Congruences satisfied by Stark units

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I. What are Stark units?

L

$\Big|_G$ abelian extension of global fields. S, T disjoint

K

finite sets of places of K such that $S \neq \emptyset$ contains all infinite and ramified primes and $U_{L,S,T}$ torsion-free.

$$U_{L,S,T} = \{x \in L^\times : w(x) = 0 \text{ for } w|_K \notin S, \\ w(x-1) > 0 \text{ for } w|_K \in T\}.$$

(S,T)-modified Artin (Dirichlet) L-function

$$\Theta_{S,T}(s) = \sum_{\chi \in \hat{G}} L_{L/K,S,T}(s, \chi^{-1}) e_\chi \in \mathbb{C}[G] \qquad (s \in \mathbb{C})$$

$$e_\chi = \frac{1}{\#G} \sum_{g \in G} \chi(g) g^{-1}$$

$$L_{L/K,S,T}(s,\chi) = \prod_{\mathfrak{p} \in T} (1 - N\mathfrak{p}^{1-s} \chi(\sigma_\mathfrak{p})) \prod_{\mathfrak{p} \notin S} (1 - N\mathfrak{p}^{-s} \chi(\sigma_\mathfrak{p}))^{-1}$$

"Refined abelian Stark conjectures" are special value conjectures for $\Theta_{S,T}$ at 0.

r splitting places in S force vanishing to order r in all L-functions.

$$\Theta_{S,T}^{(r)}(0) = \lim_{s \to 0} s^{-r} \Theta_{S,T}(s)$$

r-th term in Taylor series.

• **Stark** ($r = 1$): $\boxed{\Theta'_S(0) \leftrightarrow R(\eta)}$

$\eta \in U_{L,S}$ with special properties, R Dirichlet logarithmic regulator over splitting place.

• **Rubin[-Stark]**: $\boxed{\Theta^{(r)}_{S,T}(0) = R(\eta)}$

$\eta \in \mathbb{C} \otimes_{\mathbb{Z}} \wedge^r_{\mathbb{Z}[G]} U_{L,S,T}$ such that

$\Phi(\eta) \in \mathbb{Z}[G]$ for all $\Phi \in \wedge^r_{\mathbb{Z}[G]} \mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S,T}, \mathbb{Z}[G])$.

$R =$ Dirichlet logarithmic regulator for r splitting places,

$R(u_1 \wedge \cdots \wedge u_r) = \det(-\sum_{\sigma \in G} \ln \mid u_i^{\sigma^{-1}} \mid_w \sigma)_{i,w}$.

• **Gross** ($r = 0$): $\boxed{\Theta_{S,T}(0) \equiv h_{K,S,T} \mathrm{Reg}_{K,S,T} \pmod{I_G^{\#S}}}$

$h_{K,S,T} =$ a class number, $\mathrm{Reg}_{K,S,T} \in \mathbb{Z}[G]$ algebraic regulator $= \det((u, L_w/K_v) - 1)$, $u \in U_{K,S,T}$, $v \in S$.

• **ETNC** $\boxed{\Theta^*_S(0)\mathbb{Z}[G] = \vartheta_R(\mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet))}$

$\Theta^*_S =$ characterwise leading term

$\vartheta_R =$ trivialisation by logarithmic regulator

$C^\bullet =$ an étale cohomology complex

Kato, Kodai Math. J. (1993). Generalised by Burns and Flach.

By interpreting ETNC as existence of certain systems of units, Burns was led to combine Rubin's and Gross's conjectures.

S, T, r as above, so $\Theta^{(r)}_{S,T}(0) = R(\eta)$. Rubin:

$$\Phi(\eta) \in \mathbb{Z}[G] \text{ for all } \Phi \in \wedge^r_{\mathbb{Z}[G]} \mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S,T}, \mathbb{Z}[G])$$

Postulate a congruence for $\Phi(\eta)$ in the spirit of Gross's conjecture. $\Phi = \phi_1 \wedge \cdots \wedge \phi_r$, and

$\phi_i \in \mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S,T}, \mathbb{Z}[G]) \cong \mathrm{Hom}_{\mathbb{Z}}(U_{L,S,T}, \mathbb{Z}) \ni \phi^1_i$

Let $f_v = (-, L_w/K_v)$ local reciprocity map, $\#S = r + d + 1$.

$$\mathrm{Reg}^{\Phi} = \begin{vmatrix} \phi^1_1(u_1) & \cdots & \phi^1_1(u_{r+d}) \\ \vdots & \ddots & \vdots \\ \phi^1_r(u_1) & \cdots & \phi^1_r(u_{r+d}) \\ f_{v_1}(u_1) - 1 & \cdots & f_{v_1}(u_{r+d}) - 1 \\ \vdots & \ddots & \vdots \\ f_{v_d}(u_1) - 1 & \cdots & f_{v_d}(u_{r+d}) - 1 \end{vmatrix}$$

where u_i are a $\mathbb{Z}$-basis for $U_{K,S,T}$.

There is a natural interpretation of Reg^{Φ} in terms of maps "$\wedge^{r+d} U_{K,S,T} \xrightarrow{\Phi^1} \wedge^d U_{K,S,T} \xrightarrow{f} I^d_G/I^{d+1}_G$"

$$\mathrm{Reg}^{\Phi} = \sum (-1)^{?} \underbrace{\det(\phi_i^1(u_j))}_{r\times r \text{ det } /\mathbb{Z}} \underbrace{\det(f_i(u_j)-1)}_{d\times d \text{ det } /\mathbb{Z}[G]}.$$

So it lies in I_G^d.

Conjecture (Burns) $\boxed{\Phi(\eta) \equiv h_{K,S,T}\mathrm{Reg}^{\Phi} \pmod{I_G^{d+1}}}$

- This is a strengthening of Rubin's conjecture.
- It recovers Gross's conjecture as the special case $r = 0$.
- It is actually a consequence of ETNC (very recent work of Burns). The "E" for "equivariant" is important here. Difficult to recognise I_G^d working one character at a time. ETNC is known for absolutely abelian fields and an analogous conjecture is known for function fields with $([L:K], \mathrm{char}(K)) = 1$.
- A natural refinement exists, introducing an intermediate field E, which compares $\eta_{E/K}$ to $\eta_{L/K}$. Conjecture above is then the special case $E = K$. This is also a consequence of ETNC. It generalises Gross's "refined abelian p-adic Stark conjecture".

The conjecture in quadratic extensions

If $[L : K] = 2$ we can express $L(s, \chi)$ as ratio of zeta-functions, and then the analytic class number formula gives η explicitly.

Can use genus theory to verify Burns's conjecture when $h_{L,S,T} = 1$. The conjecture allows us to increase S to kill the class group - we also use $I_G/I_G^2 \xrightarrow[\sim]{\times(\sigma-1)^{n-1}} I_G^n/I_G^{n+1}$.

- Get new neater proof of Gross for $[L : K] = 2$.
- $I_G^n = 2^{n-1} I_G$ explains high power of 2 factor for η.

Comparison to analytic class number formula

Suppose all places in S split completely, $r = \#S - 1$.

ACNF $\iff$ $\boxed{\eta = \frac{h_{K,S,T}}{\#G^r} u_1 \wedge \cdots \wedge u_r}$, where u_i are a $\mathbb{Z}$-basis for $U_{K,S,T}$.

$\Rightarrow \Phi(\eta) = \frac{h_{K,S,T}}{\#G^r} N_G^r \det(\phi_i^1(u_j)) = \frac{h_{K,S,T}}{\#G} N_G \det(\phi_i^1(u_j))$

We have $\#G \mid h_{K,S,T}$ because L/K is unramified.

Burns: $\Phi(\eta) \equiv h_{K,S,T} \det(\phi_i^1(u_j)) \pmod{I_G}$. Image of ACNF under (arbitrary) Φ.

Relation to certain "minus-unit" conjectures

If F/K is a disjoint quadratic extension, can form minus-units with respect to $\mathrm{Gal}(F/K)$.

1) Darmon's conjecture: for $K = \mathbb{Q}$, F real, L maximal real subfield of cyclotomic field, this is a "first order analogue" of Gross's conjecture for circular minus-units.

2) Gross's conjecture on L-functions of tori.

These can be interpreted as "base-change" type statements for Burns's conjecture

$$\begin{array}{rccc} & L/K & \rightsquigarrow & LF/F \\ \text{Darmon:} & r = 1 & \rightsquigarrow & r = 2 \\ \text{Gross:} & r & \rightsquigarrow & r \end{array}$$

II. Explicit Stark units

Now focus on case $r = 1$. Let v denote the splitting place in S. Let $\#S = n + 1$. The conjecture above says there exists

$$\eta \in U_{L,S,T}$$

such that, for a certain $w \mid v$,

$$\Theta'_{L/K,S,T}(0) = -\sum_{\sigma \in G} \ln \mid \eta^{\sigma} \mid_w \sigma \quad (= R(\eta))$$

and, for all $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S,T}, \mathbb{Z}[G])$,

$$\phi(\eta) \equiv \sum_{i=1}^{n} (-1)^{i+1} \phi^1(u_i) \underbrace{\begin{vmatrix} f_{v_1}(u_1) - 1 & \cdots & f_{v_1}(u_n) - 1 \\ \vdots & \ddots & \vdots \\ f_{v_{n-1}}(u_1) - 1 & \cdots & f_{v_{n-1}}(u_n) - 1 \end{vmatrix}}_{\text{excluding } u_i} \pmod{I_G^n}$$

I consider the cases where every other place but $v =: \infty$ is finite. That is, either

1) $K = \mathbb{Q}$,

2) $K = \mathbb{Q}(\sqrt{-d})$, or

3) K is a global function field.

A = ring of ∞-integers of K, ie. (1) $\mathbb{Z}$, (2) $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$, or (3) ring of functions regular away from ∞.

Ideal $\mathfrak{m} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_n^{e_n}$. For $I \subseteq \{1, \ldots, n\}$, write $\mathfrak{m}_I = \prod_{i \in I} \mathfrak{p}_i^{e_i}$.
$H(\mathfrak{m})$ = ray class field modulo $\mathfrak{m}$ totally split at ∞, which is the field associated to the "cycle" $\mathfrak{m}$ by global class field theory.

$$\begin{aligned} \mathrm{Gal}(H(\mathfrak{m})/K) &= \mathcal{I}(\mathfrak{m})/\mathcal{P}(\mathfrak{m}) \\ &= \frac{\{I : (I, \mathfrak{m}) = 1\}}{\{(x) : x \equiv 1 \pmod{\mathfrak{m}}\}} \end{aligned}$$

$$0 \to [\mu]_{\mathfrak{m}} \to (A/\mathfrak{m})^{\times} \to \mathcal{I}(\mathfrak{m})/\mathcal{P}(\mathfrak{m}) \to \mathcal{I}/\mathcal{P} \to 0$$

is exact. Let $w(\mathfrak{m})$ be the number of roots of unity congruent to 1 (mod $\mathfrak{m}$).

Fix $T = \{\mathfrak{t}\}$ one prime coprime to $W_{H(\mathfrak{m})}\mathfrak{m}$.

There is a system of elements λ_I satisfying

- $\lambda_I \in U_{H(\mathfrak{m}_I),\mathrm{Supp}(\mathfrak{m}_I),T}$.
- Distribution relations:
  - if $i \notin I$, then $N_{H(\mathfrak{m}_{I\cup\{i\}})/H(\mathfrak{m}_I)}(\lambda_{I\cup\{i\}}^{w(I)/w(I\cup\{i\})}) = \lambda_I^{1-\sigma_{\mathfrak{p}_i}^{-1}}$
  - $N_{H(\mathfrak{m}_i)/H}(\lambda_i^{w/w(i)}) = \alpha_i \in \mathcal{O}_H$, where $\alpha_i\mathcal{O}_H = \mathfrak{p}_i^{N_\mathfrak{t}-1}\mathcal{O}_H$.
- L-function evaluation:

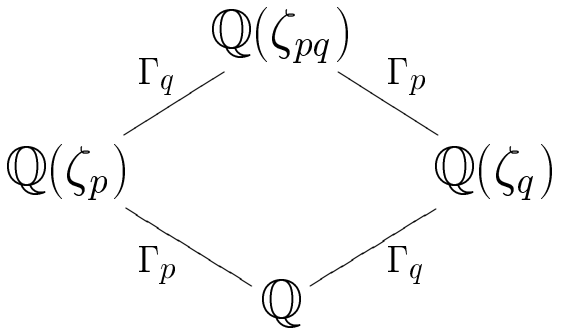
$$w(\mathfrak{m}_I)\Theta'_{H(\mathfrak{m}_I)/K,\mathfrak{m}_I,\mathfrak{t}}(0) = R(\lambda_I)$$

These units are as follows:
1) $K = \mathbb{Q}:\ \lambda_I = (1 - e^{2\pi i/m_I})^{N_\mathfrak{t}-\sigma_\mathfrak{t}}$.
2) Elliptic units, constructed from special values of Weierstrass σ-function.
3) If $\hat{\lambda}_I$ is an $\mathfrak{m}_I$-division point on a rank 1 sign-normalised Drinfeld module, then $\lambda_I = \hat{\lambda}_I^{N_\mathfrak{t}-\sigma_\mathfrak{t}}$.

We let $L = H(\mathfrak{m})^{\Delta}$, where Δ is such that

$$\mathrm{Gal}(H(\mathfrak{m})/K)/\Delta \underset{s}{\overset{}{\longrightarrow}} Cl \longrightarrow 0$$

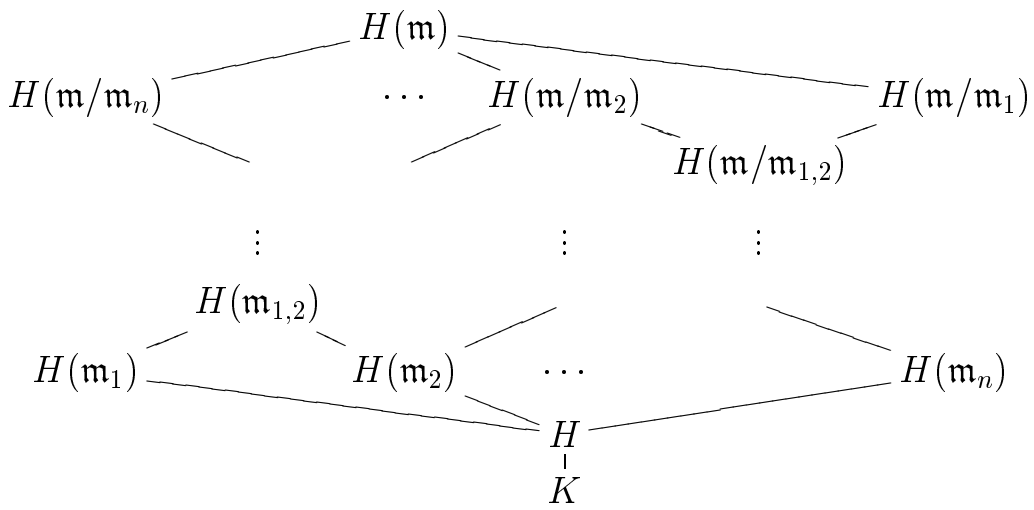
splits. Here s is a multiplicative section.
The archetype for the argument is the situation



Here, $(p, \mathbb{Q}(\zeta_{pq})_{\tilde{q}}/\mathbb{Q}_q) = ((p), \mathbb{Q}(\zeta_q)/\mathbb{Q})^{-1} = [p]_q^{-1}$.
In general, if $\Gamma_i := (A/\mathfrak{m}_i)^{\times}$, there is a surjection

$$\prod_i \Gamma_i \times s(Cl) \twoheadrightarrow G.$$

Roughly speaking, for $i \neq j$, lift σ_{p_i} on $H(\mathfrak{m}_j)^{\Delta}$ to an element $\sigma_{ij} \in G$.
$A^I =$ square matrix indexed by I of zero row sum with off-diagonal entries $\sigma_{ij} - 1$, and $A = A^{\{1,\ldots,n\}}$. If $J \subseteq I$, then A^I_J is its J-minor determinant.



Consider $N_{\Gamma_I}\lambda_I$. For $K = \mathbb{Q}$ this is always 1. In general it is a unit $\alpha_j^{\prod_{i\in I\text{-}\{j\}}(1-s(\mathfrak{p}_i)^{-1})}$ of H. Set

$$\Omega_I = \phi^1(N_{Cl}\alpha_j) \prod_{i\in I\text{-}\{j\}} (1 - s(\mathfrak{p}_i)^{-1})$$

(independent of choice of $j \in I$!)

Proposition

$$\phi(N_\Delta \lambda_\mathfrak{m}) \equiv \sum_{\substack{\emptyset\neq J \\ \subseteq\{1,\ldots,n\}}} (1-n)^{\#J-1}\Omega_J A_J \pmod{I_G^n}$$

Sketch proof - forget Δ. From expanding

$$\sum_{\gamma\in\Gamma_{\mathfrak{m}}}\sum_{\upsilon\in Cl}\phi^1(\gamma^{-1}s(\upsilon)^{-1}\lambda_{\mathfrak{m}})\prod_j([\gamma]_{\mathfrak{m}_j}-1)s(\upsilon),$$

get expression for $\phi(\lambda_{\mathfrak{m}})$ in terms of $\phi^{G_I}(N_{\Gamma_{I'}}\lambda_{\mathfrak{m}})$ for $I\subsetneq\{1,\ldots,n\}$ [where $G_I\approx\mathrm{Gal}(H(\mathfrak{m}_I)/K)$]. We can then apply distribution relations.

The proof proceeds essentially by induction on n. We find we need the relation, for each $J\subseteq\{1,\ldots,n\}$,

$$\sum_{I\subseteq\{1,\ldots,n\}}(-1)^{\#I'\cap J}(1-\#I)^{\#J-1}A^I_{I\cap J}\prod_{i\in I'\text{-}J}(\sum_{j\in I}a_{ij})=0$$

This proof is combinatorial, relying only on the fact that A is a matrix with zero row-sums. $\square$

After establishing the proposition, we relate the RHS to Reg^ϕ using global class field theory and the fact that rank $A^\times=0$, and conclude

Theorem

$$w\phi(\eta)\equiv wh_{K,S,T}\mathrm{Reg}^\phi\pmod{I_G^n}$$

The combinatorics

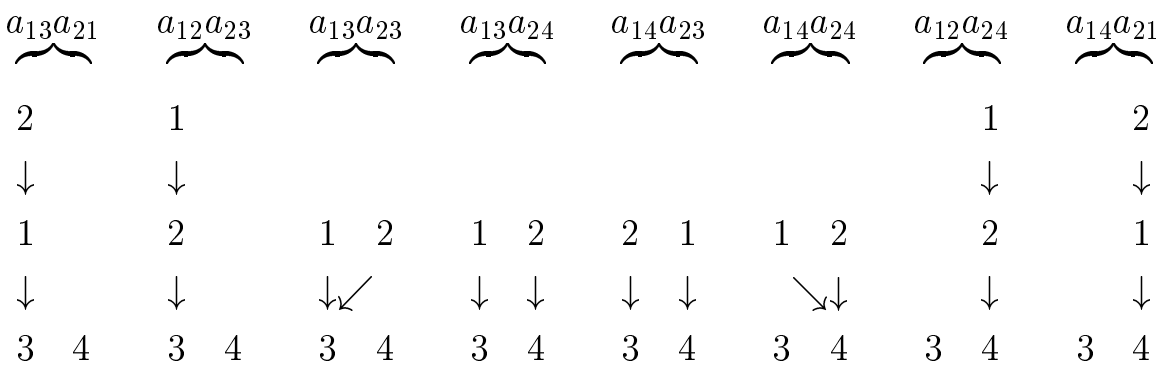
The combinatorial tricks for studying matrices of zero row sum have various applications when considering explicit units (cf. recent work of Greither and Kučera) and such matrices arise naturally when considering regulators. The main tool is

Matrix Tree Theorem (S. Chaiken and D. J. Kleitman) Let A^I be a matrix with zero row sums, indexed by I. Let $J \subseteq I$. Then we have $A^I_J = (-1)^{\#I-\#J} \sum_T A(T)$, where T runs through all arborescences on I with root J.

Example

$$A^{\{1,2,3,4\}}_{\{3,4\}} = \det \begin{pmatrix} -a_{12}-a_{13}-a_{14} & a_{12} \\ a_{21} & -a_{21}-a_{23}-a_{24} \end{pmatrix}$$

$= a_{13}a_{21}+a_{12}a_{23}+a_{13}a_{23}+a_{13}a_{24}+a_{14}a_{23}+a_{14}a_{24}+a_{12}a_{24}+a_{14}a_{21}$, corresponding to the arborescences



Let A be an $n \times n$ matrix with zero row sums. We wanted to know

$$\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{\#I' \cap J} (1 - \#I)^{\#J-1} A^I_{I \cap J} \prod_{i \in I'\text{-}J} \Big(\sum_{j \in I} a_{ij} \Big) = 0.$$

Let $F(X) = (1-X)^{\#J-1}$. Apply matrix tree theorem:

$$\text{LHS} = \sum_T A(T) \sum_{V \subseteq V^0(T)} (-1)^{n-\#J-\#V} F(n - \#V).$$

If $m = \#V^0(T)$, there are $\binom{m}{r}$ sets V of size r for $r = 0, \ldots, n$. So

$$\text{LHS} = \pm \sum_T A(T) \sum_{r=0}^{m} (-1)^r \binom{m}{r} F(n-r)$$

We use **Euler's formula**: If $0 \leq t < m$ then

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} r^t = 0.$$

Cor: If $\deg f < m$ then $\sum_{r=0}^{m} (-1)^r \binom{m}{r} f(r) = 0$.

Apply this with $f(X) = F(n - X)$. Note $\#J \leq m$ since for each "root" we have a "leaf".

The matrices are even richer in relations than we needed!