A conjectural class number formula for derivatives of abelian L-functions

Anthony Hayward
King's College, London.

Principal References

1. Karl Rubin, *A Stark conjecture "over $\mathbb{Z}$" for abelian L-functions with multiple zeros*, 1996

2. Benedict H. Gross, *On the values of abelian L-functions at $s = 0$*, 1988

3. Henri Darmon, *Thaine's method of circular units and a conjecture of Gross*, 1995

1 - Basic Setup - note that this is identical to that of Rubin's conjecture.

L/K finite abelian extension of global fields with group G.

S, T finite sets of places of K, r a nonnegative integer. S_L etc. places in L over these.

Hypothesis

1. $S \supseteq$ infinite places
2. $S \supseteq$ ramified places
3. r places in S split completely in L/K
4. $\#S \geq r+1$
5. $T \cap S = \emptyset$ and $U_{L,S,T}$ torsion free.

$U_{L,S,T} = \{S\text{-units} \equiv 1 \pmod{w}\ \forall w \in T_L\}$

Artin L-function

$$\begin{aligned} L_{L/K,S}(s,\chi) &= \prod_{v \notin S_K} (1 - N_v^{-s}\chi(\mathrm{Frob}_v))^{-1} \\ &= \sum_{\substack{I \lhd K \\ (I,S)=1}} \chi(I) N I^{-s} \end{aligned}$$

For $\chi \in \widehat{G}$, write $\quad e_\chi = \frac{1}{\#G}\sum_{g \in G} \chi(g) g^{-1}$

$$\Theta_{L/K,S,T}(s) := \delta_T(s) \sum_{\chi \in \widehat{G}} L_{L/K,S}(s,\chi^{-1}) e_\chi$$

where $\delta_T(s) = \prod_{p \in T_K}(1 - N_p^{1-s}\mathrm{Frob}_p^{-1})$. So

$$\chi(\Theta_{L/K,S,T}(s)) = L_{L/K,S,T}(s,\chi^{-1})$$

We study the rth term in the Taylor expansion at 0:

$$\Theta^r_{L/K,S,T}(0) := \lim_{s \to 0} s^{-r}\Theta_{L/K,S,T}(s).$$

Idempotent $e_r := \sum_{r(\chi)=r} e_\chi$.

Dirichlet regulator map

$$\begin{array}{rccc} \lambda_{S_L}: & U_{L,S_L} & \longrightarrow & \mathbb{R}X_{S_L} \\ & u & \longmapsto & -\sum_{w\in S_L} \ln ||u||_w \cdot w \end{array}$$

$A_{K,S,T}$ = (S,T) ray-class group of K, size

$$h_{K,S,T} = h_{K,S}\frac{\prod_{p\in T}(N_p - 1)}{(U_{K,S} : U_{K,S,T})}$$

$\lambda : \mathbb{R}U_{L,S,T} \longrightarrow \mathbb{R}X_{S_L}$ is an $\mathbb{R}[G]$ isom.

Analytic class number formula

$$\begin{aligned} \zeta_{K,S,T} &= \Theta_{K/K,S,T}(s) \\ &\equiv \pm h_{K,S,T} \det(\lambda) s^{\#S-1} \pmod{s^{\#S}} \end{aligned}$$

To generalise, relate $\Theta^r_{L/K,S,T}(0)$ to units. Choosing $W = (w_1, \ldots, w_r)$ above r splitting places leads to a $\mathbb{C}[G]$-isomorphism

$$W^* \circ \lambda_{S_L} : (\mathbb{C} \wedge^r_{\mathbb{Z}[G]} U_{L,S,T})e_r \longrightarrow \mathbb{C}[G]e_r$$

where $W^*(w_1 \wedge \cdots \wedge w_r) = 1 \in \mathbb{C}[G]$. So there is a unique element

$$\varepsilon = \varepsilon_{L/K,S,T,r,W} \in (\mathbb{C} \wedge^r_{\mathbb{Z}[G]} U_{L,S,T})e_r$$

mapping to $\Theta^r_{L/K,S,T}(0)$.

What are its integrality properties?

Take homomorphisms

$$\Phi = \phi_1 \wedge \cdots \wedge \phi_r \in \wedge^r_{\mathbb{Z}[G]} \mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S,T}, \mathbb{Z}[G])$$

Look at element $\Phi(\varepsilon) \in \mathbb{C}[G]$, where $\Phi(u_1 \wedge \cdots \wedge u_r) = \det(\phi_j(u_i))_{i,j}$.

Conjecture (Rubin) $\Phi(\varepsilon) \in \mathbb{Z}[G]$.

This conjecture, for $r = 1$ and all T, is equivalent to the abelian Stark conjecture. Note $r = 1 \implies \epsilon \in U_{L,S,T}$. We have $\epsilon = \lambda^{\delta_T}$ where $\lambda = u^{1/W_L}$ is a root of the Stark unit u.

We will demand more of $\Phi(\varepsilon)$ . . .

Taster: conjecture $\Phi(\varepsilon) \in I_G^d$, where $d = \#S - r - 1$ and $I_G =$ augmentation ideal.

Furthermore, we'll conjecture a value for $\Phi(\varepsilon)$ in I_G^d / I_G^{d+1}.

2 - Formulation of Congruence

If $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$, define $\phi^1(m)$ to be the coefficient of 1_G in $\phi(m)$, which gives $\phi^1 \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. We have

$$\phi(m) = \sum_{g \in G} \phi^1(g^{-1}m)g,$$

and $\phi \mapsto \phi^1$ is a group isomorphism.

If $0 < n \in \mathbb{Z}$ then such a ϕ^1 induces

$$\tilde{\phi} : \wedge_{\mathbb{Z}}^n M \longrightarrow \wedge_{\mathbb{Z}}^{n-1} M$$

by $\tilde{\phi}(m_1 \wedge \cdots \wedge m_n) = \sum_{i=1}^n (-1)^{i+1} \phi^1(m_i) \mathbf{m}'_i$, where $\mathbf{m}'_i$ is the wedge of all the m_j except m_i. We can iterate this so that $\Phi = \phi_1 \wedge \cdots \wedge \phi_r$ induces a map $\tilde{\Phi}$ which knocks down exterior powers by r.

For v a place of K, local reciprocity map

$$f_v = (-, L_w/K_v) : K^\times \longrightarrow G$$

Choose ordering $v_1, \ldots, v_{d+1}$ for S, excluding the r splitting places from W. Define, following Gross, a homomorphism Reg_G from $\wedge^d_{\mathbb{Z}} U_{K,S,T}$ to $\mathbb{Z}[G]/I_G^{d+1}$ by

$$u_1 \wedge \cdots \wedge u_d \longmapsto \det(f_{v_j}(u_i) - 1)_{1 \le i,j \le d}.$$

Depends on ordering of places up to sign.

$u_1, \ldots, u_{r+d}$ $\mathbb{Z}$-basis for $U_{K,S,T}$. Set

$$\mathrm{Reg}_G^\Phi = \mathrm{Reg}_G(\tilde{\Phi}(u_1 \wedge \cdots \wedge u_{r+d}))$$

N.B. independent of basis & places up to sign.

$$\mathrm{Reg}_G^{\Phi} = \begin{vmatrix} \phi_1^1(u_1) & \cdots & \phi_1^1(u_{r+d}) \\ \vdots & \ddots & \vdots \\ \phi_r^1(u_1) & \cdots & \phi_r^1(u_{r+d}) \\ f_{v_1}(u_1)-1 & \cdots & f_{v_1}(u_{r+d})-1 \\ \vdots & \ddots & \vdots \\ f_{v_d}(u_1)-1 & \cdots & f_{v_d}(u_{r+d})-1 \end{vmatrix}$$

$$= \sum_{\substack{\sigma:\sigma(1)<\ldots<\sigma(r) \\ \sigma(r+1)<\ldots<\sigma(r+d)}} \mathrm{sign}(\sigma) \overbrace{\begin{vmatrix} \phi_1^1(u_{\sigma(1)}) & \cdots & \phi_1^1(u_{\sigma(r)}) \\ \vdots & \ddots & \vdots \\ \phi_r^1(u_{\sigma(1)}) & \cdots & \phi_r^1(u_{\sigma(r)}) \end{vmatrix}}^{r\times r \text{ det. } /\mathbb{Z}} \times$$

$$\underbrace{\begin{vmatrix} f_{v_1}(u_{\sigma(r+1)})-1 & \cdots & f_{v_d}(u_{\sigma(r+d)})-1 \\ \vdots & \ddots & \vdots \\ f_{v_d}(u_{\sigma(r+1)})-1 & \cdots & f_{v_d}(u_{\sigma(r+d)})-1 \end{vmatrix}}_{d\times d \text{ det. } /\mathbb{Z}[G]}$$

where $u_1, \ldots, u_{r+d}$ is a $\mathbb{Z}$-basis for $U_{K,S,T}$

Conjecture (Burns) "$\mathrm{C}_{\equiv}(L/K,S,T,r)$"

$$\Phi(\varepsilon) \equiv \pm h_{K,S,T} \mathrm{Reg}_G^{\Phi} \pmod{I_G^{d+1}}$$

3 - Basic properties

1) $L/M/K$: $\mathsf{C}_{\equiv}(L/K, S, T, r) \Rightarrow \mathsf{C}_{\equiv}(M/K, S, T, r)$

2) $\mathsf{C}_{\equiv}(L/K, S, T, r) \Rightarrow \mathsf{C}_{\equiv}(L/K, S, T \cup \{v\}, r)$

3) $\mathsf{C}_{\equiv}(L/K, S \cup \{v\}, T, r+1) \Rightarrow \mathsf{C}_{\equiv}(L/K, S, T, r)$
if v splits completely in L/K.

4) $\mathsf{C}_{\equiv}(L/K, S, T, r) \Rightarrow \mathsf{C}_{\equiv}(L/K, S \cup \{v\}, T, r)$
for all $v \notin S \cup T$.

4 - Special cases

If $r=0$, $\varepsilon \in \mathbb{C} \wedge^0_{\mathbb{Z}[G]} U_{L,S,T} e_r$ is $\Theta_{L/K,S,T}(0) \in \mathbb{C}[G]e_r$. $\mathrm{C}_{\equiv}(L/K,S,T)$ says that modulo I_G^{d+1}, have

$$\Theta_{L/K,S,T}(0) \equiv \pm h_{K,S,T} \mathrm{Reg}_G(u_1 \wedge \cdots \wedge u_d),$$

u_i a $\mathbb{Z}$-basis for $U_{K,S,T}$. That is,

$$\Theta_{L/K,S,T}(0) \equiv \pm h_{K,S,T} \begin{vmatrix} f_{v_1}(u_1)-1 & \cdots & f_{v_1}(u_d)-1 \\ \vdots & \ddots & \vdots \\ f_{v_d}(u_1)-1 & \cdots & f_{v_d}(u_d)-1 \end{vmatrix}$$

(mod I_G^{d+1}), where $d = \#S-1$. This is $\pm$ Gross's Conjecture for this data.

If more than r places in S split completely, all the L-functions for nontrivial characters vanish to order $> r$.

$d > 0$: $\mathsf{C}_{\equiv}(L/K, S, T)$ says $0 \equiv 0$, since the zeta-function vanishes too.

$d = 0$: Only the zeta-function doesn't vanish. $\mathsf{C}_{\equiv}(L/K, S, T)$ is the homomorphic image of A.C.N.F. for K. Let u_i be a $\mathbb{Z}$-basis for $U_{K,S,T}$.
A.C.N.F. $\iff$ $\varepsilon = h_{K,S,T}/\#G^r u_1 \wedge \cdots \wedge u_r$
$\mathsf{C}_{\equiv}(L/K, S, T, r)$ $\iff$

$$\Phi(\varepsilon) \equiv \pm h_{K,S,T} \det(\phi_j^1(u_i))_{i,j}$$

5 - Some evidence

Theorem 1 (H.) $\mathrm{C}_{\equiv}(L/K, S, T, r)$ holds if L/K is quadratic.

Ingredients in the proof:

- A.C.N.F. gives formulae for ε (Rubin). Suffices for $d = 0$ case (equality in $\mathbb{Z}$).
- Can always add a split place to S and increase r.
- G cyclic $\Rightarrow$ we can add inert places to S too, for $d > 0$.
- So can increase S to kill $A_{L,S,T}$.
- Now RHS collapses to a single term. Proof of Gross's conjecture in quadratic case $\rightsquigarrow$ criterion for nonvanishing of this regulator.

Theorem 2 (H.) $2\mathrm{C}_{\equiv}(L/\mathbb{Q}, S, T, 1)$ holds with $W = \infty_L$ if L is real.

ie. $2\phi(\varepsilon) \equiv \pm 2h_{\mathbb{Q},S,T} \sum_i (-1)^{i+1} \phi^1(u_i)\mathrm{Reg}_G(\mathbf{u}'_i)$.

Ingredients in the proof:

• $L \hookrightarrow \mathbb{Q}(\zeta_m)$
• Formula for $L'_{L,S}(0, \chi) \rightsquigarrow$

$$2\varepsilon = N_{\mathbb{Q}(\zeta_m)/L}(1 - \zeta_m)^{\delta_T}$$

• Distribution relations for cyclotomic elements
• Careful induction on $\#\{p \mid m\}$
• Lower top field from $\mathbb{Q}(\zeta_m)$, add T conditions in at the end.

6 - Interpretation of Darmon's conjecture

N, S coprime integers. $K \subseteq \mathbb{Q}(\zeta_N)$ quadratic field. $\tilde{L} \subseteq \mathbb{Q}(\zeta_S)$ real.

Henri Darmon conjectured about a "circular unit" - explicit element $\alpha_S \in K(\zeta_S)$ made from cyclotomic elements. We can relate α_S to the base-change factor from $\Theta'_{\tilde{L}/\mathbb{Q},S,T}(0)$ to $\Theta^2_{\tilde{L}K/K,S,T}(0)$.

Conjecture

$$\sum_{\sigma \in \Gamma_S} \sigma^{-1}\alpha_S \otimes \sigma = \pm 2^* h_{K,S} R_S$$

in $U_{K(\zeta_S),S,T} \otimes I^n_\Gamma / I^{n+1}_\Gamma$, $n =$ number of primes dividing S which split in $K/\mathbb{Q}$. R_S is a sum of terms of the form $u_i \otimes \mathrm{Reg}$ made out of minus-units.

$$\sum_{\sigma\in\Gamma_S} \sigma^{-1}\alpha_S^{\delta_T} \otimes \sigma = \pm 2^* \frac{h_{K,S,T}}{h_{\mathbb{Q},S,T}} R_{S,T}.$$

Hit this conjecture with $\phi^1 \otimes \mathrm{id}$ to get a congruence statement in $\mathbb{Z}[G]$.

A T-modified Darmon conjecture implies a "base change" statement for $\mathsf{C}_{\equiv}$.

Theorem 3 (H.) Assume $\mathsf{C}_{\equiv}(\tilde{L}/\mathbb{Q}, S, T, 1)$ and Darmon's conjecture hold. Then can conclude $(4 \cdot 2^n)$ $\mathsf{C}_{\equiv}(\tilde{L}K/K, S, T, 2)$; ie.

$$(4\cdot 2^n)(\phi_1 \wedge \phi_2)(\varepsilon) \equiv (4\cdot 2^n)\Bigg(\pm h_{K,S,T} \sum_{i<j} (-1)^{i+1}(-1)^{j+1} \begin{vmatrix} \phi_1^1(u_i) & \phi_2^1(u_i) \\ \phi_1^1(u_j) & \phi_2^1(u_j) \end{vmatrix} \det(f_{v_k}(u_l) - 1)_{l\neq i,j} \Bigg) \pmod{I_G^{d+1}}$$

7 - Relations between abelian Stark-type conjectures