Explicit Units and the Equivariant Tamagawa Number Conjecture, II

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1 Introduction

In [1], Bley and the first named author showed that a particular case of the equivariant Tamagawa number conjecture of [9, Conj. 4] implies, in the setting of Galois extensions of number fields, the existence of elements satisfying a variety of rather explicit 'Stark-type' conditions. More recently, the relevant special case of [9, Conj. 4] has been verified for absolutely abelian fields [11] and an analogous conjecture for global function fields has been verified at all primes different from the residue characteristic [6]. In addition, the approach of [1] has been developed in [5, 6, 7] to prove refinements and generalisations of certain much-studied 'refined abelian Stark conjectures' due to *inter alia* Gross, Rubin and Tate.

In this article we further refine the approach of [1] to show that the Tamagawa number formalism predicts structural information about ideal class groups that is in general very much finer than that which can be obtained by applying the celebrated method of Thaine [26] and Rubin [23]. Indeed, the original motivation behind this article was some striking recent work of Greither and Kučera in [16] proving annihilation statements for the class groups of a restricted class of absolutely abelian fields, improving upon those that can be obtained by the method of Thaine and Rubin in that the 'special elements' involved are much better adapted to the purpose at hand than are the Sinnott circular units which the latter method would use. In fact, the special elements in [16] occur as 'roots' of the Sinnott circular units under the action of certain augmentation elements of the associated integral Galois group ring, and are constructed by rather involved combinatorial methods.

By contrast, in this article we shall combine some well known techniques and results of homological algebra with the approach of [1] to show that the existence of the special elements constructed by Greither and Kučera in [16] can be derived as a direct consequence of the validity of the relevant case of [9, Conj. 4]. By replacing the direct use of elaborate combinatorial methods with a reduction to the general formalism of [1], we obtain several rather concrete advantages. For example, our approach applies equally well in the setting of global function fields, as well as to extensions for which the associated L-functions have multiple-order zeroes at $s = 0$ (rather than only first-order zeroes, as in [16]), in which case it predicts the existence of suitable 'systems' of special elements, and it also allows us to answer a question explicitly raised by Greither and Kučera in [16] by showing that the special element described in [loc. cit., Th. 1] can be directly related to the Fitting ideals, rather than the annihilators (as in [loc. cit., Th. 2]), of certain natural class groups (see Remark 3.3). At the end of the article we also construct an explicit family of extensions showing that the cyclicity hypothesis used by Greither and Kučera in loc. cit. is indeed necessary in order to guarantee the existence of suitable special elements.

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2 The leading term conjecture

2.1 Formulation of the conjecture

Let K/k be a nontrivial abelian extension of global fields with Galois group G. Let S be a finite non-empty set of places of k containing all archimedean places (if any) and all those that ramify in the extension K/k. We write $Y_{K,S}$ for the free abelian group on the set of places $S(K)$ of K which lie above those in S and $X_{K,S}$ for the kernel of the homomorphism $Y_{K,S} \to \mathbb{Z}$ that sends each element of $S(K)$ to 1. We write $\mathcal{O}_{K,S}$ for the ring of $S(K)$-integers in K and we set $U_{K,S} := \mathrm{Spec}(\mathcal{O}_{K,S})$ and $A_{K,S} := \mathrm{Pic}(\mathcal{O}_{K,S})$.

We define the S-truncated $\mathbb{C}[G]$-valued L-function of K/k by setting

$$\theta_{K/k,S}(s) := \sum_{\chi} L_{K/k,S}(s, \chi^{-1}) e_{\chi},$$

where the index χ runs through all elements of $\mathrm{Hom}(G, \mathbb{C}^{\times})$, $L_{K/k,S}(s, \chi)$ denotes the S-truncated Artin L-function and $e_{\chi} := (1/\#G) \sum_{g \in G} \chi(g) g^{-1}$ is the idempotent at χ. Its leading term at $s = 0$ is then $\theta^*_{K/k,S}(0) := \sum_{\chi} L^*_{K/k,S}(0, \chi^{-1}) e_{\chi}$, where $L^*_{K/k,S}(0, \chi)$ is the leading coefficient in the Taylor expansion of $L_{K/k,S}(s, \chi)$ at 0. It is easily shown that $\theta^*_{K/k,S}(0)$ belongs to $\mathbb{R}[G]$, and in the remainder of this section we recall a conjectural description of the $\mathbb{Z}[G]$-submodule of $\mathbb{R}[G]$ that is generated by $\theta^*_{K/k,S}(0)$.

The key ingredient in this formula is the determinant of a perfect complex of $\mathbb{Z}[G]$-modules provided by the following result. We write $\mathfrak{D}(\mathbb{Z}[G])$ for the derived category of complexes of $\mathbb{Z}[G]$-modules. We recall that the complex $R\Gamma_{c,\text{ét}}(U_{K,S}, \mathbb{Z})$ defined in [8, (3)] is an object of $\mathfrak{D}(\mathbb{Z}[G])$ which computes the cohomology 'with compact support' of the constant étale sheaf $\mathbb{Z}$ on $U_{K,S}$.

In the sequel all unadorned tensor products are to be regarded as computed in the category of (complexes of) abelian groups.

Lemma 2.1. *There exists a complex $\tilde{\Psi}^{\cdot}_S$ of $\mathbb{Z}[G]$-modules of the form*

$$\Psi^0_S \xrightarrow{d} \Psi^1_S \to X_{K,S} \otimes \mathbb{Q}$$

which satisfies both of the following properties.

i) There exists a distinguished triangle in $\mathfrak{D}(\mathbb{Z}[G])$ of the form

$$\tilde{\Psi}^{\cdot}_S \to \mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,\text{ét}}(U_{K,S}, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-3]) \to \hat{\mathcal{O}}^{\times}_{K,S}/\mathcal{O}^{\times}_{K,S}[0] \to \tilde{\Psi}^{\cdot}_S[1]$$

where $\hat{\mathcal{O}}^{\times}_{K,S}$ denotes the profinite completion of $\mathcal{O}^{\times}_{K,S}$ and the second arrow denotes the unique morphism induced by the projection $\hat{\mathcal{O}}^{\times}_{K,S} \to \hat{\mathcal{O}}^{\times}_{K,S}/\mathcal{O}^{\times}_{K,S}$ together with the canonical identification of $H^0\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,\text{ét}}(U_{K,S}, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-3])$ with $\hat{\mathcal{O}}^{\times}_{K,S}$.

ii) Ψ^0_S is a cohomologically-trivial G-module, Ψ^1_S is a free $\mathbb{Z}[G]$-module, and the distinguished triangle in i) induces exact sequences of the form

$$0 \longrightarrow \mathcal{O}^{\times}_{K,S} \longrightarrow \Psi^0_S \xrightarrow{d} \Psi^1_S \longrightarrow X'_{K,S} \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow A_{K,S} \longrightarrow X'_{K,S} \longrightarrow X_{K,S} \longrightarrow 0. \tag{2}$$

Proof. If $A_{K,S}$ is trivial, then this result is proved in [8, Prop. 3.1] (whilst the argument of loc. cit. is phrased solely in terms of number fields, it extends immediately to the case of global function fields). In general, the explicit computation of $\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,\text{ét}}(U_{K,S},\mathbb{Z}),\mathbb{Q}/\mathbb{Z}[-3])$ made in [loc. cit., pp. 1356-1358] can be completed in just the same way even if $A_{K,S}(\cong H^1_{\text{ét}}(U_{K,S},\mathbb{G}_m))$ is non-trivial and this leads to all of the claimed results. For more details in this regard, see [6, Lem. 1, Rem. 3]. $\square$

For any commutative ring R we write Det_R for the determinant functor of [19], valued in the category $\mathcal{P}(R)$ of graded invertible R-modules, and for any object $(\mathcal{L},r)$ of $\mathcal{P}(R)$ we set $(\mathcal{L},r)^{-1} := (\mathrm{Hom}_R(\mathcal{L},R),-r)$ (which is again an object of $\mathcal{P}(R)$).

We write $\Psi^{\cdot}_S$ for the perfect complex of $\mathbb{Z}[G]$-modules which is equal to $\Psi^0_S \xrightarrow{d} \Psi^1_S$, where the first term is placed in degree 0 and the cohomology groups are identified with $\mathcal{O}^\times_{K,S}$ and $X'_{K,S}$ by means of (1). Then we obtain an isomorphism

$$\vartheta_{\Psi^{\cdot}_S,\mathbb{R}} : \mathrm{Det}_{\mathbb{R}[G]}(\Psi^{\cdot}_S\otimes\mathbb{R}) \cong (\mathbb{R}[G],0)$$

in $\mathcal{P}(\mathbb{R}[G])$ by composing the isomorphism

$$\mathrm{Det}_{\mathbb{R}[G]}(\Psi^{\cdot}_S\otimes\mathbb{R}) \cong \mathrm{Det}_{\mathbb{R}[G]}(\mathcal{O}^\times_{K,S}\otimes\mathbb{R})\otimes_{\mathcal{P}(\mathbb{R}[G])}\mathrm{Det}_{\mathbb{R}[G]}(X_{K,S}\otimes\mathbb{R})^{-1}$$

that is induced by the scalar extension of (1) with the map $\mathrm{Det}_{\mathbb{R}[G]}(\mathrm{Reg}_{K,S})\otimes_{\mathcal{P}(\mathbb{R}[G])}\mathrm{id}$ and then the evaluation pairing on the space $\mathrm{Det}_{\mathbb{R}[G]}(X_{K,S}\otimes\mathbb{R})$, where $\mathrm{Reg}_{K,S}$ denotes the $\mathbb{R}[G]$-equivariant isomorphism $\mathcal{O}^\times_{K,S}\otimes\mathbb{R}\longrightarrow X_{K,S}\otimes\mathbb{R}$ which at each $u\in\mathcal{O}^\times_{K,S}$ satisfies

$$\mathrm{Reg}_{K,S}(u) = -\sum_{w\in S(K)}\log\mid u\mid_w\cdot w,$$

with $|\cdot|_w$ denoting the normalised absolute value of each place w.

Conjecture C(K/k): *One has an equality in $\mathcal{P}(\mathbb{Z}[G])$ of the form*

$$(\theta^*_{K/k,S}(0)\cdot\mathbb{Z}[G],0) = \vartheta_{\Psi^{\cdot}_S,\mathbb{R}}(\mathrm{Det}_{\mathbb{Z}[G]}\Psi^{\cdot}_S).$$

Remark 2.2. Under the conditions of Lemma 2.1, the complex $\Psi^{\cdot}_S$ is unique to within an isomorphism in $\mathfrak{D}(\mathbb{Z}[G])$ that induces the identity map on all (non-zero) degrees of cohomology, and this can be used to show that the (graded) lattice $\vartheta_{\Psi^{\cdot}_S,\mathbb{R}}(\mathrm{Det}_{\mathbb{Z}[G]}\Psi^{\cdot}_S)$ depends only upon the pair $(K/k,S)$.

Remark 2.3. The same argument as used to prove [4, Th. 2.1.2(i)] shows that the validity of C(K/k) is independent of the chosen set S. Further, if S is large enough to ensure that $A_{K,S}=0$, then $\Psi^{\cdot}_S$ coincides with the complex described in [loc. cit., Prop. 2.1.1] and [8, Rem. following Prop. 3.1] (in the number field case) and also in [5, §2]. It follows that C(K/k) coincides with the conjectures studied in loc. cit., and is therefore equivalent (in the number field case) to the 'Equivariant Tamagawa Number Conjecture' of [9, Conj. 4(iv)] as applied to the pair $(h^0(\mathrm{Spec}\ K),\mathbb{Z}[G])$, where $h^0(\mathrm{Spec}\ K)$ is considered as a motive that is defined over k and has coefficients $\mathbb{Q}[G]$ (see [4, Th. 2.4.1] or [10, §3] for different proofs of this fact). Since G is abelian, this observation shows that C(K/k) implies the relevant special case of the 'Generalised Iwasawa Main Conjecture' formulated by Kato in [18, §3.2] (cf. [10, §2] in this regard). In addition, one knows that C(K/k) is valid if either $k=\mathbb{Q}$ (cf. [11, Th. 8.1, Rem. 8.1], [14, Th. 5.1]) or k is a global function field whose characteristic is coprime to the order of G [6, Cor. 1, Rem. 5].

Remark 2.4. If K/k is an extension of global function fields, then $\Psi^{\cdot}_S$ can also be interpreted in terms of the Weil-étale cohomology of $\mathbb{G}_m$ on $U_{K,S}$ [6, §2.2]. In particular, if $K=k$ is a global function field, then one can show that C(K/k) is equivalent to a special case of the conjecture formulated by Lichtenbaum in [20, Conj. 8.1e)].

2.2 An explicit reinterpretation

In this section we impose certain additional hypotheses on K/k in order to give an interpretation of $\mathrm{C}(K/k)$ in terms of the existence of S-units satisfying a variety of explicit conditions. This reinterpretation arises by combining the approaches of [1] and [24] and will be used in the next section to prove our main results.

For convenience we label, and hence order, the elements of S as $\{v_i : 0 \le i \le n\}$. For each index $i = 0, \dots, n$ we choose a place w_i of K above v_i, we let G_i denote the decomposition group of w_i in G, and we set $K_i := K^{G_i}$. For a place w of K and a subfield L, we write w_L for the place of L induced by w, and abbreviate $w_{i,L} := (w_i)_L$. We assume henceforth that there exists a strictly positive integer r such that G_i is trivial for each index i with $1 \le i \le r$.

We write $r(\chi)$ for the order of vanishing of the meromorphic function $L_S(s, \chi)$ at $s = 0$.

Lemma 2.5. *For a nontrivial character χ of G, the integer $r(\chi)$ is equal to the number of $v_i \in S$ such that χ is trivial on the corresponding decomposition group G_i.*

For the trivial character χ, we have $r(\chi) = n$.

Proof. This follows immediately from [25, Chap. I, Prop. I.3.4]. $\square$

For each prime p we write $\mathbb{Z}_{(p)}$ for the p-localisation of $\mathbb{Z}$.

Lemma 2.6. *Assume that $\#G$ is coprime to both $\#K^\times_{tors}$ and $\#A_{K,S}$, and that $\mathrm{C}(K/k)$ is valid. Assume also that $G_0 = G$, G_i is trivial for $1 \le i \le r$ and G_i is cyclic for $r < i \le n$, and fix a generating element g_i of G_i for each $r < i \le n$.*

Then for each $1 \le j \le n$ there exists an element ϵ_j of $\mathcal{O}^\times_{K_j,S}$ satisfying each of the following conditions.

i) If $\mathcal{E}_S$ denotes the G-submodule of $\mathcal{O}^\times_{K,S}$ generated by $\{\epsilon_j : 1 \le j \le n\}$, then $(\mathcal{O}^\times_{K,S} : \mathcal{E}_S)$ is finite and coprime to $\#G$.

ii) If $r < j \le n$, then the local symbol $(\epsilon_j, K_w/(K_j)_{w_{K_j}}) = \begin{cases} g_j & \text{if } w = w_j \\ 1 & \text{otherwise.} \end{cases}$

iii) For $1 \le i, j \le n$ define $y_{ij} \in \mathbb{R}[G] \cdot \sum_{g \in G_j} g$ by the equation

$$\frac{1}{\#G_i} \mathrm{Reg}_{K,S}(\epsilon_i) = \sum_{j=1}^{n} y_{ij}(w_j - w_0), \tag{3}$$

and let M be the $n \times n$ matrix in $\mathbb{R}[G]$ with (i,j)-th entry $y_{ij} + \delta_{ij}(g_i - 1)$. Then there exists an element x of $\bigcap_{p \mid \#G} \mathbb{Z}_{(p)}[G]^\times$ satisfying both

$$\theta^*_{K/k,S}(0) = x \det(M) \qquad \text{and} \qquad \mathbb{Z}[G]x^{-1} = \mathrm{Fitt}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S}/\mathcal{E}_S).$$

Proof. Under the sated hypotheses on K/k and S the existence of a set of elements $\{\epsilon_j : 1 \le j \le n\}$ satisfying claims i) and ii) is proved in [1, Th. 3.2a), b)]. Further, the existence of an element x of $\mathbb{Q}[G]^\times$ satisfying the last two equalities of claim iii) follows by a slight variant of the argument used to prove [loc. cit., Th. 3.2c)] (the precise details of this deduction are given in the proof of [6, Prop. 6.1, (19),(20)]). It therefore only remains for us to observe that $x \in \mathbb{Z}_{(p)}[G]^\times$ for each p dividing $\#G$, and this follows because for each such p claim i) implies that

$$Z_{(p)}[G]x^{-1} = \mathbb{Z}[G]x^{-1} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathrm{Fitt}_{\mathbb{Z}_{(p)}[G]}((\mathcal{O}^\times_{K,S}/\mathcal{E}_S) \otimes \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[G].$$

$\square$

Now consider an intermediate field $L = K^H$ between K and k, such that L/k is Galois with group $\Gamma = G/H$. We assume that the places in S are ordered so that $v_1, \ldots, v_{r_L}$ are totally split in L (i.e. $G_i \subseteq H$ for $i = 1, \ldots, r_L$, for some integer r_L) and that this is not the case for $v_{r_L+1}, \ldots, v_n$. Following the approach of [24], we observe that there exists a unique element $\eta_{L,S}$ of the space $\mathbb{R} \otimes \wedge^{r_L}_{\mathbb{Z}[G]} \mathcal{O}^\times_{L,S}$ which satisfies

$$(\wedge^{r_L}\mathrm{Reg}_{L,S})(\eta_{L,S}) = \lim_{s\to 0} s^{-r_L}\theta_{L/k,S}(s) \cdot \wedge_{i=1}^{i=r_L}(w_{i,L} - w_{0,L}) \tag{4}$$

and is such that for each character $\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)$ one has $e_\chi(\eta_{L,S}) \neq 0$ if and only if $e_\chi \cdot \lim_{s\to 0} s^{-r_L}\theta_{L/k,S}(s) \neq 0$.

Proposition 2.7. *Assume the hypotheses of Lemma 2.6 and the previous paragraph. Then in $\mathbb{Q} \otimes \wedge^{r_L}_{\mathbb{Z}[\Gamma]} \mathcal{O}^\times_{L,S}$ we have*

$$\eta_{L,S} = x \prod_{j=r_L+1}^{n} (g_j - 1) \cdot \wedge_{i=1}^{r_L} N_{K_i/K_i\cap L}\epsilon_i.$$

Proof. Let $e_H = \frac{1}{\#H}\sum_{h\in H} h$, and write $q_H : \mathbb{R}[G]e_H \to \mathbb{R}[\Gamma]$ for the $\mathbb{R}[G]$-isomorphism which sends e_H to 1. The complex characters of Γ are in bijection with those characters of G which are trivial on H, and [25, 4.2.3] shows that $L_{K/k,S}(s, \mathrm{Inf}^G_\Gamma(\chi)) = L_{L/k,S}(s, \chi)$ for each such character χ. The criteria of Lemma 2.5 show that $L_{L/k,S}(s, \chi)$ vanishes at $s = 0$ to order at least r_L, and exactly r_L if $\mathrm{Inf}^G_\Gamma\chi$ is nontrivial on each of $G_{r_L+1}, \ldots, G_n$. We write $e := \sum_\chi e_{\mathrm{Inf}^G_\Gamma(\chi)}$, where the sum is over these χ where the order of vanishing is exactly r_L. Then, setting $\theta^{r_L}_{L/k,S}(0) := \lim_{s\to 0} s^{-r_L}\theta_{L/k,S}(s)$, Lemma 2.6 iii) shows that

$$\theta^{r_L}_{L/k,S}(0) = q_H(xe\det(M)).$$

For each integer i with $0 \le i \le n$ we set $T_i := \sum_{g\in G_i} g \in \mathbb{Z}[G]$. Then we may write M in the block form

$$M = Y + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Delta \end{array}\right),$$

where Y has (i, j)-th entry $y_{ij} \in \mathbb{R}[G] \cdot T_j$ and Δ is the $(n - r_L) \times (n - r_L)$ diagonal matrix with (j, j)-th entry $g_{r_L+j} - 1$ for $1 \le j \le n - r_L$.

Lemma 2.5 shows that if j satisfies $r_L < j \le n$ then $eT_j = 0$, and so $ey_{ij} = 0$ for all $1 \le i \le n$ and $r_L < j \le n$. Therefore the matrix eM takes the form

$$eM = \left(\begin{array}{c|c} M' & * \\ \hline 0 & e\Delta \end{array}\right),$$

where M' is the r_L-square matrix with (i, j)-th entry ey_{ij}. Setting $t_S = \prod_{j=r_L+1}^{n}(g_j - 1)$, we have $e\det M = t_S \det(M')$. Now

$$\det(M') \cdot \wedge_{i=1}^{r_L}(w_{i,L} - w_{0,L}) = \wedge_{i=1}^{r_L}\left(\sum_{j=1}^{r_L} ey_{ij}(w_{j,L} - w_{0,L})\right),$$

and noting that $e(w_{j,L} - w_{0,L}) = 0$ if $j > r_L$, the right hand side of this equation is equal to $e \wedge_{i=1}^{r_L} (\sum_{j=1}^{n} y_{ij}(w_{j,L} - w_{0,L}))$.

The natural inclusion map $\mathcal{O}_{L,S}^\times \to \mathcal{O}_{K,S}^\times$ fits into a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{K,S}^\times & \xrightarrow{\mathrm{Reg}_{K,S}} & \mathbb{R}\otimes X_{K,S} \\
\uparrow & & \uparrow \\
\mathcal{O}_{L,S}^\times & \xrightarrow{\mathrm{Reg}_{L,S}} & \mathbb{R}\otimes X_{L,S},
\end{array}$$

in which the map $\mathbb{R}\otimes X_{L,S} \to \mathbb{R}\otimes X_{K,S}$ sends w_L to $(\#He_H)w$; see [25, bottom of p. 29]. Combining this with (3) implies

$$\mathrm{Reg}_{L,S}(N_{K_i/K_i\cap L}\epsilon_i) = \sum_{j=1}^n y_{ij}(w_{j,L} - w_{0,L})$$

for each index i with $1 \le i \le r_L$, and so we conclude that

$$\begin{aligned}
\det(M')\cdot \wedge_{i=1}^{r_L}(w_i - w_0) &= e\cdot \wedge_{i=1}^{r_L}\mathrm{Reg}_{L,S}(N_{K_i/K_i\cap L}\epsilon_i) \\
&= (\wedge^{r_L}\mathrm{Reg}_{L,S})(e\cdot \wedge_{i=1}^{r_L} N_{K_i/K_i\cap L}\epsilon_i).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\theta_{L/k,S}^{r_L}(0)\cdot \wedge_{i=1}^{r_L}(w_{i,L} - w_{0,L}) &= xe\det(M)\cdot \wedge_{i=1}^{r_L}(w_{i,L} - w_{0,L}) \\
&= xet_S\cdot(\wedge^{r_L}\mathrm{Reg}_{L,S})(\wedge_{i=1}^{r_L}\epsilon_i).
\end{aligned}$$

To establish the equality of Proposition 2.7 it is enough to show that $et_S = t_S$. If $r_L = n$, we have $e = 1$ and so $et_S = t_S$ is clear. If $r_L < n$ then, for each character χ inflated from Γ to G with $r(\chi) > r_L$, we have $\chi(t_S) = \prod_{j=r_L+1}^n(\chi(g_j) - 1) = 0$, since a factor in the product must be 0 by Lemma 2.5. This shows that $et_S = t_S$ in both cases and completes the proof. $\square$

3 Special cases

In this section we assume the notation and hypotheses of Lemma 2.6. We also fix a prime p and an integer $\ell > 1$ which is a power of p.

We assume given a cyclic extension K/k of degree ℓ, with $\#k_{\mathrm{tors}}^\times$ not divisible by p. We set $G := \mathrm{Gal}(K/k)$ and fix a generating element σ of G. We let $\mathrm{T} = \sum_{g\in G} g$ be the norm element in the integral group ring $\mathbb{Z}[G]$ and we denote the corresponding quotient ring $\mathbb{Z}[G]/\mathbb{Z}[G]\cdot\mathrm{T}$ by $\mathcal{A}$. We also set $e := 1 - \frac{1}{\ell}\mathrm{T}$ and observe that the G-equivariant homomorphism $\mathbb{Z}[G]e \to \mathcal{A}$ which takes e to 1 is a ring isomorphism. We set $\mathcal{A}_{(p)} := \mathcal{A}\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$.

For any G-module M we let $M^{\mathrm{T}=0}$ denote the kernel of the endomorphism of M that is induced by the action of T.

For each $i = 0, \dots, n$ we define a homomorphism $f_j : \mathcal{O}_{k,S}^\times \to \mathbb{Z}/\ell\mathbb{Z}$ by the condition that the local reciprocity symbol $(u, K_{w_i}/k_{v_i})$ is equal to $\sigma^{f_j(u)}$ for all $u \in \mathcal{O}_{k,S}^\times$. The group $\mathbb{Z}/\ell\mathbb{Z}$ acts on any group "modulo ℓ", meaning modulo ℓ-th powers.

Theorem 3.1. *Assume that $p \nmid \#k_{\mathrm{tors}}^\times$ and let K/k be a cyclic extension of order ℓ and group G. Let $S = \{v_0, \dots, v_n\}$ be a set of places of k containing all archimedean places and all which ramify in K/k. Assume that the decomposition group G_i of v_i is trivial for $i = 1, \dots, r$ and equal to G for the other values of i, with $n > r \ge 1$.*

Suppose further that $C(K/k)$ is valid. Then $\eta_{K,S} \in \overline{\wedge_{\mathbb{Z}[G]}^r\mathcal{O}_{K,S}^\times}\otimes\mathbb{Z}_{(p)}$ by Proposition 2.7. Furthermore, there exists an element ε of $\overline{\wedge_{\mathbb{Z}[G]}^r\mathcal{O}_{K,S}^\times}\otimes\mathbb{Z}_{(p)}$, respectively of $\overline{\wedge_{\mathbb{Z}[G]}^r\mathcal{O}_{K,S}^\times}$ if $\eta_{K,S} \in \overline{\wedge_{\mathbb{Z}[G]}^r\mathcal{O}_{K,S}^\times}$, which possesses each of the following properties.

i) *In* $\mathbb{Q}\otimes\wedge^r_{\mathbb{Z}[G]}\mathcal{O}^\times_{K,S}$ *one has* $\eta_{K,S}=(\sigma-1)^{n-r}(\varepsilon)$.

ii) *The congruence*

$$(\wedge^r N_{K/k})\varepsilon\equiv(f_{r+1}\wedge\cdots\wedge f_n)(\eta_{k,S})$$

holds in $\overline{\wedge^r_{\mathbb{Z}}\mathcal{O}^\times_{k,S}}\otimes\mathbb{Z}_{(p)}$ *modulo* ℓ.

iii) *Set* $S_1:=\{v_i:1\leq i\leq r\}$*, the subset of* S *consisting of places with trivial decomposition group. Then* S_1 *contains all archimedean places of* k*. Further, there exists a free* $\mathbb{Z}[G]$*-submodule* $\mathcal{E}$ *of* $\mathcal{O}^\times_{K,S}$ *of rank* r *and an integer* $t_{\mathcal{E}}$ *which is a power of* p *such that both* $\wedge^r_{\mathbb{Z}[G]}\mathcal{E}\otimes_{\mathbb{Z}[G]}\mathcal{A}_{(p)}=t_{\mathcal{E}}\mathcal{A}_{(p)}\varepsilon$ *and*

$$\begin{aligned}t_{\mathcal{E}}^{-1}(\sigma-1)^r\cdot\mathrm{Fitt}_{\mathcal{A}}((\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E})^{\mathrm{T}=0})\otimes\mathbb{Z}_{(p)}&\subseteq\mathrm{Fitt}_{\mathbb{Z}[G]}(A_{K,S})\mathcal{A}_{(p)}\\&\subseteq t_{\mathcal{E}}^{-1}\mathrm{Fitt}_{\mathcal{A}}((\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E})^{\mathrm{T}=0})\otimes\mathbb{Z}_{(p)}.\end{aligned}$$

If $r=1$*, then we may take* $t_{\mathcal{E}}=1$.

In the course of proving this result we shall actually obtain finer information than is provided by the inclusions given in iii) (cf. Lemma 3.7). However, before giving the proof we use this result to answer a question raised by Greither and Kučera in [16]. To do this, we take p odd, assume that $k=\mathbb{Q}$, and we fix $n\geq 2$ distinct rational primes $p_1,\dots,p_n$, each congruent to 1 modulo ℓ. We let $K/\mathbb{Q}$ be a cyclic extension of degree ℓ, of Galois group $G=\langle\sigma\rangle$, in which each p_i has full decomposition group and all other primes are unramified. Since such a field is totally real, the infinite place ∞ of $\mathbb{Q}$ splits completely in K, and so we set $r=1$ and $S=\{p_1,\infty,\dots,p_n\}$ (so $S_1=\{\infty\}$). Conjecture $\mathrm{C}(K/k)$ is valid by Remark 2.3. For each index i we fix a complex p_i-th root of 1, and call it ζ_{p_i}. We denote the Sinnott circular unit of conductor level by $\eta:=\mathrm{N}_{\mathbb{Q}(\zeta_{p_1\cdots p_n})/K}(1-\zeta_{p_1}\cdots\zeta_{p_n})$.

Define an n-square matrix A over $\mathbb{Z}/\ell\mathbb{Z}$ as follows. The field K is contained in $\mathbb{Q}(\zeta_{p_1\cdots p_n})$. If $i\neq j$, we let a_{ij} be the unique class in $\mathbb{Z}/\ell\mathbb{Z}$ such that $\sigma^{a_{ij}}$ is the restriction of the automorphism $\zeta_j\mapsto\zeta_j^{p_i}$. Then choose a_{ii} for each i in such a way that the matrix has zero row sums. It is easy to see that this corresponds to the definition before [16, Th. 1]. We define A_i to be the lift to $\mathbb{Z}$ of the (i,i) minor of A satisfying $0<A_i\leq\ell$.[1]

We let $\mathcal{O}_K$ denote the ring of algebraic integers $\mathcal{O}_{K,S_1}$ in K and set $A_K:=\mathrm{Pic}(\mathcal{O}_K)$.

Corollary 3.2. *Let* $K/\mathbb{Q}$ *be a cyclic extension of order a power of an odd prime* p*, and assume that the decomposition group for each prime* p_i *which ramifies in* $K/\mathbb{Q}$ *is equal to* G*. Set* $S=\{p_1,\infty,\dots,p_n\}$*. Then there exists an element* ε *of* $\mathcal{O}^\times_{K,S}$ *which satisfies each of the following conditions.*

i) $(\sigma-1)^{n-1}(\varepsilon)=\eta$.

ii) $\mathrm{N}_{K/\mathbb{Q}}(\varepsilon)^{(-1)^{n-1}}=\prod_{i=1}^n p_i^{A_i}$.

iii) $(\sigma-1)\cdot\mathrm{Ann}_{\mathcal{A}}(\mathcal{O}^\times_K/\varepsilon^{\sigma-1})\otimes\mathbb{Z}_{(p)}\subseteq\mathrm{Fitt}_{\mathcal{A}}((\sigma-1)A_K)\otimes\mathbb{Z}_{(p)}\subseteq\mathrm{Ann}_{\mathcal{A}}(\mathcal{O}^\times_K/\varepsilon^{\sigma-1})\otimes\mathbb{Z}_{(p)}$.

Remark 3.3. This answers the question raised in [16, Comment following Thm. 2], which asks about the possibility of using elements of the form ε to produce bounds on the Fitting ideal $\mathrm{Fitt}_{\mathcal{A}}((\sigma-1)A_K)\otimes\mathbb{Z}_{(p)}$.

[1] Note that we have preferred $A_i=l$ to $A_i=0$. This will ensure that ε is not a unit. Since ε may be adjusted by rational factors, as explained in the proof of Corollary 3.2 below, any length-ℓ range for A_i may be specified.

We first prove the deduction of Corollary 3.2 from Theorem 3.1, and then the Theorem.

Proof of Corollary 3.2. To identify $\eta_{K,S}$ in (4) we must choose an infinite place of K ("$w_{1,L}$" in that formula); we use the place induced by the embedding of the overlying cyclotomic field into $\mathbb{C}$ by which $\zeta_{p_1}\cdots\zeta_{p_n}$ corresponds to $e^{\frac{2\pi i}{p_1\cdots p_n}}$. The formulas for the values of L-functions of even Dirichlet characters show that $\eta^2_{K,S}=\eta$ in this situation, as in [17, §5]. Part i) of Theorem 3.2 shows that $\eta_{K,S}=(\sigma-1)^{n-1}(\varepsilon')$ for some $\varepsilon'\in\overline{\mathcal{O}^\times_{K,S}}\otimes\mathbb{Z}_{(p)}$. Since $\eta^2_{K,S}\in\overline{\mathcal{O}^\times_{K,S}}$, setting $\varepsilon=\varepsilon'^2$ gives $\pm\eta=(\sigma-1)^{n-1}(\varepsilon)$ in $\mathcal{O}^\times_{K,S}$. To see that the undetermined sign must be $+$, it is enough to take the norm to $\mathbb{Q}$ of both sides, and note that $N_{K/\mathbb{Q}}\eta=1$ by well-known properties of cyclotomic elements.

Now for the second assertion, we note that the analytic class number formula for $\mathbb{Q}$ gives $\eta^2_{\mathbb{Q},S}=p_1\wedge\cdots\wedge p_n$. Since $a_{ij}=-f_j(p_i)$, this shows that

$$(f_2\wedge\cdots\wedge f_n)(\eta^2_{\mathbb{Q},S})=\prod_{i=1}^n p_i^{(-1)^{n-1}A_i}$$

in $\mathbb{Z}_S^\times$ modulo torsion and ℓ-th powers. But the multiplicative torsion in $\mathbb{Z}_S$ consists of ℓ-th powers since ℓ is odd. So Theorem 3.1ii) shows that

$$N_{K/\mathbb{Q}}\varepsilon^{(-1)^{n-1}}=\prod_{i=1}^n p_i^{A_i}$$

in $\mathbb{Z}_S^\times$ modulo ℓ-th powers. Now the transformation $\varepsilon\mapsto y\varepsilon$ for arbitrary $y\in\mathbb{Z}_S^\times$ leaves claim i) true and multiplies $N_{K/\mathbb{Q}}\varepsilon^{(-1)^{n-1}}$ by the ℓ-th power of y. In this way ε can be adjusted to achieve the equality of claim ii) with the liftings A_i in the specified range.

Note that the norm map T from $\mathcal{O}_K^\times$ to $\mathbb{Z}^\times$ is valued in a group of order 2, and so is the zero map after tensoring with $\mathbb{Z}_{(p)}$. Therefore we may erase the 'T= 0' in Theorem 3.1iii). The $\mathbb{Z}[G]$-module $\mathcal{E}$ there is the free submodule of $\mathcal{O}^\times_{K,S}$ generated by ε, and, since each finite prime in S lies below a unique place of K, it is easy to see that $\mathcal{O}_K^\times\cap\mathcal{E}=\mathbb{Z}[G]\cdot\varepsilon^{\sigma-1}$. Hence

$$(\mathcal{O}_K^\times/\mathcal{O}_K^\times\cap\mathcal{E})^{\mathrm{T}=0}=\mathcal{O}_K^\times/\varepsilon^{\sigma-1} \tag{5}$$

and this shows that Theorem 3.1iii) implies

$$(\sigma-1)\cdot\mathrm{Ann}_{\mathcal{A}}(\mathcal{O}_K^\times/\varepsilon^{\sigma-1})\otimes\mathbb{Z}_{(p)}\subseteq\mathrm{Fitt}_{\mathbb{Z}[G]}(A_{K,S})\otimes\mathcal{A}_{(p)}\subseteq\mathrm{Ann}_{\mathcal{A}}(\mathcal{O}_K^\times/\varepsilon^{\sigma-1})\otimes\mathbb{Z}_{(p)}.$$

Our proof of Corollary 3.2 is therefore completed by means of the following observation. □

Lemma 3.4. *If $K/\mathbb{Q}$ and S are as in Corollary 3.2, then the G-modules $A_{K,S}$ and $(\sigma-1)A_K$ are naturally isomorphic.*

Proof. For any G-module M we write M^G for the submodule consisting of those elements which are fixed by the action of all elements of G. Then the exact sequence

$$0\longrightarrow A_K^G\xrightarrow{\subseteq}A_K\xrightarrow{\times(\sigma-1)}(\sigma-1)A_K\longrightarrow 0,$$

reduces us to showing that A_K^G is generated by the classes of primes above those in S.

Let I_K, resp. P_K, denote the group of fractional ideals, resp. principal fractional ideals, of $\mathcal{O}_K$. We claim that the natural morphism $I_K^G\to A_K^G$ is surjective. To show this we combine the cohomology sequence of the exact sequence $0\to P_K\to I_K\to A_K\to 0$ with the fact that

$H^1(G, I_K) = 0$ to obtain an exact sequence of the form $I_K^G \to A_K^G \to H^1(G, P_K) \to 0$. On the other hand, we may deduce that $H^1(G, P_K) = 0$ by combining the cohomology sequence of $0 \to \mathcal{O}_K^\times \to K^\times \to P_K \to 0$ together with the fact that $H^1(G, K^\times) = 0$ (Hilbert 90) and $H^2(G, \mathcal{O}_K^\times) \cong \hat{H}^0(G, \mathcal{O}_K^\times) = 0$ (since G is a cyclic group of odd order).

We now take an ideal $\mathfrak{a} := \prod_{\mathfrak{p} \in S} \mathfrak{p}^{a_\mathfrak{p}} \prod \mathfrak{q} \in I_K^G$ where no prime ideal $\mathfrak{q}$ ramifies in $K/\mathbb{Q}$. Then $\prod \mathfrak{q} \in I_K^G$ and so is of the form $m\mathcal{O}_K$ for some $m \in \mathbb{Q}$. It follows that $\mathfrak{a}$ and $\prod_{\mathfrak{p} \in S} \mathfrak{p}^{a_\mathfrak{p}}$ have the same image in A_K, as required. $\square$

Proof of Theorem 3.1. Let $r' \geq r$ be an integer and $S' = \{v'_0, \ldots, v'_{n'}\}$ be a set of places of k such that $v'_i = v_i$ for $i = 0, \ldots, r$; $v'_{r'+i} = v_{r+i}$ for $i = 1, \ldots, n - r$; and $v'_{r+1}, \ldots, v'_{r'}$ are $r' - r$ additional places which are totally split in K. By Čebotarev's density theorem, there exists such an S' with the property that $p \nmid \# A_{K,S'}$. We may now apply Lemma 2.6 with S' and r' instead of S and r, and with $g_j = \sigma$ for each index $j > r'$. We set $\varepsilon' := x \cdot \wedge_{i=1}^{i=r'} \epsilon_i \in \wedge_{\mathbb{Z}[G]}^{r'} \mathcal{O}_{K,S}^\times \otimes \mathbb{Z}_{(p)}$. Then Proposition 2.7 implies that $\eta_{K,S'} = (\sigma - 1)^{n-r}(\varepsilon')$. Now, [24, Prop. 5.2] shows that there is a $\mathbb{Z}[G]$-homomorphism $\Phi = \Phi_{K,S,S'} : \wedge_{\mathbb{Z}[G]}^{r'} \mathcal{O}_{K,S'}^\times \to \wedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{K,S}^\times$ such that $\eta_{K,S} = \Phi(\eta_{K,S'})$. Hence we can take $\varepsilon = \Phi(\varepsilon')$, and this satisfies the equality of claim i).

We next show claim ii). Proposition 2.7 gives the formulas

$$\eta_{K,S'} = x(\sigma - 1)^{n'-r'} \cdot \wedge_{i=1}^{r'} \epsilon_i; \qquad \eta_{k,S'} = x(\wedge_{i=1}^{r'} N_{K/k}\epsilon_i) \wedge (\wedge_{i=r'+1}^{n'} \epsilon_i).$$

Therefore, letting f'_j correspond to v'_j as f_j does to v_j,

$$(f'_{r'+1} \wedge \cdots \wedge f'_{n'})(\eta_{k,S'}) = x \det(f'_j(\epsilon_i))_{i,j=r'+1}^{n'} (\wedge_{i=1}^{r'} N_{K/k}\epsilon_i),$$

since the f'_j are trivial on the image of $N_{K/k}$. By Lemma 2.6ii), the determinant is that of the identity matrix, and so we obtain $(\wedge^{r'} N_{K/k})\varepsilon' \equiv (f'_{r'+1} \wedge \cdots \wedge f'_{n'})(\eta_{k,S'})$ in $\overline{\wedge_{\mathbb{Z}}^{r'} \mathcal{O}_{k,S'}^\times} \otimes \mathbb{Z}_{(p)}$ modulo ℓ. We have $\eta_{k,S} = \Phi_{k,S,S'}(\eta_{k,S'})$, and so we apply $\Phi_{k,S,S'}$ to both sides to obtain

$$\Phi_{k,S,S'}((\wedge^{r'} N_{K/k})\varepsilon') \equiv (f'_{r'+1} \wedge \cdots \wedge f'_{n'})(\eta_{k,S})$$

The definition [24, (14)] shows that $\Phi_{k,S,S'} \circ (\wedge^{r'} N_{K/k}) = (\wedge^r N_{K/k}) \circ \Phi_{K,S,S'}$, and therefore we obtain the congruence of claim ii).

Regarding claim iii) we first observe that if k is a number field, then the condition $p \nmid \# k_{\mathrm{tors}}^\times$ implies that K/k has odd degree and hence that all archimedean places belong to S_1, as claimed. We next describe the definition of the module $\mathcal{E}$. As a $\mathbb{Q}[G]$-module, we have $\mathbb{Q}\mathcal{O}_{K,S}^\times \cong \mathbb{Q}[G]^r \oplus \mathbb{Q}^{n-r}$. Hence, considering the component away from the trivial character, we have $\wedge_{\mathbb{Q}[G]e}^r e\mathbb{Q}\mathcal{O}_{K,S}^\times = \mathbb{Q}[G] \cdot e\varepsilon$, since $e_\chi \varepsilon \neq 0$ for all nontrivial χ (by its relation to the values of L-functions). Now the $\mathbb{Q}[G]e$-module $e\mathbb{Q}\mathcal{O}_{K,S}^\times$ is isomorphic to $(\mathbb{Q}[G]e)^r$ and $\wedge_{\mathbb{Q}[G]e}^r (\mathbb{Q}[G]e)^r$ is a cyclic $\mathbb{Q}[G]e$-module generated by the primitive tensor corresponding to the standard basis of $(\mathbb{Q}[G]e)^r$. We may therefore conclude that $e\varepsilon$ may be written as a primitive tensor

$$e\varepsilon = q \cdot e\alpha_1 \wedge e\alpha_2 \wedge \cdots \wedge e\alpha_r, \tag{6}$$

with $q^{-1} \in \mathbb{Z} \setminus \{0\}$ and each α_i an element of $\mathcal{O}_{K,S}^\times$. For any nontrivial character χ, we therefore have

$$e_\chi \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r \neq 0. \tag{7}$$

But expression (6) is not changed by multiplying any α_i by an element of $\mathcal{O}_{k,S}^\times$, and by doing so one may ensure that (7) holds for χ the trivial character as well. If we then take $\mathcal{E}$ to be the G-submodule of $\mathcal{O}_{K,S}^{\times}$ generated by $\{\alpha_i : 1 \leq i \leq r\}$, this module is free of rank r and the equality $\wedge^r_{\mathbb{Z}[G]}\mathcal{E} \otimes \mathcal{A}_{(p)} = t_{\mathcal{E}}\mathcal{A}_{(p)}\varepsilon$ holds with $t_{\mathcal{E}}$ the maximal power of p dividing q^{-1}.

Now if we assume $r = 1$, then we may simply take $\mathcal{E} = \mathbb{Z}_{(p)}[G]\varepsilon$, and this is free of rank 1 (adjusting ε by an element of $\mathcal{O}_{k,S}^{\times}$ if necessary, which does not affect the validity of either (i) or (ii)). Therefore we may take $t_{\mathcal{E}} = 1$ in the case $r = 1$.

We have now proved all of Theorem 3.1 apart from the inclusions of iii). In the rest of this section we deduce these via homological algebra from Conjecture C(K/k).

It is straightforward to verify that there exists a free $\mathbb{Z}[G]$-submodule $\mathcal{F}$ of Ψ_S^1 which is of rank r and projects to $\sum_{i=1}^r \mathbb{Z}[G]\cdot(w_i - w_0)$ under the composite surjection $\Psi_S^1 \to X'_{K,S} \to X_{K,S}$, where the first and second arrows are as in (1) and (2) respectively. We use the same symbol to denote the image of $\mathcal{F}$ in $X'_{K,S}$. We also let $F^{\cdot}$ denote the complex $\mathcal{E} \xrightarrow{0} \mathcal{F}$, where the first term occurs in degree 0. Then one has a tautological short exact sequence of perfect complexes

$$0 \longrightarrow F^{\cdot} \longrightarrow \Psi_S^{\cdot} \longrightarrow \Phi_S^{\cdot} \longrightarrow 0 \qquad (8)$$

where the second arrow denotes the natural 'inclusion' map and $\Phi_S^{\cdot}$ denotes the complex obtained from the middle two terms in the following exact sequence (induced by (1))

$$0 \longrightarrow \mathcal{O}_{K,S}^{\times}/\mathcal{E} \longrightarrow \Psi_S^0/\mathcal{E} \longrightarrow \Psi_S^1/\mathcal{F} \longrightarrow X'_{K,S}/\mathcal{F} \longrightarrow 0. \qquad (9)$$

We must now extend scalars from $\mathbb{Z}[G]$ to $\mathcal{A}$. For any two G-modules M and N, the 'diagonal' G-action on $M \otimes N$ is the action by which each $g \in G$ sends $m \otimes n \in M \otimes N$ to $g^{-1}m \otimes gn$. In the special case $N = \mathbb{Z}[G]$, the resulting module is isomorphic to $M \otimes \mathbb{Z}[G]$, where G acts only on the second factor (in the canonical way). The isomorphism from the former to the latter is given by taking $m \otimes g$ to $gm \otimes g$. It follows that for any finitely-generated free $\mathbb{Z}[G]$-module F, isomorphic modules are obtained from $M \otimes F$ by taking either the diagonal action, or the action on the second factor.

We now apply the exact functor $\mathcal{A} \otimes_{\mathbb{Z}} -$ to (9), endow the resulting groups with the diagonal G-action, and compare taking G-invariants and co-invariants. This gives the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & (\mathcal{A} \otimes (\mathcal{O}_{K,S}^{\times}/\mathcal{E}))^G & \to & (\mathcal{A} \otimes \Phi_S^0)^G & \to & (\mathcal{A} \otimes \Phi_S^1)^G & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & \mathcal{A} \otimes_{\mathbb{Z}[G]} \Phi_S^0 & \to & \mathcal{A} \otimes_{\mathbb{Z}[G]} \Phi_S^1 & \to & \mathcal{A} \otimes_{\mathbb{Z}[G]} (X'_{K,S}/\mathcal{F}) & \to & 0. \end{array}$$

The vertical maps here are those which are induced by the action of T and are bijective since the G-modules Φ_S^0 and Φ_S^1, and hence also $\mathcal{A} \otimes \Phi_S^0$ and $\mathcal{A} \otimes \Phi_S^1$, are cohomologically trivial. Hence one has an exact sequence of $\mathcal{A}$-modules

$$0 \longrightarrow (\mathcal{A} \otimes (\mathcal{O}_{K,S}^{\times}/\mathcal{E}))^G \longrightarrow \mathcal{A} \otimes_{\mathbb{Z}[G]} \Phi_S^0 \longrightarrow \mathcal{A} \otimes_{\mathbb{Z}[G]} \Phi_S^1 \longrightarrow \mathcal{A} \otimes_{\mathbb{Z}[G]} (X'_{K,S}/\mathcal{F}) \longrightarrow 0. \quad (10)$$

Now G acts trivially on both $\mathbb{Q} \otimes \mathcal{O}_{K,S}^{\times}/\mathcal{E}$ and $\mathbb{Q} \otimes X'_{K,S}/\mathcal{F}$, and so the second and fifth terms in the above sequence are finite. This observation implies that there is a canonical isomorphism

$$\iota : \mathbb{Q}[G]e \otimes_{\mathbb{Z}[G]} \mathrm{Det}_{\mathbb{Z}[G]}(\Phi_S^{\cdot}) \cong (\mathbb{Q}[G]e, 0).$$

Further, the approach of [12, pf. of Cor. 2] allows one to compute $\iota(\mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \Phi_S^{\cdot}))$. Indeed, after observing that $\mathcal{A}_{(p)}$ satisfies the conditions on R in [loc. cit., Lem. 5], we may deduce from the exactness of (10) an equality of the form

$$\mathrm{Fitt}_{\mathcal{A}_{(p)}}((\mathcal{A}_{(p)} \otimes (\mathcal{O}_{K,S}^{\times}/\mathcal{E}))^G)\mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \Phi_S^{\cdot}) = \mathrm{Fitt}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} (X'_{K,S}/\mathcal{F})) \quad (11)$$

(Here, when applying [loc. cit., Lem. 5], we have also used the fact that, since G is cyclic, the argument of [22, App. 1, Prop. 1] implies that $\mathrm{Fitt}_{\mathcal{A}_{(p)}}(\mathrm{Hom}_{\mathbb{Z}_{(p)}}(M, \mathbb{Q}/\mathbb{Z}_{(p)})) = \mathrm{Fitt}_{\mathcal{A}_{(p)}}(M)$ for any finite $\mathcal{A}_{(p)}$-module M.)

In the next three results we provide explicit information on each of the terms in (11).

Lemma 3.5. *If* $\mathrm{C}(K/k)$ *is valid, then* $\mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \Phi_S^{\cdot}) = \mathcal{A}_{(p)} \cdot t_{\mathcal{E}}^{-1}(\sigma-1)^{n-r}$.

Proof. Since $\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \mathcal{E}$ and $\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \mathcal{F}$ are torsion-free, whilst $\mathrm{Tor}^1_{\mathbb{Z}[G]}(\mathcal{A}_{(p)}, \Phi_S^0)$ and $\mathrm{Tor}^1_{\mathbb{Z}[G]}(\mathcal{A}_{(p)}, \Phi_S^1)$ are both finite, (8) induces an exact sequence of complexes of $\mathcal{A}_{(p)}$-modules

$$0 \to \mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} F^{\cdot} \to \mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \Psi_S^{\cdot} \to \mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \Phi_S^{\cdot} \to 0,$$

and hence also an isomorphism in $\mathcal{P}(\mathcal{A}_{(p)})$ of the form

$$\mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} F^{\cdot}) \otimes_{\mathcal{P}(\mathcal{A}_{(p)})} \mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \Phi_S^{\cdot}) \cong \mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \Psi_S^{\cdot}). \quad (12)$$

Now if $C^{\cdot}$ denotes any of the complexes $F^{\cdot}, \Psi_S^{\cdot}$ or $\Phi_S^{\cdot}$, then $\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} C^{\cdot}$ is naturally isomorphic to $\mathcal{A}_{(p)} \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C^{\cdot}$ and so $\mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} C^{\cdot})$ identifies with $\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \mathrm{Det}_{\mathbb{Z}[G]} C^{\cdot}$.

In particular, therefore, the validity of $\mathrm{C}(K/k)$ would imply that

$$(\mathbb{R}[G]e \otimes_{\mathbb{R}[G]} \vartheta_{\Psi_S^{\cdot},\mathbb{R}})(\mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \Psi_S^{\cdot})) = (\mathcal{A}_{(p)} \cdot \theta^r_{K/k,S}(0), 0).$$

We combine (12) with the isomorphism ι to identify $\mathrm{Det}_{\mathbb{Q}[G]e}(\mathbb{Q}[G]e \otimes_{\mathbb{Z}[G]} F^{\cdot})$ with $\mathrm{Det}_{\mathbb{Q}[G]e}(\mathbb{Q}[G]e \otimes_{\mathbb{Z}[G]} \Psi_S^{\cdot})$. Then, in view of (12) and the last displayed equality, the claimed result is equivalent to an equality

$$(\mathbb{R}[G]e \otimes_{\mathbb{R}[G]} \vartheta_{\Psi_S^{\cdot},\mathbb{R}})(\mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} F^{\cdot})) = (\mathcal{A}_{(p)} \cdot t_{\mathcal{E}}(\sigma-1)^{r-n}\theta^r_{K/k,S}(0), 0). \quad (13)$$

But

$$\mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} F^{\cdot}) = \mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \mathcal{E}) \otimes_{\mathcal{P}(\mathcal{A}_{(p)})} \mathrm{Det}_{\mathcal{A}_{(p)}}(\mathcal{A}_{(p)} \otimes_{\mathbb{Z}[G]} \mathcal{F})^{-1}. \quad (14)$$

Also, if we pick a basis $\alpha_1, \ldots, \alpha_r$ for the free module $\mathcal{E}$, then by what we have proved so far of Theorem 3.1iii), there exists an element u of $\mathbb{Z}_{(p)}[G]^{\times}$ such that $e \wedge_{i=1}^r \alpha_i = eut_{\mathcal{E}}(\varepsilon) = ut_{\mathcal{E}}(\sigma-1)^{r-n}e(\eta_{K,S})$. Therefore

$$\begin{aligned} e(\wedge^r \mathrm{Reg}_{K,S})(\wedge_{i=1}^r \alpha_i) &= ut_{\mathcal{E}}(\sigma-1)^{r-n}e(\wedge^r \mathrm{Reg}_{K,S})(\eta_{K,S}) \\ &= ut_{\mathcal{E}}(\sigma-1)^{r-n}e\theta^r_{K/k,S}(0) \wedge_{i=1}^r (w_i - w_0), \end{aligned}$$

the last equality following from the definition (4) of $\eta_{K,S}$. Hence the image of (14) under $\mathbb{R}[G]e \otimes_{\mathbb{R}[G]} \vartheta_{\Psi_S^{\cdot},\mathbb{R}}$ is equal to $(\mathcal{A}_{(p)} \cdot ut_{\mathcal{E}}(\sigma-1)^{r-n}\theta^r_{K/k,S}(0), 0)$, and this implies the required equality (13) because $\mathcal{A}_{(p)} \cdot u = \mathcal{A}_{(p)}$. □

Lemma 3.6. $\mathrm{Fitt}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathbb{Z}[G]} (X'_{K,S}/\mathcal{F})) = (\sigma-1)^{n-r}\mathrm{Fitt}_{\mathbb{Z}[G]}(A_{K,S})\mathcal{A}$.

Proof. We have the exact sequence $0 \to A_{K,S} \to X'_{K,S}/\mathcal{F} \to \sum_{i=r+1}^n \mathbb{Z}[G] \cdot (w_i - w_0) \to 0$ where the last nonzero term is isomorphic as a $\mathbb{Z}[G]$-module to $\mathbb{Z}^{n-r}$. Applying $\mathcal{A} \otimes_{\mathbb{Z}[G]} -$ therefore gives an exact sequence

$$0 \longrightarrow \mathcal{A} \otimes_{\mathbb{Z}[G]} A_{K,S} \longrightarrow \mathcal{A} \otimes_{\mathbb{Z}[G]} (X'_{K,S}/\mathcal{F}) \longrightarrow (\mathcal{A} \otimes_{\mathbb{Z}[G]} \mathbb{Z})^{n-r} \longrightarrow 0. \quad (15)$$

(Here we use the fact that $\mathrm{Tor}^1_{\mathbb{Z}[G]}(\mathcal{A},\mathbb{Z})=0$, as can be easily verified directly.) In addition, the exact sequence $0\longrightarrow\mathcal{A}\xrightarrow{1\mapsto\sigma-1}\mathcal{A}\longrightarrow\mathcal{A}\otimes_{\mathbb{Z}[G]}\mathbb{Z}\longrightarrow 0$ shows that [13, Lem. 3] applies to (15) and therefore implies that

$$\mathrm{Fitt}_{\mathcal{A}}(\mathcal{A}\otimes_{\mathbb{Z}[G]}(X'_{K,S}/\mathcal{F}))=\mathrm{Fitt}_{\mathcal{A}}(\mathcal{A}\otimes_{\mathbb{Z}[G]}A_{K,S})\mathrm{Fitt}_{\mathcal{A}}((\mathcal{A}\otimes_{\mathbb{Z}[G]}\mathbb{Z})^{n-r}).$$

But $\mathrm{Fitt}_{\mathcal{A}}(\mathcal{A}\otimes_{\mathbb{Z}[G]}A_{K,S})=\mathrm{Fitt}_{\mathbb{Z}[G]}(A_{K,S})\mathcal{A}$ and $\mathrm{Fitt}_{\mathcal{A}}((\mathcal{A}\otimes_{\mathbb{Z}[G]}\mathbb{Z})^{n-r})=\mathcal{A}(\sigma-1)^{n-r}$, and this implies the claimed result. □

By substituting the last two results into (11), and then cancelling the element $(\sigma-1)^{n-r}$ (which is a unit in $\mathbb{Q}[G]e$) from the resulting equalities, we obtain

$$t_{\mathcal{E}}^{-1}\mathrm{Fitt}_{\mathcal{A}_{(p)}}((\mathcal{A}_{(p)}\otimes(\mathcal{O}^\times_{K,S}/\mathcal{E}))^G)=\mathrm{Fitt}_{\mathbb{Z}[G]}(A_{K,S})\mathcal{A}_{(p)}.$$

Our proof of Theorem 3.1 will therefore be completed by the following lemma. □

Lemma 3.7. *Set $S_2:=S\setminus S_1$. Let I denote the group of fractional ideals supported above S_2, and $\langle\mathcal{E}\rangle$ the image of $\mathcal{E}$ under the natural map $\mathcal{O}^\times_{K,S}\to I$. Then $\langle\mathcal{E}\rangle$ is a free $\mathbb{Z}$-module of rank b with $b\le r$, and one has*

$$\begin{aligned}(\sigma-1)^b\cdot\mathrm{Fitt}_{\mathcal{A}}((\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E})^{\mathrm{T}=0})&\subseteq\mathrm{Fitt}_{\mathcal{A}}((\mathcal{A}\otimes(\mathcal{O}^\times_{K,S}/\mathcal{E}))^G)\\&\subseteq\mathrm{Fitt}_{\mathcal{A}}((\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E})^{\mathrm{T}=0}).\end{aligned}$$

Further, the image of the natural map $(\mathcal{A}\otimes(\mathcal{O}^\times_{K,S}/\mathcal{E}))^G\to(\mathcal{A}\otimes(I/\langle\mathcal{E}\rangle))^G$ is finite of order dividing ℓ^b, and the second, respectively first, displayed inclusion is an equality if the image of this map is trivial, respectively has order ℓ^b.

Proof. If M is any G-module, then by applying $-\otimes_{\mathbb{Z}}M$ to the augmentation sequence $0\longrightarrow\mathcal{A}\xrightarrow{1\mapsto\sigma-1}\mathbb{Z}[G]\longrightarrow\mathbb{Z}\longrightarrow 0$ and then taking fixed points under the diagonal G-action one obtains an exact sequence of G-modules of the form $0\longrightarrow(\mathcal{A}\otimes_{\mathbb{Z}}M)^G\longrightarrow M\xrightarrow{\mathrm{T}}M^G$ (where the G-action on the term $(\mathcal{A}\otimes_{\mathbb{Z}}M)^G$ is by multiplication on the first factor).

By applying this observation to each of the modules in the natural exact sequence

$$0\to\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E}\to\mathcal{O}^\times_{K,S}/\mathcal{E}\to I/\langle\mathcal{E}\rangle$$

we obtain an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & (\mathcal{A}\otimes(\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E}))^G \arrow[r] \arrow[d] & (\mathcal{A}\otimes(\mathcal{O}^\times_{K,S}/\mathcal{E}))^G \arrow[r] \arrow[d] & (\mathcal{A}\otimes(I/\langle\mathcal{E}\rangle))^G \arrow[d] \\
0 \arrow[r] & \mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E} \arrow[r] \arrow[d, "\mathrm{T}"] & \mathcal{O}^\times_{K,S}/\mathcal{E} \arrow[r] \arrow[d, "\mathrm{T}"] & I/\langle\mathcal{E}\rangle \arrow[d, "\mathrm{T}"] \\
0 \arrow[r] & (\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E})^G \arrow[r] & (\mathcal{O}^\times_{K,S}/\mathcal{E})^G \arrow[r] & I/\langle\mathcal{E}\rangle.
\end{tikzcd}$$

Now G acts trivially on the free $\mathbb{Z}$-module I so that $\ker(I/\langle\mathcal{E}\rangle\xrightarrow{\mathrm{T}}I/\langle\mathcal{E}\rangle)$ is isomorphic to a submodule of $\mathrm{cok}(\langle\mathcal{E}\rangle\xrightarrow{\cdot\ell}\langle\mathcal{E}\rangle)\cong(\mathbb{Z}/\ell\mathbb{Z})^b$ where b is the rank of the free $\mathbb{Z}$-module $\langle\mathcal{E}\rangle$. (We also note that, since $\langle\mathcal{E}\rangle$ is a quotient of $H_0(G,\mathcal{E}) \cong \mathbb{Z}^r$ one has $b \leq r$, as claimed.) The above diagram thus gives rise to an exact sequence of $\mathcal{A}$-modules of the form

$$0 \longrightarrow (\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E})^{\mathrm{T}=0} \to (\mathcal{A}\otimes(\mathcal{O}^\times_{K,S}/\mathcal{E}))^G \to (\mathbb{Z}/\ell\mathbb{Z})^b. \tag{16}$$

But $\mathbb{Z}/\ell\mathbb{Z} \cong \mathcal{A}/\mathcal{A}(\sigma-1)$ so that the exact sequence

$$0 \to \mathcal{A} \xrightarrow{1\mapsto\sigma-1} \mathcal{A} \to \mathbb{Z}/\ell\mathbb{Z} \to 0 \tag{17}$$

implies $\mathrm{Fitt}_{\mathcal{A}}((\mathbb{Z}/\ell\mathbb{Z})^b) = (\mathrm{Fitt}_{\mathcal{A}}(\mathbb{Z}/\ell\mathbb{Z}))^b = \mathcal{A}(\sigma-1)^b$ and, since G is cyclic, $\mathrm{Fitt}_{\mathcal{A}}(N) \subseteq \mathrm{Fitt}_{\mathcal{A}}(M)$ for any finite $\mathcal{A}$-modules $M \subseteq N$. Hence, the exact sequence (16) combines with well-known properties of Fitting ideals with respect to short exact sequences (cf. [21, Chap. XIX, Prop. 2.7]) to imply

$$(\sigma-1)^b\cdot\mathrm{Fitt}_{\mathcal{A}}((\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E})^{\mathrm{T}=0}) \subseteq \mathrm{Fitt}_{\mathcal{A}}((\mathcal{A}\otimes(\mathcal{O}^\times_{K,S}/\mathcal{E}))^G) \subseteq \mathrm{Fitt}_{\mathcal{A}}((\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E})^{\mathrm{T}=0}),$$

as claimed.

Finally, we observe that the image B of the map $(\mathcal{A}\otimes(\mathcal{O}^\times_{K,S}/\mathcal{E}))^G \to (\mathcal{A}\otimes(I/\langle\mathcal{E}\rangle))^G$ is isomorphic to a submodule of $(\mathbb{Z}/\ell\mathbb{Z})^b$ and hence has order dividing ℓ^b. Now if B is trivial, then (16) reduces to give an isomorphism between $(\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E})^{\mathrm{T}=0}$ and $(\mathcal{A}\otimes(\mathcal{O}^\times_{K,S}/\mathcal{E}))^G$ so that the second displayed inclusion is an equality. On the other hand, if B has order ℓ^b, then (16) is a short exact sequence and by applying [13, Lem. 3] to this sequence (which is permissible by virtue of (17)) we find that the first displayed inclusion is an equality. $\square$

We provide a condition under which the image of the map mentioned in the statement of Lemma 3.7 is trivial.

Lemma 3.8. *Let P denote the image of $\mathcal{O}^\times_{K,S}$ in I. For any group A, let $A^{\times\ell}$ denote the set of ℓ-th powers in A. If $P^{\times\ell}\cap\langle\mathcal{E}\rangle = \langle\mathcal{E}\rangle^{\times\ell}$, then*

$$\mathrm{Fitt}_{\mathcal{A}}((\mathcal{A}\otimes(\mathcal{O}^\times_{K,S}/\mathcal{E}))^G) = \mathrm{Fitt}_{\mathcal{A}}((\mathcal{O}^\times_{K,S_1}/\mathcal{O}^\times_{K,S_1}\cap\mathcal{E})^{\mathrm{T}=0}).$$

Proof. Applying the snake lemma to the exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \langle\mathcal{E}\rangle & \longrightarrow & I & \longrightarrow & I/\langle\mathcal{E}\rangle & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle\times\ell} & & \big\downarrow{\scriptstyle\times\ell} & & \big\downarrow{\scriptstyle\times\ell} & & \\
0 & \longrightarrow & \langle\mathcal{E}\rangle & \longrightarrow & I & \longrightarrow & I/\langle\mathcal{E}\rangle & \longrightarrow & 0
\end{array}$$

yields an inclusion $(\mathcal{A}\otimes(I/\langle\mathcal{E}\rangle))^G \to \langle\mathcal{E}\rangle/\langle\mathcal{E}\rangle^{\times\ell}$, under which the image of the map described in the statement of Lemma 3.7 is contained in $(P^{\times\ell}\cap\langle\mathcal{E}\rangle)/\langle\mathcal{E}\rangle^{\times\ell}$. $\square$

4 Further examples

We now show that there is no direct analogue of Corollary 3.2 in the case that $\mathrm{Gal}(K/\mathbb{Q})$ is not cyclic.

Proposition 4.1. *There exists an infinite family of non-cyclic extensions $K/\mathbb{Q}$ in which every ramified prime has decomposition subgroup equal to $G := \mathrm{Gal}(K/\mathbb{Q})$, and yet the conductor-level Sinnott unit of K cannot be expressed in the form $\prod_i \lambda_i(u_i)$ with each u_i an element of $\mathcal{O}^\times_{K,S}$ and each λ_i an element of the augmentation ideal I_G of $\mathbb{Z}[G]$, where S is the set of places of $\mathbb{Q}$ which are either archimedean or ramified in K.*

Proof. Fix an odd prime p and let p_1, p_2 be distinct rational primes congruent to 1 modulo p. For $i = 1, 2$, assume that p_i is not a p-th power modulo p_{3-i}, and let K_i be an extension of $\mathbb{Q}$ of degree p and conductor $p_i^{a_i}$ for some integer $a_i > 0$. Let K be the compositum K_1K_2, so $K/\mathbb{Q}$ has Galois group $G \cong (\mathbb{Z}/p\mathbb{Z})^2$, each p_i has decomposition subgroup equal to G, and we may take $S = \{\infty, p_1, p_2\}$. We recall that such fields have been studied in detail by Fröhlich and, in particular, that it was proved in [15] that the class number of K is coprime to p.

For a $\mathbb{Z}[G]$-module A we let $\overline{A}$ denote the quotient of A by its $\mathbb{Z}$-torsion subgroup. We also let $A^{\mathrm{T}=0}$, resp. A^{I_G}, denote the kernel of the endomorphism of A that is induced by the action of the element $\sum_{g\in G} g \in \mathbb{Z}[G]$, resp. the submodule of $A^{\mathrm{T}=0}$ that is generated by all elements of the form $i(a)$ with $a \in A$ and $i \in I_G$.

Lemma 4.2. $\overline{\mathcal{O}_{K,S}^\times}^{\mathrm{T}=0} / \overline{\mathcal{O}_{K,S}^\times}^{I_G} = H^{-1}(G, \overline{\mathcal{O}_{K,S}^\times}) \neq 0.$

Proof. The first equality follows directly from the definition of $H^{-1}(G, \overline{\mathcal{O}_{K,S}^\times})$, and so we need only show that $H^{-1}(G, \overline{\mathcal{O}_{K,S}^\times}) \neq 0$.

First note that the torsion subgroup of $\mathcal{O}_{K,S}^\times$ is $\{\pm 1\}$, which is cohomologically trivial, and hence that $H^{-1}(G, \overline{\mathcal{O}_{K,S}^\times}) \cong H^{-1}(G, \mathcal{O}_{K,S}^\times)$. Now applying the double coboundary isomorphism induced by the exact sequence (1) (which exists by our definition of S) shows $H^{-1}(G, \mathcal{O}_{K,S}^\times) \cong H^{-3}(G, X'_{K,S})$. In addition, $A_{K,S}$ is a cohomologically trivial G-module because $p \nmid \# A_{K,S}$, and so (2) implies $H^{-3}(G, X'_{K,S}) \cong H^{-3}(G, X_{K,S})$. But $X_{K,S} \cong \mathbb{Z}[G] \oplus \mathbb{Z}$ by the definition of S, and so one has $H^{-1}(G, \overline{\mathcal{O}_{K,S}^\times}) \cong H^{-3}(G, \mathbb{Z})$. On the other hand, this last group is $H_2(G, \mathbb{Z})$ and is therefore isomorphic to $\wedge^2_{\mathbb{Z}} G$ [2, Chap. V, Th. 6.4]. In our case, this gives $H_2(G, \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ which is not trivial. $\square$

We return to the proof of Proposition 4.1. Let $\eta = N_{\mathbb{Q}(\zeta_{p_1^{a_1}}\zeta_{p_2^{a_2}})/K}(1 - \zeta_{p_1^{a_1}}\zeta_{p_2^{a_2}})$ be the conductor-level Sinnott unit of K. Then, since p is odd, the argument of [3, pf. of Lem. 6.2] implies that the image of η in $\overline{\mathcal{O}_{K,S}^\times} \otimes \mathbb{Z}_p$ is a $\mathbb{Z}_p[G]$-generator of $\overline{\mathcal{O}_{K,S}^\times}^{\mathrm{T}=0} \otimes \mathbb{Z}_p$. It follows that the image of η in $\overline{\mathcal{O}_{K,S}^\times}$ generates a nontrivial subgroup of $H^{-1}(G, \overline{\mathcal{O}_{K,S}^\times})$ and hence that $\eta \notin (\mathcal{O}_{K,S}^\times)^{I_G}$, as claimed. $\square$

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