such case the composite homomorphism $\pi_1(\mathbb{F}_q,\mathbb{F}_q^c) \to \mathrm{Gal}(\mathbb{F}_q^c/\mathbb{F}_q) \to \mathbb{Z}_p$, where the first map is the canonical isomorphism and the second sends the (arithmetic) Frobenius automorphism $x \mapsto x^q$ to 1, induces an identification of $\mathrm{im}(\pi_{X,\overline{x}}) = \mathrm{im}(\pi_G)$ with an open subgroup of $\mathbb{Z}_p$. Motivated by the approach of Coates, Fukaya, Kato, Sujatha and Venjakob in [15], we therefore write $\Lambda(G)$ for the p-adic Iwasawa algebra of G, set $H := \ker(\pi_G)$ and consider the (left and right) Ore set of non-zero divisors in $\Lambda(G)$ that is defined by setting

$$S := \{f \in \Lambda(G) : \Lambda(G)/\Lambda(G)f \text{ is a finitely generated } \Lambda(H)\text{-module}\}.$$

Preliminary version of December 2010.

We write $\Lambda(G)_S$ for the localisation of $\Lambda(G)$ at S and $D^{\mathrm{p}}_S(\Lambda(G))$ for the full triangulated subcategory of the derived category of $\Lambda(G)$-modules comprising perfect complexes $C^\bullet$ for which $\Lambda(G)_S \otimes^{\mathbb{L}}_{\Lambda(G)} C^\bullet$ is acyclic. We also write $\partial_G : K_1(\Lambda(G)_S) \to K_0(\Lambda(G), \Lambda(G)_S)$ for the connecting homomorphism of relative algebraic K-theory (normalised as in [10, §1.2]) and recall that any complex $C^\bullet$ in $D^{\mathrm{p}}_S(\Lambda(G))$ has a canonical 'refined Euler characteristic' $\chi^{\mathrm{ref}}(C^\bullet)$ in $K_0(\Lambda(G), \Lambda(G)_S)$ (see §2.1).

We recall that a continuous representation of the form $\rho : G \to \mathrm{GL}_m(\mathcal{O}_\rho)$, with $\mathcal{O}_\rho$ a finite extension of $\mathbb{Z}_p$ in $\mathbb{Q}_p^c$, is said to be an 'Artin representation of G' if $\rho(G)$ is finite. Any such ρ gives rise to a flat, smooth $\mathcal{O}_\rho$-sheaf $\mathcal{L}_\rho$ on X. We further recall that for any such ρ and any element ξ of the Whitehead group $K_1(\Lambda(G)_S)$ one can define the 'leading term of ξ at ρ' as a non-zero element $\xi^*(\rho)$ of $\mathbb{Q}_p^c$ (see §2.3).

For any finite extension $\mathcal{O}$ of $\mathbb{Z}_p$, and any flat, smooth $\mathcal{O}$-sheaf $\mathcal{L}$ on X, we write $Z(X, \mathcal{L}, t)$ for the associated Zeta function in a formal variable t (see §4.1.1). We write $r_{\mathcal{L}}$ for the algebraic order of $Z(X, \mathcal{L}, t)$ at $t = 1$ and define the leading term

$$Z^*(X, \mathcal{L}, 1) := \lim_{t \to 1} (1-t)^{-r_{\mathcal{L}}} Z(X, \mathcal{L}, t) \in \mathbb{Q}_p^{c\times}.$$

The following result is then a non-abelian generalisation of a result of Emerton and Kisin [18, Cor. 1.8] and hence also of the 'main conjecture' proved by Crew in [16].

Theorem 1.1. *Let $s_X : X \to \mathrm{Spec}(\mathbb{F}_q)$ be a separated morphism of finite type. Let G be any compact p-adic Lie group as in (1) and write $f : Y \to X$ for the corresponding pro-covering of X of group G. Let $\mathcal{L}$ be a flat, smooth $\mathbb{Z}_p$-sheaf on X.*

Then the complex $R\Gamma(\mathrm{Spec}(\mathbb{F}_q)_{\text{ét}}, Rs_{X,!}f_*f^*\mathcal{L})$ belongs to $D^{\mathrm{p}}_S(\Lambda(G))$ and there exists an element $\xi_{X,G,\mathcal{L}}$ of $K_1(\Lambda(G)_S)$ with

$$\partial_G(\xi_{X,G,\mathcal{L}}) = -\chi^{\mathrm{ref}}(R\Gamma(\mathrm{Spec}(\mathbb{F}_q)_{\text{ét}}, Rs_{X,!}f_*f^*\mathcal{L}))$$

and such that for all Artin representations ρ of G one has

$$\xi^*_{X,G,\mathcal{L}}(\rho) = Z^*(X, \mathcal{L} \otimes \mathcal{L}_\rho, 1).$$

If X is an affine curve, G is abelian and $\mathcal{L}$ has abelian monodromy, then our proof of Theorem 1.1 can be reduced to the main result of Lai, Tan and the present author in [4, Appendix] and hence relies on Weil's formula for the Zeta function in terms of ℓ-adic cohomology for any prime $\ell \neq p$ and thus avoids any use of crystalline cohomology. However, our proof of Theorem 1.1 in the general case uses a result of [18] which itself relies on important aspects of the theory of F-crystals. (It is with the results of [18] in mind that we have phrased Theorem 1.1 in terms of higher direct images with proper support rather than cohomology with compact support.) The result of Theorem 1.1 is a natural p-adic analogue of the ℓ-adic ($\ell \neq p$) results proved by Witte in [42] and is also related to the p-adic results proved for elliptic curves over function fields by Ochiai and Trihan [29] and in greater generality in forthcoming work of Trihan and Vauclair.

To state a first consequence of Theorem 1.1 we introduce some notation concerning the theory of Weil-étale cohomology. For any scheme Y' of finite type over $\mathbb{F}_q$ we write $Y'_{W\acute{e}t}$ for the Weil-étale site on Y' that is defined by Lichtenbaum in [24, §2]. We also write ϕ for the geometric Frobenius automorphism $x \mapsto x^{\frac{1}{q}}$ in $\mathrm{Gal}(\mathbb{F}_p^c/\mathbb{F}_q)$, θ for the element of $H^1(\mathrm{Spec}(\mathbb{F}_q)_{W\acute{e}t}, \mathbb{Z}) = \mathrm{Hom}(\langle\phi\rangle, \mathbb{Z})$ that sends the Frobenius automorphism to 1 (and hence sends ϕ to -1) and $\theta_{Y'}$ for the pullback of θ to $H^1(Y'_{W\acute{e}t}, \mathbb{Z})$. We recall that, for any sheaf $\mathcal{F}$ on $Y'_{W\acute{e}t}$, taking cup product with $\theta_{Y'}$ gives a complex of the form

$$0 \to H^0(Y'_{W\acute{e}t}, \mathcal{F}) \xrightarrow{\cup\theta_{Y'}} H^1(Y'_{W\acute{e}t}, \mathcal{F}) \xrightarrow{\cup\theta_{Y'}} H^2(Y'_{W\acute{e}t}, \mathcal{F}) \xrightarrow{\cup\theta_{Y'}} \cdots \tag{2}$$

For any finite Galois covering $f : Y \to X$ of group $\mathcal{G}$ we set

$$Z(f,t) := \sum_{\chi\in\mathrm{Ir}(\mathcal{G})} L^{\mathrm{Artin}}(Y,\chi,t)e_\chi$$

where $\mathrm{Ir}(\mathcal{G})$ denotes the set of irreducible complex characters of $\mathcal{G}$ and for each χ we write $L^{\mathrm{Artin}}(Y,\chi,t)$ for the Artin L-function defined by Milne in [25, Exam. 13.6(b)] and e_χ for the primitive central idempotent $\chi(1)|\mathcal{G}|^{-1}\sum_{g\in\mathcal{G}}\chi(g^{-1})g$ in $\mathbb{C}[\mathcal{G}]$. Each function $L^{\mathrm{Artin}}(Y,\chi,t)$ is known to be a rational function of t and we write $r_{f,\chi}$ for its order of vanishing at $t = 1$. We further recall that the leading term

$$Z^*(f,1) := \sum_{\chi\in\mathrm{Ir}(\mathcal{G})} \lim_{t\to 1}(1-t)^{-r_{f,\chi}} L^{\mathrm{Artin}}(Y,\chi,t)e_\chi$$

of $Z(f,t)$ at $t = 1$ belongs to $\zeta(\mathbb{Q}[\mathcal{G}])^\times$, where we write $\zeta(A)$ for the centre of any ring A. We also write $\delta_\mathcal{G} : \zeta(\mathbb{Q}[\mathcal{G}])^\times \to K_0(\mathbb{Z}[\mathcal{G}], \mathbb{Q}[\mathcal{G}])$ for the 'extended boundary homomorphism' defined by Flach and the present author in [8, Lem. 9] and $\chi^{\mathrm{ref}}_\mathcal{G}(-,-)$ for the refined Euler characteristic $\chi^{\mathrm{ref}}_{\mathbb{Z}[\mathcal{G}],\mathbb{Q}[\mathcal{G}]}(-,-)$ discussed in §2.1.

By combining a special case of Theorem 1.1 with the descent formalism developed by Venjakob and the present author in [10] we will also prove the following result.

Corollary 1.2. *We assume to be given a Cartesian diagram*

$$\begin{array}{ccc} Y & \xrightarrow{j_Y} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X & \xrightarrow{j} & X' \end{array} \tag{3}$$

in which f' is a finite Galois covering of group $\mathcal{G}$, X is geometrically connected, X' and Y' are proper and j and j_Y are open immersions. We also assume that

(i) *in each degree i the group $H^i(Y'_{W\acute{e}t}, j_{Y,!}\mathbb{Z})$ is finitely generated, and*
(ii) *the complex (2) with $\mathcal{F} = j_{Y,!}\mathbb{Z}$ has finite cohomology groups.*

Then $R\Gamma(Y'_{W\acute{e}t}, j_{Y,!}\mathbb{Z})$ belongs to $D^{\mathrm{p}}(\mathbb{Z}[\mathcal{G}])$ and in $K_0(\mathbb{Z}[\mathcal{G}], \mathbb{Q}[\mathcal{G}])$ one has

$$\delta_\mathcal{G}(Z^*(f,1)) = -\chi^{\mathrm{ref}}_\mathcal{G}(R\Gamma(Y'_{W\acute{e}t}, j_{Y,!}\mathbb{Z}), \epsilon_{f,j})$$

where $\epsilon_{f,j}$ is the exact sequence of $\mathbb{Q}[\mathcal{G}]$-modules induced by (2) with $\mathcal{F} = j_{Y,!}\mathbb{Z}$.

In the case $Y' = X'$ (so $Y = X$ and $\mathcal{G}$ is trivial) the equality of Corollary 1.2 coincides with the leading term formula proved by Lichtenbaum in [24, Th. 8.2]. In fact the latter result also shows that (for any $\mathcal{G}$) the conditions (i) and (ii) in Corollary 1.2 are valid if $X = X'$ (so j is the identity) and Y' is both smooth and projective, or if Y' is a curve, or if both Y and Y' are smooth surfaces. In general, however, if Y' is either non-smooth or non-proper, then condition (i) is not always satisfied (and the validity of condition (ii) is related to Tate's conjecture on the bijectivity of the cycle-class map). In a more general setting it is natural to ask if our methods can prove a result similar to Corollary 1.2 for the theory of 'arithmetic cohomology' recently introduced by Geisser.

In the special case that X is an affine curve and $\mathbb{F}_q$ is isomorphic to the field of constants of the function field of X we show that Corollary 1.2 has the following (unconditional) consequence.

Corollary 1.3. *The central conjecture of* [3] *is valid.*

For an explicit statement of this result see Theorem 10.1. For the moment, we merely note that Corollary 1.3 is both a non-abelian generalisation of the main result of [4] and validates a natural function field analogue of an important special case of the 'equivariant Tamagawa number conjecture' formulated in [8] (for more details of this connection see [3, Rem. 2, Rem. 3]), and that the approach of [4, Rem. 3.3] also gives an interpretation of Corollary 1.3 in terms of integral congruences between families of suitably normalised leading terms of Artin L-series.

In the remainder of this section we describe several interesting consequences of Corollary 1.3. Before stating the first such result we recall that the 'Ω-Conjectures' formulated by Chinburg in [11, §4.2] are natural function field analogues of the central conjectures of Galois module theory that were formulated by Chinburg in [12, 13].

Corollary 1.4. *Let F/k be a finite Galois extension of global function fields.*

(i) *The $\Omega(3)$-Conjecture is valid for F/k.*
(ii) *If F/k is tamely ramified, then all Ω-conjectures are valid for F/k.*

Corollaries 1.3 and 1.4 are strong improvements of previous results in this area. Indeed, as far as we are aware, before now neither the central conjecture of [3] or either the $\Omega(1)$- or $\Omega(3)$-Conjecture has been proved for any non-abelian Galois extension F/k of degree divisible by p.

With F/k as in Corollary 1.4 we set $\mathcal{G} := \mathrm{Gal}(F/k)$ and fix a finite non-empty set of places Σ of k that contains all places which ramify in F/k. For any intermediate field E of F/k we write $\Sigma(E)$ for the set of places of E above those in Σ, $\mathcal{O}_{E,\Sigma}$ for the subring of E comprising all elements that are integral at places outside $\Sigma(E)$, C_E^Σ for the affine curve $\mathrm{Spec}(\mathcal{O}_{E,\Sigma})$ and $f_{E/k}^\Sigma : C_E^\Sigma \to C_k^\Sigma$ for the morphism induced by the inclusion $\mathcal{O}_{k,\Sigma} \subseteq \mathcal{O}_{E,\Sigma}$. We fix q so that $\mathbb{F}_q$ identifies with the constant field of

k, regard each $f^\Sigma_{E/k}$ as a morphism of $\mathbb{F}_q$-schemes and note that C^Σ_k is geometrically connected as a scheme over $\mathbb{F}_q$. We also fix a finite non-empty set of closed points T of C^Σ_k and define a 'T-modified' Zeta function $Z_T(f^\Sigma_{F/k}, t)$ in the usual way (see (38)). The Brumer-Stark Conjecture asserts that if $\mathcal{G}$ is abelian, then $Z_T(f^\Sigma_{F/k}, 1)$ belongs to $\mathbb{Z}[\mathcal{G}]$ and annihilates $\mathrm{Pic}^0(F)$, and we recall that this conjecture is proved by the argument of Deligne that is given in [37, Chap. V]. The next consequence of Corollary 1.3 that we state is a non-abelian generalisation of Deligne's theorem.

Before stating this result we note that for every natural number m and every matrix H in $\mathrm{M}_m(\mathbb{Z}[\mathcal{G}])$ there is a unique matrix H^* in $\mathrm{M}_m(\mathbb{Q}[\mathcal{G}])$ with $HH^* = H^*H = \mathrm{Nrd}_{\mathbb{Q}[\mathcal{G}]}(H)\cdot I_m$ and such that for every primitive central idempotent e of $\mathbb{Q}[\mathcal{G}]$ the matrix H^*e is invertible if and only if $\mathrm{Nrd}_{\mathbb{Q}[\mathcal{G}]}(H)e$ is non-zero. We then define a $\zeta(\mathbb{Z}[\mathcal{G}])$-module by setting

$$\mathcal{A}(\mathbb{Z}[\mathcal{G}]) := \{x \in \zeta(\mathbb{Q}[\mathcal{G}]) : \text{ if } d > 0 \text{ and } H \in \mathrm{M}_d(\mathbb{Z}[\mathcal{G}]) \text{ then } xH^* \in \mathrm{M}_d(\mathbb{Z}[\mathcal{G}])\}.$$

It is easy to see that $\mathcal{A}(\mathbb{Z}[\mathcal{G}]) = \zeta(\mathbb{Z}[\mathcal{G}])$ if $\mathcal{G}$ is abelian and, in general, that $|\mathcal{G}|\mathcal{M} \subseteq \mathcal{A}(\mathbb{Z}[\mathcal{G}]) \subseteq \zeta(\mathbb{Z}[\mathcal{G}])$, where $\mathcal{M}$ is the integral closure of $\zeta(\mathbb{Z}[\mathcal{G}])$ in $\zeta(\mathbb{Q}[\mathcal{G}])$ (cf. [5, Rem. 2.3.2]).

We let $\mathbb{Z}'$ be any finitely generated subring of $\mathbb{Q}$ for which the $\mathcal{G}$-module $\mathbb{Z}'\otimes\mathbb{F}_q^\times$ has finite projective dimension (so one can take $\mathbb{Z}' = \mathbb{Z}$ if, for example, the highest common factor of $q-1$ and $|\mathcal{G}|$ is invertible in $\mathbb{Z}'$). For each v in Σ we fix a place $w(v)$ of F above v and write $\mathcal{G}_{w(v)}$ for the decomposition subgroup of $w(v)$ in $\mathcal{G}$.

Corollary 1.5. *Fix $F/k, q, \Sigma$ and T as above and set $\mathcal{G} := \mathrm{Gal}(F/k)$. Then for all a in $\mathcal{A}(\mathbb{Z}[\mathcal{G}])$ the element $aZ_T(f^\Sigma_{F/k}, 1)$ belongs to $\mathbb{Z}[\mathcal{G}]$. Further, for any v in Σ and any a' in $\mathrm{Ann}_{\mathbb{Z}[\mathcal{G}]}(\mathbb{Z}[\mathcal{G}/\mathcal{G}_{w(v)}])$ the element $a'aZ_T(f^\Sigma_{F/k}, 1)$ annihilates $\mathbb{Z}'\otimes\mathrm{Pic}^0(F)$.*

We believe that a more detailed analysis of our methods is likely to show that, using the notation of Corollary 1.5, the element $aZ_T(f^\Sigma_{F/k}, 1)$ annihilates $\mathrm{Pic}^0(F)$ in all cases but we shall not pursue this possibility any further here. In fact, we also deduce Corollary 1.5 as a consequence of a more general result (Theorem 11.1) which uses values of higher-order derivatives of $Z_T(f^\Sigma_{F/k}, t)$ to construct explicit annihilators of ideal class groups.

The final consequence of Corollary 1.3 that we record here concerns a range of explicit integral refinements of Stark's Conjecture. For more details see §11.2.

Corollary 1.6. *Natural non-abelian generalisations of all of the following refinements of Stark's conjecture are valid for all global function fields.*

(i) *The Rubin-Stark Conjecture* [35, Conj. B′].
(ii) *The 'guess' formulated by Gross in* [19, top of p. 195].
(iii) *The 'refined class number formula' of Gross* [19, Conj. 4.1].
(iv) *The 'refined class number formula' of Tate* [38] *(see also* [39, $(*)$]*).*
(v) *The 'refined class number formula' of Aoki, Lee and Tan* [1, Conj. 1.1].

(vi) *The 'refined $\mathfrak{p}$-adic abelian Stark Conjecture' of Gross* [19, Conj. 7.6].

In addition to the above consequences of Corollary 1.3 we shall also derive from it (in Proposition 11.2) an explicit description of the non-commutative Fitting invariants of certain natural Weil-étale cohomology groups.

The main contents of this article is as follows. In §2 we recall relevant background material regarding the K-theory and Iwasawa algebras of non-commutative p-adic Lie groups. In §3 we recall background material regarding pro-coverings and pro-sheaves including an important observation of Witte, and also prove a technical result that will be very useful in later sections. In §4 we make several important reductions to Theorem 1.1 and use a result of Emerton and Kisin to give a full proof of it in the case that G is abelian. In §5 we describe the general strategy for the remainder of the proof of Theorem 1.1 and also recall an important algebraic result of Kakde. In §6 we describe a natural variant of the classical theory of integral group logarithms (of Oliver and M. Taylor) in the setting of certain non-commutative power series rings and in §7 we combine this construction with arguments of Kakde to prove that, in certain contexts, families of power series satisfy a range of explicit congruences. In §8 we combine the results of §7 with certain continuity arguments to complete the proof of Theorem 1.1. In §9 we prove Corollary 1.2 by combining a special case of Theorem 1.1 with the descent formalism in non-commutative Iwasawa theory developed by Venjakob and the present author and in §10 we prove Corollary 1.3 by combining a special case of Corollary 1.2 with an explicit computation of certain cup product morphisms. Finally, in §11 we derive several explicit consequences of Corollary 1.3 including Corollaries 1.5 and 1.6.

It is a great pleasure to thank Dick Gross, Kazuya Kato and John Tate for their generous encouragement and helpful conversations concerning this project. I am also very grateful to Cornelius Greither and Jan Nekovář for stimulating discussions. Most of the results described here were discussed in a lecture course given at the CRM in Barcelona in February and April 2010. I am very grateful to the CRM for providing this opportunity and to Francesc Bars for his wonderful hospitality throughout my visits.

2. K-theory and Iwasawa algebras

2.1. Modules are to be understood, unless explicitly stated otherwise, as left modules. For any associative, unital, left noetherian ring R we write $D(R)$ for the derived category of R-modules and $D^-(R)$, resp. $D^{\rm p}(R)$, for the full triangulated subcategory of $D(R)$ comprising complexes that are isomorphic to an object of the category $C^-(R)$, resp. $C^{\rm p}(R)$, of bounded above complexes of projective R-modules, resp. of bounded complexes of finitely generated projective R-modules.

For any homomorphism $R \to R'$ of rings as above one can define a relative algebraic K-group $K_0(R, R')$. We recall that if $C^\bullet$ is any object of $D^{\rm p}(R)$, then any exact sequence of R'-modules of the form

$$\epsilon:\ 0 \to \cdots \to R' \otimes_R H^i(C^\bullet) \to R' \otimes_R H^{i+1}(C^\bullet) \to R' \otimes_R H^{i+2}(C^\bullet) \to \cdots \to 0$$

gives a canonical 'refined Euler characteristic' $\chi^{\mathrm{ref}}_{R,R'}(C^\bullet, \epsilon)$ in $K_0(R, R')$ (for more details of this construction see [2]). In particular, if R' is the total quotient ring of R and $R' \otimes^{\mathbb{L}}_R C^\bullet$ is acyclic, then we write $\chi^{\mathrm{ref}}_R(C^\bullet)$ is place of $\chi^{\mathrm{ref}}(C^\bullet, \epsilon_0)$ where ϵ_0 denotes the exact sequence of zero modules. If R and R' are clear from context, then we often abbreviate $\chi^{\mathrm{ref}}_{R,R'}(C^\bullet, \epsilon)$ and $\chi^{\mathrm{ref}}_R(C^\bullet)$ to $\chi^{\mathrm{ref}}(C^\bullet, \epsilon)$ and $\chi^{\mathrm{ref}}(C^\bullet)$ respectively.

2.2. For any profinite group G and any finite extension $\mathcal{O}$ of $\mathbb{Z}_\ell$ (for any prime ℓ) we write $\Lambda_{\mathcal{O}}(\mathcal{G})$ for the $\mathcal{O}$-Iwasawa algebra $\varprojlim_U \mathcal{O}[G/U]$ of G, where U runs over the set $\mathrm{ON}(G)$ of open normal subgroups of G (partially ordered by inclusion) and the limit is taken with respect to the obvious transition morphisms. We also write $\mathrm{Q}_{\mathcal{O}}(\mathcal{G})$ for the total quotient ring of $\Lambda_{\mathcal{O}}(\mathcal{G})$, and if $\mathcal{O} = \mathbb{Z}_p$, then we omit the subscripts from both $\Lambda_{\mathcal{O}}(\mathcal{G})$ and $\mathrm{Q}_{\mathcal{O}}(\mathcal{G})$.

We fix a group G as in (1) and set $H := \ker(\pi_G)$. We also fix an algebraic closure $\mathbb{Q}_p^c$ of $\mathbb{Q}_p$ and write $\mathcal{O}$ for the valuation ring of a finite extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$. We write S for the multiplicatively closed left and right Ore set of non-zero divisors of $\Lambda_{\mathcal{O}}(G)$ described in [15, §2] and $\Lambda_{\mathcal{O}}(G)_S$ for the corresponding localisations of $\Lambda_{\mathcal{O}}(G)$ (we caution the reader that this notation does not explicitly indicate that S depends upon both $\mathcal{O}$ and G) and we often use the following canonical commutative diagram of exact sequences of abelian groups

$$\begin{array}{ccccc}
K_1(\Lambda_{\mathcal{O}}(G)) & \longrightarrow & K_1(\Lambda_{\mathcal{O}}(G)_S[\frac{1}{p}]) & \longrightarrow & K_0(\Lambda_{\mathcal{O}}(G), \Lambda_{\mathcal{O}}(G)_S[\frac{1}{p}]) \\
\Vert & & \uparrow & & \uparrow \\
K_1(\Lambda_{\mathcal{O}}(G)) & \xrightarrow{\alpha_{\mathcal{O},G}} & K_1(\Lambda_{\mathcal{O}}(G)_S) & \xrightarrow{\partial_{\mathcal{O},G}} & K_0(\Lambda_{\mathcal{O}}(G), \Lambda_{\mathcal{O}}(G)_S)
\end{array} \tag{4}$$

(for a full discussion of such exact sequences see, for example, [36, Chap. 15]). We recall that if G has no element of order p, then $K_0(\Lambda_{\mathcal{O}}(G), \Lambda_{\mathcal{O}}(G)_S)$ is isomorphic to the Grothendieck group of the category of finitely generated $\Lambda_{\mathcal{O}}(G)$-modules M with the property that $\Lambda_{\mathcal{O}}(G)_S \otimes_{\Lambda_{\mathcal{O}}(G)} M$ vanishes (for an explicit description of this isomorphism see, for example, [10, §1.2]).

2.3. Using the same notation as in diagram (1) we set $\Gamma := \pi_1^p(\mathbb{F}_q, \mathbb{F}_q^c)$ and $\Gamma_X := \mathrm{im}(\pi_{X,\overline{x}}) = \mathrm{im}(\pi_G) \subseteq \Gamma$. We also write γ for the image of the (arithmetic) Frobenius automorphism $x \mapsto x^q$ under the natural projection $\mathrm{Gal}(\mathbb{F}_q^c/\mathbb{F}_q) \to \Gamma$ and note that γ is a topological generator of Γ.

If $\mathcal{O}'$ is the valuation ring of a finite extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$ and $\mathcal{O} \subseteq \mathcal{O}'$, then each continuous representation $\rho : G \to \mathrm{GL}_n(\mathcal{O}')$ gives rise to a ring homomorphism $\Lambda_{\mathcal{O}}(G)_S \to \mathrm{M}_n(\mathcal{O}') \otimes_{\mathcal{O}'} \mathrm{Q}_{\mathcal{O}'}(\Gamma) \cong \mathrm{M}_n(\mathrm{Q}_{\mathcal{O}'}(\Gamma))$ that sends every g in G to $\rho(g) \otimes \pi_G(g)$.

This homomorphism induces a group homomorphism

$$(5)\quad \Phi_{\mathcal{O},G,S,\rho} : K_1(\Lambda_{\mathcal{O}}(G)_S) \to K_1(\mathrm{M}_n(\mathrm{Q}_{\mathcal{O}'}(\Gamma_X))) \cong K_1(\mathrm{Q}_{\mathcal{O}'}(\Gamma_X)) \\ \cong \mathrm{Q}_{\mathcal{O}'}(\Gamma_X)^\times \subseteq \mathrm{Q}_{\mathcal{O}'}(\Gamma)^\times \cong \mathrm{Q}(\mathcal{O}'[[u]])^\times$$

where $\mathcal{O}'[[u]]$ denotes the ring of power series over $\mathcal{O}'$ in the formal variable u, the first isomorphism is induced by Morita equivalence, the second by taking determinants (over $\mathrm{Q}_{\mathcal{O}'}(\Gamma_X)$) and the last by sending $\gamma - 1$ to u. We often abbreviate $\Phi_{\mathcal{O},G,S,\rho}$ to $\Phi_{G,\rho}$, believing that $\mathcal{O}$ and S will be clear from context.

The 'leading term' $\xi^*(\rho)$ at ρ of an element ξ of $K_1(\Lambda_{\mathcal{O}}(G)_S)$ is then defined to be the leading term at $u = 0$ of $\Phi_{\mathcal{O},G,S,\rho}(\xi)$. This construction applies in particular to any element ρ of the set $A(G)$ of Artin representations of G.

In the sequel we will abbreviate $\chi^{\mathrm{ref}}_{\Lambda_{\mathcal{O}}(G),\mathrm{Q}_{\mathcal{O}}(G)}(-,-)$ to $\chi^{\mathrm{ref}}_{\mathcal{O},G}(-,-)$.

3. Pro-coverings, pro-sheaves and perfect complexes

In §3.1-§3.3 we review some standard material concerning pro-coverings and perfect complexes of pro-sheaves and recall a result of Witte (the reader can find a nice treatment of all of this material in Witte's articles [41] and [42].) Then in §3.4 we prove a technical result that will be very useful in later sections.

3.1. Let X be a connected scheme over $\mathbb{F}_q$ and write $\mathbf{FEt}/X$ for the category of X-schemes that are finite and étale over X. For any geometric point $\overline{x}$ of X the functor that takes each scheme Y to the set $F_{\overline{x}}(Y) := \mathrm{Hom}_X(\overline{x}, Y)$ of geometric points of Y that lie over $\overline{x}$ gives an equivalence of categories between projective systems in $\mathbf{FEt}/X$ and the category of projective systems of finite sets upon which $\pi_1(X, \overline{x})$ acts continuously (on the left).

For any morphism $f : Y \to X$ in $\mathbf{FEt}/X$, the finite group $\mathrm{Aut}_X(Y)$ acts naturally on $F_{\overline{x}}(Y)$ (on the right) and if Y is connected, then it is said to be 'Galois over X' if for any ξ in $F_{\overline{x}}(Y)$ the map $\mathrm{Aut}_X(Y) \to F_{\overline{x}}(Y)$ given by $\sigma \mapsto \sigma \circ \xi$ is bijective.

If now G is any continuous quotient of $\pi_1(X, \overline{x})$ and $\mathrm{O}(G)$ denotes the set of open subgroups of G, then there exists a canonical projective system $(f_U : Y_U \to X, \phi_{VU})$ in $\mathbf{FEt}/X$ where U and V run over $\mathrm{O}(G)$ and for each inclusion $V \subseteq U$ the transition morphism $\phi_{VU} : Y_V \to Y_U$ is surjective. We denote this system by $(f : Y \to X, G)$, or more simply $f : Y \to X$, and refer to it as the 'pro-covering of X of group G'. We recall that each Y_U is connected and that if U is normal in G, then $f_U : Y_U \to X$ is Galois and $\mathrm{Aut}_X(Y_U)$ is canonically isomorphic to the quotient G/U.

For any closed subgroup H of G the morphisms $\{\phi_{VU}\}_{H \le V \le U}$ constitute a projective system which we denote by $f_H : Y_H \to X$. We note that this system corresponds (under the equivalence of categories described above) to the natural action of $\pi_1(X, \overline{x})$ on the set of cosets of H in G and that if H is normal, resp. open, then f_H is a pro-covering of X of group G/H, resp. can be identified with the morphism $Y_H \to X$ in

$\mathbf{FEt}/X$ discussed above. For each open subgroup U of G we write $f^U : Y \to Y_U$ for the pro-covering of Y_U of group U that is given by the projective system $\{\phi_{VU}\}_{V\in \mathrm{O}(U)}$.

3.2. Fix a prime number ℓ, a finite extension $\mathcal{O}$ of $\mathbb{Z}_\ell$ and an étale sheaf of $\mathcal{O}$-modules $\mathcal{L}$ on X. Then for any morphism $h : Y \to X$ in $\mathbf{FEt}/X$ the sheaf $h_*h^*\mathcal{L}$ is the sheaf of $\mathcal{O}$-modules that is associated to the pre-sheaf $U \mapsto \bigoplus_{\mathrm{Hom}_X(U,Y)} \mathcal{L}(U)$ where the transition morphisms $\bigoplus_{\mathrm{Hom}_X(V,Y)} \mathcal{L}(V) \to \bigoplus_{\mathrm{Hom}_X(U,Y)} \mathcal{L}(U)$ for each $\alpha : U \to V$ are given by mapping $(x_\psi)_\psi$ to $\sum_{\psi\circ\alpha} \mathcal{L}(\alpha)(x_\psi)$.

If h is Galois, then the right action of $\mathcal{G} := \mathrm{Aut}_X(Y)$ induces a right action on $\mathrm{Hom}_X(U, Y)$ and hence a left action on $h_*h^*\mathcal{L}$ by permuting the components. For the stalks at $\overline{x}$ one therefore has an isomorphism of $\mathcal{O}[\mathcal{G}]$-modules

$$(h_*h^*\mathcal{L})_{\overline{x}} \cong \bigoplus_{F_{\overline{x}}(Y)} \mathcal{F}_{\overline{x}} \cong \mathcal{O}[\mathcal{G}] \otimes_{\mathcal{O}} \mathcal{L}_{\overline{x}} \tag{6}$$

where the second map results from the fact that (since h is Galois) any choice of an element ξ of $F_{\overline{x}}(Y)$ induces an isomorphism $\sigma \mapsto \sigma \circ \xi$ of $\mathcal{G}$-sets $\mathcal{G} \cong F_{\overline{x}}(Y)$.

We now fix a pro-covering $f : Y \to X$ of X of group G as in §3.1 and a flat, smooth $\mathcal{O}$-sheaf $\mathcal{L}$ on X. We assume given a cofinal subset $\mathrm{ON}'(G)$ of $\mathrm{ON}(G)$ and for each U in $\mathrm{ON}'(G)$ a power n_U of ℓ such that n_U divides n_V whenever $V \subseteq U$ and the natural homomorphism $\Lambda_{\mathcal{O}}(G) \to \varprojlim_{U\in O'(G)}(\mathcal{O}/n_U)[G/U]$ is bijective (where the limit is taken with respect to the natural transition morphisms $(\mathcal{O}/n_V)[G/V] \to (\mathcal{O}/n_U)[G/U]$ for $V \subseteq U$). Then the above description shows that for each pair of subgroups U and V in $\mathrm{ON}'(G)$ with $V \subseteq U$ the natural morphisms of sheaves $f_{V,*}f_V^*(\mathcal{L}/n_V) \to f_{U,*}f_U^*(\mathcal{L}/n_U)$ induces an isomorphism of $(\mathcal{O}/n_U)[G/U]$-sheaves

$$(\mathcal{O}/n_U)[G/U] \otimes_{(\mathcal{O}/n_V)[G/V]} f_{V,*}f_V^*(\mathcal{L}/n_V) \cong f_{U,*}f_U^*(\mathcal{L}/n_U). \tag{7}$$

We write $f_*f^*\mathcal{L}$ or sometimes just $\mathcal{L}_G$ for the associated projective system of $\mathcal{O}$-sheaves $\{f_{U,*}f_U^*\mathcal{L}/n_U\}_{U\in \mathrm{ON}'(G)}$.

Since $s : X \to \mathrm{Spec}(\mathbb{F}_q)$ is separated we may fix a factorisation $s = s' \circ j$ with $j : X \to X'$ an open immersion and $s' : X' \to \mathrm{Spec}(\mathbb{F}_q)$ a proper morphism. We then define the direct image with proper support $Rs_!\mathcal{L}_G$ of $\mathcal{L}_G$ to be the inverse system of complexes of $\mathcal{O}$-sheaves $\{s'_*G^\bullet_{\tilde{X}}(j_!f_{U,*}f_U^*(\mathcal{L}/n_U))\}_{U\in \mathrm{ON}'(G)}$, where we write $G^\bullet_{\tilde{X}}(\mathcal{K})$ for the Godement resolution of an étale sheaf $\mathcal{K}$ on $\tilde{X}$.

We note that these constructions are independent, to within canonical isomorphisms, of the system $\{(U, n_U)\}_{U\in \mathrm{ON}'(G)}$ used above.

3.3. For any affine scheme $\mathrm{Spec}(A)$ and any complex of étale sheaves $\mathcal{F}^\bullet$ on $\mathrm{Spec}(A)$ we abbreviate $R\Gamma(\mathrm{Spec}(A)_{\text{ét}}, \mathcal{F}^\bullet)$ to $R\Gamma(A, \mathcal{F}^\bullet)$.

We fix a cofinal system $\{(U, n_U)\}_{U\in \mathrm{ON}'(G)}$ as in §3.2. Then a standard argument allows one to combine for each U in $\mathrm{ON}'(G)$ the isomorphisms (6) (with h replaced by f_U and $\mathcal{L}$ by $\mathcal{L}/n_U$) and (7) in order to deduce that there exists a bounded above complex $P^\bullet_U$ of projective $(\mathcal{O}/n_U)[G/U]$-modules that is isomorphic in $D((\mathcal{O}/n_U)[G/U])$

to $R\Gamma(X_{\text{ét}}, f_{U,*}f_U^*(\mathcal{L}/n_U))$ and such that for each $V \subseteq U$ the natural projection morphism $R\Gamma_{\text{ét}}(X_{\text{ét}}, f_{V,*}f_V^*\mathcal{L}/n_V) \to R\Gamma(X_{\text{ét}}, f_{U,*}f_U^*\mathcal{L}/n_U)$ induces an isomorphism $(\mathcal{O}/n_U)[G/U] \otimes_{(\mathcal{O}/n_V)[G/V]} P_V^\bullet \cong P_U^\bullet$ of complexes of $(\mathcal{O}/n_V)[G/V]$-modules. We write $R\Gamma(X_{\text{ét}}, \mathcal{L}_G)$ for the inverse limit $\varprojlim_U P_U^\bullet$ of any such system of complexes and note that, regarded as an object of $D(\Lambda_\mathcal{O}(G))$ in the natural way, this limit is unique up to canonical isomorphism.

By a similar construction with X and each sheaf $f_{U,*}f_U^*\mathcal{L}/n_U$ replaced by X' and $j_!f_{U,*}f_U^*\mathcal{L}/n_U = f_{U,*}f_U^*j_!\mathcal{L}/n_U$, resp. by $\mathrm{Spec}(\mathbb{F}_q)$ and $s'_*G^\bullet_{X'}(j_!f_{U,*}f_U^*\mathcal{L}/n_U)$, we define the cohomology with compact support $R\Gamma_c(X_{\text{ét}}, \mathcal{L}_G)$ and the higher direct image with proper support $R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)$. The argument of Deligne in [17, Arcata, IV, §5] shows that these complexes are each independent, up to natural isomorphism in $D(\Lambda_\mathcal{O}(G))$, of the chosen compactification $s = s' \circ j$.

We write s_c for the natural morphism $\mathrm{Spec}(\mathbb{F}_q^c) \to \mathrm{Spec}(\mathbb{F}_q)$ and recall that ϕ denotes the geometric Frobenius automorphism in $\mathrm{Gal}(\mathbb{F}_q^c/\mathbb{F}_q)$. We note that there is a natural action of ϕ on the the complex $R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G)$.

Proposition 3.1. *Let G be any continuous quotient of $\pi_1(X, \overline{x})$ and $f : Y \to X$ the corresponding pro-covering of group G. Let $\mathcal{O}$ be a finite extension of $\mathbb{Z}_\ell$ and $\mathcal{L}$ a flat and smooth $\mathcal{O}$-sheaf on X.*

(i) *$R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)$ is naturally isomorphic to $R\Gamma_c(X_{\text{ét}}, \mathcal{L}_G)$.*
(ii) *$R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)$ and $R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G)$ belong to $D^{\mathrm{p}}(\Lambda_\mathcal{O}(G))$.*
(iii) *There is a canonical exact triangle in $D^{\mathrm{p}}(\Lambda_\mathcal{O}(G))$ of the form*

$$R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G) \xrightarrow{1-\phi} R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G) \to R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)[1] \to .$$

(iv) *(Witte) If $\ell = p$ and G is any group as in (1), then $R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)$ belongs to $D^{\mathrm{p}}_S(\Lambda_\mathcal{O}(G))$.*

Proof. We first recall that $Rs_{X,!}$ can be computed as $Rs_{X',!} \circ j_!$ for a factorisation $s_X = s_{X'} \circ j$ as in §3.2. Claim (i) is thus true because for any natural number n and any sheaf of $\mathcal{O}/\ell^n$-modules $\mathcal{F}$ on X one has $R\Gamma(\mathbb{F}_q, s_{X',*}G^\bullet_{X'}(j_!\mathcal{F})) \cong R\Gamma(\mathbb{F}_q, Rs_{X',*}(j_!\mathcal{F})) \cong R\Gamma(X'_{\text{ét}}, j_!\mathcal{F})$.

Claim (ii) directly follows from [17, p. 95, Th. 4.9] and the fact that (6) implies that each stalk of $\mathcal{L}_G$ is a finitely generated projective $\Lambda_\mathcal{O}(G)$-module.

Claim (iii) is a well known consequence of the Hochschild-Serre spectral sequence and claim (iv) is proved by Witte in [42, Th. 8.1]. □

Remark 3.2. For an alternative (and more concrete) proof of Proposition 3.1(iv) in the case that X is an affine curve see [4, Prop. 4.1(iii)].

3.4. We now make a technical construction concerning the complexes introduced in §3.3 that will be very useful in later sections. We assume in this subsection that $\ell = p$ and that G has rank one as a p-adic Lie group and recall that in this case the total quotient ring $\mathrm{Q}_\mathcal{O}(G)$ of $\Lambda_\mathcal{O}(G)$ is both semisimple and Artinian.

Proposition 3.3. *We assume that $\ell = p$ and that G is any group as in (1) that has rank one as a p-adic Lie group. Let $P^\bullet$ be a complex in $C^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ that is quasi-isomorphic to $R\Gamma(\mathbb{F}_q^c, s_c^* Rs_{X,!}\mathcal{L}_G)$. Then there exists an endomorphism $\hat{\phi}$ of $P^\bullet$ which is such that $1-\hat{\phi}^i$ is injective in each degree i and there exists a commutative diagram in $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ of the form*

$$\begin{array}{ccc} P^\bullet & \xrightarrow{\iota} & R\Gamma(\mathbb{F}_q^c, s_c^* Rs_{X,!}\mathcal{L}_G) \\ {\scriptstyle 1-\hat{\phi}}\downarrow & & \downarrow{\scriptstyle 1-\phi} \\ P^\bullet & \xrightarrow{\iota} & R\Gamma(\mathbb{F}_q^c, s_c^* Rs_{X,!}\mathcal{L}_G) \end{array} \tag{8}$$

in which ι is a quasi-isomorphism. In each degree i the homomorphism $1-\hat{\phi}^i$ induces an automorphism of $\mathrm{Q}_{\mathcal{O}}(G)\otimes_{\Lambda_{\mathcal{O}}(G)} P^i$, the product

$$Z_G(1-\hat{\phi}) := \prod_{i\in\mathbb{Z}} \langle 1-\hat{\phi}^i \mid \mathrm{Q}_{\mathcal{O}}(G)\otimes_{\Lambda_{\mathcal{O}}(G)} P^i\rangle^{(-1)^{i+1}}$$

belongs to $\mathrm{im}(K_1(\Lambda_{\mathcal{O}}(G)_S)\to K_1(\mathrm{Q}_{\mathcal{O}}(G)))$ *and in* $K_0(\Lambda_{\mathcal{O}}(G),\Lambda_{\mathcal{O}}(G)_S)$ *one has*

$$\partial_{\mathcal{O},G}(Z_G(1-\hat{\phi})) = -\chi^{\mathrm{ref}}_{\mathcal{O},G}(R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)). \tag{9}$$

The proof of this result occupies the rest of this subsection. We thus assume throughout that G has rank one as a p-adic Lie group. We note that $\mathrm{Q}_{\mathcal{O}}(G) = \Lambda_{\mathcal{O}}(G)_S[\frac{1}{p}]$ is obtained by inverting all non-zero divisors in $\Lambda_{\mathcal{O}}(C)$ for any central open subgroup C of G and that for any finitely generated $\Lambda_{\mathcal{O}}(G)$-modules M and N and any integer a the group $\mathrm{Ext}^a_{\Lambda_{\mathcal{O}}(G)}(M,N)$ is a finitely generated $\Lambda_{\mathcal{O}}(C)$-module.

Proposition 3.1(i) implies that there exists an endomorphism $\hat{\phi}: P^\bullet \to P^\bullet$ in $C^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ such that there is a commutative diagram (8). This diagram combines with Proposition 3.1(ii) and (iii) to imply that in each degree i the kernel of $H^i(1-\hat{\phi})$ is a torsion $\Lambda_{\mathcal{O}}(G)$-module. Hence, by using Lemma 3.4 below (with $\epsilon = 1-\hat{\phi}$), we may change $\hat{\phi}$ by a homotopy in order to ensure $1-\hat{\phi}^i$ is injective in each degree i. Choosing $\hat{\phi}$ in this way we set $\alpha := 1-\hat{\phi}$ and find that in each degree i there is a short exact sequence of (bounded) complexes of finitely generated $\Lambda_{\mathcal{O}}(G)$-modules

$$0\to P^\bullet \xrightarrow{\alpha} P^\bullet \to \mathrm{cok}(\alpha)^\bullet \to 0 \tag{10}$$

where $\mathrm{cok}(\alpha)^i = \mathrm{cok}(\alpha^i)$ in each degree i and the differentials of $\mathrm{cok}(\alpha)^\bullet$ are induced by those of $P^\bullet$. This sequence implies that for each i the complex $\mathrm{cok}(\alpha)^i[-i]$ is isomorphic in $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ to $P^i \xrightarrow{\alpha^i} P^i$, where the first term occurs in degree $i-1$, and also satisfies

$$\chi^{\mathrm{ref}}(\mathrm{cok}(\alpha)^i[-i]) = (-1)^{i-1}(P^i,\alpha^i,P^i) = (-1)^{i+1}\partial_{\mathcal{O},G}(\langle \mathrm{id}\otimes\alpha^i | \mathrm{Q}_{\mathcal{O}}(G)\otimes_{\Lambda_{\mathcal{O}}(G)} P^i\rangle)$$

where the second equality is a consequence of our normalisation of $\partial_{\mathcal{O},G}$.

From the exact sequence (10), the commutativity of (8) and the exact triangle (3.1) one finds that $\mathrm{cok}(\alpha)^\bullet$ is isomorphic in $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ to $R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)[1]$ and hence that $-\chi^{\mathrm{ref}}(R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)) = \chi^{\mathrm{ref}}(R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)[1]) = \chi^{\mathrm{ref}}(\mathrm{cok}(\alpha)^\bullet)$ (by [2, Prop. 5.6]). Now in each degree i there is an obvious exact sequence of complexes $0 \to \mathrm{cok}(\alpha)^i[-i] \to \tau_{\le i}(\mathrm{cok}(\alpha)^\bullet) \to \tau_{\le i-1}(\mathrm{cok}(\alpha)^\bullet) \to 0$ where $\tau_{\le d}$ denotes naive truncation in degree d. By applying [2, Th. 5.7] to each of these exact sequences we obtain an equality $\chi^{\mathrm{ref}}(\mathrm{cok}(\alpha)^\bullet) = \sum_{i\in\mathbb{Z}} \chi^{\mathrm{ref}}(\mathrm{cok}(\alpha)^i[-i])$ and hence

$$\begin{aligned} Z_G(1-\hat{\phi}) &= \partial_{\mathcal{O},G}(\prod_{i\in\mathbb{Z}} \langle \mathrm{id}\otimes\alpha^i \mid \mathrm{Q}_{\mathcal{O}}(G)\otimes_{\Lambda_{\mathcal{O}}(G)} P^i\rangle^{(-1)^{i+1}}) \\ &= \sum_{i\in\mathbb{Z}} \chi^{\mathrm{ref}}(\mathrm{cok}(\alpha)^i[-i]) = \chi^{\mathrm{ref}}(\mathrm{cok}(\alpha)^\bullet) = -\chi^{\mathrm{ref}}(R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)). \end{aligned}$$

This completes the proof of Proposition 3.3.

Lemma 3.4. *Let $\epsilon : P^\bullet \to P^\bullet$ be a morphism in $C^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ for which $\ker(H^i(\epsilon))$ is $\Lambda_{\mathcal{O}}(G)$-torsion in each degree i. Then there exists a morphism $\hat{\epsilon} : P^\bullet \to P^\bullet$ in $C^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ that is homotopic to α and injective in each degree i.*

Proof. In each degree i there are tautological exact sequences of $\Lambda_{\mathcal{O}}(G)$-modules

$$(11) \qquad 0 \to Z^i \to P^i \xrightarrow{d^i} B^{i+1} \to 0\,, \;\; 0 \to B^i \to Z^i \to H^i \to 0$$

where B^i, Z^i and H^i are the corresponding coboundaries, cocycles and cohomology of $P^\bullet$. Now, since G has rank one, every submodule of a finitely generated projective $\Lambda_{\mathcal{O}}(G)[\frac{1}{p}]$-module (such as $B^i[\frac{1}{p}]$) is itself both finitely generated and projective. In particular, the first sequence of (11) splits after inverting p and so we can choose a $\Lambda_{\mathcal{O}}(G)$-submodule $\hat{B}^{i+1}$ of P^i such that $\hat{B}^{i+1}\cap Z^i = 0$ and $B^{i+1}/d^i(\hat{B}^{i+1})$ is p-torsion.

In each degree i the natural map $Z^i \to P^i/\hat{B}^{i+1}$ is injective with p-torsion cokernel and so there exists a complex of $\Lambda_{\mathcal{O}}(C)$-modules

$$\mathrm{Hom}_{\Lambda_{\mathcal{O}}(G)}(P^i/\hat{B}^{i+1}, \hat{B}^i) \xrightarrow{\eta^i} \mathrm{Hom}_{\Lambda_{\mathcal{O}}(G)}(B^i, \hat{B}^i) \to \mathrm{Ext}^1_{\Lambda_{\mathcal{O}}(G)}(H^i(P^\bullet), \hat{B}^i)$$

which has p-torsion cohomology at the central term, where η^i is induced by the composite $B^i \subset P^i \to P^i/\hat{B}^{i+1}$. The $\Lambda_{\mathcal{O}}(C)$-module $\mathrm{cok}(\eta^i)$ is therefore torsion because $\mathrm{Ext}^1_{\Lambda_{\mathcal{O}}(G)}(H^i(P^\bullet), \hat{B}^i)$ is torsion (as follows, for example, from [27, Prop. (5.4.17) and Prop. (5.5.3)(ii)] and the fact that $\hat{B}^i[\frac{1}{p}]$ is a projective $\Lambda_{\mathcal{O}}(G)[\frac{1}{p}]$-module).

Since $B^{i+1}/d^i(\hat{B}^{i+1})$ is p-torsion the map $\mathrm{Hom}_{\Lambda_{\mathcal{O}}(G)}(B^i, \hat{B}^i) \to \mathrm{Hom}_{\Lambda_{\mathcal{O}}(G)}(\hat{B}^i, B^i)$ sending k to $d^{i-1}\circ k\circ d^{i-1}$ has p-torsion kernel and cokernel. As each module $\mathrm{cok}(\eta^a)$ is $\Lambda_{\mathcal{O}}(C)$-torsion we may therefore choose k^i in $\mathrm{Hom}_{\Lambda_{\mathcal{O}}(G)}(P^i/\hat{B}^{i+1}, \hat{B}^i)$ so that the cokernel of $d^{i-1}\circ\alpha^i + d^{i-1}\circ k^i\circ d^{i-1} \in \mathrm{Hom}_{\Lambda_{\mathcal{O}}(G)}(\hat{B}^i, B^i)$ is $\Lambda_{\mathcal{O}}(C)$-torsion. In each degree i we then set $\hat{\alpha}^i := \alpha^i - (d^{i-1}\circ k^i + k^{i+1}\circ d^i)$.

This gives a morphism of complexes $\hat{\alpha}$ that is homotopic to α via the homotopy $\{-k^i\}_{i\in\mathbb{Z}}$ and such that, in each degree i, the quotient $B^{i+1}/d^i(\hat{\alpha}^i(\hat{B}^{i+1}))$ is $\Lambda_{\mathcal{O}}(C)$-torsion. Since the cokernel of each map $d^i : \hat{B}^{i+1} \to B^i$ is p-torsion the cokernel of each map $\hat{\alpha}^{i+1} : B^{i+1} \to B^{i+1}$ is also $\Lambda_{\mathcal{O}}(C)$-torsion. But $H^i(\hat{\alpha}) = H^i(\alpha)$ is also $\Lambda_{\mathcal{O}}(C)$-torsion in each degree i and so, by an easy exercise involving the sequences (11), one finds that $\mathrm{cok}(\hat{\alpha}^i)$ is $\Lambda_{\mathcal{O}}(C)$-torsion in each degree i. By considering dimensions over the field $\mathrm{Q}(C)$ it follows that $\ker(\hat{\alpha}^i)$ is $\Lambda_{\mathcal{O}}(C)$-torsion in each degree i. Since each $\Lambda_{\mathcal{O}}(C)$-module P^i is finitely generated and projective it follows that in each degree i the homomorphism $\hat{\alpha}^i$, resp. $\mathrm{Q}_{\mathcal{O}}(G) \otimes_{\Lambda_{\mathcal{O}}(G)} \hat{\alpha}^i$, is injective, resp. bijective, as required. □

Remark 3.5. The final equality of Proposition 3.3 combines with the sequences in (4) to imply that the existence of a diagram (8) determines $Z_G(1-\hat{\phi})$ up to multiplication by an element of $\mathrm{im}(\alpha_{\mathcal{O},G})$. An alternative approach of Witte in [42] relies on an interpretation of $K_1(\Lambda_{\mathcal{O}}(G)_S)$ in terms of Waldhausen K-theory due to Muro and Tonks [26] and can be used to show that $Z_G(1-\hat{\phi})$ (and a natural analogue for higher rank G) is uniquely determined by a diagram of the form (8). However, since we only use this fact (via the results of Emerton and Kisin in §4.2) in the case that G is abelian and in this case it can be proved directly as in [17, p. 115, Cor. 1.13], the concrete approach of Proposition 3.3 allows us to avoid any use of Waldhausen K-theory and the rather delicate constructions that it requires.

4. Some reduction steps

In this section we make several important reductions to Theorem 1.1 by combining the general approach introduced in [6] together with results of [3] and of Emerton and Kisin [18].

4.1. In this subsection we fix s_X and G as in Theorem 1.1 and for any open subgroup U of G we write s_{Y_U} for the induced structure morphism $s_X f_U : Y_U \to \mathrm{Spec}(\mathbb{F}_q)$.

We recall that a p-adic Lie group is said to be 'p-elementary' if it is a direct product of the form $H \times P$ with H a finite abelian group of order prime to p and P a pro-p p-adic Lie group.

The main result of this subsection is then the following.

Proposition 4.1. *Theorem 1.1 is valid as stated if for each open subgroup U of G it is valid with s_X and G replaced by s_{Y_U} and U/J respectively, where J runs over all open subgroups of $\ker(\pi_G)$ that are normal in U and such that the quotient U/J is a (rank one) p-elementary group.*

We shall deduce this result from the general reduction result of [6, Cor. 8.3]. However, before doing so, we need to make several preliminary observations. To do this we fix a pro-covering $(f : Y \to X, G)$ as in Theorem 1.1, a finite extension $\mathcal{O}$ of $\mathbb{Z}_p$ and a flat, smooth $\mathcal{O}$-sheaf $\mathcal{L}$ on X.

4.1.1. We first describe the relevant functorial properties of $R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)$ and $R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G)$ under change of G.

Lemma 4.2.

(i) *If U is an open subgroup of G, then the complex $R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)$ is naturally isomorphic to $R\Gamma(\mathbb{F}_q, Rs_{Y_U,!}(f_U^*\mathcal{L})_U)$ in $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(U))$.*

(ii) *If H is a closed normal subgroup of G, then $f_H : Y_H \to X$ is a Galois pro-covering of group $Q := G/H$ and there is a natural isomorphism in $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(Q))$ of the form $\Lambda_{\mathcal{O}}(Q) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G) \cong R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_Q)$.*

(iii) *Claims (i) and (ii) remain valid if one replaces each occurrence of $\mathbb{F}_q$ by $\mathbb{F}_q^c$ and each occurrence of Rs by s_c^*Rs.*

Proof. We fix a factorisation $s_X = s_{X'} \circ j$ as in §3.2 and set $s := s_X$ and $s' := s_{X'}$.

To prove claim (i) we recall that one can construct (via, for example, the explicit approach used by Witte in (Step 2 of) the proof of [41, Prop. 4.4.3]) a commutative diagram

(12)
$$\begin{tikzcd}
Y_U \arrow[r, "j_U"] \arrow[d, "f_U"'] & Y'_U \arrow[d, "f'_U"] \arrow[ddr, "s'_U"] & \\
X \arrow[r, "j"] \arrow[drr, "s"'] & X' \arrow[dr, "s'"] & \\
 & & \mathrm{Spec}(\mathbb{F}_q)
\end{tikzcd}$$

in which j_U is an open immersion and f'_U, and hence also s'_U, are proper. In particular, since $s_{Y_U} = s \circ f_U$ we can compute the functor $Rs_{Y_U,!}$ as $R(s' \circ f'_U)_* \circ j_{U,!}$. In addition, since f_U and f'_U are both proper, for any torsion étale sheaf $\mathcal{T}$ on Y'_U there are canonical identifications $j_! f_{U,*}\mathcal{T} = f'_{U,*} j_{U,!}\mathcal{T}$ and hence $G^\bullet_X(j_! f_{U,*}\mathcal{T}) = f'_{U,*}G^\bullet_{X'_U}(j_{U,!}\mathcal{T})$ (cf. the proofs of [41, Props 4.3.14(i) and 4.3.16(i)]). Now $Rs_!\mathcal{L}_G$ identifies with the projective system $\{s'_*G^\bullet_{X'}(j_! f_{V,*}f_V^*\mathcal{L}/n_V)\}_{V\in \mathrm{O}(U)}$ and for each V in $\mathrm{O}(U)$ one has $f_V = \phi_{VU} \circ f_U$ and hence $f_{V,*}f_V^*\mathcal{L}/n_V = f_{U,*}\mathcal{L}_{VU}$ with $\mathcal{L}_{VU} := \phi_{VU,*}\phi_{VU}^* f_U^*\mathcal{L}/n_V$. For each such V one therefore has $s'_*G^\bullet_{X'}(j_! f_{V,*}f_V^*\mathcal{L}/n_V) = s'_*G^\bullet_{X'}(j_! f_{U,*}\mathcal{L}_{VU}) = (s' \circ f'_U)_*G^\bullet_{X'_U}(j_{U,!}\mathcal{L}_{VU})$ and this induces an identification of projective systems $Rs_!\mathcal{L}_G = R(s \circ f_U)_!(f_U^*\mathcal{L})_U$, as required to prove claim (i).

The first assertion of claim (ii) is clear. We next recall the cofinal system $\{(U, n_U)\}$ fixed in §3.2 and for each U in $\mathrm{ON}'(G)$ we set $\Lambda_U := \mathbb{Z}_p/n_U$. Then the second assertion of claim (ii) is true because if $P_U^\bullet$ is any complex $C^p(\Lambda_U[G/U])$ that is isomorphic in $D(\Lambda_U[G/U])$ to $\Gamma(\mathbb{F}_p, s_*G^\bullet_{X'}(j_! f_{U,*}f_U^*(\mathcal{L}/n_U))$ and U' is any subgroup in $\mathrm{ON}'(G)$ that contains HU, then $\Lambda_{U'}[G/U'] \otimes_{\Lambda_U[G/U]} P_U^\bullet$ belongs to $C^p(\Lambda_{U'}[G/U'])$ and is isomorphic in $D(\Lambda_{U'}[G/U'])$ to

$$
\begin{aligned}
&\Lambda_{U'}[G/U'] \otimes_{\Lambda_U[G/U]} \Gamma(\mathbb{F}_q, s_* G^\bullet_{X'}(j_! f_{U,*} f_U^*(\mathcal{L}/n_U)))\\
\cong&\Gamma(\mathbb{F}_q, s_* G^\bullet_{X'}(j_!(\Lambda_{U'}[G/U'] \otimes_{\Lambda_U[G/U]} f_{U,*} f_U^*(\mathcal{L}/n_U))))\\
\cong&\Gamma(\mathbb{F}_q, s_* G^\bullet_{X'}(j_!(f_{U',*} f_{U'}^*(\mathcal{L}/n_{U'}))))
\end{aligned}
$$

where the last isomorphism is induced by the fact that the natural morphism of sheaves $\Lambda_{U'}[G/U'] \otimes_{\Lambda_U[G/U]} f_{U,*} f_U^*(\mathcal{L}/n_U) \to f_{U',*} f_{U'}^*(\mathcal{L}/n_{U'})$ is an isomorphism.

To prove claim (iii) one can use the following commutative diagram (in which both squares are Cartesian)

$$
\begin{tikzcd}
X^c \arrow[r, "j^c"] \arrow[d, "t_X"'] & X'^c \arrow[r, "s_{X'^c}"] \arrow[d, "t_{X'}"'] & \mathrm{Spec}(\mathbb{F}_q^c) \arrow[d, "s_c"] \\
X \arrow[r, "j"] & X' \arrow[r, "s_{X'}"] & \mathrm{Spec}(\mathbb{F}_q).
\end{tikzcd}
$$

In this diagram the composition $s_{X'^c} j^c$ is a compactification of the structure morphism $s_{X^c} := s_X t_X$. One also knows that $s_c^* s_{X',*} = s_{X'^c,*} t_{X'}^*$ (by [41, Lem. 4.3.1]) and that for any torsion sheaf $\mathcal{F}$ on X taking global sections of the natural morphism of complexes

$$
\begin{aligned}
s_c^* s_{X',*} G^\bullet_{X'}(j_!\mathcal{F}) = s_{X'^c,*} t_{X'}^* G^\bullet_{X'}(j_!\mathcal{F}) &= s_{X'^c,*} t_{X'}^* t_{X',*} G^\bullet_{X'^c}(t_{X'}^* j_!\mathcal{F})\\
&= s_{X'^c,*} t_{X'}^* t_{X',*} G^\bullet_{X'^c}(j_!^c t_X^*\mathcal{F}) \to s_{X'^c,*} G^\bullet_{X'^c}(j_!^c t_X^*\mathcal{F})
\end{aligned}
$$

induces an isomorphism in $D(\Lambda_{\mathcal{O}}(G))$ since both complexes $\Gamma(\mathbb{F}_q^c, s_c^* s_{X',*} G^\bullet_{X'}(j_!\mathcal{F}))$ and $\Gamma(\mathbb{F}_q^c, s_{X'^c,*} G^\bullet_{X'^c}(j_!^c t_X^*\mathcal{F}))$ compute the cohomology with compact support of $t_X^*\mathcal{F}$ (cf. [41, Prop. 4.6.5(2)]). This observation implies that one can prove claim (iii) by simply repeating the arguments given above but with $\mathbb{F}_q, X, Y_U$ and $\mathcal{L}$ replaced by $\mathbb{F}_q^c, X^c, Y_U \times_X X^c$ and $t_X^*\mathcal{L}$ respectively. □

4.1.2. We next describe the relevant functorial properties of Zeta functions. To do this we write X^0 for the set of closed points of X and for each x in X^0 we write $\mathrm{d}(x)$ for the degree of x over $\mathbb{F}_q$. We choose an identification of the residue field of x with the subfield $\mathbb{F}_{q^{\mathrm{d}(x)}}$ of $\mathbb{F}_q^c$, and denote by $\overline{x}$ an $\mathbb{F}_q^c$-valued point of X lying over x. Then for any complete commutative noetherian $\mathbb{Z}_p$-algebra A and any flat, smooth A-sheaf $\mathcal{L}$ on X the stalk $\mathcal{L}_{\overline{x}}$ is a free A-module equipped with an action of $\phi^{\mathrm{d}(x)}$. For any bounded complex of such sheaves $\mathcal{L}^\bullet$ we may therefore define

$$
Z_A(X, \mathcal{L}^\bullet, t) := \prod_{i\in\mathbb{Z}} \prod_{x\in X^0} \mathrm{det}_A(1 - \phi^{\mathrm{d}(x)} t^{\mathrm{d}(x)} \mid \mathcal{L}^i_{\overline{x}})^{(-1)^{i+1}} \in 1 + tA[[t]].
$$

In the case $A = \mathbb{Z}_p$ we write $Z(X, \mathcal{L}^\bullet, t)$ in place of $Z_A(X, \mathcal{L}^\bullet, t)$.

Lemma 4.3. *Let $\mathcal{L}$ be a flat, smooth $\mathbb{Z}_p$-sheaf on X.*

(i) *If U is an open subgroup of G, then for each representation ρ' in $A(U)$ one has $Z(X, \mathcal{L}\otimes\mathcal{L}_{\mathrm{Ind}_U^G\rho'}, t) = Z(Y_U, f_U^*\mathcal{L}\otimes\mathcal{L}_{\rho'}, t)$.*

(ii) *If Q is a quotient of G, then for each ρ in $A(Q)$ one has $Z(X, \mathcal{L}\otimes\mathcal{L}_\rho, t) = Z(X, \mathcal{L}\otimes\mathcal{L}_{\mathrm{Inf}_Q^G\rho}, t)$.*

Proof. We set $Y := Y_U$, $\mathcal{F} := \mathcal{L}\otimes\mathcal{L}_{\mathrm{Ind}_U^G\rho'}$ and $\mathcal{F}' := f_U^*\mathcal{L}\otimes\mathcal{L}_{\rho'}$ and for each x in X^0 we write $Y(x)$ for the set of points of Y above x. Then to prove claim (i) it is enough to show that for each such x one has

$$\det_{\mathbb{Z}_p}(1-\phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)} \mid \mathcal{F}_{\overline{x}}) = \prod_{y\in Y(x)}\det_{\mathbb{Z}_p}(1-\phi^{\mathrm{d}(y)}t^{\mathrm{d}(y)} \mid \mathcal{F}'_{\overline{y}}). \tag{13}$$

Now there is a natural isomorphism of sheaves on Y of the form $f_U^*\mathcal{F}\cong f_U^*\mathcal{L}\otimes f_U^*\mathcal{L}_{\mathrm{Ind}_U^G\rho'}\cong f_U^*\mathcal{L}\otimes(\Lambda(G)\otimes_{\Lambda(U)}\mathcal{L}_{\rho'})\cong\Lambda(G)\otimes_{\Lambda(U)}\mathcal{F}'$ and for any $\overline{y}_0$ in $F_{\overline{x}}(Y)$ the map $g\mapsto g(\overline{y}_0)$ induces a bijection between the set of cosets of U in G and the set $F_{\overline{x}}(Y)$. One therefore has a decomposition

$$\mathcal{F}_{\overline{x}}\cong(f_U^*\mathcal{F})_{\overline{y}_0}\cong\bigoplus_{\overline{y}\in F_{\overline{x}}(Y)}\mathcal{F}'_{\overline{y}}.$$

Further, if y is the point of Y below $\overline{y}$ and we set $e_{y/x} := \mathrm{d}(y)/\mathrm{d}(x)$, then the action of $\phi_x := \phi^{\mathrm{d}(x)}$ induces isomorphisms $\mathcal{F}'_{\phi_x^m(\overline{y})}\cong\mathcal{F}'_{\overline{y}}$ for each integer m in the set $\Sigma_{y/x} := \{m : 1\le m\le e_{y/x}\}$. In particular, one has $\phi_x^{e_{y/x}}(\overline{y}) = \overline{y}$ and the isomorphism $\mathcal{F}'_{\phi_x^{e_{y/x}}(\overline{y})}\cong\mathcal{F}'_{\overline{y}}$ is induced by the action of $\phi_x^{e_{y/x}} = \phi^{\mathrm{d}(y)}$. One can therefore decompose $\mathcal{F}_{\overline{x}}$ as a direct sum

$$\mathcal{F}_{\overline{x}} = \bigoplus_{y\in Y(x)}\bigoplus_{F_{\overline{x}}(y)}\mathcal{F}'_{\overline{y}} \tag{14}$$

where $F_{\overline{x}}(y)$ is the set of geometric points of Y over y and $\overline{y}$ is a choice of element of $F_{\overline{x}}(y)$ and on each module $\bigoplus_{F_{\overline{x}}(y)}\mathcal{F}'_{\overline{y}}$ the endomorphism $1-\phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)}$ acts via the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & -\phi^{\mathrm{d}(y)}t^{\mathrm{d}(x)} \\ -t^{\mathrm{d}(x)} & 1 & 0 & \dots & 0 \\ 0 & -t^{\mathrm{d}(x)} & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -t^{\mathrm{d}(x)} & 1 \end{pmatrix}$$

Now by using elementary row operations this matrix is reduced to an upper triangular matrix in which the last diagonal entry is $1-\phi^{\mathrm{d}(y)}t^{\mathrm{d}(y)}$ and all other diagonal entries are equal to 1 and so one has

$$\det_A(1-\phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)} \mid \bigoplus_{F_{\overline{x}}(y)}\mathcal{F}_{\overline{y}}) = \det_A(1-\phi^{\mathrm{d}(y)}t^{\mathrm{d}(y)} \mid \mathcal{F}'_{\overline{y}}).$$

The required equality (13) thus follows from the decomposition (14).

Claim (ii) follows immediately from the fact that there is a natural identification $\mathcal{L}_\rho = \mathcal{L}_{\mathrm{Inf}_Q^G \rho}$ of sheaves on X. $\square$

4.1.3. We now relate the diagram (1) to the analogous diagrams that are obtained by replacing X by Y_U for any open subgroup U of G, and G by any suitable quotient of U.

For any diagram (1) we set $d_X := |\Gamma/\Gamma_X| = |\Gamma/\operatorname{im}(\pi_G)|$.

Lemma 4.4. *Let Q be a rank one subquotient of G of the form U/J where J is any open subgroup of $\ker(\pi_G)$ that is normal in U (and so U is open in G). Fix a geometric point $\overline{y}_U$ of Y_U above $\overline{x}$. Then there exists a natural analogue of the commutative diagram (1) in which $\pi_1(X,\overline{x})$ and G are replaced by $\pi_1(Y_U,\overline{y}_U)$ and Q respectively and one has $d_{Y_U} = d_X|\Gamma_X/\pi_G(U)|$.*

Proof. Since $J \subseteq \ker(\pi_G)$ the restriction of π_G to U factors through a homomorphism $\pi_Q : Q \to \pi_1^p(\mathbb{F}_q, \mathbb{F}_q^c)$. It therefore suffices to show that there exists a commutative diagram of homomorphisms

$$\begin{tikzcd}
\pi_1(Y_U,\overline{y}_U) \arrow[rrr, "\pi_{Y_U,\overline{y}_U}"] \arrow[dr, "\pi'"] \arrow[ddr, "\pi'_{Q,1}"'] & & & \pi_1^p(\mathbb{F}_q,\mathbb{F}_q^c) \\
& \pi_1(X,\overline{x}) \arrow[urr, "\pi_{X,\overline{x}}"] \arrow[dr, "\pi'_G"] & & \\
& U \arrow[r, "\subseteq"] \arrow[dr, "\pi'_{Q,2}"'] & G \arrow[uur, "\pi_G"] & \\
& & Q \arrow[uuur, "\pi_Q"'] &
\end{tikzcd}$$

in which $\pi_{X,\overline{x}}, \pi'_G$ and π_G are as in diagram (1), π' is the natural homomorphism and $\pi'_{Q,1}$ and $\pi'_{Q,2}$ are both surjective. Indeed, by setting $\pi'_Q := \pi'_{Q,2} \circ \pi'_{Q,1}$, such a diagram provides the required analogue of (1) and also shows that $\Gamma_{Y_U} := \operatorname{im}(\pi_{Y_U,\overline{y}_U}) = \operatorname{im}(\pi_Q) = \pi_G(U)$ so $d_{Y_U} := |\Gamma/\Gamma_{Y_U}| = |\Gamma/\operatorname{im}(\pi_G)||\operatorname{im}(\pi_G)/\pi_G(U)| = d_X|\Gamma_X/\pi_G(U)|$, as claimed.

To verify the existence of a diagram as above we note that the upper and left hand triangles are the only parts of the diagram that do not obviously both exist and commute. It thus suffices to note that the commutativity of the upper triangle can be shown by comparing the explicit definitions of $\pi_{Y_U,\overline{y}_U}, \pi_{X,\overline{x}}$ and π' (cf. [21, Exp. V, §7]) and using the fact that for any finite étale covering $Z \to \operatorname{Spec}(\mathbb{F}_q)$ there is a natural isomorphism $Y_U \times_X (X \times_{\operatorname{Spec}(\mathbb{F}_q)} Z) \cong Y_U \times_{\operatorname{Spec}(\mathbb{F}_q)} Z$ of $\operatorname{Spec}(\mathbb{F}_q)$-schemes, and that the existence of a homomorphism $\pi'_{Q,1}$ making the left hand triangle commute follows from the fact $\operatorname{im}(\pi')$ is equal to the stabiliser of $\overline{y}_U$ in $\pi_1(X,\overline{x})$ whilst U is equal to the stabiliser of $\overline{y}_U$ in G. $\square$

4.1.4. We next interpret Theorem 1.1 in terms of the formalism introduced in [6]. To do this we note that the element $\gamma_X := \gamma^{d_X}$ is a topological generator of Γ_X and hence that $\underline{G} := (G, \pi_G, \gamma_X)$ is a triple of the sort defined in [6, §1.1.1].

For any representations ρ_1 and ρ_2 in $A(G)$ there are natural isomorphisms of sheaves $\mathcal{L}_{\rho_1+\rho_2} \cong \mathcal{L}_{\rho_1} \otimes \mathcal{L}_{\rho_2}$ and, if ρ_1 is isomorphic to ρ_2, also $\mathcal{L}_{\rho_1} \cong \mathcal{L}_{\rho_2}$. This gives equalities of functions $Z(X, \mathcal{L} \otimes \mathcal{L}_{\rho_1+\rho_2}, t) = Z(X, \mathcal{L} \otimes \mathcal{L}_{\rho_1}, t)Z(X, \mathcal{L} \otimes \mathcal{L}_{\rho_2}, t)$ and if ρ_1 is isomorphic to ρ_2 also $Z(X, \mathcal{L} \otimes \mathcal{L}_{\rho_1}, t) = Z(X, \mathcal{L} \otimes \mathcal{L}_{\rho_2}, t)$.

Thus, if for any G as in (1), any sheaf $\mathcal{L}$ and any ρ in $A(G)$ we write $\xi^*_{\mathcal{L},\rho}$ for the leading term of $Z(X, \mathcal{L} \otimes \mathcal{L}_\rho, t)$ at $t = 1$ and $r_\rho(X, \mathcal{L}_G)$ for the integer $r_\rho(R\Gamma(\mathbb{F}_q, Rs_{X,!}(\mathcal{L}_G))[1])$ defined in [6, §2.1.1], then the data $\{d_X^{-r_\rho(X,\mathcal{L}_G)}\xi^*_{\mathcal{L},\rho}\}_{\rho\in A(G)}$ is 'leading term interpolation data' in the sense of [6, §1.2.1].

We abbreviate the statement $\mathrm{MC}^*(\underline{G}, R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)[1], \{d_X^{-r_\rho(X,\mathcal{L})}\xi^*_{\mathcal{L},\rho}\}_\rho)$ formulated in [6, §1.2.2] to $\mathrm{MC}^*(X, G, \mathcal{L})$.

Lemma 4.5. *Theorem 1.1 is valid if and only if* $\mathrm{MC}^*(X, G, \mathcal{L})$ *is valid.*

Proof. In view of Proposition 3.1(iv), it suffices to prove that there exists an element $\xi_{X,G,\mathcal{L}}$ as in Theorem 1.1 if and only if $\mathrm{MC}^*(X, G, \mathcal{L})$ is valid.

For each ρ in $A(G)$ the homomorphism $\Phi_{\mathcal{O},\underline{G},S,\rho}$ defined in [6, (3)] is obtained from the homomorphism $\Phi_{\mathcal{O},G,S,\rho}$ we defined in (5) by replacing the composite map $\mathrm{Q}_{\mathcal{O}'}(\Gamma_X)^\times \subseteq \mathrm{Q}_{\mathcal{O}'}(\Gamma)^\times \cong \mathrm{Q}(\mathcal{O}'[[u]])^\times$ by the isomorphism $\mathrm{Q}_{\mathcal{O}'}(\Gamma_X)^\times \cong \mathrm{Q}(\mathcal{O}'[[u]])^\times$ that sends $\gamma_X - 1 = \gamma^{d_X} - 1$ to u. This implies in particular that for each ξ in $K_1(\Lambda(G)_S)$ one has $\xi^*(\rho) = d_X^{r_\rho(\xi)}\xi^*_{\underline{G}}(\rho)$ where $\xi^*_{\underline{G}}(\rho)$ is the leading term defined in [6, §1.1.3] and $r_\rho(\xi)$ the integer defined in [6, §2.1.1].

If now ξ is any element of $K_1(\Lambda(G)_S)$ with $\partial_G(\xi) = \chi^{\mathrm{ref}}(R\Gamma(\mathbb{F}_p, Rs_{X,!}(\mathcal{L}_G))[1]) = -\chi^{\mathrm{ref}}(R\Gamma(\mathbb{F}_p, Rs_{X,!}(\mathcal{L}_G)))$, then for every ρ in $A(G)$ one has $r_\rho(\xi) = r_\rho(X, \mathcal{L}_G)$ and hence $\xi^*_{\underline{G}}(\rho) = d_X^{-r_\rho(\xi)}\xi^*(\rho) = d_X^{-r_\rho(X,\mathcal{L})}\xi^*(\rho)$. This implies ξ validates $\mathrm{MC}^*(X, G, \mathcal{L})$ if and only if it is a suitable choice for the element $\xi_{X,G,\mathcal{L}}$ in Theorem 1.1, as required. □

4.1.5. We are now ready to prove Proposition 4.1. We thus consider rank one subquotients of G of the form $Q = U/J$ where J is an open subgroup of $\ker(\pi_G)$ that is normal in U. Then U is open in G and from Lemma 4.2 there is a natural isomorphism in $D^{\mathrm{p}}(\Lambda(Q))$ of the form

$$\Lambda(Q) \otimes^{\mathbb{L}}_{\Lambda(U)} \mathrm{res}^{\Lambda(G)}_{\Lambda(U)} R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G) \cong R\Gamma(\mathbb{F}_q, Rs_{Y_U,!}(f_U^*\mathcal{L})_U). \tag{15}$$

We fix a representation ψ in $A(Q)$ and set $\mathrm{I}^G_Q(\psi) := \mathrm{Ind}^G_U\mathrm{Inf}^U_Q\psi$. By combining the isomorphism (15) with the result of [6, Lem. 3.6(iv)] (with ξ taken to be any element with $\partial_G(\xi) = \chi^{\mathrm{ref}}(R\Gamma(\mathbb{F}_q, Rs_{X,!}(\mathcal{L}_G))[1])$) one finds that $r_{\mathrm{I}^G_Q(\psi)}(X, \mathcal{L}) = r_\psi(Y_U, (f_U^*\mathcal{L})_U)$. From Lemma 4.3 it therefore follows that

$$d_X^{-r_{\mathrm{I}^G_Q(\psi)}(X,\mathcal{L})}\xi^*_{\mathcal{L},\mathrm{I}^G_Q(\psi)} = d_X^{-r_\psi(Y_U,(f_U^*\mathcal{L})_U)}\xi^*_{f_U^*\mathcal{L},\psi}. \tag{16}$$

In addition, the result of [6, Cor. 8.3] combines with Lemma 4.5 and the isomorphism (15) to imply that Theorem 1.1 is valid if for all subquotients $Q = U/J$ as above there exists an element ξ_Q of $K_1(\Lambda(Q)_S)$ with

$$\partial_Q(\xi_Q) = \chi^{\mathrm{ref}}(R\Gamma(\mathbb{F}_q, Rs_{Y_U,!}(f_U^*\mathcal{L})_U)[1]) \tag{17}$$

and such that for all ψ in $A(Q)$ one has

$$(\xi_Q)^*_{\underline{Q}}(\psi) = d_Q^{-r_\psi(Y_U,(f_U^*\mathcal{L})_U)} d_X^{-r_{\mathrm{I}_Q^G(\psi)}(X,\mathcal{L})} \xi^*_{\mathcal{L},\mathrm{I}_Q^G(\psi)} = (d_Q d_X)^{-r_\psi(Y_U,(f_U^*\mathcal{L})_U)} \xi^*_{f_U^*\mathcal{L},\psi} \tag{18}$$

where d_Q denotes the index of $\ker(\pi_G)U$ in G and $\underline{Q}$ the triple $(Q, \pi_Q, \gamma_X^{d_Q})$ and the second equality follows from (16). Now $d_Q = |\,\mathrm{im}(\pi_G)/\pi_G(U)| = |\Gamma_X/\pi_G(U)|$ and so Lemma 4.4 implies $d_Q d_X = d_{Y_U}$. The argument of Lemma 4.5 (with X, G and $\mathcal{L}$ replaced by Y_U, Q and $f_U^*\mathcal{L}$ respectively) therefore shows that the existence of an element ξ_U satisfying (17) and (18) is equivalent to the validity of Theorem 1.1 with X, G and $\mathcal{L}$ replaced by Y_U, Q and $f_U^*\mathcal{L}$ respectively, and so completes the proof of Proposition 4.1.

4.2. In this subsection we assume that G is abelian and of rank one and deduce this case of Theorem 1.1 from a result of Emerton and Kisin.

4.2.1. Since G is assumed to be abelian and of rank one it is isomorphic to a direct product $P' \times P$ with P' finite abelian of order prime to p and P a rank one abelian pro-p p-adic Lie group (which we may regard as both a subgroup and quotient of G). The algebra $\Lambda(G)$ is therefore isomorphic to the product $\prod_{\psi\in P'^*} \Lambda_{\mathbb{Z}_p[\psi]}(P)$ where $P'^* := \mathrm{Hom}(P', \mathbb{Q}_p^{c\times})$ and $\mathbb{Z}_p[\psi]$ is the finite (unramified) extension of $\mathbb{Z}_p$ generated by $\{\psi(g) : g \in P'\}$. In addition, for each flat, smooth $\mathbb{Z}_p$-sheaf $\mathcal{L}$ on X there is an isomorphism $\mathcal{L}_G \cong \oplus_{\psi\in P'^*}(\mathcal{L}\otimes\mathcal{L}_\psi)_P$ and for each irreducible representation ρ in $A(G)$ an isomorphism of sheaves $\mathcal{L}\otimes\mathcal{L}_\rho \cong (\mathcal{L}\otimes\mathcal{L}_{\rho|P'})\otimes\mathcal{L}_{\rho|P}$.

Given these facts, it is clear that the validity of Theorem 1.1 in the case that $G = P' \times P$ is abelian follows by applying (Proposition 4.1 and) the following result with $G = P$ and $\mathcal{F} = \mathcal{L}\otimes\mathcal{L}_\psi$ for each $\psi \in P'^*$.

Theorem 4.6. *Let G be a rank one abelian pro-p p-adic Lie group as in (1). Fix a finite unramified extension $\mathcal{O}$ of $\mathbb{Z}_p$ and a flat and smooth $\mathcal{O}$-sheaf $\mathcal{F}$ on X. Then the complex $R\Gamma(\mathbb{F}_p, Rs_{X,!}\mathcal{F}_G)$ belongs to $D^{\mathrm{p}}_S(\Lambda_{\mathcal{O}}(G))$ and there exists a unique element $\xi_{\mathcal{F}}$ of $K_1(\Lambda_{\mathcal{O}}(G)_S)$ with*

$$\partial_{\mathcal{O},G}(\xi_{\mathcal{F}}) = -\chi^{\mathrm{ref}}_{\mathcal{O},G}(R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{F}_G)) \tag{19}$$

and such that for all representations ρ in $A(G)$ one has

$$\xi^*_{\mathcal{F}}(\rho) = Z^*_{\mathcal{O}}(X, \mathcal{F}\otimes\mathcal{L}_\rho, 1). \tag{20}$$

Remark 4.7. If X is an affine curve and $\mathcal{F}$ has abelian monodromy, then (after translating from the language of graded invertible modules to that of relative algebraic K-theory in the natural way) the argument of Lai, Tan and the present author in [4, §A.2] can be used to give a proof of Theorem 4.6 that relies solely on Weil's formula for the Zeta function in terms of ℓ-adic cohomology for any prime $\ell \neq p$ (and hence avoids any use of crystalline cohomology).

If an element $\xi_{\mathcal{F}}$ exists as in Theorem 1.1, then it is unique as a consequence of [6, Lem. 3.4 and Rem. 3.5]. In the rest of this subsection we will deduce its existence from a special case of [18, Cor. 1.8].

4.2.2. We start with a convenient reduction.

Lemma 4.8. *It is enough to prove Theorem 4.6 in the case that $q = p$.*

Proof. We write $\tilde{X}$ for the scheme X regarded as defined over $\mathbb{F}_p$. Then it is enough to show that an element $\xi_{\mathcal{F}}$ satisfies (19) and (20) if and only if it satisfies the analogous equalities that are obtained by replacing X by $\tilde{X}$.

We write $s_q : \mathrm{Spec}(\mathbb{F}_q) \to \mathrm{Spec}(\mathbb{F}_p)$ for the morphism induced by the inclusion $\mathbb{F}_p \subseteq \mathbb{F}_q$ and set $s_{\tilde{X}} := s_q s_X$. Since s_q is proper any compactification $s_X = s'_X j$ of s_X gives rise to a compactification $s_{\tilde{X}} = (s_q s'_X)j$ of $s_{\tilde{X}}$. From the natural isomorphisms $R\Gamma(\mathbb{F}_p, Rs_{\tilde{X},!}\mathcal{F}_G) = R\Gamma(\mathbb{F}_p, R(s_q \circ s'_X)(j_!\mathcal{F}_G)) \cong R\Gamma(\mathbb{F}_p, Rs_q(Rs'_X(j_!\mathcal{F}_G))) \cong R\Gamma(\mathbb{F}_q, Rs'_X(j_!\mathcal{F}_G)) = R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{F}_G)$ it is thus clear that the validity of (19) does not change if we replace X by $\tilde{X}$.

To consider (20) we write m for the degree of $\mathbb{F}_q$ over $\mathbb{F}_p$ and set $\tilde{\Gamma} := \pi_1^p(\mathbb{F}_p, \mathbb{F}_p^c)$. Then for each ρ in $A(G)$ one has $Z_{\mathcal{O}}(\tilde{X}, \mathcal{F} \otimes \mathcal{L}_\rho, t) = Z_{\mathcal{O}}(X, \mathcal{F} \otimes \mathcal{L}_\rho, t^m)$ and hence

$$Z^*_{\mathcal{O}}(\tilde{X}, \mathcal{F} \otimes \mathcal{L}_\rho, 1) = m^{r'_\rho} Z^*_{\mathcal{O}}(X, \mathcal{F} \otimes \mathcal{L}_\rho, 1)$$

where r'_ρ denotes the order of vanishing of $Z_{\mathcal{O}}(\tilde{X}, \mathcal{F} \otimes \mathcal{L}_\rho, t)$ at $t = 1$.

We now write $\tilde{\pi}_G$ for the homomorphism $G \to \tilde{\Gamma}$ obtained from the analogous diagram to (1) with X replaced by $\tilde{X}$ and $\tilde{\Phi}_{\mathcal{O},G,S,\rho}$ for the homomorphism $K_1(\Lambda_{\mathcal{O}}(G)_S) \to Q(\mathcal{O}'[[u]])^\times$ defined as in (5) but with the roles of π_G, $\Gamma = \pi_1^p(\mathbb{F}_q, \mathbb{F}_q^c)$ and γ replaced by $\tilde{\pi}_G$, $\tilde{\Gamma}$ and the image $\tilde{\gamma}$ of the Frobenius automorphism $x \mapsto x^p$ in $\tilde{\Gamma}$. Then, since $\tilde{\pi}_G$ is the composite of π_G and the natural inclusion $\Gamma \cong \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_q^c) \subseteq \mathrm{Gal}(\mathbb{F}_p/\mathbb{F}_p^c) \cong \tilde{\Gamma}$ and $\gamma = \tilde{\gamma}^m$, one has $\tilde{\Phi}_{\mathcal{O},G,S,\rho} = [m] \circ \Phi_{\mathcal{O},G,S,\rho}$ with $[m]$ the endomorphism of $Q(\mathcal{O}'[[u]])^\times$ that sends u to $(1+u)^m - 1$. This implies that for any element $\xi_{\mathcal{F}}$ the order of vanishing r_ρ of $\tilde{\Phi}_{\mathcal{O},G,S,\rho}(\xi_{\mathcal{F}})$ at $u = 0$ is equal to the order of vanishing of $\Phi_{\mathcal{O},G,S,\rho}(\xi_{\mathcal{F}})$ at $u = 0$ and also that the corresponding leading terms are related by $\tilde{\Phi}_{\mathcal{O},G,S,\rho}(\xi_{\mathcal{F}})^*(0) = m^{r_\rho}\Phi_{\mathcal{O},G,S,\rho}(\xi_{\mathcal{F}})^*(0)$. Comparing this equality with the displayed formula above it is clear that the equalities (20) for X and $\tilde{X}$ are equivalent provided that $r_\rho = r'_\rho$ for all ρ in $A(G)$. Since the equalities $r_\rho = r'_\rho$ are an immediate consequence of Lemma 4.9 below (with $\xi_{\mathcal{F}} = Z_{\Lambda_{\mathcal{O}}(G)}(X, \mathcal{F}_G, 1)$), this completes the proof of the claimed result. $\square$

4.2.3. We now assume that G and $\mathcal{O}$ are as in the statement of Theorem 4.6 and (following Lemma 4.8) that $q = p$. In this case the ring $\Lambda_{\mathcal{O}}(G)$ is local and we write $\mathfrak{m}$ for its maximal ideal and $\Lambda_{\mathcal{O}}(G)\langle t\rangle$ for the $\mathfrak{m}$-adic completion of the polynomial ring $\Lambda_{\mathcal{O}}(G)[t]$. We also write $\mathcal{F}_G^\bullet$ for the complex comprising $\mathcal{F}_G$ placed in degree 0. Then the function $L(X, \mathcal{F}_G^\bullet, t)$ defined in [18, (1.7)] is equal to $Z_{\Lambda_{\mathcal{O}}(G)}(X, \mathcal{F}_G, t)$. Further, if $\overline{z}$ denotes the geometric point $\mathrm{Spec}(\mathbb{F}_p^c)$ of $\mathrm{Spec}(\mathbb{F}_p)$, then [25, Chap. II, Th. 3.2(a)] (with $\pi = s_c$) implies that the complex of stalks $(Rs_{X,!}\mathcal{F}_G)^\bullet_{\overline{z}}$ identifies with $R\Gamma(\mathbb{F}_p^c, s_c^* Rs_{X,!}\mathcal{F}_G)$. From [17, p. 115, Cor. 1.13] it follows that

$$L(\mathbb{F}_p, Rs_{X,!}\mathcal{F}_G^\bullet, t) = \prod_{i\in\mathbb{Z}} \mathrm{det}_{\Lambda_{\mathcal{O}}(G)}(1 - \hat{\phi}^i \cdot t \mid P^i)^{(-1)^{i+1}} \in 1 + t\Lambda_{\mathcal{O}}(G)[[t]]$$

where $L(\mathbb{F}_p, Rs_{X,!}\mathcal{F}_G^\bullet, t)$ is defined as in [18] and $\hat{\phi}$ and $P^\bullet$ are chosen as in Proposition 3.3. In particular, Proposition 3.3 implies that the series $L(\mathbb{F}_p, Rs_{X,!}\mathcal{F}_G^\bullet, t)$ converges at $t = 1$ to an element of $\Lambda_{\mathcal{O}}(G)_S^\times \cong \mathrm{im}(K_1(\Lambda_{\mathcal{O}}(G)_S) \to K_1(\mathrm{Q}_{\mathcal{O}}(G)))$ and also satisfies

$$\partial_{\mathcal{O},G}(L(\mathbb{F}_p, Rs_{X,!}\mathcal{F}_G^\bullet, 1)) = -\chi^{\mathrm{ref}}_{\mathcal{O},G}(R\Gamma(\mathbb{F}_p, Rs_{X,!}\mathcal{F}_G)). \tag{21}$$

In addition, the result of Emerton and Kisin in [18, Cor. 1.8] implies that

$$Z_{\Lambda_{\mathcal{O}}(G)}(X, \mathcal{F}_G^\bullet, t) = \upsilon(X, \mathcal{F}_G^\bullet, t) L(\mathbb{F}_p, Rs_{X,!}\mathcal{F}_G^\bullet, t) \in 1 + t\Lambda_{\mathcal{O}}(G)[[t]] \tag{22}$$

for an element $\upsilon(X, \mathcal{F}_G^\bullet, t)$ of $1 + \mathfrak{m}t\Lambda_{\mathcal{O}}(G)\langle t\rangle$.

Any series in $1 + \mathfrak{m}t\Lambda_{\mathcal{O}}(G)\langle t\rangle$ converges at $t = 1$ to give an element of $\Lambda_{\mathcal{O}}(G)^\times$ and so (22) implies that $Z_{\Lambda_{\mathcal{O}}(G)}(X, \mathcal{F}_G^\bullet, t)$ converges at $t = 1$ to the value $\xi_{\mathcal{F}} := Z_{\Lambda_{\mathcal{O}}(G)}(X, \mathcal{F}_G^\bullet, 1) = \upsilon(1) L(\mathbb{F}_p, Rs_{X,!}\mathcal{F}_G^\bullet, 1) \in \Lambda_{\mathcal{O}}(G)_S^\times$, where we set $\upsilon(1) := \upsilon(X, \mathcal{F}_G^\bullet, 1)$. Thus, since $\upsilon(1)$ belongs to $\Lambda_{\mathcal{O}}(G)^\times$, the equality (21) implies that

$$\begin{aligned}\partial_{\mathcal{O},G}(\xi_{\mathcal{F}}) &= \partial_{\mathcal{O},G}(\upsilon(1) L(\mathbb{F}_p, Rs_{X,!}\mathcal{F}_G^\bullet, 1)) \\ &= \partial_{\mathcal{O},G}(L(\mathbb{F}_p, Rs_{X,!}\mathcal{F}_G^\bullet, 1)) = -\chi^{\mathrm{ref}}_{\mathcal{O},G}(R\Gamma(\mathbb{F}_p, Rs_{X,!}\mathcal{F}_G)).\end{aligned}$$

In addition, if for each ρ in $A(G)$ we write (as in the proof of Lemma 4.8) r_ρ for the order of vanishing of the function $\Phi_{G,\rho}(\xi_{\mathcal{F}})$ at $u = 0$, then Lemma 4.9 below implies that $\xi_{\mathcal{F}}$ satisfies

$$\begin{aligned}\xi^*_{\mathcal{F}}(\rho) &:= \Phi_{G,\rho}(\xi_{\mathcal{F}})^*(0)\\ &= \lim_{u\to 0} u^{-r_\rho} Z_{\mathcal{O}}(X, \mathcal{F}\otimes\mathcal{L}_\rho, (1+u)^{-1})\\ &= \lim_{t\to 1}(1-t)^{-r_\rho}t^{r_\rho}Z_{\mathcal{O}}(X,\mathcal{F}\otimes\mathcal{L}_\rho,t)\\ &= \lim_{t\to 1}(1-t)^{-r_\rho}Z_{\mathcal{O}}(X,\mathcal{F}\otimes\mathcal{L}_\rho,t)\\ &= Z^*_{\mathcal{O}}(X,\mathcal{F}\otimes\mathcal{L}_\rho,1)\end{aligned}$$

where the third equality follows by setting $t := (1+u)^{-1}$ (so $u = (1-t)t^{-1}$). The given element $\xi_{\mathcal{F}}$ therefore validates Theorem 4.6, as required.

Lemma 4.9. *Fix an Artin representation $\rho : G \to \mathrm{GL}_m(\mathcal{O}')$, with $\mathcal{O} \subseteq \mathcal{O}'$, and identify $\mathrm{Q}_{\mathcal{O}'}(\Gamma)$ with $\mathrm{Q}(\mathcal{O}'[[u]])$ by setting $u := \gamma - 1$. Then $Z_{\mathcal{O}'}(X, \mathcal{F}\otimes_{\mathcal{O}}\mathcal{L}_\rho, (1+u)^{-1})$ is well defined as an element of $\mathrm{Q}(\mathcal{O}'[[u]])$ and satisfies $Z_{\mathcal{O}'}(X, \mathcal{F}\otimes_{\mathcal{O}}\mathcal{L}_\rho, (1+u)^{-1}) = \Phi_{\mathcal{O},G,S,\rho}(Z_{\Lambda_{\mathcal{O}}(G)}(X, \mathcal{F}_G, 1))$.*

Proof. By mimicking the construction of $\Phi_{\mathcal{O},G,S,\rho}$ in §2.3 one can define a composite homomorphism

$$\begin{aligned}\Phi_{\mathcal{O},G,\rho,t} : \Lambda_{\mathcal{O}}(G)[[t]]^\times \cong K_1(\Lambda_{\mathcal{O}}(G)[[t]]) &\to K_1(\mathrm{M}_m(\mathrm{Q}(\Lambda_{\mathcal{O}'}(\Gamma_X)[[t]])))\\ &\cong K_1(\mathrm{Q}(\Lambda_{\mathcal{O}'}(\Gamma_X)[[t]])) \cong \mathrm{Q}(\Lambda_{\mathcal{O}'}(\Gamma_X)[[t]])^\times\\ &\qquad\subseteq \mathrm{Q}(\Lambda_{\mathcal{O}'}(\Gamma)[[t]])^\times \cong \mathrm{Q}(\mathcal{O}'[[u,t]])^\times.\end{aligned}$$

For each $x \in X^0$ the inverse limit over all open normal subgroups U of G of the isomorphism (6) with $\mathcal{L} = \mathcal{F}$ and $h = f_U$ constitutes an isomorphism of $\Lambda_{\mathcal{O}}(G)[\phi^{\mathrm{d}(x)}]$-modules between $\mathcal{F}_{G,\overline{x}}$ and $\Lambda_{\mathcal{O}}(G)\otimes_{\mathcal{O}}\mathcal{F}_{\overline{x}}$, where on the latter module $\Lambda_{\mathcal{O}}(G)$ acts by multiplication on the left on $\Lambda_{\mathcal{O}}(G)$ and $\phi^{\mathrm{d}(x)}$ acts diagonally (via its natural action on $\mathcal{F}_{\overline{x}}$ and its actions on the sets $F_{\overline{x}}(Y_U)$ that are induced by its action on $\overline{x}$). By using this explicit description of each stalk $\mathcal{F}_{G,\overline{x}}$ one finds that $\Phi_{\mathcal{O},G,\rho,t}$ sends the element $Z_{\Lambda_{\mathcal{O}}(G)}(X, \mathcal{F}^\bullet_G, t)$ of $1 + t\Lambda_{\mathcal{O}}(G)[[t]] \subset \Lambda_{\mathcal{O}}(G)[[t]]^\times$ to the series

$$\theta(u,t) := \prod_{x\in X^0} \det{}_{\mathcal{O}'[[u]]}(1-\phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)} \mid \mathcal{O}'[[u]]\otimes_{\mathcal{O}'}\mathcal{L}_{\rho,\overline{x}}\otimes_{\mathcal{O}'}\mathcal{F}_{\overline{x}})^{-1} \in \mathcal{O}'[[u,t]],$$

where $\phi^{\mathrm{d}(x)}$ acts diagonally on the indicated tensor product, in the natural way on the stalks $\mathcal{L}_{\rho,\overline{x}}$ and $\mathcal{F}_{\overline{x}}$ and as multiplication by $(1+u)^{-\mathrm{d}(x)}$ on $\mathcal{O}'[[u]]$.

We next note that if $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}'$ then for any element $\varpi(u)$ of $1+\mathfrak{p}u\mathcal{O}'\langle u\rangle$ the series $\varpi((1+u)^{-1})$ is well-defined as an element of $1+\mathfrak{p}u\mathcal{O}'[[u]]$. In particular, from the equality (22) with $\Lambda_{\mathcal{O}}(G)$ and $\mathcal{L}_G$ replaced by $\mathcal{O}'$ and $\mathcal{F}\otimes\mathcal{L}_\rho$ respectively, it follows that $Z_{\mathcal{O}'}(X,\mathcal{F}\otimes\mathcal{L}_\rho,(1+u)^{-1}) = \theta(0,(1+u)^{-1})$ is well defined as an element of $\mathrm{Q}(\mathcal{O}'[[u]])$. On the other hand, the given action of $\phi^{\mathrm{d}(x)}$ on $\mathcal{O}'[[u]]$

implies that $\theta(0,(1+u)^{-1})$ is equal to $\theta(u,1)$ and hence that in $\mathrm{Q}(\mathcal{O}'[[u]])$ one has

$$\begin{aligned}\Phi_{\mathcal{O},G,S,\rho}(Z_{\Lambda_{\mathcal{O}}(G)}(X,\mathcal{F}^\bullet_G,1)) &= \Phi_{\mathcal{O},G,\rho,t}(Z_{\Lambda_{\mathcal{O}}(G)}(X,\mathcal{F}^\bullet_G,t))|_{t=1}= \theta(u,1)\\ &= \theta(0,(1+u)^{-1}) = Z(X,\mathcal{F}\otimes\mathcal{L}_\rho,(1+u)^{-1}),\end{aligned}$$

as claimed. □

5. The general strategy

The results of §4 imply that it suffices to prove Theorem 1.1 in the case that G is a p-elementary group of rank one and in addition that this result is valid if G is abelian. We quickly review the strategy introduced by the present author and Kato that can be used to deduce Theorem 1.1 in the general case.

5.1. We assume that G has rank one and that $\Sigma(G)$ is any set of (rank one abelian) subquotients of G of the form U^{ab} with U an open subgroup of G that has the following property:

(∗) For each ρ in $A(G)$ there is a finite subset $\{U_i^{\mathrm{ab}} : i \in I\}$ of $\Sigma(G)$ and for each index i an integer m_i and a degree one representation ρ_i of U_i^{ab} such that there is an isomorphism of (virtual) representations $\rho \cong \sum_{i\in I} m_i \cdot \mathrm{Ind}_{U_i}^G \mathrm{Inf}_{U_i^{\mathrm{ab}}}^{U_i}\rho_i$.

We fix a finite unramified extension $\mathcal{O}$ of $\mathbb{Z}_p$ and for each $\mathcal{A} = U^{\mathrm{ab}}$ in $\Sigma(G)$ we write $q_{\mathcal{O},\mathcal{A},S}$ for the composite homomorphism

$$K_1(\Lambda_{\mathcal{O}}(G)_S) \to K_1(\Lambda_{\mathcal{O}}(U)_S) \to K_1(\Lambda_{\mathcal{O}}(\mathcal{A})_S) \to \Lambda_{\mathcal{O}}(\mathcal{A})_S^\times \subset \mathrm{Q}_{\mathcal{O}}(\mathcal{A})^\times$$

where the first arrow is the natural restriction map, the second the natural projection and the third is the isomorphism induced by taking determinants over $\Lambda_{\mathcal{O}}(\mathcal{A})_S$. We then define a composite homomorphism

$$\Delta_{\mathcal{O},\Sigma(G)} : K_1(\Lambda_{\mathcal{O}}(G)) \to K_1(\Lambda_{\mathcal{O}}(G)_S) \to \prod_{\mathcal{A}\in\Sigma(G)} \mathrm{Q}_{\mathcal{O}}(\mathcal{A})^\times$$

where the first arrow is induced by the inclusion $\Lambda_{\mathcal{O}}(G) \subset \Lambda_{\mathcal{O}}(G)_S$ and the second is the diagonal map $\prod_{\mathcal{A}} q_{\mathcal{O},\mathcal{A},S}$. We abbreviate $q_{\mathbb{Z}_p,\mathcal{A},S}$ and $\Delta_{\mathbb{Z}_p,\Sigma(G)}$ to $q_{\mathcal{A},S}$ and $\Delta_{\Sigma(G)}$.

We fix a flat, smooth $\mathbb{Z}_p$-sheaf $\mathcal{L}$ on X and, following Theorem 4.6, for each $\mathcal{A} = U^{\mathrm{ab}}$ in $\Sigma(G)$ we write $\xi_{\mathcal{A},\mathcal{L}}$ for the unique element of $K_1(\Lambda(\mathcal{A})_S) \cong \Lambda(\mathcal{A})_S^\times$ that validates Theorem 1.1 with X, G and $\mathcal{L}$ replaced by Y_U, U^{ab} and $f_U^*\mathcal{L}$ respectively.

For details of more general versions of the following result see [6, Th. 2.2 and Rem. 2.5] and [23, Prop. 2.5].

Proposition 5.1. *Theorem 1.1 is valid if for every p-elementary group G of rank one, and any set of subquotients $\Sigma(G)$ that satisfies property (∗), there exists an element ξ of $K_1(\Lambda(G)_S)$ which satisfies the following two conditions.*

(i) $\partial_G(\xi) = -\chi^{\mathrm{ref}}(R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G))$.

(ii) *The element* $(q_{\mathcal{A}}(\xi)^{-1}\xi_{\mathcal{A},\mathcal{L}})_{\mathcal{A}}$ *belongs to* $\mathrm{im}(\Delta_{\Sigma(G)})$.

Proof. Let v be any element of $K_1(\Lambda(G))$ with $\Delta_{\Sigma(G)}(v) := (q_{\mathcal{A}}(\xi)^{-1}\xi_{\mathcal{A},\mathcal{L}})_{\mathcal{A}}$ and write u for the image of v under the natural homomorphism $K_1(\Lambda(G)) \to K_1(\Lambda(G)_S)$. Then we claim that the element $\xi_{X,G,\mathcal{L}} := u\xi$ validates the conditions of Theorem 1.1.

Firstly, since u belongs to $\ker(\partial_G)$, condition (i) implies that

$$\partial_G(\xi_{X,G,\mathcal{L}}) = \partial_G(u\xi) = \partial_G(\xi) = -\chi^{\mathrm{ref}}(R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)).$$

To show that $\xi_{X,G,\mathcal{L}}$ has the necessary interpolation properties we fix an Artin representation ρ of G. Then the isomorphism $\rho \cong \sum_{i\in I} m_i \cdot \mathrm{Ind}_{U_i}^G \mathrm{Inf}_{U_i^{\mathrm{ab}}}^{U_i}\rho_i$ given by condition $(*)$ implies that

$$\begin{aligned}\Phi_{G,\rho}(\xi_{X,G,\mathcal{L}}) &= \prod_{i\in I}\Phi_{G,\mathrm{Ind}_{U_i}^G\mathrm{Inf}_{\mathcal{A}_i}^{U_i}(\rho_i)}(\xi_{X,G,\mathcal{L}})^{m_i}\\ &= \prod_{i\in I}\Phi_{\mathcal{A}_i,\rho_i}(q_{\mathcal{A}_i,S}(\xi_{X,G,\mathcal{L}}))^{m_i} = \prod_{i\in I}\Phi_{\mathcal{A}_i,\rho_i}(\xi_{\mathcal{A}_i,\mathcal{L}})^{m_i}\end{aligned}$$

where we set $\mathcal{A}_i := U_i^{\mathrm{ab}}$, the first displayed equality is obvious, the second follows from [6, Lem. 3.6(i)] and the last is valid because $q_{\mathcal{A}_i,S}(\xi_{X,G,\mathcal{L}}) = \xi_{\mathcal{A}_i,\mathcal{L}}$. Thus, since each element $\xi_{\mathcal{A}_i,\mathcal{L}}$ is assumed to validate Theorem 1.1 with X, G and $\mathcal{L}$ replaced by Y_{U_i}, $\mathcal{A}_i$ and $f_{U_i}^*\mathcal{L}$ respectively, one has

$$\begin{aligned}\xi^*_{X,G,\mathcal{L}}(0) := \Phi_{G,\rho}(\xi_{X,G,\mathcal{L}})^*(0) &= \prod_{i\in I}\Phi_{\mathcal{A}_i,\rho_i}(\xi_{\mathcal{A}_i,\mathcal{L}})^*(0)^{m_i}\\ &= \prod_{i\in I} Z^*(Y_{U_i}, f_{U_i}^*\mathcal{L}\otimes\mathcal{L}_{\rho_i}, 1)^{m_i}\\ &= Z^*(X, \mathcal{L}\otimes\mathcal{L}_{\sum_{i\in I} m_i\cdot\mathrm{Ind}_{U_i}^G\mathrm{Inf}_{U_i^{\mathrm{ab}}}^{U_i}(\rho_i)}, 1)\\ &= Z^*(X, \mathcal{L}\otimes\mathcal{L}_\rho, 1),\end{aligned}$$

where the fourth equality follows from Lemma 4.3. □

5.2. We now recall Kakde's explicit description of a set $\Sigma(G)$ as above and also of the associated group $\mathrm{im}(\Delta_{\Sigma(G)})$. In fact, we shall assume (as we may) that G is a p-elementary group of rank one, with a corresponding direct product decomposition $G = P' \times P$ and then, just as in §4.2, we use the induced decompositions $\Lambda(G) = \prod_{\psi\in P'^*}\Lambda_{\mathbb{Z}_p[\psi]}(P)$ and $\mathcal{L}_G \cong \oplus_{\psi\in P'^*}(\mathcal{L}\otimes\mathcal{L}_\psi)_P$. In particular, after replacing $G, \Lambda(G)$ and $\mathcal{L}$ by $P, \Lambda_{\mathcal{O}}(P)$ and $\mathcal{L}\otimes\mathcal{L}_\psi$ we may (and will) henceforth assume that G is a pro-p group of rank one.

We fix a lift $\tilde{\Gamma}$ of Γ in G and use this to identify G with the semi-direct product $H \rtimes \Gamma$ with $H := \ker(\pi_G)$. We also fix a natural number e such that $\tilde{\Gamma}^{p^e}$ is central in G and set $\overline{G} := G/\tilde{\Gamma}^{p^e}$ and $R := \Lambda_{\mathcal{O}}(\tilde{\Gamma}^{p^e})$. The algebra $\Lambda_{\mathcal{O}}(G)$ can be identified with the twisted group ring $R[\overline{G}]^\tau$ with multiplication $(h\tilde{\gamma}^a)^\tau(h'\tilde{\gamma}^b)^\tau = \tilde{\gamma}^{p^e[\frac{a+b}{p^e}]}(h\tilde{\gamma}^a\cdot h'\tilde{\gamma}^b)^\tau$

where $[x]$ denotes the greatest integer less than or equal to x and for any g in G we write g^τ for its image in $R[\overline{G}]^\tau$.

For any subgroup P of $\overline{G}$ we write U_P for the inverse image of P in G, write $N_{\overline{G}}P$ for the normaliser of P in $\overline{G}$ and set $W_{\overline{G}}P := N_{\overline{G}}P/P$. We write $C(\overline{G})$ for the set of cyclic subgroups of $\overline{G}$ and note that for any P in $C(\overline{G})$ the group U_P is a rank one abelian subquotient of G. For each P in $C(\overline{G})$ we write T_P for the image of the endomorphism of $R[P]^\tau$ given by $x \mapsto \sum_{g\in W_{\overline{G}}(P)} g^\tau x(g^\tau)^{-1}$. For any subgroups P and P' of $\overline{G}$ with $[P',P'] \le P \le P'$ we write

$$\mathrm{nr}_P^{P'} : \Lambda_{\mathcal{O}}(U_{P'}^{\mathrm{ab}})^\times \to \Lambda_{\mathcal{O}}(U_P/[U_{P'},U_{P'}])^\times,$$
$$\pi_P^{P'} : \Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}}) \to \Lambda_{\mathcal{O}}(U_P/[U_{P'},U_{P'}])$$

for the natural norm and projection maps respectively.

For any non-trivial cyclic subgroup P of $\overline{G}$ we fix a homomorphism $\omega_P : P \to \mathbb{Q}_p^{c\times}$ of order p. We also fix a homomorphism $\omega_1 := \omega_{\{1\}} : \tilde{\Gamma}^{p^e} \to \mathbb{Q}_p^{c\times}$ of order p and use the same symbol to denote the endomorphism of $\Lambda_{\mathcal{O}}(U_P)^\times$ that sends each $g \in U_P$ to $\omega_P(g)g$. For each subgroup P of $\overline{G}$ we define a map α_P from $\Lambda_{\mathcal{O}}(U_P)_S^\times$ to itself in the following way: if P is the trivial subgroup $\{1\}$, we set $\alpha_{\{1\}}(x) := x^p\varphi(x)^{-1}$; if P is non-trivial and cyclic we set $\alpha_P(x) := x^p(\prod_{k=0}^{k=p-1}\omega_P^k(x))^{-1}$; if P is not cyclic, we set $\alpha_P(x) := x^p$. For each P in $C(\overline{G})$ we write $C_P(\overline{G})$ for the set of cyclic subgroups P' of $\overline{G}$ with $P'^p = P$ and $P' \neq P$.

Finally we note that the group G, and hence also $\overline{G}$ (since $\tilde{\Gamma}^{p^e}$ is central), acts on the set $\{U_P^{\mathrm{ab}} : P \le \overline{G}\}$ by conjugation.

Proposition 5.2. *(Kakde) Let G be a rank one pro-p group. Then the set $\Sigma(G) := \{U_P^{\mathrm{ab}} : P \le \overline{G}\}$ satisfies the condition $(*)$ in §5.1. Further, an element $(\xi_{\mathcal{A}})_{\mathcal{A}}$ of $\prod_{\mathcal{A}\in\Sigma(G)}\Lambda_{\mathcal{O}}(\mathcal{A})^\times$ belongs to $\mathrm{im}(\Delta_{\mathcal{O},\Sigma(G)})$ if and only if it satisfies all of the following three conditions.*

(i) *For all subgroups P and P' of $\overline{G}$ with $[P',P'] \le P \le P'$ one has $\mathrm{nr}_P^{P'}(\xi_{U_{P'}^{\mathrm{ab}}}) = \pi_P^{P'}(\xi_{U_P^{\mathrm{ab}}})$.*

(ii) *For all subgroups P of $\overline{G}$ and all g in $\overline{G}$ one has $\xi_{gU_P^{\mathrm{ab}}g^{-1}} = g\xi_{U_P^{\mathrm{ab}}}g^{-1}$.*

(iii) *For all $P \in C(\overline{G})$ one has $\alpha_P(\xi_{U_P^{\mathrm{ab}}}) \equiv \prod_{P'\in C_P(\overline{G})}\alpha_{P'}(\xi_{U_{P'}^{\mathrm{ab}}}) \pmod{pT_P}$.*

Proof. The first claim follows directly from the proof of [22, Prop. 86] and the second is the result of [22, Th. 48]. □

6. Logarithms for power series

The first step in our proof of Theorem 1.1 will be to prove an analogue of the congruences that occur in Proposition 5.2 for certain kinds of power series. To do this we need to introduce a variant of the logarithmic techniques introduced by Oliver and M. Taylor and this is what we describe in this section. The constructions that we make here are analogues of those first made by Ritter and Weiss in [34] (and then later by Kakde in [22]).

In this section we use the notation and hypotheses of §5.2 and hence assume, in particular, that G is pro-p. We set $R := \Lambda_{\mathcal{O}}(\tilde{\Gamma}^{p^e})$ and $R_t := R[[t]]$. For any subgroup P of $\overline{G}$ we set $R_{P,t} := \Lambda_{\mathcal{O}}(U_P)[[t]] = R_t[P]^\tau$ and write κ_P for the R_t-linear map $R_t[P]^\tau \to R_t[\mathrm{Conj}(P)]^\tau$ which for each g in P sends g^τ to its conjugacy class in P. We also write $J(A)$ for the Jacobson radical of any ring A.

6.1. In the next result we use the p-adic logarithm series $\mathrm{Log}(1+x) := \sum_{i\geq 1}(-1)^{i+1}\frac{x^i}{i}$.

Proposition 6.1. *For each subgroup P of $\overline{G}$ the association $1+x \mapsto \mathrm{Log}(1+x)$ for x in $J(R_{P,t})$ induces a canonical homomorphism $\log_{P,t} : K_1(R_{P,t}) \to R_t[\mathrm{Conj}(P)]^\tau[\frac{1}{p}]$.*

Proof. We claim first that for any $x \in J(R_{P,t})$ the series $\mathrm{Log}(1+x)$ converges to an element of $J(R_{P,t})[\frac{1}{p}]$. To see this note that, since $\mathcal{O}/\mathbb{Z}_p$ is unramified, $J(R_{P,t})$ is the kernel of the natural homomorphism $\Lambda_{\mathcal{O}}(U_P)[[t]] \to (\mathcal{O}/p)(\Gamma)$ and so is generated by t, p and the augmentation ideal $I_{H'}$ of $\Lambda_{\mathcal{O}}(H')[[t]]$ where we write H' for the kernel $H \cap U_P$ of the natural composite homomorphism $U_P \subset G \to \Gamma$. But, since H' is a finite p-group, one has $I_{H'}^n \subseteq p\Lambda_{\mathcal{O}}(H')$ for any large enough integer n. This implies that for any natural number m there exists an integer m' such that $J(R_{P,t})^n \subseteq (p,X)^m R_{P,t}$ for all $n \geq m'$. For any given natural number M it follows that the coefficient of t^M in x^i/i converges to 0 as i tends to infinity. It is therefore clear that for any $x \in J(R_{P,t})$ the series $\mathrm{Log}(1+x)$ converges to an element of $R_{P,t}[\frac{1}{p}] = R_t[P]^\tau[\frac{1}{p}]$.

To construct $\log_{P,t}$ one can now copy arguments of Oliver. Firstly, the argument used by Oliver to prove [30, Lem. 2.7] shows that for any x, y in $J(R_{P,t})$ one has

$$\mathrm{Log}((1+x)(1+y)) \equiv \mathrm{Log}(1+x) + \mathrm{Log}(1+y) \pmod{[R_{P,t}[\frac{1}{p}], J(R_{P,t})[\frac{1}{p}]]}$$

and then the argument of [30, Th. 2.8] implies that the association $1+x \mapsto \mathrm{Log}(1+x)$ for x in $J(R_{P,t})$ induces a well-defined homomorphism

$$\log'_{P,t} : K_1(R_{P,t}, J(R_{P,t})) \to (J(R_{P,t})/[R_{P,t}, J(R_{P,t})])[\frac{1}{p}].$$

Finally we note that $R_{P,t}/J(R_{P,t})$ is isomorphic to $\mathbb{F}_p$ and hence that there is an exact sequence

$$1 \to K_1(R_{P,t}, J(R_{P,t})) \to K_1(R_{P,t}) \to K_1(\mathbb{F}_p). \tag{23}$$

In particular, since $K_1(\mathbb{F}_p) \cong \mathbb{F}_p^\times$ is finite and $(J(R_{P,t})/[R_{P,t}, J(R_{P,t})])[\frac{1}{p}]$ is torsion-free, it is clear that $\log'_{P,t}$ extends uniquely to give the desired isomorphism $\log_{P,t}$. □

Before stating the next result we note that if U_P is abelian, then $K_1(R_{P,t})$ and $R_t[\mathrm{Conj}(P)]^\tau$ identify with $R_{P,t}^\times$ and $R_{P,t}$ respectively.

Lemma 6.2. *Assume that U_P is abelian and let I be any ideal of $\Lambda_{\mathcal{O}}(U_P) = R[P]^\tau$ that belongs to $p\Lambda_{\mathcal{O}}(U_P)$. Then $\log_{P,t}$ induces an isomorphism between the groups $1 + I[[t]]$ and $I[[t]]$.*

Proof. We fix x in $I[[t]]$ and for each pair of non-negative integers m and n we write $c_n^m(x)$ for the coefficient of t^m in x^n.

We first claim that $\log_{P,t}$ maps $1 + I[[t]]$ to $I[[t]]$. To see that note that, since $I \subseteq pR[P]^\tau$ one has $I^p \subseteq pI$ and hence $I^n \subset nI$ for all natural numbers n. In particular, since $c_n^m(x) \in I^n$, one has $c_n^m(x)/n \in I$ for all $n \geq 1$. As the series $\sum_{n\geq 1}(-1)^{n+1}c_n^m(x)/n$ converges (by the argument in the proof of Proposition 6.1) it must therefore converge to an element of I. It follows that the series $\log_{P,t}(1 + x) = \sum_{m\geq 1}(\sum_{n\geq 1}(-1)^{n+1}c_n^m(x)/n)t^m$ belongs to $I[[t]]$, as claimed.

We next claim that for each x in I the exponential series $\exp_{P,t}(x) := \sum_{i\geq 0} \frac{x^i}{i!}$ converges to an element of $1 + I[[t]]$. The key point here is that $I \subseteq pR[P]^\tau$ so $I \subset J(R[P]^\tau)$ and $I^p \subset pI \cdot J(R[P]^\tau)$. By the same argument as given in [30, p. 52] it thus follows that for any non-negative integer k and any natural number n with $p^k \leq n < p^{k+1}$ one has $I^n \subset n!I \cdot J(R[P]^\tau)^k$ and hence that $c_n^m(x)/n!$ belongs to $I \cdot J(R[P]^\tau)^k$. This implies that the series $\sum_{n\geq 1} c_n^m(x)/n!$ converges to an element of I and hence that the series $\exp_{P,t}(x) = 1 + \sum_{m\geq 0}(\sum_{n\geq 1} c_n^m(x)/n!)t^m$ converges to an element of $1 + I[[t]]$.

Lastly we note that standard power series identities imply that $\log_{P,t} \circ \exp_{P,t}(x) = x$ and $\exp_{P,t} \circ \log_{P,t}(1 + x) = 1 + x$ for all x in I and hence that $\log_{P,t}$ induces an isomorphism from $1 + I[[t]]$ to $I[[t]]$, as claimed. □

6.2. For each pair of subgroups P and P' of $\overline{G}$ with $P \leq P'$ we write

$$\theta_{P,t}^{P'} : K_1(\Lambda_{\mathcal{O}}(U_{P'})[[t]]) = K_1(R_{P',t}) \to K_1(R_{P,t}) = K_1(\Lambda_{\mathcal{O}}(U_P)[[t]])$$

for the standard restriction map on K-groups. We also define an R_t-linear map

$$\mathrm{Res}_{P,t}^{P'} : R_t[\mathrm{Conj}(P')]^\tau \to R_t[\mathrm{Conj}(P)]^\tau$$

by setting, for each g in P',

$$\mathrm{Res}_{P,t}^{P'}(\kappa_{P'}(g^\tau)) := \sum_x \kappa_P((x^\tau)^{-1}(gx)^\tau)$$

where x runs over all elements in a given set of left coset representatives of P in P' which satisfy $xgx^{-1} \in P$. (It is straightforward to check that this recipe does indeed give a well-defined homomorphism.)

We will use the following functorial property of the functions $\log_{P,t}$ under change of group P.

Lemma 6.3. *For each subgroup P of $\overline{G}$ the following diagram commutes.*

$$\begin{array}{ccc} K_1(\Lambda_{\mathcal{O}}(G)[[t]]) & \xrightarrow{\log_{\overline{G},t}} & R_t[\mathrm{Conj}(\overline{G})]^\tau[\frac{1}{p}] \\ {\scriptstyle\theta^{\overline{G}}_{P,t}}\big\downarrow & & {\scriptstyle\mathrm{Res}^{\overline{G}}_{P,t}}\big\downarrow \\ K_1(\Lambda_{\mathcal{O}}(U_P)[[t]]) & \xrightarrow{\log_{P,t}} & R_t[\mathrm{Conj}(U)]^\tau[\frac{1}{p}]. \end{array}$$

Proof. The same argument as used by Oliver and L. Taylor to prove [31, Th. 1.4] shows that for all ξ in $\Lambda_{\mathcal{O}}(G)[[t]] = R_t[\overline{G}]^\tau$ one has

$$\log_{P,t}(\theta^{\overline{G}}_{P,t}(1+p\xi)) = \mathrm{Res}^{\overline{G}}_{P,t}(\log_{\overline{G},t}(1+p\xi)). \tag{24}$$

Now for any $x \in J(R_t[\overline{G}]^\tau)$ the proof of Proposition 6.1 shows that there exists an element $\tilde{\xi}$ of $R_t[\overline{G}]^\tau$ with

$$(1+x)^{p^n} = 1 + p\xi' + x^{p^n} = 1 + p\xi + t^{p^n}\xi' = (1+t^{p^n}\xi')[1+p\xi(1+t^{p^n}\xi')^{-1}].$$

But $t^{p^n}\xi' \in J(R_t[\overline{G}]^\tau)$ and $\xi(1+t^{p^n}\xi')^{-1} \in R_t[\overline{G}]^\tau$ and so (24) implies

$$\begin{aligned} & p^n\log_{P,t}(\theta^{\overline{G}}_{P,t}(1+x)) \\ =& \log_{P,t}(\theta^{\overline{G}}_{P,t}((1+x)^{p^n})) \\ =& \log_{P,t}(\theta^{\overline{G}}_{P,t}(1+p\xi(1+t^{p^n}\xi')^{-1})) + \log_{P,t}(\theta^{\overline{G}}_{P,t}(1+t^{p^n}\xi')) \\ =& \mathrm{Res}^{\overline{G}}_{P,t}(\log_{\overline{G},t}(1+p\xi(1+t^{p^n}\xi')^{-1})) + \log_{P,t}(\theta^{G}_{P,t}(1+t^{p^n}\xi')) \\ =& \mathrm{Res}^{\overline{G}}_{P,t}(\log_{\overline{G},t}((1+x)^{p^n})) + \log_{P,t}(\theta^{G}_{P,t}(1+t^{p^n}\xi')) - \mathrm{Res}^{G}_{P,t}(\log_{\overline{G},t}((1+t^{p^n}\xi'))) \\ =& p^n\mathrm{Res}^{\overline{G}}_{P,t}(\log_{\overline{G},t}(1+x)) + \log_{P,t}(\theta^{G}_{P,t}(1+t^{p^n}\xi')) - \mathrm{Res}^{\overline{G}}_{P,t}(\log_{\overline{G},t}((1+t^{p^n}\xi'))). \end{aligned}$$

It is clear that $\log_{\overline{G},t}(1+t^{p^n}\xi')$ belongs to $t^{p^n}R_t[\mathrm{Conj}(P)]^\tau[\frac{1}{p}]$ and easy to see that

$$\theta^{\overline{G}}_{P,t}(1+t^{p^n}\xi') \equiv 1 + t^{p^n}\mathrm{Res}^{\overline{G}}_{P,t}\xi' \pmod{K_1(R_t[P]^\tau, t^{p^n}R_t[P]^\tau)}.$$

This implies that $\log_{P,t}(\theta^{\overline{G}}_{P,t}(1+t^{p^n}\xi'))$ and $\mathrm{Res}^{\overline{G}}_{P,t}(\log_{\overline{G},t}((1+t^{p^n}\xi')))$ both belong to $t^{p^n}R_t[\mathrm{Conj}(P)]^\tau[\frac{1}{p}]$ and hence, in conjunction with the above formula, that

$$\log_{P,t}(\theta^{\overline{G}}_{P,t}(1+x)) \equiv \mathrm{Res}^{\overline{G}}_{P,t}(\log_{\overline{G},t}(1+x)) \ (\mathrm{mod}\, t^{p^n}R_t[\mathrm{Conj}(P)]^\tau[\frac{1}{p}]).$$

Since this is true for all natural numbers n it follows that $\log_{P,t}(\theta^{\overline{G}}_{P,t}(1+x)) = \mathrm{Res}^{\overline{G}}_{P,t}(\log_{\overline{G},t}(1+x))$ and hence that the diagram in the statement of the lemma commutes if one replaces $K_1(\Lambda_{\mathcal{O}}(G)) = K_1(R_t[\overline{G}]^\tau)$ by $K_1(R_t[\overline{G}]^\tau, J(R_t[\overline{G}]^\tau))$. However, the index of $K_1(R_t[\overline{G}]^\tau, J(R_t[G]^\tau))$ in $K_1(R_t[G]^\tau)$ is finite (by (23)) and the group $R_t[\mathrm{Conj}(P)]^\tau[\frac{1}{p}]$ is torsion-free and so the given diagram must also commute, as claimed. □

6.3. We now define φ_t to be the map on $R_t = \Lambda_{\mathcal{O}}(\tilde{\Gamma}^{p^e})[[t]]$ that is equal to the Frobenius automorphism Fr on $\mathcal{O}$ and the p-th power map on $\tilde{\Gamma}^{p^e}$ and in addition sends each term t^i to t^{ip}. We extend this map, again denoted φ_t, to $R_t[\mathrm{Conj}(\overline{G})]^\tau$ by mapping $\kappa(g^\tau)$ to $\kappa((g^\tau)^p)$.

We then set $L_{G,t} := (1 - p^{-1}\varphi_t) \circ \log_{\overline{G},t}$.

Proposition 6.4. *One has* $\mathrm{Im}(L_{G,t}) \subseteq R_t[\mathrm{Conj}(\overline{G})]^\tau$.

Proof. The index of $K_1(R_t[\overline{G}]^\tau, J(R_t[\overline{G}]^\tau))$ in $K_1(R_t[\overline{G}]^\tau)$ is finite and prime to p (by (23)) and so it is enough to prove that $L_{G,t}(K_1(R_t[\overline{G}]^\tau, J(R_t[\overline{G}]^\tau))) \subseteq R_t[\mathrm{Conj}(\overline{G})]$. By a result of Vaserstein [40] every element of $K_1(R_t[\overline{G}]^\tau, J(R_t[\overline{G}]^\tau))$ is the image $\langle 1-x\rangle$ of an element $1-x$ with $x \in J(R_t[\overline{G}]^\tau)$. Now for any such x one has

$$L_{G,t}(\langle 1-x\rangle) := -\sum_{i\geq 1}\frac{x^i}{i} + \sum_{j\geq 1}\frac{\varphi_t(x)}{ip} = -\sum_{\substack{i\geq 1\\ p\nmid i}}\frac{x^i}{i} + \sum_{k\geq 1}\frac{x^{pk}-\varphi(x^k)}{pk}$$

and so it suffices to prove that $\sum_{k\geq 1}(pk)^{-1}(x^{pk} - \varphi_t(x^k))$ belongs to $R_t[\mathrm{Conj}(\overline{G})]^\tau$. Since every prime other than p is invertible in R_t it is thus enough to prove that $p^{-n}(y^{p^n} - \varphi_t(y^{p^n})) \in R_t[\mathrm{Conj}(\overline{G})]^\tau$ for all natural numbers n and all $y \in R_t[\overline{G}]^\tau$. This containment can be proved by simply mimicking the proof of [30, the first assertion of Th. 6.2]. Indeed, only two points are worth mentioning in our context: for any set of elements $\{g_{i_j} : 1 \leq j \leq q\}$ of $\overline{G}$ one has $\kappa_{\overline{G}}(\prod_{j=1}^{j=q} g_{i_j}^\tau) = \kappa_{\overline{G}}(g_{i_q}^\tau \prod_{j=1}^{j=q-1} g_{i_j}^\tau)$ (since $\kappa_{\overline{G}}(g^\tau h^\tau) = \kappa_{\overline{G}}(h^\tau g^\tau)$ for all g and h in $\overline{G}$) and for any element $r = \sum_{i,j\geq 0} c_{ij}(\gamma^{p^e}-1)^i t^j$ of R_t with $c_{ij} \in \mathcal{O}$ for all i and j, one has

$$\varphi_t(r) = \sum_{i,j\geq 0}\mathrm{Fr}(c_{ij})(\gamma^{p^{e+1}}-1)^i t^{pj} \equiv \sum_{i,j\geq 0} c_{ij}^p(\gamma^{p^e}-1)^{pi}t^{pj} \equiv r^p \pmod{pR_t}.$$

□

7. Congruences for power series

In this section we continue to use the notation and hypotheses of §5.2.

7.1. For any subgroups P and P' of $\overline{G}$ with $[P', P'] \leq P \leq P'$ we write

$$\mathrm{nr}_{P,t}^{P'} : \Lambda_{\mathcal{O}}(U_{P'}^{\mathrm{ab}})[[t]]^\times \to \Lambda_{\mathcal{O}}(U_P/[U_{P'}, U_{P'}])[[t]]^\times$$
$$\pi_{P,t}^{P'} : \Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}})[[t]] \to \Lambda_{\mathcal{O}}(U_P/[U_{P'}, U_{P'}])[[t]]$$

for the natural norm and projection maps respectively. For any subgroup P of $\overline{G}$ we also write $\theta_{P,t}^{\overline{G},\mathrm{ab}}$ for the natural composite homomorphism

$$K_1(\Lambda_{\mathcal{O}}(G)[[t]]) \xrightarrow{\theta_{P,t}^{\overline{G}}} K_1(\Lambda_{\mathcal{O}}(U_P)[[t]]) \to K_1(\Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}})[[t]]) \cong \Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}})[[t]]^\times$$

where the isomorphism is induced by taking determinants over $\Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}})[[t]]$.

For any non-trivial cyclic subgroup P of $\overline{G}$ we write $\omega_{P,t}$ for the endomorphism of $\Lambda_{\mathcal{O}}(U_P)[[t]]^\times$ that sends each element gt^i with g in U_P to $\omega_P(g)gt^i$. We also write $\omega_{1,t} := \omega_{\{1\},t}$ for the endomorphism of $\Lambda_{\mathcal{O}}(U_{\{1\}})[[t]]^\times = \Lambda_{\mathcal{O}}(\tilde{\Gamma}^{p^e})[[t]]^\times$ that sends each gt^i to $\omega_P(g)gt^i$ for g in Γ^{p^e} and $i \geq 0$. For each subgroup P of $\overline{G}$ we define a map $\alpha_{P,t}$ from $\Lambda_{\mathcal{O}}(U_P)[[t]]^\times$ to itself in the following way: if P is the trivial subgroup $\{1\}$, we set $\alpha_{\{1\},t}(x) := x^p\varphi_t(x)^{-1}$; if P is non-trivial and cyclic we set $\alpha_{P,t}(x) := x^p(\prod_{k=0}^{k=p-1}\omega_{P,t}^k(x))^{-1}$; if P is not cyclic, we set $\alpha_{P,t}(x) := x^p$.

The main result that we shall prove in this section is the following

Theorem 7.1. *Fix an element $\xi(t)$ of $K_1(\Lambda_{\mathcal{O}}(G)[[t]])$ and for all subgroups P of $\overline{G}$ set $\xi_P(t) := \theta_{P,t}^{\overline{G},\mathrm{ab}}(\xi(t)) \in \Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}})[[t]]^\times$.*

(i) *For all subgroups P and P' of $\overline{G}$ with $P \leq P'$ and $[P',P'] \leq P$ one has $\mathrm{nr}_{P,t}^{P'}(\xi_{P'}(t)) = \pi_{P,t}^{P'}(\xi_P(t))$.*
(ii) *For all subgroups P of $\overline{G}$ and all g in $\overline{G}$ one has $\xi_{gPg^{-1}}(t) = g\xi_P(t)g^{-1}$.*
(iii) *For all $P \in C(G)$ one has $\alpha_{P,t}(\xi_P(t)) \equiv \prod_{P'\in C_P(G)}\alpha_{P',t}(\xi_{P'}(t)) \pmod{pT_P[[t]]}$.*

Our proof of this result combines the logarithmic constructions made in §6 with a straightforward modification of arguments of Kakde in [22].

7.2. Before proving Theorem 7.1 we first recall how the homomorphisms $\theta_{P,t}^{P',\mathrm{ab}}$ can be computed explicitly.

To do this we set $n_{P',P} := [P' : P] = [U_{P'} : U_P]$. We also note that, since $\Lambda_{\mathcal{O}}(U_{P'})[[t]]$ is a local ring, the natural homomorphism $q_{P'}$ from $\Lambda_{\mathcal{O}}(U_{P'})[[t]]^\times$ to $K_1(\Lambda_{\mathcal{O}}(U_{P'})[[t]])$ is surjective. For any ξ in $K_1(\Lambda_{\mathcal{O}}(U_{P'})[[t]])$ we therefore choose an element $\tilde{\xi}$ of $\Lambda_{\mathcal{O}}(U_{P'})[[t]]^\times$ with $q_{P'}(\tilde{\xi}) = \xi$ and a set of left coset representatives $C(P,P') := \{c_i : 1 \leq i \leq n_{P',P}\}$ of U_P in $U_{P'}$ and write $M_{C(P',P)}(\tilde{\xi})$ for the matrix in $\mathrm{M}_{n_{P':P}}(\Lambda_{\mathcal{O}}(U_P)[[t]])$ of the automorphism of the $\Lambda(U_P)[[t]]$-module $\Lambda_{\mathcal{O}}(U_{P'})[[t]] = \bigoplus_{i=1}^{i=n_{P',P}}\Lambda_{\mathcal{O}}(U_P)[[t]]c_i$ that is given by multiplying on the right by $\tilde{\xi}$. Then $\theta_{P,t}^{P',\mathrm{ab}}(\xi)$ is equal to $\det(\pi_P^{n_{P':P}}(M_{C(P,P')}(\tilde{\xi}))) \in \Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}})[[t]]^\times$ where $\pi_{P',P}$ is the natural projection $\mathrm{M}_{n_{P',P}}(\Lambda_{\mathcal{O}}(U_P)[[t]]) \to \mathrm{M}_{n_{P',P}}(\Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}})[[t]])$.

It is now clear that Theorem 7.1(i) is valid because the diagram given below commutes, where $\pi_{P',t}$ and $\pi_{P,t}$ denote the natural maps. (Note that the upper quadrilateral in this diagram obviously commutes, whilst the lower quadrilateral commutes because $C(P,P')$ can also be regarded as an $\Lambda_{\mathcal{O}}(U_P/[U_{P'},U_{P'}])[[t]]$-basis of $\Lambda_{\mathcal{O}}(U_{P'}^{\mathrm{ab}})[[t]]$.) Indeed, the commutativity of this diagram implies that the image under $\mathrm{nr}_{P,t}^{P'}$ of the projection of $\tilde{\xi}$ to $\Lambda_{\mathcal{O}}(U_{P'}^{\mathrm{ab}})[[t]]$ is equal to $\pi_{P,t}^{P'}(\det(\pi_P^{n_{P',P}}(M_{C(P,P')}(\tilde{\xi})))) = \pi_{P,t}^{P'}(\theta_{P,t}^{P',\mathrm{ab}}(\xi))$, as required.

$$\begin{tikzcd}
K_1(\Lambda_{\mathcal{O}}(G)[[t]]) \arrow[d, "\theta^{\overline{G}}_{P',t}"'] \arrow[rrd, "\theta^{\overline{G}}_{P,t}"] & & \\
K_1(\Lambda_{\mathcal{O}}(U_{P'})[[t]]) \arrow[d, "\pi_{P',t}"'] \arrow[rrd, "\theta^{P',\mathrm{ab}}_{P,t}"] & & K_1(\Lambda_{\mathcal{O}}(U_P)[[t]]) \arrow[d, "\pi_{P,t}"] \\
\Lambda_{\mathcal{O}}(U^{\mathrm{ab}}_{P'})[[t]]^\times \arrow[rd, "\mathrm{nr}^{P'}_{P,t}"] & & \Lambda_{\mathcal{O}}(U^{\mathrm{ab}}_{P})[[t]]^\times \arrow[ld, "\pi^{P'}_{P,t}"'] \\
& \Lambda_{\mathcal{O}}(U_P/[U_{P'},U_{P'}])[[t]]^\times. &
\end{tikzcd}$$

To prove Theorem 7.1(ii) we first compute explicitly each term $\xi_P(t) = \theta^{\overline{G},\mathrm{ab}}_{P,t}(\xi(t))$ by using the recipe above (with ξ equal to $\xi(t)$). We then set $C := C(P,\overline{G})$ and note that for any g in $\overline{G}$ the set $gCg^{-1} := \{gc_ig^{-1} : 1 \le i \le [P' : P]\}$ is a set of left coset representatives of $gU_Pg^{-1} = U_{gPg^{-1}}$ in G. It is easily checked that $M_{gCg^{-1}}(g\tilde{\xi}g^{-1}) = gM_C(\tilde{\xi})g^{-1}$. Since $q_{\overline{G}}(g\tilde{\xi}g^{-1}) = q_{\overline{G}}(\tilde{\xi}) = \xi(t)$ it therefore follows that

$$\xi_{gPg^{-1}}(t) = \det(\pi_{\overline{G},gPg^{-1}}(gM_C(\tilde{\xi})g^{-1})) = g\det(\pi_{\overline{G},P}(M_C(\tilde{\xi})))g^{-1} = g\xi_P(t)g^{-1},$$

as required to prove claim (ii).

7.3. To prove Theorem 7.1(iii) we need a preliminary result. In this result we use the following notation: for each subgroup P of $\overline{G}$ we define $\eta_{P,t}$ to be the R_t-linear endomorphism of $R_t[P]^\tau$ which for each $g \in P$ sends g^τ to 0, resp. g^τ, if P is cyclic and g is not a generator of P, resp. otherwise.

Lemma 7.2. *For all x in $K_1(\Lambda_{\mathcal{O}}(G)[[t]])$ and all P in $C(\overline{G})$ one has*

$$\log_{P,t}\Big(\frac{\alpha_{P,t}(\theta^G_{P,t}(x))}{\prod_{P' \in C_P(\overline{G})} \alpha_{P',t}(\theta^G_{P',t}(x))}\Big) = p(\eta_{P,t} \circ \mathrm{Res}^{\overline{G}}_{P,t})(L_{\overline{G},t}(x)).$$

Proof. It is enough to prove that there is a commutative diagram of the form

$$\begin{tikzcd}
K_1(\Lambda_{\mathcal{O}}(G)[[t]]) \arrow[r, "\log_{\overline{G},t}"] \arrow[d, "(\theta^G_{P,t})_P"'] & R_t[\mathrm{Conj}(\overline{G})]^\tau \arrow[d, "(\mathrm{Res}^{\overline{G}}_{P,t})_P"'] \arrow[rdd] & \\
\prod_{P\in C(\overline{G})} K_1(\Lambda_{\mathcal{O}}(U_P)[[t]]) \arrow[r, "(\log_{P,t})_P"] \arrow[rrd] & \prod_{P\in C(\overline{G})} R_t[P]^\tau[\frac{1}{p}] \arrow[rd] & \\
& & \prod_{P\in C(\overline{G})} R_t[P]^\tau[\frac{1}{p}]
\end{tikzcd}$$

where the upper diagonal arrow sends x to $(\eta_{P,t}(\mathrm{Res}^{\overline{G}}_{P,t}((1-p^{-1}\varphi_t)(x)))_P$ and the lower diagonal arrow sends $(x_P)_P$ to $(p^{-1}\log_{Q,t}(\frac{\alpha_{Q,t}(x_Q)}{\prod_{P'\in C_Q(G)}\alpha_{P',t}(x_{P'})}))_Q$.

Lemma 6.3 for each $P\in C(G)$ implies that the square in the diagram commutes and so it suffices to prove that the central diagonal arrow can be chosen so that the two triangles commute. The methods of Kakde show that this is possible if the central diagonal arrow sends each element $(x_P)_P$ to the element with P-component $1-\sum_{P'\in C_P(G)}\varphi_t(x_{P'})-\delta_P p^{-1}\varphi_t(x_P)$ with δ_P equal to 1, resp. 0, if P is non-trivial, resp. trivial. Indeed, with this definition, the commutativity of the upper triangle follows from natural analogues of [22, Lem. 78] and the first commutative diagram in [22, Lem. 74], whilst the commutativity of the lower triangle follows from a natural analogue of [22, Lem. 76].

To obtain these analogues one makes the following notational changes. In the proof of [22, Lem. 74] one replaces the terms $\Lambda_{\mathcal{O}}(U_P)$, log, α_P, η_P and ω_P by $\Lambda_{\mathcal{O}}(U_P)[[t]]=R_t[P]^\tau$, $\log_{P,t}$, $\alpha_{P,t}$, $\eta_{P,t}$ and $\omega_{P,t}$ respectively. With these changes of notation the argument proceeds exactly as in loc. cit.. In the argument of [22, Lem. 76] one replaces $\Lambda_{\mathcal{O}}(\Gamma^{p^e})$, $\Lambda_{\mathcal{O}}(U_C^{\mathrm{ab}}),\Lambda_{\mathcal{O}}(U_P)$ and φ by R_t, $\Lambda_{\mathcal{O}}(U_C^{\mathrm{ab}})[[t]],\Lambda_{\mathcal{O}}(U_P)[[t]]=R_t[P]^\tau$ and φ_t respectively. One must also replace the maps β^G_P and β^G by $\beta^{\overline{G}}_{P,t}$ and $\beta^{\overline{G}}=(\beta^{\overline{G}}_{P,t})_{P\le\overline{G}}$ respectively, where $\beta^{\overline{G}}_{P,t}:R_t[\mathrm{Conj}(\overline{G})]^\tau\to R_t[P^{\mathrm{ab}}]^\tau$ is defined in the same way as $\beta^{\overline{G}}_P$ in [22, Def. 50] but with $R=\Lambda_{\mathcal{O}}(\Gamma^{p^e})$ replaced by R_t. We also replace v^G_P by the homomorphism $v^{\overline{G}}_{P,t}:\prod_{C\le\overline{G}}\Lambda_{\mathcal{O}}(U_C^{\mathrm{ab}})[[t]]\to\Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}})[[t]]$ which is defined as in [22, Def. 75] but with $\mathrm{ver}^{P'}_P$ replaced by the $\Lambda(\tilde{\Gamma}^{p^e})$-linear map $\mathrm{ver}^{P'}_{P,t}:R_t[P']^\tau\to R_t[P]^\tau$ which acts as p-th power on P' and sends each term t^i to t^{ip}. It then suffices to check that the analogues of the diagrams in [22, Lem. 76] commute on each element of the form $\kappa_{\overline{G}}(g^\tau)t^i$ with $g\in\overline{G}$ and $i\ge 0$, and this is done by the same argument as in loc. cit. (after taking into account the above changes of notation). Finally, in following the argument of [22, Lem. 78] one must make the changes described above and also replace v^G, resp. u^G, by the endomorphism $(v^{\overline{G}}_{P,t})_{P\le\overline{G}}$ of $\prod_{P\le\overline{G}}\Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}})[[t]]$, resp. by the endomorphism of $\prod_{C\le\overline{G}}\Lambda_{\mathcal{O}}(U_C^{\mathrm{ab}})[[t]]^\times$ defined just as in [22, Def. 77] but with $\mathrm{ver}^{P'}_P$ now replaced by $\mathrm{ver}^{\overline{P}'}_{P,t}$. □

Now, since $pT_P[[t]]\subset pR_t[\overline{G}]^\tau$ the result of Lemma 6.2 with $I=pR[\overline{G}]^\tau$ combines with Proposition 6.4 and Lemma 7.2 to imply that the congruence of Theorem 7.1 is valid provided that $\eta_{P,t}\circ\mathrm{Res}^{\overline{G}}_{P,t}$ maps $R_t[\mathrm{Conj}(\overline{G})]^\tau$ to $T_P[[t]]$. Since it is clear that $\eta_{P,t}$ preserves $T_P[[t]]$, the required containment therefore follows directly from the next result.

Lemma 7.3. $\mathrm{Res}^{\overline{G}}_{P,t}(R_t[\mathrm{Conj}(\overline{G})]^\tau)\subseteq T_P[[t]]$.

Proof. A typical element of $R_t[\mathrm{Conj}(\overline{G})]^\tau$ has the form $x:=\sum_{i\ge0}(\sum_{g\in\overline{G}}r_{g,i}\kappa(g))t^i$, with $r_{g,i}\in R$ for all g and i. Now $\mathrm{Res}^{\overline{G}}_{P,t}=\mathrm{Res}^H_{P,t}\circ\mathrm{Res}^{\overline{G}}_{H,t}$ with $H:=N_{\overline{G}}(P)$ and so

$\mathrm{Res}^{\overline{G}}_{P,t}(x) = \sum_{i\geq 0}(\sum_{g\in\overline{G}} r_{g,i}\mathrm{Res}^{H}_{P,t}(\mathrm{Res}^{\overline{G}}_{H,t}(\kappa(g))))t^i$. Further, for each $h, h' \in H$ one has $h^{-1}h'h \in P$ if and only if $h' \in P$ and for any such h' one has

$$\mathrm{Res}^{H}_{P,t}(\kappa_H(h')) = \begin{cases} \sum_{x\in W_{\overline{G}}(P)} \kappa_P((x^\tau)^{-1}h^\tau x^\tau), & \text{if } h' \in P, \\ 0, & \text{otherwise.} \end{cases}$$

This implies that each element $r_{g,i}\mathrm{Res}^{H}_{P,t}(\mathrm{Res}^{\overline{G}}_{H,t}(\kappa(g)))$ belongs to T_P as required. $\square$

8. The proof of Theorem 1.1

8.1. If η is an endomorphism of a finitely generated projective $\Lambda_{\mathcal{O}}(G)$-module M, then for any series $f(\eta,t) := \sum_{j\geq 0}\sum_{i\geq 0} m_{ij}\eta^i t^j$ in $\mathbb{Z}[\eta][[t]]$ one obtains a well-defined automorphism of the associated finitely generated projective $\Lambda_{\mathcal{O}}(G)[[t]]$-module $M_t := \mathcal{O}[[t]]\otimes_{\mathcal{O}} M$ by setting $f(\eta,t)(\lambda\otimes m) := \sum_{j\geq 0}\sum_{i\geq 0} m_{ij}t^j\lambda\otimes\eta^i(m)$ for each $\lambda\in\mathcal{O}[[t]]$ and $m \in M$. We write $\langle f(\eta,t) \mid M_t\rangle$ for the corresponding element of $K_1(\Lambda_{\mathcal{O}}(G)[[t]])$.

In particular, for any flat, smooth $\mathcal{O}$-sheaf $\mathcal{L}$ on X we obtain well-defined elements of $K_1(\Lambda_{\mathcal{O}}(G)[[t]])$ by setting

$$Z_G(X,\mathcal{L}_G,t) := \langle \prod_{x\in X^0}(1-\phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)})^{-1} \mid (\mathcal{L}_{G,\overline{x}})_t\rangle$$

and

$$Z_G(1-\hat{\phi},t) := \prod_{i\in\mathbb{Z}}\langle 1-\hat{\phi}^i t \mid P^i_t\rangle^{(-1)^{i+1}}$$

where the endomorphism $\hat{\phi}$ and complex $P^\bullet$ in $C^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ are chosen as in Proposition 3.3.

Lemma 8.1. *We fix a subgroup P of $\overline{G}$ and set $Q := U_P^{\mathrm{ab}}$.*

(i) $\theta^{\overline{G},\mathrm{ab}}_{P,t}(Z_G(X,\mathcal{L}_G,t)) = Z_{\Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}})}(Y_{U_P}, f_U^*\mathcal{L}_Q, t)$.

(ii) *There exists an endomorphism $\hat{\phi}_Q$ of a complex $P^\bullet_Q$ in $C^{\mathrm{p}}(\Lambda_{\mathcal{O}}(Q))$ that satisfies all of the following conditions:*

  (a) *$\hat{\phi}_Q$ lies in a commutative diagram of the form (8) but with $R\Gamma(\mathbb{F}^c_q, s_c^*Rs_{X,!}\mathcal{L}_G)$ replaced by $R\Gamma(\mathbb{F}^c_q, s_c^*Rs_{Y_U,!}(f_U^*\mathcal{L})_Q)$;*

  (b) *$\theta^{\overline{G},\mathrm{ab}}_{P,t}(Z_G(1-\hat{\phi},t)) = Z_Q(1-\hat{\phi}_Q,t)$ in $K_1(\Lambda_{\mathcal{O}}(Q)[[t]]) \cong \Lambda_{\mathcal{O}}(Q)[[t]]^\times$;*

  (c) *each endomorphism $1-\hat{\phi}^i_Q$ induces an automorphism of $\mathrm{Q}_{\mathcal{O}}(Q)\otimes_{\Lambda_{\mathcal{O}}(Q)} P^i_Q$ and one has $\theta^{\overline{G},\mathrm{ab}}_{P}(Z_G(1-\hat{\phi})) = Z_Q(1-\hat{\phi}_Q)$ in $K_1(\mathrm{Q}_{\mathcal{O}}(Q)) \cong \mathrm{Q}_{\mathcal{O}}(Q)^\times$.*

Proof. We set $U := U_P$. For any endomorphism α of a $\Lambda_{\mathcal{O}}(G)$-module, or complex of $\Lambda_{\mathcal{O}}(G)$-modules, M we write α_Q for the induced endomorphism $\mathrm{id}\otimes\alpha$ of $M_Q := \Lambda_{\mathcal{O}}(Q)\otimes_{\Lambda_{\mathcal{O}}(U)}\mathrm{res}^{\Lambda_{\mathcal{O}}(G)}_{\Lambda_{\mathcal{O}}(U)}M$.

Then claim (i) is valid because

$$
\begin{aligned}
\theta^{\overline{G},\mathrm{ab}}_{P,t}(Z_G(X,\mathcal{L}_G,t)) &= \det{}_{\Lambda(Q)[[t]]}(\prod_{x\in X^0}(1-\phi_Q^{\mathrm{d}(x)}t^{\mathrm{d}(x)})^{-1}\mid(\mathcal{L}_{G,\overline{x}})_{Q,t})\\
&= \prod_{x\in X^0}\det{}_{\Lambda_{\mathcal{O}}(Q)[[t]]}((1-\phi_Q^{\mathrm{d}(x)}t^{\mathrm{d}(x)})^{-1}\mid(\mathcal{L}_{G,\overline{x}})_{Q,t})\\
&= \prod_{x\in X^0}\det{}_{\Lambda_{\mathcal{O}}(Q)}(1-\phi_Q^{\mathrm{d}(x)}t^{\mathrm{d}(x)}\mid(\mathcal{L}_{G,\overline{x}})_{Q})^{-1}\\
&= \prod_{y\in Y_U^0}\det{}_{\Lambda_{\mathcal{O}}(Q)}(1-\phi_Q^{\mathrm{d}(y)}t^{\mathrm{d}(y)}\mid\Lambda(Q)\otimes_{\Lambda(U)}(f_U^*\mathcal{L})_{U,\overline{y}})^{-1}\\
&= Z_{\Lambda_{\mathcal{O}}(U_P^{\mathrm{ab}})}(Y_U,(f_U^*\mathcal{L})_Q,t)
\end{aligned}
$$

where the fourth equality follows from the same type of argument as used in Lemma 4.3(i).

Regarding claim (ii) we first observe that Lemma 4.2(iii) implies there are natural isomorphisms in $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(Q))$

$$
\begin{aligned}
\Lambda(Q)\otimes^{\mathbb{L}}_{\Lambda(U)}\mathrm{res}^{\Lambda_{\mathcal{O}}(G)}_{\Lambda_{\mathcal{O}}(U)}R\Gamma(\mathbb{F}_q^c,s_c^*Rs_{X,!}\mathcal{L}_G) &\cong \Lambda_{\mathcal{O}}(Q)\otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(U)}R\Gamma(\mathbb{F}_q^c,s_c^*Rs_{Y_U,!}(f_U^*\mathcal{L})_U)\\
&\cong R\Gamma(\mathbb{F}_q^c,s_c^*Rs_{Y_U,!}(f_U^*\mathcal{L})_Q),
\end{aligned}
$$

and hence that $\hat{\phi}_Q$ and $P_Q^\bullet$ do indeed lie in a commutative diagram of the form described in claim (a). Claim (b) is then true because

$$
\begin{aligned}
\theta^{\overline{G},\mathrm{ab}}_{P,t}(Z_G(1-\hat{\phi},t)) &= \prod_{i\in\mathbb{Z}}\theta^{\overline{G},\mathrm{ab}}_{P,t}(\langle 1-\hat{\phi}^i t\mid P_t^i\rangle)^{(-1)^{i+1}}\\
&= \prod_{i\in\mathbb{Z}}\det{}_{\Lambda(Q)[[t]]}(1-\hat{\phi}_Q^i t\mid P_{Q,t}^i)^{(-1)^{i+1}}\\
&= Z_Q(1-\hat{\phi}_Q,t).
\end{aligned}
$$

It is also clear that each endomorphism $1-\hat{\phi}_Q^i=(1-\hat{\phi}^i)_Q$ induces an automorphism of $Q_{\mathcal{O}}(Q)\otimes_{\Lambda_{\mathcal{O}}(Q)}P_Q^i$ and hence that the element $Z_Q(1-\hat{\phi}_Q)$ in claim (c) is well-defined. The equality in claim (c) then follows by an exact analogue of the argument used to prove claim (b). □

8.2. We are now ready to prove Theorem 1.1. In view of the observations made at the beginning of §5.2, we may (and will) assume that G is a rank one pro-p group and that $\mathcal{L}$ is a flat, smooth $\mathcal{O}$-sheaf on X for a finite unramified extension $\mathcal{O}$ of $\mathbb{Z}_p$.

By combining the equality (9) with the result of Proposition 3.1(iv) and the exact commutative diagram (4) one finds that the element $Z_G(1-\hat{\phi})$ belongs to the image of $K_1(\Lambda_{\mathcal{O}}(G)_S)\to K_1(Q_{\mathcal{O}}(G))$. In this subsection we shall prove Theorem 1.1 by showing that any pre-image ξ of $Z_G(1-\hat{\phi})$ in $K_1(\Lambda_{\mathcal{O}}(G)_S)$ satisfies the conditions of

Proposition 5.1 with respect to the set of subgroups $\Sigma(G)$ described in Proposition 5.2.

That such an element ξ satisfies condition (i) of Proposition 5.1 follows immediately from the equality (9). To discuss condition (ii) we first recall that for each subgroup P of $\overline{G}$ Theorem 4.6 implies that $\xi_{U_P^{\rm ab}} := Z_{\Lambda_{\mathcal{O}}(U_P^{\rm ab})}(Y_{U_P}, (f_{U_P}^*\mathcal{L})_{U_P^{\rm ab}}, 1)$ is the (unique) element of $K_1(\Lambda_{\mathcal{O}}(U_P^{\rm ab})) = \Lambda_{\mathcal{O}}(U_P^{\rm ab})^\times$ that validates Theorem 1.1 with $X, G, \Lambda(G)$ and $\mathcal{L}$ replaced by $Y_{U_P}, U_P^{\rm ab}, \Lambda_{\mathcal{O}}(U_P^{\rm ab})$ and $f_{U_P}^*\mathcal{L}$ respectively. Thus, since Lemma 8.1(ii)(c) implies $\theta_P^{\overline{G},{\rm ab}}(\xi) = Z_{U_P^{\rm ab}}(1-\hat{\phi}_{U_P^{\rm ab}})$, we can deduce from the equality (22) that

$$\theta_P^{\overline{G},{\rm ab}}(\xi)\xi_{U_P^{\rm ab}} = Z_{\Lambda_{\mathcal{O}}(U_P^{\rm ab})}(Y_{U_P}, f_{U_P}^*\mathcal{L}, 1) Z_{U_P^{\rm ab}}(1-\hat{\phi}_{U_P^{\rm ab}})^{-1} = \upsilon_P(1) \in \Lambda_{\mathcal{O}}(U_P^{\rm ab})^\times$$

where we set $\upsilon_P(t) := \upsilon(Y_{U_P}, f_{U_P}^*\mathcal{L}, t) \in 1 + t\mathfrak{m}\Lambda_{\mathcal{O}}(U_P^{\rm ab})\langle t\rangle$ and $\mathfrak{m}$ denotes the maximal ideal of the local ring $\Lambda_{\mathcal{O}}(U_P^{\rm ab})$. To verify condition (ii) of Proposition 5.1 it therefore suffices to use the criteria of Proposition 5.2 to show that the element $(\upsilon_P(1))_{P\le\overline{G}}$ belongs to ${\rm im}(\Delta_{\mathcal{O},\Sigma(G)})$.

To do this we set $\xi(t) := Z_G(1-\hat{\phi}, t)^{-1} Z_G(X, \mathcal{L}_G, t) \in \Lambda_{\mathcal{O}}(G)[[t]]^\times$. Then Lemma 8.1 implies that for all subgroups P of $\overline{G}$ one has

$$\theta_{P,t}^{\overline{G},{\rm ab}}(\xi(t)) = Z_{U_P^{\rm ab}}(1-\hat{\phi}_{U_P^{\rm ab}}, t)^{-1} Z_{\Lambda_{\mathcal{O}}(U_P^{\rm ab})}(Y_{U_P}, \mathcal{L}_{U_P^{\rm ab}}, t) = \upsilon_P(t).$$

From Theorem 7.1 it follows that the series $\upsilon_P(t)$ for $P\le\overline{G}$ satisfy all of the explicit conditions that are stated in that result. But each series $\upsilon_P(t)$ converges at $t=1$ and so to deduce that the limit elements $\upsilon_P(1)$ for $P\le\overline{G}$ satisfy the conditions of Proposition 5.2, as is required to complete the proof of Theorem 1.1, we now need only apply the following result with each $f_P(t)$ equal to the series $\upsilon_P(t)$.

Lemma 8.2. *For each subgroup P of $\overline{G}$ we assume to be given a series $f_P(t)$ in $\Lambda_{\mathcal{O}}(U_P^{\rm ab})[[t]]^\times$ that converges at $t=1$ to an element of $\Lambda_{\mathcal{O}}(U_P^{\rm ab})^\times$.*

- (i) *For all subgroups P and P' of $\overline{G}$ with $P\le P'$ and $[P',P']\le P$ the series ${\rm nr}_{P,t}^{P'}(f_{P'}(t))$ converges at $t=1$ to the value ${\rm nr}_P^{P'}(f_{P'}(1))$.*
- (ii) *For all subgroups P and P' of $\overline{G}$ with $P\le P'$ and $[P',P']\le P$ the series $\pi_{P,t}^{P'}(f_P(t))$ converges at $t=1$ to the value $\pi_P^{P'}(f_P(1))$.*
- (iii) *For all subgroups P of $\overline{G}$ and all g in $\overline{G}$ the series $gf_P(t)g^{-1}$ converges at $t=1$ to the value $gf_P(1)g^{-1}$.*
- (iv) *For all P in $C(\overline{G})$ the series $\alpha_{P,t}(f_P(t))$ converges at $t=1$ to $\alpha_P(f_P(1))\in\Lambda_{\mathcal{O}}(U_P)^\times$.*
- (v) *If P belongs to $C(\overline{G})$ and $\alpha_{P,t}(f_P(t)))\equiv\prod_{P'\in C_P(\overline{G})}\alpha_{P',t}(f_{P'}(t)) \pmod{pT_P[[t]]}$, then also $\alpha_P(f_P(1))\equiv\prod_{P'\in C_P(G)}\alpha_{P'}(f_{P'}(1)) \pmod{pT_P}$.*

Proof. In the context of claim (i) we set $R' := \Lambda_{\mathcal{O}}(U_P/[U_{P'},U_{P'}])$ and $R := \Lambda_{\mathcal{O}}(U_P^{\rm ab})$ and fix a set of coset representatives $\mathcal{U} := \{u_i : 1\le i\le n\}$ of P in P'. Then $\mathcal{U}$ can be regarded both as an R-basis of R' and as an $R[[t]]$-basis of $R'[[t]]$. This implies that $\mathrm{nr}_{P,t}^{P'}(f_{P'}(t))$ is equal to the determinant of the matrix $M_{P',t}$ in $\mathrm{M}_n(R[[t]])$ that represents, with respect to $\mathcal{U}$, the endomorphism of $R'[[t]]$ given by multiplication by $f_{P'}(t)$ whilst $\mathrm{nr}_P^{P'}(f_{P'}(1))$ is equal to the determinant of the matrix $M_{P'}$ in $\mathrm{M}_n(R)$ that represents, with respect to $\mathcal{U}$, the endomorphism of R' given by multiplication by $f_{P'}(1)$. An easy check shows that each entry of $M_{P',t}$ converges at $t = 1$ to the corresponding entry of $M_{P',1}$ and hence that $\det(M_{P',t})$ converges at $t = 1$ to the value $\det(M_{P',1})$, as required to prove claim (i).

Claim (ii) is an obvious consequence of the definitions of $\pi_{P,t}^{P'}$ and $\pi_P^{P'}$ and claim (iii) follows immediately from the fact that the map $x \mapsto gxg^{-1}$ is continuous on $\Lambda_{\mathcal{O}}(G)$.

Claim (iv) is a consequence of the fact that for all P in $C(\overline{G})$ each map $\omega_P^a : \Lambda_{\mathcal{O}}(P)^\times \to \Lambda_{\mathcal{O}}(P)^\times$, and hence also the map $\alpha_P : \Lambda_{\mathcal{O}}(P)^\times \to \Lambda_{\mathcal{O}}(P)^\times$, is continuous. To consider claim (v) we write $\Delta(t)$ for the series $\alpha_{P,t}(f_P(t)) - \prod_{P'\in C_P(G)} \alpha_{P',t}(f_{P'}(t))$ in $\Lambda_{\mathcal{O}}(P)[[t]]$. Then the hypothesis of claim (v) is that $\Delta(t)$ belongs to $pT_P[[t]]$ and so Lemma 8.3 below implies that $\Delta(t) = p\sum_{i=1}^{i=n} \Delta_i(t)T(g_i)$ with $\Delta_i(t) \in R_t$ for each index i. Since claim (iv) implies that $\Delta(t)$ converges at $t = 1$ it follows that each series $\Delta_i(t)$ converges at $t = 1$ (to an element of R) and hence that the element $\Delta(1) = p\sum_{i=1}^{i=n} \Delta_i(1)T(g_i)$ belongs to pT_P, as required to prove claim (v). □

Lemma 8.3. *For each g in P set $T(g) := \sum_{w\in W_{\overline{G}}(P)} w^{-1}gw \in \mathcal{O}[P]$. Then there exists a subset $\{g_i : 1 \le i \le n\}$ of P such that the set $\{T(g_i) : 1 \le i \le n\}$ is an R_t-basis of $T_P[[t]]$.*

Proof. It is clear that the R_t-module $T_P[[t]]$ is spanned by the set $\mathcal{B} := \{T(g) : g \in P\}$ and so it suffices to show this set is linearly independent over R_t. Set $n := |\mathcal{B}|$ and choose a minimal subset $\{g_i : 1 \le i \le n\}$ of P with $\mathcal{B} = \{T(g_i) : 1 \le i \le n\}$. Since for each $i \ne j$ one has $T(g_i) \ne T(g_j)$ the sets $\{w^{-1}g_iw : w \in W_{\overline{G}}(P)\}$ and $\{w^{-1}g_jw : w \in W_{\overline{G}}(P)\}$ are disjoint. Thus, if for each i we write n_i for the number of summands in $T(g_i)$ that are equal to g_i, then in any expression of the form $x = \sum_{j=1}^{j=n} r_jT(g_j)$, with $r_j \in R_t$ for each j, the coefficient of each element g_i is equal to n_ir_i. Hence if $x = 0$, then $r_i = 0$ for all i, as required. □

This completes our proof of Theorem 1.1.

Remark 8.4. At each geometric point $\overline{x}$ of X the $\Lambda(G)$-module $\mathcal{L}_{G,\overline{x}}$ is isomorphic to $\Lambda(G) \otimes_{\mathbb{Z}_p} \mathcal{L}_{\overline{x}}$ and so is free of rank, d say, independent of $\overline{x}$. In particular, if we fix a $\Lambda(G)$-basis of $\mathcal{L}_{G,\overline{x}}$ and write $M(\phi^{\mathrm{d}(x)})$ for the matrix that corresponds to the action of $\phi^{\mathrm{d}(x)}$ on $\mathcal{L}_{G,\overline{x}}$, then after choosing an ordering of X^0 we obtain an element of $\mathrm{GL}_d(\Lambda(G)[[t]])$ by setting $\hat{Z}_G(X, \mathcal{L}_G, t) := \prod_{x\in X^0}(\mathrm{Id} - M(\phi^{\mathrm{d}(x)})t^{\mathrm{d}(x)})^{-1}$.

The natural homomorphism $\mathrm{GL}_d(\Lambda(G)[[t]]) \to K_1(\Lambda(G)[[t]])$ sends $\hat{Z}_G(X, \mathcal{L}_G, t)$ to the element $Z_G(X, \mathcal{L}_G, t)$ that plays a key role in the proof of Theorem 1.1. However, the validity of Theorem 1.1 does not imply that $\hat{Z}_G(X, \mathcal{L}_G, t)$ converges at $t = 1$. Indeed, if $\hat{Z}_G(X, \mathcal{L}_G, t)$ does converge at $t = 1$ (for any choice of $\Lambda(G)$-bases of the stalks $\mathcal{L}_{G,\overline{x}}$ and any ordering of X^0), then it can be used to generalise the approach to Theorem 1.1 discussed in Remark 4.7.

9. The proof of Corollary 1.2

In this section we fix a Cartesian diagram (3) and assume both that X is geometrically connected and the conditions (i) and (ii) in Corollary 1.2 are satisfied. We note in particular that $\Gamma_X = \Gamma$ under these hypotheses (by [21, Exp. V, Prop. 6.9]). We also abbreviate the Euler characteristics $\chi^{\rm ref}_{\mathbb{Z}[\mathcal{G}],\mathbb{Q}[\mathcal{G}]}(-,-)$ and $\chi^{\rm ref}_{\mathbb{Z}_\ell[\mathcal{G}],\mathbb{Q}_\ell[\mathcal{G}]}(-,-)$ for each prime ℓ that are discussed in §2.1 to $\chi^{\rm ref}_{\mathcal{G}}(-,-)$ and $\chi^{\rm ref}_{\mathcal{G},\ell}(-,-)$ respectively.

9.1. For each prime ℓ we consider the complex of $\mathbb{Z}_\ell[\mathcal{G}]$-modules

$$0 \to H^0_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell) \xrightarrow{\beta^0_{f,\ell}} H^1_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell) \xrightarrow{\beta^1_{f,\ell}} H^2_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell) \xrightarrow{\beta^2_{f,\ell}} \cdots \tag{25}$$

where $\beta^i_{f,\ell}$ denotes the composite homomorphism

$$\begin{aligned} H^i_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell) \cong H^i(\mathbb{F}_q, Rs_{X,!}f_*f^*\mathbb{Z}_\ell) \to H^i(\mathbb{F}^c_q, s^*_cRs_{X,!}f_*f^*\mathbb{Z}_\ell) \\ \to H^{i+1}(\mathbb{F}_q, Rs_{X,!}f_*f^*\mathbb{Z})_\ell) \cong H^{i+1}_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell) \end{aligned}$$

where the arrows denote the maps that occur in the cohomology sequence of the exact triangle in Proposition 3.1(iii) with G and $\mathcal{L}$ replaced by $\mathcal{G}$ and $\mathbb{Z}_\ell$ respectively.

We also write $\delta_{\mathcal{G},\ell}$ for the composite homomorphism

$$\zeta(\mathbb{Q}_\ell[\mathcal{G}])^\times \to K_1(\mathbb{Q}_\ell[\mathcal{G}]) \to K_0(\mathbb{Z}_\ell[\mathcal{G}], \mathbb{Q}_\ell[\mathcal{G}])$$

where the first map is induced by the inverse of the (bijective) reduced norm map $\mathrm{Nrd}_{\mathbb{Q}_\ell[\mathcal{G}]}$ and the second map is the standard connecting homomorphism of relative K-theory.

We fix an isomorphism $\iota : \mathbb{C} \cong \mathbb{C}_p$ and for each χ in $\mathrm{Ir}(\mathcal{G})$ we use this isomorphism to identify the composite character $\iota \circ \chi$ with any fixed choice of representation in $A(\mathcal{G})$ that has character $\iota \circ \chi$.

Proposition 9.1. *The complex $R\Gamma(X'_{W\text{ét}}, f'_*f'^*j_!\mathbb{Z})$ belongs to $D^{\rm p}(\mathbb{Z}[\mathcal{G}])$ and for each prime ℓ the complex $R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell)$ belongs to $D^{\rm p}(\mathbb{Z}_\ell[\mathcal{G}])$. Further, the cohomology groups of the complex (25) are finite and Corollary 1.2 is valid if and only if in $K_0(\mathbb{Z}_p[\mathcal{G}], \mathbb{Q}_p[\mathcal{G}])$ one has*

$$\delta_{\mathcal{G},p}(Z^*_{\mathcal{G}}(X,1)) = -\chi^{\rm ref}_{\mathcal{G},p}(R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_p), \beta_{f,p}) \tag{26}$$

where we set

$$Z^*_{\mathcal{G}}(X,1) := \sum_{\chi\in\mathrm{Ir}(\mathcal{G})} Z^*(X, \mathcal{L}_{\iota\circ\chi}, 1)e_{\iota\circ\chi} \in \zeta(\mathbb{Q}^c_p[\mathcal{G}])^\times$$

and write $\beta_{f,p}$ for the exact sequence of $\mathbb{Q}_p[\mathcal{G}]$-modules induced by (25) with $\ell = p$.

Proof. Since f'_* is exact (as f' is finite), $f'^*j_! = j_{Y,!}f^*$ and $f'^*\mathbb{Z} = \mathbb{Z}$ on $Y'_{W\text{ét}}$ there are natural isomorphisms $R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z}) \cong R\Gamma(Y'_{W\text{ét}}, f'^*j_!\mathbb{Z}) \cong R\Gamma(X'_{W\text{ét}}, f'_*f'^*j_!\mathbb{Z})$ in $D(\mathbb{Z}[\mathcal{G}])$. In particular, condition (i) of Corollary 1.2 implies that each module $H^i(R\Gamma(X'_{W\text{ét}}, f'_*f'^*j_!\mathbb{Z}))$ is a finitely generated abelian group and so the criterion of [17, Rapport, Lem. 4.5.1] shows that $R\Gamma(X'_{W\text{ét}}, f'_*f'^*j_!\mathbb{Z})$ belongs to $D^{\mathrm{p}}(\mathbb{Z}[\mathcal{G}])$ if and only if it has finite Tor-dimension. In view of the isomorphisms proved in Lemma 9.2 below it is therefore enough to show that $R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell)$ belongs to $D^{\mathrm{p}}(\mathbb{Z}_\ell[\mathcal{G}])$ for each prime ℓ. But at each ℓ the stalk of $f_*f^*\mathbb{Z}_\ell$ at any geometric point of X is isomorphic to $\mathbb{Z}_\ell[\mathcal{G}]$, regarded as a (finitely generated, flat) left $\mathbb{Z}_\ell[\mathcal{G}]$-module in the natural way, and so [17, Rapport, Th. 4.9] implies that $R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell)$ belongs to $D^{\mathrm{p}}(\mathbb{Z}_\ell[\mathcal{G}])$.

We now set $x := \delta_{\mathcal{G}}(Z^*(f,1)) + \chi^{\mathrm{ref}}_{\mathcal{G}}(R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z}), \epsilon_{f,j}) \in K_0(\mathbb{Z}[\mathcal{G}], \mathbb{Q}[\mathcal{G}])$ and for each prime ℓ write $\pi_\ell : K_0(\mathbb{Z}[\mathcal{G}], \mathbb{Q}[\mathcal{G}]) \to K_0(\mathbb{Z}_\ell[\mathcal{G}], \mathbb{Q}_\ell[\mathcal{G}])$ for the natural map. Then, since $\bigcap_\ell \ker(\pi_\ell) = 0$ (where the intersection runs over all primes), Corollary 1.2 is valid if and only if $\pi_\ell(x)$ vanishes for every ℓ.

For each prime ℓ we write θ_ℓ for the element of the group $H^1(\mathrm{Spec}(\mathbb{F}_q)_{\text{ét}}, \mathbb{Z}_\ell) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\mathbb{F}_q^c/\mathbb{F}_q), \mathbb{Z}_\ell)$ that sends the Frobenius automorphism to 1 (and hence sends ϕ to -1). For each scheme Z over $\mathbb{F}_q$ we write $\theta_{Z,\ell}$ for the pullback of θ_ℓ to $H^1(Z_{\text{ét}}, \mathbb{Z}_\ell)$. Taking cup product with $\theta_{Z,\ell}$ then gives for any sheaf $\mathcal{F}$ of $\mathbb{Z}_\ell[\mathcal{G}]$-modules on Z a complex of $\mathbb{Z}_\ell[\mathcal{G}]$-modules of the form

$$0 \to H^0_c(Z_{\text{ét}}, \mathcal{F}) \xrightarrow{\cup\theta_{Z,\ell}} H^1_c(Z_{\text{ét}}, \mathcal{F}) \xrightarrow{\cup\theta_{Z,\ell}} H^2_c(Z_{\text{ét}}, \mathcal{F}) \xrightarrow{\cup\theta_{Z,\ell}} \cdots \tag{27}$$

Now from Lemma 9.2 below we know that condition (i) of Corollary 1.2 implies that the complex (27) with $Z = X$ and $\mathcal{F} = f_*f^*\mathbb{Z}_\ell$ has finite cohomology groups and moreover that

$$\begin{aligned}\pi_\ell(\chi^{\mathrm{ref}}_{\mathcal{G}}(R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z}), \epsilon_{f,j})) &= \chi^{\mathrm{ref}}_{\mathcal{G},\ell}(\mathbb{Z}_\ell \otimes R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z}), \mathbb{Q}_\ell \otimes_{\mathbb{Q}} \epsilon_{f,j})\\ &= \chi^{\mathrm{ref}}_{\mathcal{G},\ell}(R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell), \epsilon_{f,\ell})\\ &= \chi^{\mathrm{ref}}_{\mathcal{G},\ell}(R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell), \beta_{f,\ell}).\end{aligned}$$

Here we write $\epsilon_{f,\ell}$ and $\beta_{f,\ell}$ for the exact sequences of $\mathbb{Q}_\ell[\mathcal{G}]$-modules that are induced by (27) with $Z = X$ and $\mathcal{F} = f_*f^*\mathbb{Z}_\ell$ and by (25) respectively, the second equality uses the (obvious) fact that the identifications of Lemma 9.2 induce an equality $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} \epsilon_{f,j} = \epsilon_{f,\ell}$ and the final equality is valid because, as originally observed by Rapoport and Zink in [33, 1.2] (see also [9, §3.2.1]), the description on the level of complexes of taking cup product with $\theta_{X,\ell}$ shows that $\epsilon_{f,\ell}$ and $\beta_{f,\ell}$ coincide.

On the other hand, if ι_ℓ denotes the natural inclusion $\zeta(\mathbb{Q}[\mathcal{G}])^\times \subset \zeta(\mathbb{Q}_\ell[\mathcal{G}])^\times$, then the explicit definition of $\delta_{\mathcal{G}}$ implies that the following diagram commutes

$$\begin{array}{ccc} \zeta(\mathbb{Q}[\mathcal{G}])^\times & \xrightarrow{\delta_{\mathcal{G}}} & K_0(\mathbb{Z}[\mathcal{G}],\mathbb{Q}[\mathcal{G}]) \\ {\scriptstyle\iota_\ell}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi_\ell} \\ \zeta(\mathbb{Q}_\ell[\mathcal{G}])^\times & \xrightarrow{\delta_{\mathcal{G},\ell}} & K_0(\mathbb{Z}_\ell[\mathcal{G}],\mathbb{Q}_\ell[\mathcal{G}]). \end{array}$$

It follows that $\pi_\ell(x)$ vanishes if and only if in $K_0(\mathbb{Z}_\ell[\mathcal{G}],\mathbb{Q}_\ell[\mathcal{G}])$ one has

$$\delta_{\mathcal{G},\ell}(Z^*(f,1)) = -\chi^{\mathrm{ref}}_{\mathcal{G},\ell}(R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell), \beta_{f,\ell}). \tag{28}$$

For each prime $\ell \neq p$ this equality is proved in Proposition 9.3 below. It is therefore clear that Corollary 1.2 is valid if and only if (28) is valid in the case $\ell = p$. The statement of the proposition is thus true because $L^{\mathrm{Artin}}(Y,\chi,t) = Z(X,\mathcal{L}_{\iota\circ\chi},t)$ for each χ in $\mathrm{Ir}(\mathcal{G})$ so $Z(f,t) = Z_{\mathcal{G}}(X,t)$ and hence $Z^*(f,1) = Z^*_{\mathcal{G}}(X,1)$. □

Lemma 9.2. *For every prime ℓ there are natural isomorphisms in $D^-(\mathbb{Z}_\ell[\mathcal{G}])$ of the form*

$$\begin{aligned}\mathbb{Z}_\ell[\mathcal{G}] \otimes^{\mathbb{L}}_{\mathbb{Z}[\mathcal{G}]} R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z}) &\cong \mathbb{Z}_\ell[\mathcal{G}] \otimes^{\mathbb{L}}_{\mathbb{Z}[\mathcal{G}]} R\Gamma(X'_{W\text{ét}}, f'_*f'^*j_!\mathbb{Z}) \\ &\cong R\Gamma(X'_{\text{ét}}, f'_*f'^*j_!\mathbb{Z}_\ell) \cong R\Gamma(X'_{\text{ét}}, j_!f_*f^*\mathbb{Z}_\ell) \cong R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell).\end{aligned}$$

Proof. The complex $R\Gamma(X'_{\text{ét}}, j_!f_*f^*\mathbb{Z}_\ell)$ identifies with $R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell)$ by the very definition of compactly supported étale cohomology, and the functor $\mathbb{Z}_\ell[\mathcal{G}] \otimes_{\mathbb{Z}[\mathcal{G}]} -$ identifies with the exact functor $\mathbb{Z}_\ell \otimes -$. Since we have already shown that the complexes $R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z})$ and $R\Gamma(X'_{W\text{ét}}, f'_*f'^*j_!\mathbb{Z})$ are naturally isomorphic it is therefore enough to prove that there are natural isomorphisms in $D(\mathbb{Z}_\ell[\mathcal{G}])$ of the form $\mathbb{Z}_\ell \otimes R\Gamma(X'_{W\text{ét}}, f'_*f'^*j_!\mathbb{Z}) \cong R\Gamma(X'_{\text{ét}}, f'_*f'^*j_!\mathbb{Z}_\ell) \cong R\Gamma(X_{\text{ét}}, j_!f_*f^*\mathbb{Z}_\ell)$.

To construct such an isomorphism we set $\mathcal{L} := f'_*f'^*j_!\mathbb{Z}$ and $\mathcal{L}/\ell^n := f'_*f'^*j_!(\mathbb{Z}/\ell^n)$ for each natural number n. We also fix an object $P^\bullet$ of $C^-(\mathbb{Z}[\mathcal{G}])$ that is isomorphic in $D^-(\mathbb{Z}[\mathcal{G}])$ to $R\Gamma(X'_{\text{Wét}}, \mathcal{L})$ and an object $Q^\bullet$ of $C^-(\mathbb{Z}_\ell[\mathcal{G}])$ isomorphic in $D(\mathbb{Z}_\ell[\mathcal{G}])$ to $R\Gamma(X_{\text{ét}}, j_!f_*f^*\mathbb{Z}_\ell)$. For each natural number n we set $\Lambda_n := \mathbb{Z}/\ell^n[\mathcal{G}]$. Then the exact sequence $0 \to P^\bullet \xrightarrow{\times \ell^n} P^\bullet \to P^\bullet/\ell^n \to 0$ combines with the natural exact triangle

$$R\Gamma(X'_{\text{Wét}}, \mathcal{L}) \xrightarrow{\times \ell^n} R\Gamma(X'_{\text{Wét}}, \mathcal{L}) \to R\Gamma(X'_{\text{Wét}}, \mathcal{L}/\ell^n) \to R\Gamma(X'_{\text{Wét}}, \mathcal{L})[1]$$

to give an isomorphism $P^\bullet/\ell^n \cong R\Gamma(X'_{\text{Wét}}, \mathcal{L}/\ell^n)$ in $D^{\mathrm{p}}(\Lambda_n)$. In a similar way there is an isomorphism $Q^\bullet/\ell^n \cong R\Gamma(X_{\text{ét}}, j_!f_*f^*\mathbb{Z}/\ell^n)$ in $D(\Lambda_n)$. Next we note that there are natural isomorphisms in $D(\Lambda_n)$

$$R\Gamma(X'_{\text{Wét}}, \mathcal{L}/\ell^n) \cong R\Gamma(X'_{\text{ét}}, \mathcal{L}/\ell^n) \cong R\Gamma(X'_{\text{ét}}, j_!f_*f^*(\mathbb{Z}/\ell^n))$$

where the first isomorphism is by [24, Prop. 2.4(g)] and the second is because $f_*j_{Y,!}\mathcal{T} = j_!f_{U,*}\mathcal{T}$ for any torsion sheaf (by the general observation made just after diagram (12)) and hence $\mathcal{L}/\ell^n = f'_*f'^*j_!(\mathbb{Z}/\ell^n) = f'_*j_{Y,!}f^*(\mathbb{Z}/\ell^n) = j_!f_*f^*(\mathbb{Z}/\ell^n)$.

The observations above imply that there is a natural isomorphism $\alpha_n : Q^\bullet/\ell^n \cong P^\bullet/\ell^n$ in $D^-(\Lambda_n)$ and it may be shown that, as n varies, the isomorphisms α_n are such that the diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & Q^\bullet/\ell^n & \xrightarrow{\pi_n^Q} & Q^\bullet/\ell^{n-1} & \longrightarrow & \cdots \\
 & & {\scriptstyle\alpha_n}\big\downarrow & & {\scriptstyle\alpha_{n-1}}\big\downarrow & & \\
\cdots & \longrightarrow & P^\bullet/\ell^n & \xrightarrow{\pi_n^P} & P^\bullet/\ell^{n-1} & \longrightarrow & \cdots
\end{array}$$

commutes in $D^-(\Lambda_n)$, where π_n^Q and π_n^P denote the natural quotient maps. Since however $Q^\bullet/\ell^n$ consists of projective Λ_n-modules and $P^\bullet/\ell^n$ of Λ_n-modules we can realize each α_n as an actual map of complexes. Moreover, $\alpha_{n-1}\circ\pi_n^Q$ will then be homotopic to $\pi_n^P\circ\alpha_n$, i.e.

$$\alpha_{n-1}\circ\pi_n^Q - \pi_n^P\circ\alpha_n = d\circ h + h\circ d$$

for some map $h : Q^\bullet/\ell^n \to P^\bullet/\ell^{n-1}[-1]$. But, for each i, the projection $P^i/\ell^n \to P^i/\ell^{n-1}$ is surjective and Q^i/ℓ^n is a projective Λ_n-module and so we can lift h to a map $h' : Q^\bullet/\ell^n \to P^\bullet/\ell^n[-1]$. If we then replace α_n by $\alpha_n + d\circ h' + h'\circ d$, the above diagram will actually be a commutative diagram of maps of complexes. So by induction we may assume that, taken together, the maps α_n constitute a map of inverse systems of complexes. The inverse limit of such a compatible system then gives an isomorphism in $D^-(\mathbb{Z}_\ell[\mathcal{G}])$ of the form

$$\begin{aligned}
R\Gamma(X_{\text{ét}}, j_!f_*f^*\mathbb{Z}_\ell) \cong Q^\bullet \cong \varprojlim_n Q^\bullet/\ell^n \cong \varprojlim_n P^\bullet/\ell^n \\
\cong \mathbb{Z}_\ell\otimes P^\bullet \cong \mathbb{Z}_\ell\otimes R\Gamma(X'_{W\text{ét}}, f'_*f'^*j_!\mathbb{Z}),
\end{aligned}$$

as required. $\square$

Proposition 9.3. *The equality (28) is valid for each prime $\ell \neq p$.*

Proof. In each degree i we define $\mathbb{Q}_\ell[\mathcal{G}]$-modules $V_\ell^i := \mathbb{Q}_\ell\otimes_{\mathbb{Z}_\ell} H^i(\mathbb{F}_q^c, s_c^*Rs_{X,!}f_*f^*\mathbb{Z}_\ell)$ and $V_\ell^{i,0} := \ker(1-\phi \mid V_\ell^i)$. Then, as the complex (25) has finite cohomology groups, the composite tautological homomorphism $V_\ell^{i,0}\subseteq V_\ell^i \to \operatorname{cok}(1-\phi\mid V_\ell^i)$ is bijective (so that, in the standard terminology, the action of $1-\phi$ on V_ℓ^i is 'semisimple at 0'). We may therefore fix a $\mathbb{Q}_\ell[\mathcal{G}][\phi]$-equivariant direct complement D_ℓ^i to $V_\ell^{i,0}$ in V_ℓ^i. By applying [10, Prop. 5.10] (or, equivalently, [3, Prop. 3.1]) to the exact triangle of Proposition 3.1(iii) with G replaced by $\mathcal{G}$, $\mathcal{O}=\mathbb{Z}_\ell$ and $\mathcal{L}=\mathbb{Z}_\ell$ we then obtain an equality

$$\begin{aligned}
(29)\qquad -\chi^{\text{ref}}_{\mathcal{G},\ell}(R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell), \beta_{f,\ell}) &= -\sum_{i\in\mathbb{Z}}(-1)^i\delta_{G,\ell}(\mathrm{Nrd}_{\mathbb{Q}_\ell[\mathcal{G}]}(1-\phi\mid D_\ell^i)) \\
&= \delta_{G,\ell}\Big(\prod_{i\in\mathbb{Z}}\mathrm{Nrd}_{\mathbb{Q}_\ell[\mathcal{G}]}(1-\phi\mid D_\ell^i)^{(-1)^{i+1}}\Big).
\end{aligned}$$

We next claim that the result of Grothendieck in [20] implies that for each prime $\ell \neq p$ there is an equality in $\zeta(\mathbb{Q}_\ell[\mathcal{G}])[[t]]$

$$\prod_{i\in\mathbb{Z}} \mathrm{Nrd}_{\mathbb{Q}_\ell[\mathcal{G}]}(1-\phi\cdot t : V_\ell^i)^{(-1)^{i+1}} = Z(f,t). \tag{30}$$

To explain this equality we fix an identification of $\mathbb{C}$ with $\mathbb{C}_\ell$ and for each χ in $\mathrm{Ir}(\mathcal{G})$ a finite extension Ω_χ of $\mathbb{Q}_\ell$ and a representation $\mathcal{G} \to \mathrm{Aut}_{\Omega_\chi}(V_\chi)$ of character χ. Then our definition of $Z(f,t)$ in §1 implies that (30) is valid provided that for each χ in $\mathrm{Ir}(\mathcal{G})$ one has an equality in $\Omega_\chi[[t]]$

$$L^{\mathrm{Artin}}(Y,\chi,t) = \prod_{i\in\mathbb{Z}} \det_{\Omega_\chi}(1-\phi\cdot t : \mathrm{Hom}_{\Omega_\chi[\mathcal{G}]}(V_\chi, \Omega_\chi\otimes_{\mathbb{Q}_\ell} V_\ell^i))^{(-1)^{i+1}}.$$

But, since each space $\mathrm{Hom}_{\Omega_\chi[\mathcal{G}]}(V_\chi, \Omega_\chi\otimes_{\mathbb{Q}_\ell} V_\ell^i)$ identifies with $H_c^i(X^c_{\text{ét}}, E_\chi)$ with E_χ the sheaf of Ω_χ-vector spaces on X that is defined by the contragredient of χ, the last displayed equality follows directly from the exposition of Grothendieck's results given by Milne in [25, Chap. VI, Th. 13.3 and Exam. 13.6(b)].

Now in each degree i there is a direct sum decomposition of $\mathbb{Q}_\ell[\mathcal{G}][\phi]$-modules $V_\ell^i = D_\ell^i \oplus V_\ell^{i,0}$ and hence an equality in $\zeta(\mathbb{Q}_\ell[\mathcal{G}])[[t]]$

$$\begin{aligned}\mathrm{Nrd}_{\mathbb{Q}_\ell[\mathcal{G}]}(1-\phi\cdot t : V_\ell^i) &= \mathrm{Nrd}_{\mathbb{Q}_\ell[\mathcal{G}]}(1-\phi\cdot t : D_\ell^i)\mathrm{Nrd}_{\mathbb{Q}_\ell[\mathcal{G}]}(1-t : V_\ell^{i,0})\\ &= \mathrm{Nrd}_{\mathbb{Q}_\ell[\mathcal{G}]}(1-\phi\cdot t : D_\ell^i)\sum_{\chi\in\mathrm{Ir}(\mathcal{G})}(1-t)^{r_\chi^i}e_\chi\end{aligned}$$

with $r_\chi^i := \dim_{\Omega_\chi}(\mathrm{Hom}_{\Omega_\chi[\mathcal{G}]}(V_\chi, \Omega_\chi\otimes_{\mathbb{Q}_\ell} V_\ell^{i,0}))$. Taken in conjunction with (30) these equalities imply that $Z^*(f,1) = \prod_{i\in\mathbb{Z}}\mathrm{Nrd}_{\mathbb{Q}_\ell[\mathcal{G}]}(1-\phi \mid D_\ell^i)^{(-1)^{i+1}}$ and by substituting this formula into (29) we deduce that the equality (28) is valid for every prime $\ell \neq p$, as required. □

Remark 9.4.

(i) The occurrence of the unit series $\upsilon(X,\mathcal{F}_G,t)$ in the formula (22) of Emerton and Kisin implies that there is no direct analogue of Grothendieck's formula (30) in the case that $\ell = p$ and hence prevents one from using [10, Prop. 5.10] to give a more direct proof of Theorem 1.1.

(ii) The result of Proposition 9.3 can also be proved by combining the main result of Witte in [42] together with the descent formalism developed by Venjakob and the present author in [10].

9.2. We write H_1 for the kernel of the projection homomorphism $\pi_1(X,\overline{x}) \to \mathcal{G}$. Using the notation of diagram (1) we also set $H_2 := \ker(\pi_{X,\overline{x}})$ and note that this group corresponds to the pro-covering $f_\infty : X_\infty \to X$ of group $\Gamma_X = \Gamma \cong \mathbb{Z}_p$. We set $H := H_1 \cap H_2$ and $G := \pi_1(X,\overline{x})/H$ and write $\tilde{f} : \tilde{Y} \to X$ for the corresponding pro-covering of group G. Then the group G lies in a commutative diagram (1) with $\ker(\pi_G)$ equal to the finite group H_2/H and in this subsection we shall deduce the required equality (26) by combining Theorem 1.1 together with the isomorphism $R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_p) \cong R\Gamma(X_{\text{ét}}, j_!f_*f^*\mathbb{Z}_p) \cong R\Gamma(\mathbb{F}_q, Rs_{X,!}f_*f^*\mathbb{Z}_p)$ coming from Lemma 9.2 and the descent formalism developed in [10].

We set $\mathcal{E} := \mathcal{L}_G = f_*f^*\mathbb{Z}_p$ and also define auxiliary pro-sheaves $\tilde{\mathcal{E}} := \tilde{f}_*\tilde{f}^*\mathbb{Z}_p$ and $\mathcal{E}_\infty := f_{\infty,*}f_\infty^*\mathbb{Z}_p$ and complexes $K^\bullet := R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{E})$ and $\tilde{K}^\bullet := R\Gamma(\mathbb{F}_q, Rs_{X,!}\tilde{\mathcal{E}})$. We endow the complex $\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\tilde{K}^\bullet$ with a left action of $\mathbb{Z}_p[\mathcal{G}]\times\Lambda(G)$ in such a way that $(x,g)(x'\otimes k^i) = xx'(\overline{g})^{-1}\otimes g(k^i)$ for all x and x' in $\mathbb{Z}_p[\mathcal{G}]$, g in G and k^i in $\tilde{K}^i$, where $\overline{g}$ denotes the image of g under the natural projection homomorphism $G\to\mathcal{G}$. With this action Lemma 4.2 gives a natural isomorphism $\mathbb{Z}_p[\mathcal{G}]\otimes^{\mathbb{L}}_{\Lambda(G)}\tilde{K}^\bullet \cong K^\bullet \cong R\Gamma_c(X_{\text{ét}},\mathcal{E})$ in $D^{\mathrm{p}}(\mathbb{Z}_p[\mathcal{G}])$ and hence induces an exact triangle in $D(\mathbb{Z}_p[\mathcal{G}])$

$$\begin{aligned}(31)\quad \Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\tilde{K}^\bullet) &\xrightarrow{(1-\gamma)\otimes\mathrm{id}} \Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\tilde{K}^\bullet)\\ &\to R\Gamma_c(X_{\text{ét}},\mathcal{E})\to\Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\tilde{K}^\bullet)[1]\end{aligned}$$

where, as before, γ denotes the image in Γ of the Frobenius automorphism $x\mapsto x^q$. We thereby obtain a complex of $\mathbb{Z}_p[\mathcal{G}]$-modules

$$(32)\qquad 0\to H^0_c(X_{\text{ét}},\mathcal{E})\xrightarrow{\hat{\beta}^0_{f,p}} H^1_c(X_{\text{ét}},\mathcal{E})\xrightarrow{\hat{\beta}^1_{f,p}} H^2_c(X_{\text{ét}},\mathcal{E})\xrightarrow{\hat{\beta}^2_{f,p}}\cdots$$

in which each $\hat{\beta}^i_{f,p}$ is equal to the composite homomorphism

$$H^i_c(X_{\text{ét}},\mathcal{E})\to H^{i+1}(\Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\tilde{K}^\bullet))\to H^{i+1}_c(X_{\text{ét}},\mathcal{E})$$

where both arrows are induced by the long exact cohomology sequence of (31).

The next result applies the descent formalism of [10] in our present setting.

Lemma 9.5. *If the cohomology groups of (32) are finite in each degree, then in $K_0(\mathbb{Z}_p[\mathcal{G}],\mathbb{Q}_p[\mathcal{G}])$ one has $\delta_{\mathcal{G},p}(Z^*_{\mathcal{G}}(X,1)) = -\chi^{\mathrm{ref}}_{\mathcal{G},p}(R\Gamma_c(X_{\text{ét}},\mathcal{E}),\hat{\beta}_{f,p})$ where $\hat{\beta}_{f,p}$ is the exact sequence of $\mathbb{Q}_p[\mathcal{G}]$-modules that is induced by (32).*

Proof. We note first that, in view of the normalisations that we fixed in §2.1, for each complex $C^\bullet$ in $D^{\mathrm{p}}_S(\Lambda(G))$ one has $-\chi^{\mathrm{ref}}(C^\bullet) = \chi(C^\bullet)$, where the latter Euler characteristic is as defined in [10, §1.4]. In addition, if the cohomology groups of (32) are finite in each degree, then the complex $\tilde{K}^\bullet$ is, in terms of the terminology used in [10], semisimple at each irreducible representation of $\mathcal{G}$ over $\mathbb{Q}_p^c$. Given these facts and our explicit definition of the leading term $Z^*_{\mathcal{G}}(X,1)$, the claimed equality follows directly upon applying the descent formalism of [10, Th. 2.2] to the result of Theorem 1.1 with f replaced by $\tilde{f}$. (In making this deduction note that the triangle (31) differs from the corresponding triangle $\Delta(\mathrm{tw}_{\mathcal{G}}(\tilde{K}^\bullet),\gamma)$ defined in [10, §5.2.4] in that we use the morphism induced by $1-\gamma$ rather than $\gamma-1$. However, by using the explicit formula of [10, Lem. 5.5(iv)], one can take account of this difference by omitting the terms $(-1)^{r_{\mathcal{G}}(\tilde{K}^\bullet)}$ that occur in the formula of [10, Th. 2.2] and this is what we have done.) $\square$

To deduce the required equality (26) from Lemma 9.5 it suffices to show that the sequences $\beta_{f,p}$ and $\hat{\beta}_{f,p}$ coincide. Indeed, if this is true then $\hat{\beta}_{f,p}$ is exact so that the cohomology groups of (32) are finite in each degree and in addition the terms $\chi^{\mathrm{ref}}_{\mathcal{G},p}(R\Gamma_c(X_{\text{ét}},\mathcal{E}),\hat{\beta}_{f,p})$ and $\chi^{\mathrm{ref}}_{\mathcal{G},p}(R\Gamma_c(X_{\text{ét}},\mathcal{E}),\beta_{f,p})$ are equal. Our proof of Corollary 1.2 is therefore completed by the following result.

Lemma 9.6. *There are morphisms of exact triangles in $D(\mathbb{Z}_p[\mathcal{G}])$ of the form*

$$\begin{tikzcd}
\Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\tilde{K}^\bullet) \arrow[r, "(1-\gamma)\otimes\mathrm{id}"] & \Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\tilde{K}^\bullet) \arrow[r] & K^\bullet \arrow[r] & {} \\
R\Gamma(\mathbb{F}_q,Rs_{X,!}(\mathcal{E}_\infty\otimes_{\mathbb{Z}_p}\mathcal{E})) \arrow[u, "(-\gamma^{-1}\otimes\mathrm{id})\circ\alpha_1"] \arrow[r] & R\Gamma(\mathbb{F}_q,Rs_{X,!}(\mathcal{E}_\infty\otimes_{\mathbb{Z}_p}\mathcal{E})) \arrow[u, "\alpha_1"] \arrow[r] & K^\bullet \arrow[u, "\mathrm{id}"] \arrow[r] & {} \\
R\Gamma(\mathbb{F}_q^c,s_c^*Rs_{X,!}\mathcal{E})[-1] \arrow[u, "\alpha_2"] \arrow[r, "1-\phi"] & R\Gamma(\mathbb{F}_q^c,s_c^*Rs_{X,!}\mathcal{E})[-1] \arrow[u, "\alpha_2"] \arrow[r] & K^\bullet \arrow[u, "\mathrm{id"] \arrow[r] & {}.
\end{tikzcd} \tag{33}$$

in which the upper row is (31), the central row is induced by the natural short exact sequence of sheaves $0\to\mathcal{E}_\infty\otimes_{\mathbb{Z}_p}\mathcal{E}\xrightarrow{(1-\phi)\otimes\mathrm{id}}\mathcal{E}_\infty\otimes_{\mathbb{Z}_p}\mathcal{E}\to\mathcal{E}\to 0$ and the lower row is the -1-shift of the exact triangle in Proposition 3.1(iii) with $\mathcal{O}=\mathbb{Z}_p$ and $\mathcal{L}=\mathbb{Z}_p$. In particular, the sequences $\beta_{f,p}$ and $\hat{\beta}_{f,p}$ coincide.

Proof. For each continuous quotient Q of G we write $\Lambda(Q)^\#$ for the set $\Lambda(Q)$ regarded as a $\Lambda(Q\times\pi_1(X,\overline{x}))$-module in such a way that $(x,g)(x')=xx'\kappa_Q(g)^{-1}$ for all $x\in\Lambda(Q), x'\in\Lambda(Q)^\#$ and $g\in\pi_1(X,\overline{x})$ where κ_Q is the natural homomorphism $\pi_1(X,\overline{x})\to Q$.

The modules $\Lambda(\Gamma)\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\Lambda(G)^\#$ and $\Lambda(\Gamma)^\#\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\mathcal{G}]^\#$ are then endowed with continuous actions of $\Lambda(\Gamma\times\mathcal{G}\times G\times\pi_1(X,\overline{x}))$ and $\Lambda(\Gamma\times\mathcal{G}\times\pi_1(X,\overline{x}))$ which satisfy $(a,b,c,d)(x\otimes y\otimes z)=ax\otimes byc^{-1}\otimes cz\kappa_G(d)^{-1}$ and $(a,b,d)(x\otimes y)=ax\kappa_\Gamma(d)^{-1}\otimes y\kappa_{\mathcal{G}}(d)^{-1}$ for all $a\in\Gamma, b\in\mathcal{G}, c\in G, d\in\pi_1(X,\overline{x}), x\in\Lambda(\Gamma), y\in\mathbb{Z}_p[\mathcal{G}]$ and $z\in\Lambda(G)$. With respect to these actions one has a commutative diagram of short exact sequences of left $\mathbb{Z}_p[\mathcal{G}]\times\Lambda(\pi_1(X,\overline{x}))$-modules

$$\begin{tikzcd}
\Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\Lambda(G)^\#) \arrow[r, hook, "\theta_1"] & \Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\Lambda(G)^\#) \arrow[r, two heads, "\theta_2"] & \mathbb{Z}_p[\mathcal{G}]^\# \\
\Lambda(\Gamma)^\#\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\mathcal{G}]^\# \arrow[u, "(-\gamma^{-1}\otimes\mathrm{id})\circ\hat{\alpha}_1"] \arrow[r, hook, "\theta_3"] & \Lambda(\Gamma)^\#\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\mathcal{G}]^\# \arrow[u, "\hat{\alpha}_1"] \arrow[r, two heads, "\theta_4"] & \mathbb{Z}_p[\mathcal{G}]^\# \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

where for each $x\in\Lambda(\Gamma)^\#, y\in\mathbb{Z}_p[\mathcal{G}]^\#$ and $z\in\Lambda(G)^\#$ one has $\theta_1(x\otimes_{\Lambda(G)}(y\otimes_{\mathbb{Z}_p}z))=(1-\gamma)x\otimes_{\Lambda(G)}(y\otimes_{\mathbb{Z}_p}z)$, $\theta_2(x\otimes_{\Lambda(G)}(y\otimes_{\mathbb{Z}_p}z))=\epsilon(x)y\pi(z)$, $\hat{\alpha}_1(x\otimes y)=x\otimes_{\Lambda(G)}(y\otimes_{\mathbb{Z}_p}1)$, $\theta_3(x\otimes_{\mathbb{Z}_p}y)=(1-\gamma^{-1})x\otimes_{\mathbb{Z}_p}y$ and $\theta_4(x\otimes_{\mathbb{Z}_p}y)=\epsilon(x)y$ with $\epsilon:\Lambda(\Gamma)\to\mathbb{Z}_p$ and $\pi:\Lambda(G)^\#\to\mathbb{Z}_p[\mathcal{G}]^\#$ the natural homomorphisms.

Converting the above diagram into a diagram of sheaves on X (via the correspondence discussed in [25, Chap. V, Rem. 1.2(b)]) we obtain a commutative diagram of of short exact sequences of sheaves of $\mathbb{Z}_p[\mathcal{G}]$-modules of the form

$$\begin{tikzcd}
0 \arrow[r] & \Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\tilde{\mathcal{E}}) \arrow[r, "(1-\gamma)\otimes\mathrm{id}"] & \Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\tilde{\mathcal{E}}) \arrow[r] & \mathcal{E} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}_\infty\otimes_{\mathbb{Z}_p}\mathcal{E} \arrow[u, "(-\gamma^{-1}\otimes\mathrm{id})\circ\hat{\alpha}'_1"] \arrow[r, "(1-\phi)\otimes\mathrm{id}"] & \mathcal{E}_\infty\otimes_{\mathbb{Z}_p}\mathcal{E} \arrow[u, "\hat{\alpha}'_1"] \arrow[r, "\theta'_4"] & \mathcal{E} \arrow[u, "\mathrm{id}"] \arrow[r] & 0.
\end{tikzcd}$$

The upper morphism of exact triangles in (33) now results by applying the functor $R\Gamma(\mathbb{F}_q, Rs_{X,!}-)$ to this diagram and noting $R\Gamma(\mathbb{F}_q, Rs_{X,!}(\Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p}\tilde{\mathcal{E}})))$ is naturally isomorphic to $\Lambda(\Gamma)\otimes_{\Lambda(G)}(\mathbb{Z}_p[\mathcal{G}]\otimes_{\mathbb{Z}_p} R\Gamma(\mathbb{F}_q, Rs_{X,!}\tilde{\mathcal{E}}))$ in $D(\Lambda(\Gamma))$.

To complete the construction of (33) it remains to define the morphism α_2 in such a way that the two lower squares commute. To do this we note that $\mathcal{E}_\infty\otimes_{\mathbb{Z}_p}\mathcal{E}$ is naturally isomorphic to the projective system $f_{\infty,*}f^*_\infty\mathcal{E}$ (cf. [42, Prop. 6.3]) and hence that $R\Gamma(\mathbb{F}_q, Rs_{X,!}(\mathcal{E}_\infty\otimes_{\mathbb{Z}_p}\mathcal{E}))$ can be computed (in the manner described in §3.3) as the inverse limit $\varprojlim_n R\Gamma(\mathbb{F}_{p^n}, s_n^*Rs_{X,!}\mathcal{E}_n)$ with respect to the natural corestriction morphisms $\kappa_n : R\Gamma(\mathbb{F}_{p^{n+1}}, s^*_{n+1}Rs_{X,!}\mathcal{L}_{n+1}) \to R\Gamma(\mathbb{F}_{p^n}, s^*_nRs_{X,!}\mathcal{L}_n)$, where for each natural number n we write s_n for the natural morphism $\mathrm{Spec}(\mathbb{F}_{p^n}) \to \mathrm{Spec}(\mathbb{F}_q)$ and set $\mathcal{E}_n := \mathcal{E}/p^n$. For each natural number n the analogue of the exact triangle of Proposition 3.1(iii) in which ϕ is replaced by ϕ^{p^n} and $\mathcal{L}_G$ by $\mathcal{E}_n$ induces a morphism $\delta_n : R\Gamma(\mathbb{F}^c_q, s^*_cRs_{X,!}\mathcal{E}_n)[-1] \to R\Gamma(\mathbb{F}_{p^n}, s^*_nRs_{X,!}\mathcal{E}_n)$ in $D((\mathbb{Z}_p/p^n)[\mathcal{G}])$ and as n varies these morphisms lie in commutative diagrams in $D((\mathbb{Z}_p/p^n)[\mathcal{G}])$

$$\begin{tikzcd}
R\Gamma(\mathbb{F}^c_q, s^*_cRs_{X,!}\mathcal{E}_{n+1})[-1] \arrow[r, "\delta_{n+1}"] \arrow[d, "\pi_n"] & R\Gamma(\mathbb{F}_{p^{n+1}}, s^*_{n+1}Rs_{X,!}\mathcal{E}_{n+1}) \arrow[r, "\delta'_{n+1}"] \arrow[d, "\kappa_n"] & R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{E}_{n+1}) \arrow[d, "\pi'_n"] \\
R\Gamma(\mathbb{F}^c_q, s^*_cRs_{X,!}\mathcal{E}_n)[-1] \arrow[r, "\delta_n"] & R\Gamma(\mathbb{F}_{p^n}, s^*_nRs_{X,!}\mathcal{E}_n) \arrow[r, "\delta'_n"] & R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{E}_n)
\end{tikzcd}$$

where π_n and π'_n are induced by the natural projection $\mathcal{E}_{n+1} \to \mathcal{E}_n$, each δ'_n is the natural corestriction morphism and the composite $\delta'_n \circ \delta_n$ is equal to the morphism that is induced by the triangle of Proposition 3.1(iii) with $\mathcal{L}_G$ replaced by $\mathcal{E}_n$. We define α_2 to be the inverse limit of δ_n with respect to the transition morphisms given by the first commutative square in the above diagram and it is then straightforward to check that the lower squares in (33) commute.

It remains to show that the sequences $\beta_{f,p}$ and $\hat{\beta}_{f,p}$ coincide. Using the notation that occurs in (33) in each degree i we set $V^{i,0}_1 := \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\ker(H^i((1-\gamma)\otimes\mathrm{id}))$, $V^i_{1,0} := \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathrm{cok}(H^i((1-\gamma)\otimes\mathrm{id}))$, $V^{i,0}_2 := \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\ker(H^i(1-\phi))$ and $V^i_{2,0} := \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathrm{cok}(H^i(1-\phi))$. In particular, under the given hypotheses we may (and will) identify $V_1^{i,0}$ with $V_{1,0}^i$ and $V_2^{i,0}$ with $V_{2,0}^i$. Then in each degree i the diagram (33) gives a commutative diagram of the form

$$\begin{tikzcd}
\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_c^i(X_{\text{ét}}, \mathcal{E}) \arrow[r] & V_1^{i+1,0} \arrow[r, "\text{id}"] & V_{1,0}^{i+1} \arrow[r] & \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_c^{i+1}(X_{\text{ét}}, \mathcal{E}) \\
\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_c^i(X_{\text{ét}}, \mathcal{E}) \arrow[r] \arrow[u, equal] & V_2^{i,0} \arrow[r, "-\text{id}"] \arrow[u, "-H^{i+1}(\alpha_1\circ\alpha_2)"'] & V_{2,0}^i \arrow[r] \arrow[u, "H^{i+1}(\alpha_1\circ\alpha_2)"'] & \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_c^{i+1}(X_{\text{ét}}, \mathcal{E}) \arrow[u, equal]
\end{tikzcd}$$

in which all unlabeled maps are induced by the exact sequences of cohomology of the corresponding row of (33). Now the upper composite homomorphism in this diagram is by definition equal to the homomorphism $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{\beta}_{f,p}^i$ that occurs in the sequence $\hat{\beta}_{f,p}$ and, because of the -1-shift in the lower row of (33), the lower composite homomorphism is equal to the homomorphism $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \beta_{f,p}^i$ that occurs in the sequence $\beta_{f,p}$. From the commutativity of the latter diagram it thus follows that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{\beta}_{f,p}^i = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \beta_{f,p}^i$ in each degree i and hence that the sequences $\hat{\beta}_{f,p}$ and $\beta_{f,p}$ coincide, as required. □

10. The proof of Corollary 1.3

10.1. We first quickly review the notation introduced just after Corollary 1.4. For any global function field E we write C_E for the unique geometrically irreducible smooth projective curve with function field E. We fix a finite Galois extension of such fields F/k and set $\mathcal{G} := \mathrm{Gal}(F/k)$. We also fix a finite non-empty set of places Σ of k that contains all places which ramify in F/k and, with E denoting either k or F, we write C_E^Σ for the affine curve $\mathrm{Spec}(\mathcal{O}_{E,\Sigma})$, $j_E : C_E^\Sigma \to C_E$ for the corresponding open immersion and $f : C_F^\Sigma \to C_k^\Sigma$ and $f' : C_F \to C_k$ for the morphisms induced by the inclusion $k \subseteq F$. We fix q so that $\mathbb{F}_q$ identifies with the constant field of k and regard all of the above morphisms as morphisms of $\mathbb{F}_q$-schemes. We also write $\mathcal{O}_{F,\Sigma}^\times$ for the unit group of $\mathcal{O}_{F,\Sigma}$, set $B_{F,\Sigma} := \bigoplus_w \mathbb{Z}$ where w runs over $\Sigma(F)$ and write $B_{F,\Sigma}^0$ for the kernel of the homomorphism $B_{F,\Sigma} \to \mathbb{Z}$ sending $(n_w)_w$ to $\sum_w n_w$. We note that each of the groups $\mathcal{O}_{F,\Sigma}, \mathcal{O}_{F,\Sigma}^\times$, $B_{F,\Sigma}$ and $B_{F,\Sigma}^0$ has a natural action of $\mathcal{G}$.

10.2. It is known that the complex $K_{F,\Sigma}^\bullet := R\Gamma(C_{F,\mathrm{W\acute{e}t}}^\Sigma, \mathbb{G}_m)$ belongs to $D^{\mathrm{p}}(\mathbb{Z}[\mathcal{G}])$, is acyclic outside degrees 0 and 1 and is such that the explicit computations of [7] induce identifications of $H^0(K_{F,\Sigma}^\bullet)$ and $\mathbb{Q} \otimes_{\mathbb{Z}} H^1(K_{F,\Sigma}^\bullet)$ with $\mathcal{O}_{F,\Sigma}^\times$ and $\mathbb{Q} \otimes_{\mathbb{Z}} B_{F,\Sigma}^0$ respectively (cf. [3, Lem. 1]).

We identify each place w of F with the corresponding (closed) point of C_F and write val_w for the associated valuation on F and $\mathrm{d}(w)$ for the degree of w over $\mathbb{F}_q$. We let $\mathrm{D}_{F,\Sigma}^q : \mathcal{O}_{F,\Sigma}^\times \to Y_{F,\Sigma}^0$ denote the homomorphism which sends each element u of $\mathcal{O}_{F,\Sigma}^\times$ to $(\mathrm{val}_w(u)\mathrm{d}(w))_{w\in\Sigma(F)}$. Then the induced map $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{D}_{F,\Sigma}^q$ is bijective and so induces an exact sequence of $\mathbb{Q}[\mathcal{G}]$-modules

$$\epsilon_{F,\Sigma} : 0 \to \mathbb{Q} \otimes_{\mathbb{Z}} H^0(K^\bullet_{F,\Sigma}) \xrightarrow{\mathbb{Q}\otimes_{\mathbb{Z}}\mathrm{D}^q_{F,\Sigma}} \mathbb{Q} \otimes_{\mathbb{Z}} H^1(K^\bullet_{F,\Sigma}) \xrightarrow{0} \mathbb{Q} \otimes_{\mathbb{Z}} H^2(K^\bullet_{F,\Sigma}) \xrightarrow{0} \cdots$$

We write $x \mapsto x^{\#}$ for the $\mathbb{Q}$-linear involution of $\zeta(\mathbb{Q}[\mathcal{G}])$ that is induced by setting $g^{\#} := g^{-1}$ for each g in $\mathcal{G}$.

We can now give a more explicit statement of Corollary 1.3.

Theorem 10.1. $\delta_{\mathcal{G}}(Z^*(f,1)^{\#}) = \chi^{\mathrm{ref}}_{\mathcal{G}}(R\Gamma(C^{\Sigma}_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m), \epsilon_{F,\Sigma})$.

Remark 10.2. Let m be the degree of $\mathbb{F}_q$ over $\mathbb{F}_p$. Then $Z(f,t)$ and $\mathrm{D}^q_{F,\Sigma}$ are related to the function $Z_{F/k,\Sigma}(t)$ and homomorphism $D_{F,\Sigma}$ that occur in [3, Lem. 2] by the equalities $Z_{F/k,\Sigma}(t) = Z(f,t^m)$ and $\mathrm{D}_{F,\Sigma} = [m] \circ \mathrm{D}^q_{F,\Sigma}$ where $[m]$ denotes the endomorphism of $Y^0_{F,\Sigma}$ given by multiplication by m. This implies (by an argument similar to that used in Lemma 4.8) that the equality in Theorem 10.1 is equivalent to the equality of [3, (3)] and hence that [3, Lem. 2] implies Theorem 10.1 verifies the central conjecture (Conj. C(F/k)) of [3]. The computation of [3, Prop. 4.1] also then implies that Theorem 10.1 is equivalent to the function field case of [5, Conj. LTC(F/k)]. Finally, we recall that the argument of [24, proof of Th. 6.5] gives a canonical isomorphism in $D^{\mathrm{p}}(\mathbb{Z}[\mathcal{G}])$

$$\Delta : R\Gamma(C^{\Sigma}_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m) \cong R\operatorname{Hom}_{\mathbb{Z}}(R\Gamma(C_{F,\mathrm{W\acute{e}t}}, j_{F,!}\mathbb{Z}), \mathbb{Z}[-2])$$

where the linear dual complex is endowed with the contragredient action of $\mathcal{G}$. The isomorphism Δ combines with Lemma 9.2 to show that Theorem 10.1 generalises the main result (Theorem 3.1) of [4].

10.3. In this subsection we prove Theorem 10.1. For each x in $\zeta(\mathbb{Q}[\mathcal{G}])^\times$ one can show that $\delta_{\mathcal{G}}(x^{\#}) = -\psi^*(\delta_{\mathcal{G}}(x))$. Here we write ψ^* for the involution of $K_0(\mathbb{Z}[\mathcal{G}], \mathbb{Q}[\mathcal{G}])$ which sends (P,μ,Q) to $(\operatorname{Hom}_{\mathbb{Z}}(P,\mathbb{Z}), \operatorname{Hom}_{\mathbb{Q}}(\mu,\mathbb{Q})^{-1}, \operatorname{Hom}_{\mathbb{Z}}(Q,\mathbb{Z}))$ for each finitely generated projective $\mathbb{Z}[\mathcal{G}]$-modules P and Q and each isomorphism of $\mathbb{Q}[\mathcal{G}]$-modules $\mu : \mathbb{Q} \otimes_{\mathbb{Z}} P \to \mathbb{Q} \otimes_{\mathbb{Z}} Q$, where the linear duals are endowed with the contragredient action of $\mathcal{G}$ and so are projective $\mathbb{Z}[\mathcal{G}]$-modules. Given this equality, the isomorphism $R\operatorname{Hom}_{\mathbb{Z}}(\Delta, \mathbb{Z}[-2])$ implies that Theorem 10.1 is valid if and only if one has

$$\delta_{\mathcal{G}}(Z^*(f,1)) = -\chi^{\mathrm{ref}}_{\mathcal{G}}(R\Gamma(C_{F,\mathrm{W\acute{e}t}}, j_{F,!}\mathbb{Z}), \epsilon^*_{F,\Sigma}) \tag{34}$$

where $\epsilon^*_{F,\Sigma}$ is the exact sequence of $\mathbb{Q}[\mathcal{G}]$-modules obtained by taking the $\mathbb{Q}$-linear dual of $\epsilon_{F,\Sigma}$ and using $\operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} H^i(\Delta), \mathbb{Q})$ to identify $\operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} H^i(C^{\Sigma}_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m), \mathbb{Q})$ with $\mathbb{Q} \otimes_{\mathbb{Z}} H^{2-i}(C_{F,\mathrm{W\acute{e}t}}, j_{F,!}\mathbb{Z})$.

We shall deduce (34) from Corollary 1.2 with our present choice of f (so that $Y = C^{\Sigma}_F$, $Y' = C_F$, $X = C^{\Sigma}_k$, $X' = C_k$, $j = j_k$ and $j_Y = j_F$). Now X is geometrically connected and, since Y' is a curve, the conditions (i) and (ii) of Corollary 1.2 are satisfied (by [24, Th. 8.2]) and so that result implies $\delta_{\mathcal{G}}(Z^*(f,1)) =$

$-\chi_{\mathcal{G}}^{\mathrm{ref}}(R\Gamma(C_{F,\mathrm{W\acute{e}t}}, j_{F,!}\mathbb{Z}), \epsilon_{f,j_k})$. To deduce (34) it is thus enough to show that the exact sequences $\epsilon^*_{F,\Sigma}$ and ϵ_{f,j_k} coincide. This is in turn a direct consequence of the following result.

Lemma 10.3. *There is a commutative diagram of* $\mathbb{Q}[\mathcal{G}]$*-modules*

$$\begin{array}{ccc} \mathbb{Q}\otimes_{\mathbb{Z}} H^1(C_{F,\mathrm{W\acute{e}t}}, j_{F,!}\mathbb{Z}) & \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(H^1(\Delta),\mathbb{Q})} & \mathrm{Hom}_{\mathbb{Z}}(B^0_{F,\Sigma}, \mathbb{Q}) \\ {\scriptstyle \mathbb{Q}\otimes_{\mathbb{Z}}(\cup\theta_Y)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathbb{Z}}(\mathrm{D}^q_{F,\Sigma},\mathbb{Q})} \\ \mathbb{Q}\otimes_{\mathbb{Z}} H^2(C_{F,\mathrm{W\acute{e}t}}, j_{F,!}\mathbb{Z}) & \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(H^0(\Delta),\mathbb{Q})} & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{F,\Sigma}, \mathbb{Q}). \end{array}$$

Proof. It is enough to prove that the diagram commutes after applying $\mathbb{Q}_\ell\otimes_{\mathbb{Q}} -$ for any prime ℓ. We therefore fix a prime ℓ and, following Lemma 9.2, we consider $H^i(C_{k,\mathrm{\acute{e}t}}, j_!\mathcal{F}_\ell)$ with $j := j_k$ and $\mathcal{F}_\ell := f_*f^*\mathbb{Z}_\ell$ rather than $H^i(C_{F,\mathrm{W\acute{e}t}}, j_{F,!}\mathbb{Z})$.

We write Z^Σ for the complement of C_k^Σ in C_k and write $i : Z^\Sigma \to C_k$ for the natural closed immersion. If y is a place of any subfield of F, then we write $\kappa(y)$ for the residue field of y and set $Z_y := \mathrm{Spec}(\kappa(y))$. For each $y \in \Sigma$ we also write $i_y : Z_y \to C_k$ for the natural closed immersion and $S_y(F)$ for the set of places of F above y. Now $\mathcal{F}_\ell = j^*\mathcal{F}'_\ell$ with $\mathcal{F}'_\ell := f'_*f'^*\mathbb{Z}_\ell$ so there is a natural exact sequence of $\mathbb{Z}_\ell[\mathcal{G}]$-sheaves $0 \to j_!\mathcal{F}_\ell \to \mathcal{F}'_\ell \to i_*i^*\mathcal{F}'_\ell \to 0$ on C_k and hence a composite morphism in $D(\mathbb{Z}_\ell[\mathcal{G}])$ of the form

$$\xi : \bigoplus_{y\in\Sigma} R\Gamma(Z_{y,\mathrm{\acute{e}t}}, i_y^*\mathcal{F}'_\ell) \cong R\Gamma(C_{k,\mathrm{\acute{e}t}}, i_*i^*\mathcal{F}'_\ell) \to R\Gamma(C_{k,\mathrm{\acute{e}t}}, j_!\mathcal{F}_\ell)[1].$$

This morphism induces in turn a commutative diagram of $\mathbb{Z}_\ell[\mathcal{G}]$-modules

$$\begin{array}{ccccc} \bigoplus_{y\in\Sigma} H^0(Z_{y,\mathrm{\acute{e}t}}, i_y^*\mathcal{F}'_\ell) & \xrightarrow{H^0(\xi)} & H^1(C_{k,\mathrm{\acute{e}t}}, j_!\mathcal{F}_\ell) & \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(H^1(\Delta),\mathbb{Q}_\ell)} & \mathrm{Hom}_{\mathbb{Z}}(B^0_{F,\Sigma}, \mathbb{Q}_\ell) \\ \big\downarrow{\scriptstyle (-1)\times(\cup\theta_{Z_y,\ell})_y} & & \big\downarrow{\scriptstyle \cup\theta_{Y,\ell}} & & \\ \bigoplus_{y\in\Sigma} H^1(Z_{y,\mathrm{\acute{e}t}}, i_y^*\mathcal{F}'_\ell) & \xrightarrow{H^1(\xi)} & H^2(C_{k,\mathrm{\acute{e}t}}, j_!\mathcal{F}_\ell) & \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(H^0(\Delta),\mathbb{Q}_\ell)} & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{F,\Sigma}, \mathbb{Q}_\ell) \end{array} \tag{35}$$

where the factor -1 occurs in the left vertical homomorphism because ξ maps to the 1-shift of $R\Gamma(C_{k,\mathrm{\acute{e}t}}, j_!\mathcal{F}_\ell)$.

Now each complex $\mathcal{E}^\bullet_y := R\Gamma(Z_{y,\mathrm{\acute{e}t}}, i_y^*\mathcal{F}'_\ell)$ is canonically isomorphic to the direct sum over w in $S_y(F)$ of the complexes $\mathbb{Z}_\ell[\mathcal{G}_w/\mathcal{G}^0_w]^\# \xrightarrow{1-\sigma_w} \mathbb{Z}_\ell[\mathcal{G}_w/\mathcal{G}^0_w]^\#$ where the first term occurs in degree 0, $\mathcal{G}_w$ and $\mathcal{G}^0_w$ are the decomposition and inertia groups of w in $\mathcal{G}$ and σ_w the frobenius automorphism in $\mathcal{G}_w/\mathcal{G}^0_w \cong \mathrm{Gal}(\kappa(w)/\kappa(v))$. To compute the groups $H^i(\mathcal{E}^\bullet_y)$ explicitly we must use the conventions of [7] (since they underlie the explicit descriptions of the cohomology of $K^\bullet_{F,\Sigma}$ given above). We therefore use the isomorphism $\mathcal{E}^\bullet_y \cong \bigoplus_{w\in S_y(F)} R\Gamma(Z_{y,\mathrm{\acute{e}t}}, \mathbb{Z}_\ell[\mathcal{G}_w/\mathcal{G}^0_w])$ induced by the morphism (of complexes of $\mathbb{Z}_\ell[\mathcal{G}_w/\mathcal{G}_w^0]$-modules)

$$\begin{array}{ccc} \mathbb{Z}_\ell[\mathcal{G}_w/\mathcal{G}_w^0]^\# & \xrightarrow{1-\sigma_w} & \mathbb{Z}_\ell[\mathcal{G}_w/\mathcal{G}_w^0]^\# \\ \Big\| & & \Big\downarrow{\scriptstyle -\sigma_w} \\ \mathbb{Z}_\ell[\mathcal{G}_w/\mathcal{G}_w^0] & \xrightarrow{1-\sigma_w} & \mathbb{Z}_\ell[\mathcal{G}_w/\mathcal{G}_w^0]. \end{array} \tag{36}$$

We then identify $H^0(Z_{y,\text{ét}}, \mathbb{Z}_\ell[\mathcal{G}_w/\mathcal{G}_w^0])$ and $H^1(Z_{y,\text{ét}}, \mathbb{Z}_\ell[\mathcal{G}_w/\mathcal{G}_w^0])$ with $H^0(Z_{w,\text{ét}}, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$ and $H^1(Z_{w,\text{ét}}, \mathbb{Z}_\ell) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\kappa(w)^c/\kappa(w)), \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$ in the natural way, where the last map evaluates each homomorphism at the topological generator $\phi^{-\mathrm{d}(w)}$ of $\mathrm{Gal}(\kappa(w)^c/\kappa(w))$. By passing to cohomology in this diagram, we obtain a commutative diagram

$$\begin{array}{ccccc} \bigoplus_y(\bigoplus_w \mathbb{Q}_\ell) & \xrightarrow{\cong} & \bigoplus_y \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} H^0(\mathcal{E}_y^\bullet) & \xrightarrow{a_0} & \mathrm{Hom}_{\mathbb{Z}}(B^0_{F,\Sigma}, \mathbb{Q}_\ell) \\ {\scriptstyle a_2}\Big\downarrow & & \Big\downarrow{\scriptstyle (-1)\times(\cup\theta_{Z_y,\ell})_y} & & \\ \bigoplus_y(\bigoplus_w \mathbb{Q}_\ell) & \xrightarrow{\cong} & \bigoplus_y \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} H^1(\mathcal{E}_y^\bullet) & \xrightarrow{a_1} & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{F,\Sigma}, \mathbb{Q}_\ell) \end{array} \tag{37}$$

where y runs over Σ and w over $S_y(F)$, a_0 and a_1 are the homomorphisms induced by the upper and lower rows of (35) respectively and a_2 is defined to make the first square commute. Now the composite upper, resp. lower, horizontal map in (37) is equal to $\mathrm{Hom}_{\mathbb{Z}}(a_3, \mathbb{Q}_\ell)^{-1}$ with a_3 equal to the natural map $B^0_{F,\Sigma} \subset B_{F,\Sigma} \cong \bigoplus_y(\bigoplus_w \mathbb{Z})$, resp. with a_3 equal to the map $\mathcal{O}^\times_{F,\Sigma} \to \bigoplus_y(\bigoplus_w \mathbb{Z})$ sending u to $(-\mathrm{val}_w(u))_w$ where the minus sign occurs because the morphism (36) induces $-\mathrm{id}_{\mathbb{Z}_\ell}$ on cohomology in degree 1. In particular, since the upper horizontal map in (37) is surjective the existence of a commutative diagram as claimed will follow from (35) and (37) provided that a_2 sends each element $(x_w)_w$ to $(-\mathrm{d}(w)x_w)_w$ and this is clear since $(-1) \times \theta_\ell$ maps $\phi^{-\mathrm{d}(w)}$ to $-\mathrm{d}(w)$. □

11. Completion of the proofs

In this section we derive several explicit consequences of Theorem 10.1 including Corollaries 1.4, 1.5 and 1.6. We fix a finite Galois extension of global function fields F/k, fix q so that $\mathbb{F}_q$ identifies with the constant field of k and set $\mathcal{G} := \mathrm{Gal}(F/k)$.

11.1. In [3, Th. 4.1] it is shown that [3, Conj. C(F/k)] implies the validity of Chinburg's $\Omega(3)$-Conjecture for F/k. Corollary 1.4(i) is therefore a direct consequence of Theorem 10.1. To prove Corollary 1.4(ii) we now assume that F/k is tamely ramified. In this case Chinburg has proved the validity of his '$\Omega(2)$-Conjecture' (this follows upon combining [11, §4.2, Th. 4] with [14, Cor. 4.10]). In view of Corollary 1.4(i) we therefore need only recall that Chinburg has also proved that if the $\Omega(2)$-Conjecture and $\Omega(3)$-Conjecture are both valid for an extension F/k, then the $\Omega(1)$-Conjecture is automatically valid for F/k (this is a consequence of [11, §4.1, Th. 2 and the remarks which follow it]).

This completes the proof of Corollary 1.4.

11.2. We now fix a finite Galois cover $f : C_F^\Sigma \to C_k^\Sigma$ of affine curves (over $\mathbb{F}_q$), as in Corollary 1.5. We also fix a finite non-empty set of closed points of C_k^Σ and then set

$$Z_T(f,t) = (\prod_{t\in T} \mathrm{Nrd}_{\mathbb{Q}[\mathcal{G}]}([1 - \mathrm{N}t\cdot \mathrm{Fr}_{\mathfrak{t}}^{-1}]_r))Z(f,t) \tag{38}$$

where $\mathfrak{t}$ is any (closed) point of C_F^Σ above t, and for any x in $\mathbb{Q}[\mathcal{G}]$ we write $[x]_r$ for the endomorphism $y \mapsto yx$ of $\mathbb{Q}[\mathcal{G}]$, regarded as a left $\mathbb{Q}[\mathcal{G}]$-module in the obvious way. For each integer m we then define an 'm-th order Stickelberger function' by setting

$$Z_T^{(m)}(f,t) := (\sum_{\chi\in \mathrm{Ir}(\mathcal{G})} (1-t)^{-m\chi(1)}e_\chi)Z_T(f,t).$$

For each homomorphism ϕ in $\mathrm{Hom}_G(\mathcal{O}_{F,\Sigma}^\times, X_{K,\Sigma})$ we also define a $\zeta(\mathbb{Q}[\mathcal{G}])$-valued regulator by setting $R(\phi) := \mathrm{Nrd}_{\mathbb{Q}[\mathcal{G}]}((\mathrm{D}^q_{F,\Sigma})^{-1}\circ(\mathbb{Q}\otimes\phi))$.

In the next result we label, and so order, the elements of Σ as $\{v_i : 0 \le i \le |S|-1\}$. (This labeling is convenient but does not effect the validity of any of our results.) We fix a non-negative integer r and assume that every place in $\Sigma_r := \{v_i : 1 \le i \le r\}$ splits completely in F/k. We recall that in this case the function $Z_T^{(r)}(f,t)$ is holomorphic at $t = 1$ (cf. [5, Lem. 2.2.1]).

For each subgroup $\mathcal{H}$ of $\mathcal{G}$ we write $I(\mathcal{H})$ for the two-sided ideal of $\mathbb{Z}[\mathcal{G}]$ generated by the set $\{h-1 : h\in\mathcal{H}\}$. We write $\mathcal{G}_j$ for the decomposition subgroup of any place of F above v_j and consider the following set of matrices

$$\mathfrak{M}_\Sigma(\mathcal{G}) := \{M = (M_{ij}) \in \mathrm{M}_d(\mathbb{Z}[\mathcal{G}]) : d \ge |\Sigma|-1,\ r < j < |\Sigma| \Rightarrow M_{ij}\in I(\mathcal{G}_j)\}.$$

Theorem 11.1. *Fix ϕ in $\mathrm{Hom}_{\mathcal{G}}(\mathcal{O}_{F,\Sigma}^\times, X_{F,\Sigma})$ and a in $\mathcal{A}(\mathbb{Z}[\mathcal{G}])$.*

- (i) *$Z_T^{(r)}(f,1)R(\phi)$ is a finite integral linear combination of elements $\mathrm{Nrd}_{\mathbb{Q}[\mathcal{G}]}(M)$ with M in $\mathfrak{M}_\Sigma(\mathbb{Z}[\mathcal{G}])$ and so $aZ_T^{(r)}(f,1)R(\phi)$ belongs to $\mathbb{Z}[\mathcal{G}]$.*
- (ii) *$aZ_T^{(r)}(f,1)R(\phi)$ annihilates $\mathbb{Z}^*\otimes \mathrm{Cl}(\mathcal{O}_{F,\Sigma_r\cup\{v\}})$ for any $v\in\Sigma\setminus\Sigma_r$.*
- (iii) *Corollary 1.5 is valid.*

Proof. After allowing for the differences in normalisations that are used here and in [5] (as discussed in Remark 10.2), claims (i) and (ii) follow directly by combining Corollary 1.3 with [5, Th. 4.1.1].

To prove claim (iii) we now deduce Corollary 1.5 from claim (ii). If $r = 0$, then Σ_0 is empty and so we may apply this case of claim (ii) to any non-empty set Σ for which f is étale. Further, one has $Z_T^{(0)}(f,1)R(\phi) = Z_T(f,1)$ (cf. the proof of [5, Prop. 3.5.1]) and so claim (ii) asserts that $aZ_T(f,1)$ annihilates $\mathbb{Z}^*\otimes\mathrm{Cl}(\mathcal{O}_{F,\{v\}})$. We write $B_{F,v}$ for the free abelian group on the places of F above v and note that $B_{F,v}$ is isomorphic, as a G-module, to $\mathbb{Z}[G/G_w]$. If $B^0_{F,v}$ is the kernel of the homomorphism

$B_{F,v} \to \mathbb{Z}$ sending each place above v to 1, then there is a natural exact sequence of G-modules $B^0_{F,v} \to \mathrm{Pic}^0(F) \to \mathrm{Cl}(\mathcal{O}_{F,\{v\}})$.

Since $aZ_T(f,1)$ annihilates $\mathbb{Z}' \otimes \mathrm{Cl}(\mathcal{O}_{F,\{v\}})$ and any a' in $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}[G/G_w])$ annihilates $B^0_{F,v}$ the above sequence implies $a'aZ_T(f,1)$ annihilates $\mathbb{Z}' \otimes \mathrm{Pic}^0(F)$, as claimed by Corollary 1.5. □

If $\mathcal{G}$ is abelian, then $\mathcal{A}(\mathbb{Z}[\mathcal{G}]) = \mathbb{Z}[\mathcal{G}]$ (so we can choose $a = 1$ in Theorem 11.1) and the argument of [5, Prop. 3.4.1] shows that this case of Theorem 11.1(i) is a refinement of the Rubin-Stark Conjecture [35, Conj. B′]. Theorem 11.1(i) therefore constitutes a natural non-commutative generalisation of [35, Conj. B′], as claimed by Corollary 1.6(i). Further, the results of [5, Th. 7.5.1 and Rem. 7.5.2(ii)] combine with Corollary 1.3 to imply that the matrices $M(\phi)$ in Theorem 11.1(i) can be chosen so that their columns satisfy certain mutual congruence relations which, in the case that $\mathcal{G}$ is abelian, constitute natural 'higher order' generalisations of the $\mathfrak{p}$-adic abelian Stark conjecture of Gross [19, Conj. 7.6], the 'guess' formulated by Gross in [19, top of p. 195], and of the refined class number formulas conjectured by Gross [19, Conj. 4.1], by Tate [39, $(*)$] and by Aoki, Lee and Tan [1, Conj. 1.1]. This completes the proof of Corollary 1.6.

11.3. As a final application of Theorem 10.1 we show that it gives information about the Galois structure of certain Weil-étale cohomology groups. Before stating this result we recall that a natural non-commutative generalisation of the classical notion of Fitting ideal was introduced by Parker in his thesis [32] and that such 'non-commutative Fitting invariants' have more recently been developed extensively by Nickel [28]. In the sequel we shall (for the reader's convenience) use the terminology of [28, §3]. We recall that $\mathbb{F}_q$ identifies with the constant field of k and continue to use the morphism of $\mathbb{F}_q$-schemes $f : C^\Sigma_F \to C^\Sigma_k$ that was introduced in §11.2.

Proposition 11.2. *Let $\mathbb{Z}'$ be any finitely generated subring of $\mathbb{Q}$ for which the $\mathcal{G}$-module $\mathbb{Z}' \otimes \mathbb{F}_q^\times$ is cohomologically-trivial (this is automatically the case if, for example, the highest common factor of $q-1$ and $|\mathcal{G}|$ is invertible in $\mathbb{Z}'$). Then there are non-commutative Fitting invariants $\mathcal{F}_{\mathbb{Z}[\mathcal{G}]}(\mathbb{F}_q^\times)$ and $\mathcal{F}_{\mathbb{Z}[\mathcal{G}]}(H^1(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m))$ for which*

$$Z(f,1) \cdot \mathbb{Z}' \otimes_{\mathbb{Z}} \mathcal{F}_{\mathbb{Z}[\mathcal{G}]}(\mathbb{F}_q^\times) = \mathbb{Z}' \otimes_{\mathbb{Z}} \mathcal{F}_{\mathbb{Z}[\mathcal{G}]}(H^1(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m)).$$

In particular, if $\mathcal{G}$ is abelian, then in $\mathbb{Q}[\mathcal{G}]$ one has

$$Z(f,1) \cdot \mathbb{Z}' \otimes_{\mathbb{Z}} \mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{F}_q^\times) = \mathbb{Z}' \otimes_{\mathbb{Z}} \mathrm{Fit}_{\mathbb{Z}[G]}(H^1(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m)).$$

Proof. We set $K^\bullet := R\Gamma(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m)$ and define an idempotent $e_0 := \sum_\chi e_\chi$ of $\zeta(\mathbb{Q}[\mathcal{G}])$ where in the sum χ runs over all elements of $\mathrm{Ir}(G)$ for which the module $e_\chi(\mathbb{Q}^c \otimes H^0(K^\bullet))$ vanishes. We also set $\mathfrak{A} := \mathbb{Z}'[G]e$ and $A := \mathbb{Q}[G]e$ and for any G-module M write $M_{\mathfrak{A}}$ for the $\mathfrak{A}$-module $\mathfrak{A} \otimes_{\mathbb{Z}[\mathcal{G}]} M$.

Since $K^\bullet$ belongs to $D^{\mathrm{p}}(\mathbb{Z}[\mathcal{G}])$, is acyclic outside degrees 0 and 1 and is such that $H^0(K^\bullet)$ identifies with $\mathcal{O}^\times_{F,\Sigma}$ the natural inclusion $\mathbb{F}_q^\times \subset \mathcal{O}^\times_{F,\Sigma}$ induces an exact triangle

$(\mathbb{F}_q^\times)_{\mathfrak{A}}[0] \to \mathfrak{A}\otimes^{\mathbb{L}}_{\mathbb{Z}[\mathcal{G}]} K^\bullet \to H^1(K^\bullet)_{\mathfrak{A}}[-1] \to (\mathbb{F}_q^\times)_{\mathfrak{A}}[1]$ in $D^{\rm p}(\mathfrak{A})$ in which all cohomology groups of all complexes that occur are finite. This triangle implies that in $K_0(\mathfrak{A}, A)$ one has

$$\chi^{\rm ref}_{\mathfrak{A},A}(\mathfrak{A}\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} K^\bullet, 0) = -\chi^{\rm ref}_{\mathfrak{A},A}((\mathbb{F}_q^\times)_{\mathfrak{A}}[-1], 0) + \chi^{\rm ref}_{\mathfrak{A},A}(H^1(K^\bullet)_{\mathfrak{A}}[-1], 0).$$

Now for each character χ in $\mathrm{Ir}(\mathcal{G})$ the order of vanishing of $e_\chi Z(f,t)$ at $t = 1$ is equal to $\chi(1)^{-1}\dim_{\mathbb{Q}^c}(e_\chi(\mathbb{Q}^c \otimes H^0(K^\bullet)))$ (this follows from [37, p. 111]) and so one has $Z(f,1) = Z^*(f,1)e \in \zeta(A)^\times$. Thus, if we write $\pi_{\mathfrak{A}}$ for the natural scalar extension homomorphism $K_0(\mathbb{Z}[\mathcal{G}], \mathbb{Q}[\mathcal{G}]) \to K_0(\mathfrak{A}, A)$, then the equality of Theorem 10.1 combines with the last displayed equation (and the naturality with respect to change of algebra of the extended boundary homomorphism) to imply that

$$\begin{aligned}\delta_{\mathfrak{A}}(Z(f,1)) + \chi^{\rm ref}_{\mathfrak{A},A}((\mathbb{F}_q^\times)_{\mathfrak{A}}[-1],0) =&\delta_{\mathfrak{A}}(Z^*(f,1)e) + \chi^{\rm ref}_{\mathfrak{A},A}((\mathbb{F}_q^\times)_{\mathfrak{A}}[-1],0)\\ =&\pi_{\mathfrak{A}}(\delta_G(Z^*(f,1))) + \chi^{\rm ref}_{\mathfrak{A},A}((\mathbb{F}_q^\times)_{\mathfrak{A}}[-1],0)\\ =&\pi_{\mathfrak{A}}(\chi^{\rm ref}_{\mathbb{Z}[\mathcal{G}],\mathbb{Q}[\mathcal{G}]}(K^\bullet, \epsilon_{F,\Sigma})) + \chi^{\rm ref}_{\mathfrak{A},A}((\mathbb{F}_q^\times)_{\mathfrak{A}}[-1],0)\\ =&\chi^{\rm ref}_{\mathfrak{A},A}(\mathfrak{A}\otimes^{\mathbb{L}}_{\mathbb{Z}[\mathcal{G}]} K^\bullet, 0) + \chi^{\rm ref}_{\mathfrak{A},A}((\mathbb{F}_q^\times)_{\mathfrak{A}})[-1],0)\\ =&\chi^{\rm ref}_{\mathfrak{A},A}(H^1(K^\bullet)_{\mathfrak{A}}[-1],0).\end{aligned}$$

Since the finite $\mathfrak{A}$-modules $(\mathbb{F}_q^\times)_{\mathfrak{A}}$ and $H^1(K^\bullet)_{\mathfrak{A}}$ are of projective dimension at most one, an easy exercise comparing the definitions of Fitting invariants $\mathcal{F}_{\mathbb{Z}[\mathcal{G}]}(-)$ and refined Euler characteristics $\chi^{\rm ref}_{\mathfrak{A},A}(-,-)$ (which we leave to the reader) shows the last displayed equality implies that $Z(f,1)\cdot\mathcal{F}_{\mathfrak{A}}((\mathbb{F}_q^\times)_{\mathfrak{A}}) = \mathcal{F}_{\mathfrak{A}}(H^1(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m)_{\mathfrak{A}})$. The first equality of the proposition is therefore true because [28, Lem. 3.4] implies that $\mathcal{F}_{\mathfrak{A}}(M_{\mathfrak{A}}) = \mathfrak{A}\otimes_{\mathbb{Z}[\mathcal{G}]}\mathcal{F}_{\mathbb{Z}[\mathcal{G}]}(M)$ for any finitely generated $\mathcal{G}$-module M.

Now if $\mathcal{G}$ is abelian, then for every finitely generated $\mathcal{G}$-module M the Fitting invariant $\mathcal{F}_{\mathbb{Z}[\mathcal{G}]}(M)$ corresponds to the classical Fitting ideal $\mathrm{Fit}_{\mathbb{Z}[\mathcal{G}]}(M)$ in such a way that equalities of Fitting invariants correspond to equalities of Fitting ideals (by [28, Rem. 1(ii)]). The second equality of the proposition therefore follows directly from the first equality and the fact that $\mathbb{F}_q^\times$ is cyclic so $\mathrm{Fit}_{\mathbb{Z}[\mathcal{G}]}(\mathbb{F}_q^\times) = \mathrm{Ann}_{\mathbb{Z}[\mathcal{G}]}(\mathbb{F}_q^\times)$. □

References

[1] N. Aoki, J. Lee, K-S. Tan, A refinement for a conjecture of Gross, preprint 2005.

[2] M. Breuning, D. Burns, Additivity of Euler characteristics in relative algebraic K-theory, Homology, Homotopy and Applications **7** (2005) 11-36.

[3] D. Burns, On the values of equivariant Zeta functions of curves over finite fields, Doc. Math. **9** (2004) 357-399.

[4] D. Burns, Congruences between derivatives of geometric L-functions (with an appendix by D. Burns, K. F. Lai and K-S. Tan), to appear in Invent. math.

[5] D. Burns, On derivatives of Artin L-series, to appear in Invent. math.

[6] D. Burns, On main conjectures in non-commutative Iwasawa theory and related conjectures, submitted for publication.

[7] D. Burns, M. Flach, On Galois structure invariants associated to Tate motives, Amer. J. Math. **120** (1998) 1343-1397.
[8] D. Burns, M. Flach, Tamagawa numbers for motives with (non-commutative) coefficients, Doc. Math. **6** (2001) 501-570.
[9] D. Burns, O. Venjakob, Leading terms of Zeta isomorphisms and p-adic L-functions in non-commutative Iwasawa theory, Doc. Math., Extra Volume (2006) 165-209.
[10] D. Burns, O. Venjakob, On descent theory and main conjectures in non-commutative Iwasawa theory, J. Inst. Math. Jussieu **10** (2011) 59-118.
[11] Ph. Cassou-Noguès, T. Chinburg, A. Fröhlich, M. J. Taylor, L-functions and Galois modules, (notes by D. Burns and N. P. Byott), In: 'L-functions and Arithmetic', J. Coates, M. J. Taylor (eds.), London Math. Soc. Lecture Note Series **153**, 75-139, Cambridge Univ. Press, Cambridge, 1991.
[12] T. Chinburg, On the Galois structure of algebraic integers and S-units, Invent. math. **74** (1983) 321-349.
[13] T. Chinburg, Exact sequences and Galois module structure, Ann. Math. **121** (1985) 351-376.
[14] T. Chinburg, Galois module structure of de Rham cohomology, J. Th. Nombres Bordeaux **4** (1991) 1-18.
[15] J. Coates, T. Fukaya, K. Kato, R. Sujatha, O. Venjakob, The GL_2 main conjecture for elliptic curves without complex multiplication, Publ. IHES **101** (2005) 163-208.
[16] R. Crew, L-functions of p-adic characters and geometric Iwasawa theory, Invent. math. **88** (1987) 395-403.
[17] P. Deligne, Seminaire de Geomtrie Algebrique du Bois-Marie, SGA$4\frac{1}{2}$, Lecture Note Math. **569**, Springer, New York, 1977.
[18] M. Emerton, M. Kisin, Unit L-functions and a conjecture of Katz, Ann. Math. **153** (2001) 329-354.
[19] B. H. Gross, On the value of abelian L-functions at $s = 0$, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math., **35** (1988) 177-197.
[20] A. Grothendieck, Formule de Lefschetz et rationalité des fonctions L, in Sém. Bourbaki vol. 1965-1966, Benjamin (1966), exposé 306.
[21] A. Grothendieck, Revêtements Etales et Groupe Fondamental, Lecture Note Math. **224**, Springer, New York, 1971.
[22] M. Kakde, The main conjecture of Iwasawa theory for totally real fields, submitted for publication.
[23] K. Kato, Iwasawa theory of totally real fields for Galois extensions of Heisenberg type, preprint, 2006.
[24] S. Lichtenbaum, The Weil-étale topology on schemes over finite fields, Compositio Math. **141** (2005) 689-702.
[25] J. S. Milne, Etale cohomology, Princeton Univ. Press, Princeton, 1980.
[26] F. Muro, A. Tonks, The 1-type of a Waldhausen K-theory spectrum, Adv. Math. **216** (2007) 178-211.
[27] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of number fields, Springer-Verlag, Berlin, 2000.
[28] A. Nickel, Non-commutative Fitting invariants and annihilation of class groups, J. Algebra **323** (2010) 2756-2778.
[29] T. Ochiai, F. Trihan, On the Selmer groups of abelian varieites over functions fields of characteristic $p > 0$, Math. Proc. Cam. Phil. Soc. **146** (2009) 23-43.

[30] R. Oliver, Whitehad Groups of Finite Groups, LMS Lecture Note Series **132**, Cambridge Univ. Press, Cambridge, 1988.
[31] R. Oliver, L. Taylor, Logarithmic descriptions of Whitehead groups and class groups for p-groups, Mem. Amer. Math. Soc. **392** 1988.
[32] A. Parker, Equivariant Tamagawa numbers and non-commutative Fitting invariants, Ph.D. Thesis, King's College London, 2007.
[33] M. Rapoport, Th. Zink, Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik, Invent. math. **68** (1982) 21-101.
[34] J. Ritter, A. Weiss, Toward equivariant Iwasawa theory, Part III, Math. Ann. **336** (2006) 27-49.
[35] K. Rubin, A Stark Conjecture 'over $\mathbb{Z}$' for abelian L-functions with multiple zeros, Ann. Inst. Fourier **46** (1996) 33-62.
[36] R. G. Swan, Algebraic K-theory, Lecture Note in Math. **76**, Springer, 1968.
[37] J. Tate, Les Conjectures de Stark sur les Fonctions L d'Artin en $s = 0$ (notes par D.Bernardi et N. Schappacher), Progress in Math., **47**, Birkhäuser, Boston, 1984.
[38] J. Tate, Letter to Joongul Lee, 22 July 1997.
[39] J. Tate, Refining Gross's conjecture on the values of abelian L-functions, Contemp. Math. **358** (2004) 189-192.
[40] L. N. Vaserstein, On the stabilisation of the general linear group over a ring, Mat. Sb. **79** (1969) 405-424; translation - Math. USSR Sbornik **8** (1969) 383-400.
[41] M. Witte, Noncommutative Iwasawa main conjecture for varieties over finite fields, PhD. thesis, Universität Leipzig, 2008. Available at http://www.dart-europe.eu/full.php?id=162794.
[42] M. Witte, On a noncommutative Iwasawa main conjecture for varieties over finite fields, preprint 2010.

King's College London, Department of Mathematics, London WC2R 2LS, U.K.
E-mail address: `david.burns@kcl.ac.uk`