A class number formula for higher derivatives of abelian L-functions

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Abstract

Gross and Rubin have made conjectures about special values of equivariant L-functions associated to abelian extensions of global fields. We describe a common refinement, due to Burns, and give evidence in favour of this conjecture for quadratic extensions and cyclotomic fields. We also note that the statement provides a new interpretation of further conjectures of Darmon and Gross.

1. Introduction

For K/k a finite abelian extension of global fields with Galois group G, and S and T finite disjoint sets of places of k such that S contains all infinite places, one defines an equivariant L-function $\Theta_{K/k,S,T}(s)$ for $s \in \mathbb{C}$, valued in $\mathbb{C}[G]$. When $K = k$, this is the (S-truncated, T-modified) zeta function of the field k. Dirichlet's analytic class number formula tells about the properties of this zeta function at the point $s = 0$, specifically its order of vanishing and its leading term. This latter is the product of a transcendental "regulator" term, formed from the units of k, with $h_{k,S,T}/w_{k,S,T}$, a ratio of integer invariants related to the arithmetic in the field k.

Towards the end of the 1970s, Stark conjectured analogues of these properties for more general L-functions. In particular, for abelian extensions he proposed an integrality statement for $\Theta'_{K/k,S,T}(0)$. Of the work which followed this, we note the paper [Ru96] of Rubin, where he made a conjecture which extended Stark's to higher orders of vanishing. Rubin's conjecture has the property that it tends to strengthen as the order of vanishing increases; indeed for the zeroth derivative $\Theta_{K/k,S,T}(0)$, where it states $\Theta_{K/k,S,T}(0) \in \mathbb{Z}[G]$, it follows easily from theorems of Deligne, Ribet and Weil. On the other hand, Gross in [Gr88] had made a conjecture of a different kind for this very element, in which he relates it to the class number $h_{k,S,T}$ and a certain group-ring valued regulator. However, for higher orders of vanishing Gross's conjecture becomes trivial.

In this paper we study a conjecture of Burns, 2.6 below, which unites these two approaches. It represents a strengthening of Rubin's conjecture which is precisely in the spirit of Gross, and it specialises to Gross's conjecture for the zeroth derivative $\Theta_{K/k,S,T}(0)$. The formulation was inspired by work in [Bu01], where it is shown that the Equivariant Tamagawa Number Conjecture, as formulated by Burns and Flach in [BF01], implies, for a certain class of extensions, a stronger variant of this Conjecture 2.6. The statement here proposes a generalisation of this to arbitrary abelian extensions.

In sections 2 to 4, we state the conjecture and give some elementary properties and special cases, including a proof for quadratic extensions. We then go on to use the theory of Dirichlet L-functions and cyclotomic fields to study the conjecture for certain extensions of number fields. In particular, we give evidence in the case of a real abelian extension of $\mathbb{Q}$.

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We also discuss two conjectures bearing a family resemblance to Gross's but concerning "minus-units" relative to a quadratic extension. These are due to Darmon ([Da95], dealing with an explicit "circular unit" related to first derivatives of L-functions) and Gross ([Gr88], Conjecture 8.8, which has more general hypotheses and concerns the values of the L-functions). In each context we interpret these conjectures as rather striking "base-change"-type statements for Burns's conjecture, which transport it from an extension $\tilde{L}/k$ to an extension $\tilde{L}K/K$, where K is a quadratic extension of k.

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2. Notation and Statement

2.1 Basic Setup

Let F be a global field, S a nonempty finite set of places of F containing all archimedean places.

We define $\mathcal{O}_{F,S} := \{\alpha \in F : v(\alpha) \geqslant 0 \text{ for all } v \notin S\}$, the ring of S-integers of F, and $U_{F,S} = \mathcal{O}_{F,S}^{\times}$, the S-units. The S-class group $A_{F,S}$ is defined to be the Picard group of $\mathcal{O}_{F,S}$, and fits into the exact sequence

$$0 \longrightarrow U_{F,S} \longrightarrow F^{\times} \longrightarrow \bigoplus_{\mathfrak{p} \notin S} \mathfrak{p}^{\mathbb{Z}} \longrightarrow A_{F,S} \longrightarrow 0. \tag{1}$$

Now let T be a finite set of places of F, disjoint from S. The subgroup of $U_{F,S}$ consisting of those S-units congruent to 1 modulo every prime in T is denoted $U_{F,S,T}$. The S ray-class group modulo T, denoted $A_{F,S,T}$, is the quotient of the group of fractional ideals of $\mathcal{O}_{F,S}$ prime to T by the subgroup of principal ideals with a generator congruent to 1 modulo each prime in T. The class groups fit into an exact sequence

$$0 \longrightarrow U_{F,S,T} \longrightarrow U_{F,S} \longrightarrow \prod_{\mathfrak{p} \in T} \mathbb{F}_{\mathfrak{p}}^{\times} \longrightarrow A_{F,S,T} \longrightarrow A_{F,S} \longrightarrow 0, \tag{2}$$

where $\mathbb{F}_{\mathfrak{p}}$ denotes the residue field of F at $\mathfrak{p}$. For any finite place $\mathfrak{p}$ of F, we let $N_{\mathfrak{p}}$ be the size of $\mathbb{F}_{\mathfrak{p}}$. Define the S- and (S,T)- class numbers $h_{F,S} = \#A_{F,S}$, $h_{F,S,T} = \#A_{F,S,T}$. Then

$$h_{F,S,T} = h_{F,S} \cdot \frac{\prod_{\mathfrak{p} \in T}(N\mathfrak{p} - 1)}{(U_{F,S} : U_{F,S,T})} = h_{F,S}\big(\prod_{\mathfrak{p} \in T} \mathbb{F}_{\mathfrak{p}}^{\times} : \widetilde{U_{F,S}}\big), \tag{3}$$

where $\widetilde{U_{F,S}}$ denotes the image of $U_{F,S}$ in the residue fields.

For the rest of §2, we fix an abelian extension of global fields K/k with Galois group G, and a nonnegative integer r. Let $S = S_k$ and $T = T_k$ be finite sets of places of k, and define S_K and T_K to be the sets of places of K dividing places in S_k and T_k respectively. We will abbreviate U_{K,S_K,T_K} to $U_{K,S,T}$, and do similarly for the class groups and class numbers. Let S_1 be a subset of S. We assume S, S_1 and T satisfy

Hypothesis 2.1. i) S contains all archimedean places of k;

ii) S contains the places which ramify in K/k;

iii) S_1 consists of r places which split completely in K/k;

iv) $\#S \geqslant r + 1$;

v) $T \cap S = \emptyset$ and $U_{K,S,T}$ is torsion-free.

We write $\#S = r + d + 1$, so $U_{k,S,T}$ is a free abelian group of rank $r + d$. Note that our setup closely follows [Ru96].

2.2 The equivariant L-function

For a finite unramified place v of k let Frob_v be the (arithmetic) Frobenius of the residue extension corresponding to w/v for a place w of K dividing v. Because K/k is abelian, this is a well-defined element of G.

For a character χ of the Galois group G of the extension K/k, write

$$e_\chi(K/k) := \frac{1}{\#G}\sum_{g\in G}\chi(g)g^{-1}$$

for the corresponding idempotent in $\mathbb{C}[G]$. Define the S-truncated abelian (Artin) L-function of χ by

$$L_{K/k,S}(s,\chi) = \prod_{v\notin S_k}(1-\chi(\mathrm{Frob}_v)N_v^{-s})^{-1}.$$

The product converges for $\mathrm{Re}\, s > 1$ and it is well-known that the function can be meromorphically extended to all of $\mathbb{C}$. The L-functions combine to give the S-truncated, T-modified equivariant L-function $\mathbb{C} \longrightarrow \mathbb{C}[G]$, as defined, for example, in [Ta84], Chapter IV, §1:

$$\begin{aligned}\Theta_{K/k,S,T}(s) &:= \Big(\prod_{t\in T_k}(1-N_t^{1-s}\mathrm{Frob}_t^{-1})\Big)\sum_{\chi\in\hat{G}}L_{K/k,S_k}(s,\chi^{-1})e_\chi(K/k)\\ &= \Big(\prod_{t\in T_k}(1-N_t^{1-s}\mathrm{Frob}_t^{-1})\Big)\Big(\prod_{v\notin S_k}(1-N_v^{-s}\mathrm{Frob}_v^{-1})\Big)^{-1}.\end{aligned}\tag{4}$$

Because of the assumption that r places in S_k split completely in K/k, we see by Proposition I.3.4 of [Ta84] that each $L_{K/k,S_k}(s,\chi^{-1})$ vanishes to order at least r at $s=0$. Write $e_r = \sum_\chi e_\chi$, where the sum is over all those characters for which the order of vanishing is exactly r. The r-th term in the Taylor expansion is given by

$$\Theta^r_{K/k,S,T}(0) := \lim_{s\to 0} s^{-r}\Theta_{K/k,S,T}(s).$$

It satisfies $\Theta^r_{K/k,S,T}(0) = \Theta^r_{K/k,S,T}(0)e_r$.

2.3 Special values and units

We set $Y_{S_K} := \{\sum_{v\in S_K} n_v v : n_v \in \mathbb{Z}\}$, the free abelian group on S_K, and $X_{S_K} := \{\sum_{v\in S_K} n_v v \in Y_{S_K} : \sum_{v\in S_K} n_v = 0\}$ its augmentation subgroup.

Define absolute values at places v of K as follows:

$$|a|_v = \begin{cases}|a| & \text{if } K_v = \mathbb{R},\\ |a|^2 & \text{if } K_v = \mathbb{C},\\ N_v^{-v(a)} & \text{for } v \text{ a finite place,}\end{cases}$$

where the valuation v is normalised so that its image is $\mathbb{Z}$.

For any $\mathbb{Z}[G]$-module M and any ring R, $RM := R\otimes_{\mathbb{Z}} M$ will denote the $R[G]$-module obtained from M by extending scalars to R. The logarithmic regulator map is defined by:

$$\begin{array}{rccc}\lambda_{S_K}: & U_{K,S} & \longrightarrow & \mathbb{R}X_{S_K}\\ & u & \longmapsto & -\sum_{v\in S_K}\ln|u|_v v\end{array}$$

It is well-known that this induces an $\mathbb{R}[G]$-module isomorphism $\mathbb{R}U_{K,S}\to\mathbb{R}X_{S_K}$. Its extension to a map $\wedge^n_G U_{K,S} \longrightarrow \mathbb{R}\wedge^n_G X_{S_K}$ will be written $\lambda^{(n)}_{S_K}$.

Let us recall the analytic class number formula of Dirichlet. $U_{k,S,T}$ has $\mathbb{Z}$-rank $r+d$, the same as X_{S_k}. Choose a basis $u_1,\dots,u_{r+d}$ for $U_{k,S,T}$ modulo torsion. Order the elements of S_k

as $v_1, \ldots, v_{r+d+1}$; then $v_1 - v_{r+d+1}, \ldots, v_{r+d} - v_{r+d+1}$ is a basis for X_{S_k}. The map $\lambda_{S_k}^{(r+d)}$ gives us a real determinant with respect to these bases. The determinant may be calculated as

$$R_{k,S,T} = \det\big(-\ln|u_i|_{v_j}\big)_{1 \leqslant i,j \leqslant r+d}.$$

The choice of the ordering of S_k only affects the determinant up to sign. In this paper we will choose to ignore systematically all questions of signs of regulators.

Dirichlet's analytic class number formula (see [Gr88], (1.6)) says that the meromorphic function $\zeta_{k,S,T}$ has a zero of exact order $\#S_k - 1$ at 0, and that the coefficient of the leading term in the Taylor expansion here is

$$-\frac{h_{k,S,T}|R_{k,S,T}|}{\#(U_{k,S,T})_{tors}}.$$

We now relate $\Theta^r_{K/k,S,T}(0)$ to the S-units of K. Let W be an r-tuple $(w_1, \ldots, w_r)$ where w_i is a place of K chosen above $v_i \in S_{1,k}$. Define $w_i^* \in \mathrm{Hom}_G(Y_{S_K}, \mathbb{Z}[G])$ on $w' \in S_K$ by $w_i^*(w') = \sum_\gamma \gamma$, summed over the elements γ of G with $\gamma w_i = w'$. Set $W^* = w_1^* \wedge \cdots \wedge w_r^* \in \wedge^r_G \mathrm{Hom}_G(Y_{S_K}, \mathbb{Z}[G])$. Then Remark 2 of [Po99], §1.6 shows that

$$W^* \circ \lambda_{S_K}^{(r)} : (\mathbb{C} \wedge^r_G U_{K,S,T})e_r \longrightarrow \mathbb{C}[G]e_r$$

is a $\mathbb{C}[G]$-isomorphism. Hence there is a unique element[1]

$$\eta = \eta_{K/k,S,T,r,W} \in \mathbb{C} \wedge^r_G U_{K,S,T}e_r$$

such that $W^* \circ \lambda_{S_K}^{(r)}(\eta) = \Theta^r_{K/k,S,T}(0)$. If we choose another place $w \in S_K - S_{1,K}$ and set $\mathbf{b} := (w_1 - w) \wedge \cdots \wedge (w_r - w)$, then we have

$$\Theta^r_{K/k,S,T}(0) \wedge^r_G X_{S_K} = \Theta^r_{K/k,S,T}(0)\mathbb{Z}[G]\mathbf{b}, \qquad \lambda_{S_K}^{(r)}(\eta) = \Theta^r_{K/k,S,T}(0)\mathbf{b}.$$

We refer to [Ru96], Lemma 2.6(ii) and the proof of Proposition 2.4, for the proof of this.

We are interested in integrality properties of this $\eta_{K/k,S,T,r,W}$, which we will test using elements $\Phi \in \wedge^r_G \mathrm{Hom}_{\mathbb{Z}[G]}(U_{K,S,T}, \mathbb{Z}[G])$. It will always suffice for our purposes to assume Φ is a primitive tensor $\phi_1 \wedge \cdots \wedge \phi_r$ (or $1 \in \mathbb{Z}[G]$ if $r = 0$), by linearity of our statements. Then $\Phi(u_1 \wedge \cdots \wedge u_r)$ means $\det(\phi_j(u_i))_{i,j}$. The element Φ induces a $\mathbb{C}$-linear map $\mathbb{C}\wedge^r_G U_{K,S,T} \longrightarrow \mathbb{C}[G]$, and we consider $\Phi(\eta) \in \mathbb{C}[G]$. We propose to strengthen the following conjecture, which is Conjecture B′ of [Ru96]:

CONJECTURE 2.2 (RUBIN). For every $\Phi \in \wedge^r_G \mathrm{Hom}_{\mathbb{Z}[G]}(U_{K,S,T}, \mathbb{Z}[G])$, we have $\Phi(\eta) \in \mathbb{Z}[G]$.

2.4 Formulation of the conjecture

Let $\mathrm{aug} : \mathbb{Z}[G] \longrightarrow \mathbb{Z}$ be the augmentation homomorphism, and write I_G for its kernel, the augmentation ideal of $\mathbb{Z}[G]$. Assume Rubin's conjecture holds. Burns's conjecture puts further conditions on the group ring element $\Phi(\eta)$, by proposing a congruence for $\Phi(\eta)$ modulo I_G^{d+1}.

For G any abelian group and M, N any $\mathbb{Z}[G]$-modules, one may make the group $\mathrm{Hom}_G(M,N) := \mathrm{Hom}_{\mathbb{Z}[G]}(M,N)$ into a $\mathbb{Z}[G]$-module with G-action given by $(g\alpha)(m) = g\alpha(m)$ for $g \in G, \alpha \in \mathrm{Hom}_G(M,N), m \in M$. There is a canonical isomorphism of abelian groups

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_G(M,\mathbb{Z}[G]) \\ \phi^1 & \longmapsto & \big(x \mapsto \sum_{g\in G} \phi^1(g^{-1}x)g\big). \end{array} \tag{5}$$

We write $\phi \mapsto \phi^1$ for the inverse of this isomorphism.

[1]Rubin in [Ru96] denotes this by ε instead of η. Throughout, we will omit subscripts from η which are clear from the context.

In §1.2 of [Ru96], Rubin observes that for any $\mathbb{Z}$-module M and any $n > 0$, every $h \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ induces a homomorphism

$$\begin{array}{cccc} \tilde{h}: & \wedge_{\mathbb{Z}}^{n} M & \longrightarrow & \wedge_{\mathbb{Z}}^{n-1} M \\ & m_1 \wedge \cdots \wedge m_n & \longmapsto & \sum_{i=1}^{n} (-1)^{i+1} h(m_i) m_1 \wedge \cdots \wedge \hat{m}_i \wedge \cdots \wedge m_r, \end{array}$$

where $\hat{}$ means "omit". This construction can be iterated to obtain

$$\begin{array}{ccc} \wedge^r \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\wedge^n M, \wedge^{n-r} M) \\ h_1 \wedge \cdots \wedge h_n & \longmapsto & \tilde{h_1} \circ \cdots \circ \tilde{h_n}. \end{array}$$

If $\Phi = \phi_1 \wedge \cdots \wedge \phi_r \in \wedge_{\mathbb{Z}[G]}^{r} \mathrm{Hom}_{\mathbb{Z}[G]}(U_{K,S,T}, \mathbb{Z}[G])$, we define $\tilde{\Phi}$ to be the map from $\wedge_{\mathbb{Z}}^{r+d} U_{k,S,T}$ to $\wedge_{\mathbb{Z}}^{d} U_{k,S,T}$ thus obtained from $\phi_1^1, \ldots, \phi_r^1$.

Definition 2.3. Let $r \leqslant n$ be nonnegative integers. Define a set of permutations

$$\begin{bmatrix} n \\ r \end{bmatrix} := \left\{ \sigma \in S_n : \begin{array}{l} \sigma(1) < \sigma(2) < \ldots < \sigma(r) \text{ and} \\ \sigma(r+1) < \sigma(r+2) < \ldots < \sigma(n) \end{array} \right\}.$$

Note that the cardinality of this set is the binomial coefficient $\binom{n}{r}$. Each element corresponds to choosing a subset of r elements from $\{1, \ldots, n\}$, and associates to it a sign, $\mathrm{sign}(\sigma)$.

Lemma 2.4. *We have the formula*

$$\tilde{\Phi}(u_1 \wedge \cdots \wedge u_{r+d}) = \sum_{\sigma \in \left[\begin{smallmatrix} r+d \\ r \end{smallmatrix}\right]} \mathrm{sign}(\sigma) \det\left(\phi_j^1(u_{\sigma(i)})\right)_{1 \leqslant i,j \leqslant r} u_{\sigma(r+1)} \wedge \cdots \wedge u_{\sigma(r+d)}.$$

The proof is routine. □

Now, following Gross, we define a group ring-valued regulator. Let the places in $S_k - S_{1,k}$ be denoted $v'_1, \ldots, v'_{d+1}$. For each v'_i, local class field theory gives us a local reciprocity map

$$f_{v'_i} : k^\times \longrightarrow G,$$

coming from the reciprocity map in the local extension $K_w/k_{v'_i}$ for a place w of K above v'_i. We compose this with the isomorphism

$$\begin{array}{ccc} G & \longrightarrow & I_G/I_G^2 \\ g & \longmapsto & g-1 \end{array}$$

to get a homomorphism to the additive group I_G/I_G^2. We now define the Gross-style regulator homomorphism (cf. [Gr88], (2.2)):

$$\begin{array}{cccc} \mathrm{Reg}_G = \mathrm{Reg}_{K/k,S,r}^{(v'_1,\ldots,v'_d)} : & \wedge_{\mathbb{Z}}^{d} U_{k,S,T} & \longrightarrow & \mathbb{Z}[G]/I_G^{d+1} \\ & u_1 \wedge \cdots \wedge u_d & \longmapsto & \det\left(f_{v'_j}(u_i) - 1\right)_{1 \leqslant i,j \leqslant d}. \end{array}$$

We will vary the subscripts of Reg according to any clarification needed in context. Note that here we have chosen to exclude v'_{d+1}. So we need:

Proposition 2.5. *The homomorphism above does not depend on the choice of which of the v'_i to exclude, or the ordering of the v'_i, up to sign.*

Proof. By the product formula of global class field theory, we have $\prod_{v \in S} f_v(x) = 1$ for all $x \in \mathcal{O}_{k,S}^\times$. Because $v \in S_1$ split completely in K/k, $f_v(x) = 1$ for these v. Hence

$$\prod_{v' \in S - S_1} f_{v'}(x) = 1.$$

Now choose $j \in \{1, \ldots, d\}$. In the determinant $\mathrm{Reg}_G(u_1 \wedge \cdots \wedge u_d)$, adding every other column to the column corresponding to v'_j and using the product formula shows that the i-th entry in column j is congruent to $-(f_{v'_{d+1}}(u_i)-1) \pmod{I_G^2}$. So the determinant becomes $-\mathrm{Reg}_G(u_1\wedge\cdots\wedge u_d)$ calculated with respect to the places $v'_1,\ldots,v'_{j-1},v'_{d+1},v'_{j+1},\ldots,v'_d$. Reordering these can only change the sign again. $\square$

Let $u_1,\ldots,u_{r+d}$ be a $\mathbb{Z}$-basis for $U_{k,S,T}$. We set $\mathrm{Reg}_G^\Phi = \mathrm{Reg}_G(\tilde{\Phi}(u_1\wedge\cdots\wedge u_{r+d}))$, and note that this is independent of the choice of basis up to sign. We have by Lemma 2.4 that

$$\mathrm{Reg}_G^\Phi = \sum_{\sigma\in\left[{r+d \atop r}\right]} \mathrm{sign}(\sigma)\det\left(\phi_j^1(u_{\sigma(i)})\right)_{1\leqslant i,j\leqslant r}\det\left(f_{v'_j}(u_{\sigma(r+i)})-1\right)_{1\leqslant i,j\leqslant d},$$

where $\left[{r+d \atop r}\right]$ was defined in Definition 2.3. The conjecture we will discuss is as follows.

Conjecture 2.6 (Burns). Let $K/k, S\supseteq S_1, T, r$ satisfy Hypothesis 2.1. Assume that Rubin's conjecture holds for this data, so that for every $\Phi\in\wedge^r_{\mathbb{Z}[G]}\mathrm{Hom}_{\mathbb{Z}[G]}(U_{K,S,T},\mathbb{Z}[G])$, we have $\Phi(\eta)\in\mathbb{Z}[G]$. Then this element satisfies

$$\Phi(\eta)\equiv \pm h_{k,S,T}\mathrm{Reg}_G^\Phi \pmod{I_G^{d+1}}.$$

Note that this conjecture implies $\Phi(\eta)\in I_G^d$, an "order of vanishing" statement which generalises [Gr88] (4.2) (via Proposition 3.9 below). For more on the formulation of Conjecture 2.6, including a method for specifying the sign in the congruence, see [Bu03].

3. Basic properties of the conjecture

3.1 Varying the data

Firstly, we wish to check that Conjecture 2.6 will remain true if we lower the top field K. We note a useful result about the unit groups.

Lemma 3.1. *For any $K/k, S, T$ such that K/k is Galois and $U_{K,S,T}$ is torsion-free, the quotient $U_{K,S,T}/U_{k,S,T}$ is also torsion-free.*

Proof. Suppose $u\in U_{K,S,T}$ is such that $u^n\in U_{k,S,T}$ for some $n>0$. This means that for all $\sigma\in\mathrm{Gal}(K/k)$, we have $(u^n)^{\sigma-1}=1$. But this is $(u^{\sigma-1})^n$. Hence for all $\sigma\in\mathrm{Gal}(K/k)$, $u^{\sigma-1}$ is a torsion element of $U_{K,S,T}$ and so is 1. Hence $u\in k$ as required. $\square$

Proposition 3.2. *Let $L/K/k$ be a tower of finite extensions, with L/k and K/k abelian with groups Γ and $G=\Gamma/H$ respectively. If Conjecture 2.6 holds for $L/k, S\supseteq S_1, T$ then it holds for $K/k, S\supseteq S_1, T$.*

Proof. It is clear, using Proposition IV.1.8 of [Ta84], that $\eta_{K/k}=(\wedge^r N_{L/K})\eta_{L/k}$.

The inclusion $U_{K,S,T}\hookrightarrow U_{L,S,T}$ and the $\mathbb{Z}[G]$-module isomorphism

$$\begin{array}{ccc}\mathbb{Z}[\Gamma]^H & \xrightarrow{\sim} & \mathbb{Z}[G]\\ N_{L/K} & \longmapsto & 1\end{array}$$

induce a surjective map

$$\mathrm{Hom}_\Gamma(U_{L,S,T},\mathbb{Z}[\Gamma])\longrightarrow\mathrm{Hom}_G(U_{K,S,T},\mathbb{Z}[G])$$

whereby each ϕ in the second group can be lifted to a $\hat{\phi}$ in the first in such a way that the projection of $\hat{\phi}(u)$ to $\mathbb{Z}[G]$ is $\phi(N_{L/K}u)$ for all $u\in U_{L,S,T}$. This follows by applying [Ru96], Proposition 1.1, to the exact sequence of $\mathbb{Z}$-torsion free Γ-modules given by Lemma 3.1, and using [Ru96], diagram (16).

Take $\Phi\in\wedge^r_G\mathrm{Hom}_G(U_{K,S,T},\mathbb{Z}[G])$ and lift it to $\hat{\Phi}\in\wedge^r_\Gamma\mathrm{Hom}_\Gamma(U_{L,S,T},\mathbb{Z}[\Gamma])$ componentwise. Now

$$\hat{\Phi}(\eta_{L/k})\equiv\pm h_{k,S,T}\mathrm{Reg}_\Gamma(\tilde{\hat{\Phi}}(\mathbf{u}))\pmod{I_\Gamma^{d+1}},$$

where $\mathbf{u} = u_1 \wedge \cdots \wedge u_{r+d}$, u_i a $\mathbb{Z}$-basis of $U_{k,S,T}$. Passing to the quotient in this congruence, and noting that $\tilde{\tilde{\Phi}} = \tilde{\Phi}$, we get

$$\Phi(\eta_{K/k}) \equiv \pm h_{k,S,T} \mathrm{Reg}_G(\tilde{\Phi}(\mathbf{u})) \pmod{I_G^{d+1}},$$

as required. ☐

We now look at enlarging T.

Proposition 3.3. *Suppose Conjecture 2.6 holds for $K/k, S \supseteq S_1, T$, and v is a place of k not in S or T. Set $T' = T \cup \{v\}$. Then Conjecture 2.6 holds for $K/k, S \supseteq S_1, T'$.*

Proof. The definition of $\Theta_{K/k,S,T}$ shows that $\eta_{T'} = \eta_T^{1-N_v \mathrm{Frob}_v^{-1}}$.

We will adapt [Po02], proof of Proposition 5.3.1. Let $\phi_1', \ldots, \phi_r'$ be in $\mathrm{Hom}_G(U_{K,S,T'}, \mathbb{Z}[G])$ and set $\Phi' = \phi_1' \wedge \cdots \wedge \phi_r'$. Popescu proves that there exist $\phi_i \in \mathrm{Hom}_G(U_{K,S,T}, \mathbb{Z}[G])$, $\alpha_i \in \mathbb{Z}[G]$ and $\phi_0 \in \mathrm{Hom}_G(U_{K,S,T'}, \mathbb{Z}[G])$ such that $\phi_i' = \phi_i + \alpha_i \phi_0$ for all $i = 1, \ldots, r$. Let $\delta_v = 1 - N_v \mathrm{Frob}_v^{-1}$, then it is clear that the map $\delta_v \phi_0 : x \mapsto \phi_0(\delta_v x)$ is in $\mathrm{Hom}_G(U_{K,S,T}, \mathbb{Z}[G])$. Popescu shows that

$$\Phi'(\eta_{T'}) = \Psi(\eta_T), \tag{6}$$

where $\Psi \in \wedge_G^r \mathrm{Hom}_G(U_{K,S,T}, \mathbb{Z}[G])$ is given by

$$\delta_v(\phi_1 \wedge \cdots \wedge \phi_r) + \sum_{i=1}^{r} \alpha_i \phi_1 \wedge \cdots \wedge \phi_{i-1} \wedge \delta_v \phi_0 \wedge \phi_{i+1} \wedge \cdots \wedge \phi_r. \tag{7}$$

Now let $\mathbf{u} = u_1 \wedge \cdots \wedge u_{r+d}$ be the wedge of a basis of $U_{k,S,T}$ and $\mathbf{u}' = u_1' \wedge \cdots \wedge u_{r+d}'$ similarly for $U_{k,S,T'}$. We have $(U_{k,S,T} : U_{k,S,T'})\mathbf{u} = \mathbf{u}'$ in $\wedge_{\mathbb{Z}}^{r+d} U_{k,S,T}$. Apply $\tilde{\Psi}$ to both sides of this equality. We note that $(\delta_v \phi_0)^1(u_i') = \mathrm{aug}(\delta_v)\phi_0^1(u_i')$ since $u_i' \in U_{k,S,T'}$. Note also that $\mathrm{aug}(\delta_v) = -(N_v - 1)$. By the form of (7), this shows that

$$(U_{k,S,T} : U_{k,S,T'})\tilde{\Psi}(\mathbf{u}) = -(N_v - 1)\widetilde{\Phi'}(\mathbf{u}') \qquad \text{in } \wedge_{\mathbb{Z}}^d U_{k,S,T}.$$

But $(U_{k,S,T} : U_{k,S,T'})$ divides $(N_v - 1)$. Since the group in which the equality holds is torsion-free, we may cancel $(U_{k,S,T} : U_{k,S,T'})$ from both sides. Furthermore, by (3),

$$h_{k,S,T'} = \frac{(N_v - 1)}{(U_{k,S,T} : U_{k,S,T'})} h_{k,S,T}.$$

So we have $h_{k,S,T}\tilde{\Psi}(\mathbf{u}) = -h_{k,S,T'}\widetilde{\Phi'}(\mathbf{u}')$ in $\wedge_{\mathbb{Z}}^d U_{k,S,T}$. Applying Reg_G to this and using (6) gives the result. ☐

We next look at changing S. We will use:

Lemma 3.4. *Let k, S, T be such that $U_{k,S,T}$ is torsion-free. Suppose $u_1, \ldots, u_n$ is a $\mathbb{Z}$-basis for $U_{k,S,T}$. Let v be a place of k not in S or T. Take $u' \in U_{k,S\cup\{v\},T}$ such that $v(u')$ is minimal positive. Then $u_1, \ldots, u_n, u'$ is a basis for $U_{k,S\cup\{v\},T}$.*

Proof. Let $u \in U_{k,S\cup\{v\},T}$. Then there exists $a \in \mathbb{Z}$ such that $v(u) = av(u')$. Then $u/u'^a \in U_{k,S,T}$, so we see $u_1, \ldots, u_n, u'$ generates $U_{k,S\cup\{v\},T}$. Linear independence follows from considering the valuations at v. ☐

Lemma 3.5. *Using the notation of the previous Lemma, we have*

$$h_{k,S\cup\{v\},T} \cdot v(u') = h_{k,S,T}.$$

Proof. Write $S' = S \cup \{v\}$ for short. The result follows from the analytic class number formula as follows. If $n = \#S - 1$, we have

$$\zeta_{k,S',T}(s) \equiv h_{k,S',T}R_{k,S',T}s^{n+1} \pmod{s^{n+2}},$$
$$\zeta_{k,S,T}(s) \equiv h_{k,S,T}R_{k,S,T}s^{n} \pmod{s^{n+1}},$$

and the leading terms are related by

$$h_{k,S',T}R_{k,S',T} = \lim_{s\to 0}\frac{1-N_v^{-s}}{s}h_{k,S,T}R_{k,S,T}.$$

Hence $h_{k,S',T}R_{k,S',T} = (\ln N_v)h_{k,S,T}R_{k,S,T}$. On the other hand, the definition of the regulator, and the fact that $v(u_i) = 0$ for $i = 1, \dots, n$ shows that $R_{k,S',T} = \ln|u'|_v R_{k,S,T} = v(u')(\ln N_v)R_{k,S,T}$. This gives the result. □

PROPOSITION 3.6. *Let $K/k, S, T, r$ be data satisfying Hypothesis 2.1, and let v be a place of k not in S or T and which splits completely in K/k. Set $S' = S \cup \{v\}$ and $S_1' = S_1 \cup \{v\}$. Suppose Conjecture 2.6 holds for $K/k, S' \supseteq S_1', T, r+1$. Then Conjecture 2.6 holds for $K/k, S \supseteq S_1, T, r$.*

Proof. Choose bases as in Lemma 3.4. Note that "n"$= r + d$ in the notation of that lemma, and define $u_{r+d+1} = u'$. We choose w above v to go into W. By [Ru96], Proposition 5.2, we have $\eta_S = \check{w}(\eta_{S'})$, where $\check{w} \in \mathrm{Hom}_G(U_{K,S',T}, \mathbb{Z}[G])$ is defined by

$$\check{w}(u) = \sum_{g\in G} w(g^{-1}u)g.$$

Take $\Phi \in \wedge_G^r \mathrm{Hom}_G(U_{K,S,T}, \mathbb{Z}[G])$. The hypothesis that $U_{K,S,T}$ and $U_{K,S',T}$ are torsion-free implies that the map $\mathrm{Hom}_G(U_{K,S',T}, \mathbb{Z}[G]) \to \mathrm{Hom}_G(U_{K,S,T}, \mathbb{Z}[G])$ (restriction) is surjective (cf. [Ru96], Proposition 1.1 (ii)). So we may lift Φ componentwise to $\Phi' \in \wedge_G^r \mathrm{Hom}_G(U_{K,S',T}, \mathbb{Z}[G])$. Then

$$\Phi(\eta_S) = (\Phi' \circ \check{w})(\eta_{S'}) \equiv \pm h_{k,S',T}\mathrm{Reg}_G(\tilde{\Phi}(\tilde{\check{w}}(u_1 \wedge \cdots \wedge u'))) \pmod{I_G^{d+1}},$$

and $\tilde{\check{w}}(u_1 \wedge \cdots \wedge u') = \sum_{i=1}^{r+d+1}(-1)^{i+1}w(u_i)u_1 \wedge \cdots \wedge u_{i-1} \wedge u_{i+1} \wedge \cdots \wedge u_{r+d+1}$. But $u_1, \dots, u_{r+d}$ are S-units so this collapses to $\pm w(u')u_1 \wedge \cdots \wedge u_{r+d}$. Now because v splits completely in K/k, we have $w(u') = v(u')$. Hence

$$\Phi(\eta_S) \equiv \pm v(u')h_{k,S',T}\mathrm{Reg}_G(\tilde{\Phi}(u_1 \wedge \cdots \wedge u_{r+d})) \pmod{I_G^{d+1}},$$

which by Lemma 3.5 is what we want. □

PROPOSITION 3.7. *Suppose Conjecture 2.6 holds for $K/k, S \supseteq S_1, T, r$. Let v be a place of k not in S or T, and set $S' = S \cup \{v\}$. Then Conjecture 2.6 holds for $K/k, S' \supseteq S_1, T, r$.*

Proof. Again we have $n = r+d$ and define $u_{r+d+1} = u'$. We note that, because S satisfies Hypothesis 2.1, v is unramified in K/k. Therefore the Artin symbol at v can be calculated by the formula

$$f_v(x) = (\mathrm{Frob}_v)^{v(x)},$$

for all x in $k^\times$. The definition of $\Theta_{K/k,S,T}$ shows that $\eta_{S'} = \eta_S^{1-\mathrm{Frob}_v}$. Take $\Phi = \phi_1 \wedge \cdots \wedge \phi_r \in \wedge_G^r \mathrm{Hom}_{\mathbb{Z}[G]}(U_{K,S',T}, \mathbb{Z}[G])$, then Conjecture 2.6 for S' asks for:

$$\Phi(\eta_{S'}) \equiv \pm h_{k,S',T}\mathrm{Reg}_G(\tilde{\Phi}(u_1 \wedge \cdots \wedge u_{r+d+1})) \pmod{I_G^{d+2}}.$$

We choose the places $v_1, \dots, v_{d+1}$ appearing in Reg_G by taking the set $S - S_1$ of places not designated as splitting, excluding one place, then adding $v = v_{d+1}$. We have[2]

$$\mathrm{Reg}_G(\tilde{\Phi}(u_1 \wedge \cdots \wedge u_{r+d+1})) = \sum_{\sigma \in \left[{r+d+1 \atop r}\right]} \mathrm{sign}(\sigma)\det\left(\phi_j^1(u_{\sigma(i)})\right)\det\left(f_{v_j}(u_{\sigma(r+i)}) - 1\right).$$

---

[2]for the definition of $\left[{r+d+1 \atop r}\right]$ see Definition 2.3.

Let us consider two cases of σ. If $r+d+1 \in \{\sigma(1),\dots,\sigma(r)\}$ then in the corresponding term, the column in the second determinant corresponding to v is all zeros, since $u_1,\dots,u_{r+d}$ are S-units. The other possibility is that $r+d+1=\sigma(r+d+1)$. Then this same column is all zeros apart from the bottom-right entry, which is

$$f_v(u_{r+d+1})-1=(\mathrm{Frob}_v)^{v(u')}-1\equiv v(u')(\mathrm{Frob}_v-1)\pmod{I_G^2}.$$

Hence $\mathrm{Reg}_{K/k,S'}(\tilde{\Phi}(u_1\wedge\cdots\wedge u_{r+d+1}))=\pm v(u')(\mathrm{Frob}_v-1)\mathrm{Reg}_{K/k,S}(\tilde{\Phi}(u_1\wedge\cdots\wedge u_{r+d}))$, where in the second expression we may consider Φ as being restricted to $U_{K,S,T}$. So Conjecture 2.6 for S' now reads, using Lemma 3.5,

$$(1-\mathrm{Frob}_v)\Phi(\eta_S)\equiv\pm(\mathrm{Frob}_v-1)h_{k,S,T}\mathrm{Reg}_G(\tilde{\Phi}(u_1\wedge\cdots\wedge u_{r+d}))\pmod{I_G^{d+2}}.$$

Therefore if Conjecture 2.6 holds for S, it holds for S'. □

On the other hand, the proof shows that we have the following, possibly weaker, implication in the other direction.

Proposition 3.8. *Suppose Conjecture 2.6 holds for $K/k, S\cup\{v\}\supseteq S_1, T, r$. Then we have, for the data $K/k, S\supseteq S_1, T, r$,*

$$(1-\mathrm{Frob}_v)\Phi(\eta)\equiv(1-\mathrm{Frob}_v)(\pm h_{k,S,T}\mathrm{Reg}_G^{\Phi})\pmod{I_G^{d+1}},$$

in the notation of Conjecture 2.6. That is, we obtain the image of the congruence in the next level of the augmentation filtration under multiplication by $(1-\mathrm{Frob}_v)$.

3.2 Special cases

We study the behaviour of the conjecture for some interesting special cases of the data.

Proposition 3.9. *Suppose $r=0$ and $K/k, S\supseteq\emptyset, T, 0$ satisfies Hypothesis 2.1, ie. we designate no places as splitting in K/k. Then Conjecture 2.6 is equivalent to Conjecture 4.1 of [Gr88], up to sign.*

Proof. The element $\eta\in\mathbb{C}\wedge_G^0 U_{K,S,T}e_r=\mathbb{C}[G]e_r$ is characterised by $\eta=\Theta_{K/k,S,T}(0)$. Taking $\Phi=1\in\mathbb{Z}[G]$, Conjecture 2.6 now reads:

$$\Theta_{K/k,S,T}(0)\equiv\pm h_{k,S,T}\mathrm{Reg}_G(u_1\wedge\cdots\wedge u_d)\pmod{I_G^{d+1}}.$$

This is a sign-indifferent version of Gross's conjecture for the extension K/k and sets S and T. □

Proposition 3.10. *Conjecture 2.6 holds when more than r places in S split completely.*

Proof. We adapt the method of [Ru96], Proposition 3.1.

Note that all the S-truncated L-functions corresponding to nontrivial characters vanish to order greater than r at $s=0$ (see [Ta84], Proposition I.3.4). If $\#S>r+1$ then this is true for the trivial character too, and so η is the identity. On the other hand, if $\#S>r+1$ then our Gross-style regulators in Conjecture 2.6 can be calculated with respect to a totally split place and so are all zero. Hence Conjecture 2.6 says $0\equiv 0$.

Now assume $\#S=r+1$. Let $u_1,\dots,u_r$ be a $\mathbb{Z}$-basis for $U_{k,S,T}$. In his proof of [Ru96], Proposition 3.1, Rubin shows that, by the analytic class number formula,

$$\eta=\frac{h_{k,S,T}}{\#G^r}u_1\wedge\cdots\wedge u_r,$$

(for which we might have to invert a unit u_1 to get the sign right). We apply $\Phi=\phi_1\wedge\cdots\wedge\phi_r\in\wedge_G^r\mathrm{Hom}_G(U_{K,S,T},\mathbb{Z}[G])$ to η. Note that, because $u_i\in k$, $\phi_j(u_i)=\phi_j^1(u_i)N_G$, where $N_G=\sum_{g\in G}g$. We obtain

$$\Phi(\eta)=\pm\frac{h_{k,S,T}}{(\#G)^r}N_G^r\det(\phi_j^1(u_i))=\pm\frac{h_{k,S,T}}{\#G}N_G\det(\phi_j^1(u_i)).$$

Rubin (loc. cit.) argues by class field theory that $\#G \mid h_{k,S,T}$, so this is an element of $\mathbb{Z}[G]$. Since N_G has augmentation $\#G$, reducing this equation mod I_G gives us Conjecture 2.6. $\square$

COROLLARY 3.11. *If $k/k, S, T, r$ satisfy Hypothesis 2.1, then Conjecture 2.6 is true for this data.*

Proof. This is because all places in S split completely. $\square$

PROPOSITION 3.12. *Assume $K/k, S \supseteq S_1, T, r$ satisfy Hypothesis 2.1 and furthermore that $\#S \geqslant r+2$. Assume Conjecture 2.2 holds for this data. Then we have $\Phi(\eta) \in I_G$.*

Proof. This is [Bu01], Theorem 4.4 (iii). We reproduce the proof. Since $\zeta_{k,S}$ vanishes to order $r+1$ at $s=0$, $\Theta^r_{K/k,S,T}(0)$ lies in $\mathbb{C}I_G$. Hence $N_{K/k}\eta = 1$, and so $\Phi(\eta) \in \mathbb{C}I_G$. Now if Conjecture 2.2 holds, we have $\Phi(\eta) \in \mathbb{C}I_G \cap \mathbb{Z}[G] = I_G$, as required. $\square$

4. Quadratic Extensions

In this section we take $K/k, S, T, r$ with K/k quadratic with group G generated by τ. We will assume, using Proposition 3.10, that exactly r places S_1 split in K/k.

Perhaps the most involved arguments of Rubin in [Ru96] and Gross in [Gr88] are to verify their respective conjectures in this situation. We adapt their methods to prove:

THEOREM 4.1. *Let $K/k, S \supseteq S_1, T, r$ be data satisfying Hypothesis 2.1, with $[K : k] = 2$. Then Conjecture 2.6 holds.*

Remark 4.2. This result provides a new proof of the validity of Gross's conjecture ([Gr88], Conjecture 4.1) for quadratic extensions. Its proof avoids the technicalities and special cases considered by Gross in [Gr88], §6, by using the extra functorial properties of Conjecture 2.6 with respect to increase of S. For comparison, note that §4.2 below corresponds to the known case "$n=0$" of Gross's conjecture ([Gr88] (4.3)), and that in §4.3 below the sign of the regulator is irrelevant.

4.1 Cohomology of $U_{K,S,T}$

Let $u_1, \ldots, u_{r+d+r}$ be a basis of $U_{K,S,T}$ such that $u_1, \ldots, u_{r+d}$ is a basis of $U_{k,S,T}$, which is possible by Lemma 3.1. The relevant structure of this basis is closely related to the Galois cohomology of the G-module $U_{K,S,T}$. Our first result in this direction is the following (cf. [Ru96], Theorem 3.5, proof).

LEMMA 4.3. *If $H^1(G, U_{K,S,T}) \neq 0$ then we can assume $N_{K/k}u_{r+d+1} = 1$.*

Proof. Take $u \in U_{K,S,T}$ representing a nontrivial element of $H^1(G, U_{K,S,T}) = U^-_{K,S,T}/U^{1-\tau}_{K,S,T}$, where $U^-_{K,S,T}$ is the set of (S,T)-units of norm 1. Write $u = \epsilon \prod_i u^{\alpha_i}_{r+d+i}$, where $\epsilon \in U_{k,S,T}$, and write ϵ_i for the norm $u^{1+\tau}_{r+d+i} \in U_{k,S,T}$. Then $u^{1-\tau}_{r+d+i} = u^2_{r+d+i}\epsilon_i^{-1}$. Therefore we can assume each α_i is 0 or 1. But they cannot all be 0. Hence u can go into a basis of $U_{K,S,T}$. $\square$

LEMMA 4.4. *We have*

$$\frac{\#\hat{H}^0(G, U_{K,S,T})}{\#H^1(G, U_{K,S,T})} = 2^d.$$

Proof. Note the left-hand side is the Herbrand quotient $h(U_{K,S,T})$ of the $\mathbb{Z}[G]$-module $U_{K,S,T}$ in the sense of [Se79], Ch. VIII, §4. The composite isomorphism of $\mathbb{Q}[G]$-modules

$$\mathbb{Q}U_{K,S,T} \cong \mathbb{Q}X_{S_K} \cong \mathbb{Q}[G]^r \oplus \mathbb{Q}^d$$

implies that there is an injection of $U_{K,S,T}$ into $\mathbb{Z}[G]^r \oplus \mathbb{Z}^d$ with finite cokernel. Then $h(U_{K,S,T}) = h(\mathbb{Z}[G])^r h(\mathbb{Z})^d = 2^d$ as required. $\square$

Lemma 4.5. *If $H^1(G, U_{K,S,T}) = 0$, we can assume that $u_i = N_{K/k} u_{r+d+i}$ for $i = 1, \dots, r$.*

Proof. Write $Nu_{r+d+j} = \prod_{i=1}^{r+d} u_i^{\alpha_{ji}}$ for $j = 1, \dots, r$. We may perform the following operations on the $r \times (r+d)$ matrix (α_{ji}): elementary column operations, which correspond to swapping and multiplying the units in the basis of $U_{k,S,T}$, and elementary row operations, which correspond to swapping and multiplying the units $u_{r+d+1}, \dots, u_{r+d+r}$.

Thus we can put (α_{ji}) into diagonal form with integers $a_1, \dots, a_r$ on the diagonal. Now suppose some a_i is even, so $Nu_{r+d+i} = \epsilon^2$ for some ϵ in $U_{k,S,T}$. Then $u_{r+d+i}\epsilon^{-1}$ is a unit in $K - k$ with norm 1. But the group $U_{K,S,T}^{1-\tau} \subseteq U_{k,S,T} U_{K,S,T}^2$. So $u_{r+d+i}\epsilon^{-1}$ represents a nontrivial element of

$$U_{K,S,T}^- / U_{K,S,T}^{1-\tau} = H^1(G, U_{K,S,T}).$$

This is a contradiction to our assumption. We conclude that each a_i is odd. Now replacing u_{r+d+i} by $u_{r+d+i} u_i^{-[a_i/2]}$ gives us the result. ∎

In our situation, noting that S contains a place not splitting in K, Lemma 3.4 of [Ru96] states:

Lemma 4.6 (Rubin). i) $h_{k,S,T} \mid h_{K,S,T}$

ii) $\# H^1(G, U_{K,S,T}) \mid h_{k,S,T}$

iii) *If $\hat{H}^0(G, U_{K,S,T})$ and $H^1(G, U_{K,S,T})$ are trivial then* $h_{k,S,T} \equiv h_{K,S,T}/h_{k,S,T} \pmod 2$.

We write $\epsilon_- = u_{r+d+1} \wedge \cdots \wedge u_{r+d+r}$. In proving his conjecture for quadratic extensions ([Ru96], proof of Proposition 3.5), Rubin uses the analytic class number formula to express η in terms of this element; we make extensive use of his formulae, which are quoted below.

4.2 The case $d = 0$

We assume $d = 0$, that is, $\#S = r + 1$, and show that the congruence in Conjecture 2.6 holds. We require:

$$\Phi(\eta) \equiv \pm h_{k,S,T} \begin{vmatrix} \phi_1^1(u_1) & \cdots & \phi_r^1(u_1) \\ \vdots & \ddots & \vdots \\ \phi_1^1(u_r) & \cdots & \phi_r^1(u_r) \end{vmatrix} \pmod{I_G}. \tag{8}$$

Note that this is an equality in $\mathbb{Z}[G]/I_G \cong \mathbb{Z}$.

Assume first that $H^1(G, U_{K,S,T}) = 0$. Then Rubin shows in [Ru96], proof of Proposition 3.5, that

$$\eta = \pm \left(\frac{h_{K,S,T}}{h_{k,S,T}} \frac{(1-\tau)}{2} \pm h_{k,S,T} \frac{(1+\tau)}{2} \right) \epsilon_-.$$

Lemma 4.6 shows the $\mathbb{Q}[G]$-factor here lies in $\mathbb{Z}[G]$, and we note its augmentation is $\pm h_{k,S,T}$. Now we can assume $u_{r+d+i}^{1+\tau} = u_i$ for $i = 1, \dots, r$ by Lemma 4.5, so

$$\phi(u_{r+d+i}) = \phi^1(u_{r+d+i}) - \tau\phi^1(u_{r+d+i}) + \tau\phi^1(u_i) \equiv \phi^1(u_i) \pmod{I_G}.$$

Hence (8) is satisfied in this case.

Now assume $H^1(G, U_{K,S,T}) \neq 0$. Let $\bar{u}_1, \dots, \bar{u}_r \in U_{K,S,T}$ such that $N_{K/k}\bar{u}_i$ is a basis for $NU_{K,S,T}$. Set $\epsilon_+ = \bar{u}_1 \wedge \cdots \wedge \bar{u}_r$. Rubin shows (loc. cit.) that

$$\eta = \pm \frac{h_{k,S,T}}{\#\hat{H}^0(G, U_{K,S,T})} \frac{(1+\tau)}{2} \epsilon_+ \pm \frac{h_{K,S,T}}{h_{k,S,T}} \frac{(1-\tau)}{2} \epsilon_-,$$

where again the $\mathbb{Q}[G]$-factors are in $\mathbb{Z}[G]$. By Lemma 4.3 we can assume $N_{K/k} u_{r+d+1} = 1$, and then

$$\phi_j(u_{r+d+1}) = \phi_j^1(u_{r+d+1}) + \tau\phi_j^1(u_{r+d+1}^\tau) \equiv \phi_j^1(N_{K/k} u_{r+d+1}) = 0 \pmod{I_G},$$

so $\Phi(\epsilon_-) \equiv 0 \pmod{I_G}$. On the other hand, $\phi_j(\bar{u}_i) \equiv \phi_j^1(N_{K/k}\bar{u}_i) \pmod{I_G}$ and the index of the group generated by the $N_{K/k}\bar{u}_i$ in $U_{k,S,T}$ is $\#\hat{H}^0(G, U_{K,S,T})$. This shows that (8) holds in this case too.

This verifies Theorem 4.1 in the case $d = 0$.

4.3 The case $d > 0$

Now we assume $d > 0$, ie. $\#S > r + 1$. For $d > 0$, I_G^d/I_G^{d+1} is a group of order 2, so the congruence statement in Conjecture 2.6 only concerns in which power of the augmentation ideal the terms lie. We have $(1-\tau)^n = 2^{n-1}(1-\tau)$ for $n > 0$. Note that the map

$$\begin{array}{ccc} I_G^d/I_G^{d+1} & \longrightarrow & I_G^{d+1}/I_G^{d+2} \\ x & \longmapsto & (1-\tau)x \end{array}$$

is a bijection. We have the following freedom to increase S:

LEMMA 4.7. *Let $K/k, S, T, r$ satisfy Hypothesis 2.1 with K/k a quadratic extension. Assume $d = \#S - r - 1 > 0$. Let v be a place of k not in S or T, and set $S' = S \cup \{v\}$, $S_1' = S_1 \cup \{v\}$. Then either of the following conditions implies Conjecture 2.6 for $K/k, S, T, r$.*

i) *v splits in K/k, and Conjecture 2.6 holds for $K/k, S' \supseteq S_1', T, r+1$;*
ii) *v is inert in K/k, and Conjecture 2.6 holds for $K/k, S' \supseteq S_1, T, r$.*

Proof. This follows from Propositions 3.6 and 3.8, given the structure of the augmentation filtration. □

Rubin shows in [Ru96], proof of Proposition 3.5, that for the case K/k quadratic and $d > 0$ we have

$$\eta = \pm 2^d \frac{h_{K,S,T}}{h_{k,S,T}} \frac{(1-\tau)}{2} \epsilon_-.$$

We note that $2^{d-1}(1-\tau) \in I_G^d$.

It turns out that we only need to consider the congruence statement under the following cohomological assumption[3].

LEMMA 4.8. *Suppose $d > 0$ and $H^1(G, U_{K,S,T}) = 0$. Then the congruence of Conjecture 2.6 is implied by the following statement:*

$$2^{d-1}(1-\tau)\frac{h_{K,S,T}}{h_{k,S,T}} \equiv h_{k,S,T}\mathrm{Reg}_G(u_{r+1} \wedge \cdots \wedge u_{r+d}) \pmod{I_G^{d+1}}. \tag{9}$$

Proof. We apply Lemma 4.5 to assume that $u_i = N_{K/k}u_{r+d+i}$ for $i = 1, \ldots r$. Then $\phi_j(u_{r+d+i}) \equiv \phi_j^1(u_i) \pmod{I_G}$. Thus

$$\Phi(\eta) \equiv 2^{d-1}(1-\tau)\frac{h_{K,S,T}}{h_{k,S,T}} \begin{vmatrix} \phi_1^1(u_1) & \cdots & \phi_r^1(u_1) \\ \vdots & \ddots & \vdots \\ \phi_1^1(u_r) & \cdots & \phi_r^1(u_r) \end{vmatrix} \pmod{I_G^{d+1}}$$

For the right-hand side of the congruence in Conjecture 2.6, we note that if any u_i, $1 \leqslant i \leqslant r$, appears in the argument of Reg_G, then the corresponding term is 0 (because u_i is a norm from K, and therefore in the kernel of all the local reciprocity maps). So the right-hand side collapses to a

[3] In fact it is easy to show that both sides of the congruence vanish if this assumption is not satisfied.

single term:

$$h_{k,S,T}\begin{vmatrix} \phi_1^1(u_1) & \cdots & \phi_r^1(u_1) \\ \vdots & \ddots & \vdots \\ \phi_1^1(u_r) & \cdots & \phi_r^1(u_r) \end{vmatrix} \mathrm{Reg}_G(u_{r+1}\wedge\cdots\wedge u_{r+d}).$$

This gives the result. □

Next we identify a condition for the non-vanishing of the regulator. We need an auxiliary lemma. For Tate's theory of the group cohomology of finite cyclic groups, we refer to [Se79], Ch. VIII, §4.

Lemma 4.9. *Suppose $h_{k,S}=1$. Then there is an isomorphism*

$$\frac{U_{k,S}\cap NK^\times}{NU_{K,S}} \cong A_{K,S}^G.$$

Proof. (cf. [Gr88] p191) We have the exact sequence (from (1)):

$$0 \longrightarrow K^\times/U_{K,S} \longrightarrow \bigoplus_{\mathfrak{P}\notin S_K} \mathfrak{P}^{\mathbb{Z}} \longrightarrow A_{K,S} \longrightarrow 0.$$

Considering the decompositions of primes shows that $H^1(G,\bigoplus_{\mathfrak{P}\notin S_K}\mathfrak{P}^{\mathbb{Z}})=0$. Then taking cohomology gives an exact sequence

$$0 \longrightarrow H^0(G,K^\times/U_{K,S}) \longrightarrow \bigoplus_{\mathfrak{p}\notin S_k}\mathfrak{p}^{\mathbb{Z}} \stackrel{0}{\longrightarrow} A_{K,S}^G \longrightarrow H^1(G,K^\times/U_{K,S}) \longrightarrow 0.$$

Observe that $k^\times/U_{k,S}$ injects into $H^0(G,K^\times/U_{K,S})$, and (1) for k shows that it surjects onto $\bigoplus_{\mathfrak{p}\notin S_k}\mathfrak{p}^{\mathbb{Z}}$, because the S-class group of k is trivial. This is why the map marked 0 is zero. Therefore we have $A_{K,S}^G \cong H^1(G,K^\times/U_{K,S})$.

On the other hand, applying Hilbert's Theorem 90 ([Se79], Ch. X, Proposition 2) and Tate cohomology to the short exact sequence

$$0 \longrightarrow U_{K,S} \longrightarrow K^\times \longrightarrow K^\times/U_{K,S} \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow H^1(G,K^\times/U_{K,S}) \longrightarrow \hat{H}^0(G,U_{K,S}) \longrightarrow \hat{H}^0(G,K^\times).$$

Therefore

$$H^1(G,K^\times/U_{K,S}) \cong \ker(U_{k,S}/NU_{K,S} \longrightarrow k^\times/NK^\times) = U_{k,S}\cap NK^\times/NU_{K,S}.$$

This completes the proof. □

Lemma 4.10. *Suppose $h_{K,S,T}=1$. Then* $\mathrm{Reg}_G(u_{r+1}\wedge\cdots\wedge u_{r+d}) \not\equiv 0 \pmod{I_G^{d+1}}$.

Proof. By (3) and Lemma 4.6 (part (i)), we have that $h_{k,S}=1$, and then Corollary 2 of [Ri65] shows that $H^1(G,U_{K,S})=0$. Therefore $\hat{H}^0(G,U_{K,S})$ is a 2-torsion group with 2^d elements by Lemma 4.4, and hence is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d$.

Similarly to [Gr88] p191, we define a homomorphism

$$f : U_{k,S} \longrightarrow G^{S-S_1} \cong (\mathbb{Z}/2\mathbb{Z})^{d+1}$$

by the local reciprocity maps f_v. Then, by the product formula,

$$\mathrm{im}\, f \subseteq V := \left\{(g_v)_{v\in S-S_1} : \prod g_v = 1\right\} \cong (\mathbb{Z}/2\mathbb{Z})^d.$$

Now $u\in U_{k,S}$ is in $\ker f$ if and only if u is a local norm at all the places in $S-S_1$, and we note it is automatically a norm at all other places. Since K/k is cyclic, u is a local norm everywhere if and only if it is a global norm. So $\ker f = U_{k,S} \cap NK^\times$. Since $h_{K,S,T} = 1$, $A^G_{K,S} = 0$ and so Lemma 4.9 shows that $\ker f = NU_{K,S}$. On the other hand, $\hat{H}^0(G, U_{K,S}) = U_{k,S}/NU_{K,S} \cong (\mathbb{Z}/2\mathbb{Z})^d$, so we have $f(U_{k,S}) = V$.

We note that the form of our regulator and the choice of our unit basis show that the non-vanishing of our regulator

$$\mathrm{Reg}_G(u_{r+1} \wedge \cdots \wedge u_{r+d}) \not\equiv 0 \pmod{I_G^{d+1}}$$

is equivalent to saying $f(U_{k,S,T}) = V$.

The reduction map $U_{K,S} \longrightarrow \prod_{\mathfrak{P}\in T_K} \mathbb{F}_{\mathfrak{P}}^\times$ is surjective, by (2) for K and the assumption that $h_{K,S,T} = 1$. Also, the norm in an extension of finite fields is surjective. Hence $\ker f = NU_{K,S}$ surjects onto $\prod_{\mathfrak{p}\in T} \mathbb{F}_{\mathfrak{p}}^\times$. The sequence (2) for k shows that this latter is isomorphic by the reduction map to $U_{k,S}/U_{k,S,T}$. So every element of $U_{k,S}$ can be written as the product of something in $\ker f$ by something in $U_{k,S,T}$, which gives our result. □

Now consider $A_{K,S,T}$, the S_K ray class group modulo T_K. Let $S'_K = \{w_1, \ldots, w_n\}$ be a set of primes of $\mathcal{O}_{K,S}$ coprime to T_K whose classes generate this group. Set S' to be the set of places of k lying below these. If S contains S', then $h_{K,S,T} = 1$. On the other hand:

Lemma 4.11. *If $h_{K,S,T} = 1$, then the congruence of Conjecture 2.6 holds.*

Proof. By Lemma 4.6 (part (i)) we have that $h_{k,S,T} = 1$, which also shows by part (ii) of the same lemma that $H^1(G, U_{K,S,T}) = 0$. So by Lemma 4.8 it is sufficient to show (9) holds. Lemma 4.10 shows that the right-hand side of (9) is not zero. On the other hand $h_{K,S,T}/h_{k,S,T} = 1$ so the left-hand side is not zero either. □

Now by Lemma 4.7, we may assume $S' \subseteq S$, increasing r by the number of split primes in $S' - S$. Then $h_{K,S,T} = 1$, so Lemma 4.11 implies that the congruence of Conjecture 2.6 holds.

We have verified Theorem 4.1 in all cases.

5. Real abelian extensions of $\mathbb{Q}$

In real abelian extensions of $\mathbb{Q}$, the infinite place splits and the Stark unit is known to be essentially a cyclotomic element. In this section we show that Conjecture 2.6 can be verified (up to a factor of 2 on each side) using the theory of cyclotomic elements.

5.1 Determination of the special unit

Suppose F is a totally real, nontrivial, abelian extension of $\mathbb{Q}$ with group G and conductor m. We consider Conjecture 2.6 for the extension $F/\mathbb{Q}$. By the Kronecker-Weber theorem, F is contained in $\mathbb{Q}(\zeta_m)$.

We set $S_\mathbb{Q} = \{p \mid m\} \cup \{\infty\}$, $S_{1,\mathbb{Q}} = \{\infty\}$ and $r = 1$ in the notation of §2, noting that the infinite place does indeed split completely in the extension $F/\mathbb{Q}$ because F is real. Let ∞_F be the infinite place of F induced by the embedding

$$\begin{array}{ccc} \mathbb{Q}(\zeta_m) & \longrightarrow & \mathbb{C} \\ \zeta_m & \longmapsto & e^{2\pi i/m}. \end{array}$$

Set $\beta = 1 - \zeta_m$ and, in the notation of §2, $W = (\infty_F)$.

Lemma 5.1. $W^*(\lambda_{S_F}(N_{\mathbb{Q}(\zeta_m)/F}\beta)) = 2\Theta^1_{F/\mathbb{Q},S_\mathbb{Q},\emptyset}(0)$.

Proof. We have

$$W^*(\lambda_{S_F}(N_{\mathbb{Q}(\zeta_m)/F}\beta)) = -\sum_{\sigma\in G} \ln|\sigma N_{\mathbb{Q}(\zeta_m)/F}\beta|\sigma^{-1}.$$

On the other hand, the value at $s=0$ of the L-function of an even Dirichlet character χ defined modulo m is given by:

$$L(0,\chi)=0, \qquad L'(0,\chi) = -\frac{1}{2}\sum_{i=1}^{m-1}\chi(i)\ln|1-\zeta_m^i|, \tag{10}$$

which holds whether or not m is the conductor of χ. The result follows easily by combining these formulae. □

Let $T = T_{\mathbb{Q}}$ be as required by Hypothesis 2.1, ie. T contains a prime of odd residue characteristic. Then the T-correction factor is

$$\delta_T = \prod_{v\in T}(1 - N_v\mathrm{Frob}_v^{-1}), \qquad \text{ie.} \quad \Theta^1_{F/\mathbb{Q},S,T}(0) = \delta_T\Theta^1_{F/\mathbb{Q},S,\emptyset}(0).$$

We have $W^*(\lambda_{S_F}(\delta_T N_{\mathbb{Q}(\zeta_m)/F}\beta)) = 2\Theta^1_{F/\mathbb{Q},S,T}(0)$, so (in the notation of §2) $\delta_T N_{\mathbb{Q}(\zeta_m)/F}\beta = 2\eta_{F/\mathbb{Q}}$ in $\mathbb{C}U_{F,S,T}$.

Because of this, we wish to study the properties of the cyclotomic elements $(1-\zeta_m)$. The next lemma summarises their well-known distribution properties.

LEMMA 5.2. *For each positive integer n, set $\zeta_n = e^{2\pi i/n}$, and define the norm element of the integral group ring of $\Gamma_n := \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ by $N_n := \sum_{g\in\Gamma_n} g$. Take positive integers p, f, r such that p is prime, $f>1$ and $p \nmid f$. By linear disjointness of $\mathbb{Q}(\zeta_{p^r})$ and $\mathbb{Q}(\zeta_f)$ as extensions of $\mathbb{Q}$, there is a natural inclusion $\Gamma_{p^r} \hookrightarrow \Gamma_{p^r f}$. Let $\sigma_{a,b}$ denote the automorphism of $\mathbb{Q}(\zeta_b)$ sending ζ_b to ζ_b^a for a coprime to b. Then we have:*

i) $(1-\zeta_{p^r f})^{N_{p^r}} = (1-\zeta_f)^{(1-\sigma_{p,f}^{-1})}$;
ii) $(1-\zeta_{p^r})^{N_{p^r}} = p$;
iii) *If two distinct primes divide n, then* $(1-\zeta_n)^{N_n} = 1$ □

These well-known facts follow from the factorisation of $X^{p^r}-1 \in \mathbb{C}[X]$.

5.2 Relations between determinants of certain matrices

The following linear algebra result will be useful in the following section. Fix a commutative ring with 1, and call it R. Let B be a finite set of positive integers, and for each $i,j\in B$ with $i\neq j$ fix $a_{ij}\in R$. For each $I\subseteq B$, let A^I be the square matrix indexed by I with (i,j)-th entry a_{ij} for $i\neq j$ and $-\sum_{k\in I\text{-}\{j\}} a_{ik}$ for $i=j$, so that A^I is a matrix with row-sum zero. Let A^I_i be the (i,i)-th minor determinant of A^I for $i\in I$.

PROPOSITION 5.3. *For each $i\in B$,*

$$\sum_{\{i\}\subseteq I\subseteq B} A^I_i \prod_{j\in B-I}\sum_{k\in I} a_{jk} = 0.$$

Proof. The proof uses trees in an analogous way to [GK03], proof of Theorem 8. If J is a finite set, then a *tree* T on J consists of the set of vertices J and edges between them which form a connected graph with no loops. A choice of a vertex $r\in J$ to be the "root" $\sqrt{T}$ of T induces a direction on each edge such that the out-degree of r is 0 and the out-degree of all other vertices is 1. For a directed tree T on a subset of B, define $A(T) := \prod_{(i\to j)\in T} a_{ij}$.

Since the row-sums of A^I are zero, the Kirchhoff-Tutte theorem (see [Tu48], or [GK02], Theorem 4) states that:

$$A_i^I = (-1)^{\#I-1} \sum_{\substack{T \text{ tree on } I: \\ \sqrt{T}=i}} A(T).$$

We also note that

$$\prod_{j \in B-I} \sum_{k \in I} a_{jk} = \sum_{f:(B-I)\to I} \prod_{j \in B-I} a_{j,f(j)}.$$

Hence the left-hand side in the desired equality is

$$\sum_{\{i\}\subseteq I \subseteq B} \sum_{\substack{T_I \text{ tree on } I: \\ \sqrt{T}=i}} \sum_{f_I:(B-I)\to I} (-1)^{\#I-1} A(T_I) \prod_{j \in B-I} a_{j,f_I(j)}.$$

For each tree T on B with root $i \in B$, we calculate the coefficient of $A(T)$ in the above sum. If V is a subset of the set of vertices of in-degree 0 in T, removing the vertices V and the edges attached to them gives a tree T_I on $B - V =: I$. Defining $f_I : (B - I) \to I$ by the relation $(j \to f_I(j)) \in T$, the pair (T, V) corresponds bijectively to the index (I, T_I, f_I) from the sum, whose term is $(-1)^{\#I} A(T)$. But given T, there are as many V with $\#I$ even as $\#I$ odd. Hence the term for T is 0. $\square$

5.3 A congruence statement for cyclotomic elements

Let $m > 1$ and write $m = p_1^{a_1} p_2^{a_2} \ldots p_{d+1}^{a_{d+1}}$. Write $\beta_m = 1 - \zeta_m$ for the associated cyclotomic element.

If $p \mid m$, and $\mathfrak{p}$ is a place of $\mathbb{Q}(\zeta_m)$ above p, we let $f_p(x)$ denote the Artin symbol $(x, \mathbb{Q}(\zeta_m)_\mathfrak{p}/\mathbb{Q}_p)$ for all nonzero $x \in \mathbb{Z}$. It is a simple exercise in the global class field theory of cyclotomic fields to show the following

Lemma 5.4. *If $j \neq i$, $f_{p_j}(p_i)^{-1}$ is given by the automorphism $\sigma_{p_i, p_j^{a_j}}$ of $\mathbb{Q}(\zeta_m)$ defined by:*

$$\begin{aligned} \zeta_{p_j^{a_j}} &\mapsto \zeta_{p_j^{a_j}}^{p_i} \\ \zeta_{p_k^{a_k}} &\mapsto \zeta_{p_k^{a_k}} \qquad \text{for } k \neq j. \quad \square \end{aligned}$$

In the notation of the previous section, we set $B = \{1, \ldots, d+1\}$ and $a_{ij} = f_{p_j}(p_i) - 1 \in \mathbb{Z}[\Gamma]$, where $\Gamma := \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. Then there are defined certain elements $A_i^I \in \mathbb{Z}[\Gamma]$ for $i \in I \subseteq B$. Set $S = \{p \mid m\} \cup \{\infty\}$, a finite set of places of $\mathbb{Q}$. We prove the following congruence statement for β_m:

Proposition 5.5. *For all $\phi : U_{\mathbb{Q}(\zeta_m),S} \longrightarrow \mathbb{Z}[\Gamma]$, we have*

$$\phi(\beta_m) \equiv \sum_{i=1}^{d+1} \phi^1(p_i) A_i^B \pmod{I_\Gamma^{d+1}}.$$

Proof. (cf. [Da95], Theorem 4.2) By induction on $d+1$. If $d+1 = 1$ we have $m = p_1^{a_1}$. Then

$$\phi(\beta_m) = \sum_{g \in \Gamma_m} \phi^1(g^{-1}\beta_m) g \equiv \sum_{g \in \Gamma_m} \phi^1(g^{-1}\beta_m) \pmod{I_\Gamma},$$

and this is $\phi^1(N_{p_1^{a_1}} \beta_m) = \phi^1(p_1)$ by Lemma 5.2. Hence the claim is true for $d+1 = 1$.

Now assume it is true for $d+1 = 1, 2, \ldots, n$. Set $d+1 = n+1 > 1$. For $I \subseteq \{1, \ldots, d+1\}$, write

$\Gamma_I = \prod_{i\in I}(\mathbb{Z}/p_i^{a_i}\mathbb{Z})^* \hookrightarrow \Gamma_m$, $\Gamma_i = \Gamma_{\{i\}}$ and $m(I) = \prod_{i\in I} p_i^{a_i}$. We have the following equality in $\mathbb{Z}[\Gamma]$:

$$\begin{aligned}
&\sum_{g_1\in\Gamma_1}\cdots\sum_{g_{d+1}\in\Gamma_{d+1}} \phi^1((g_1^{-1}\dots g_{d+1}^{-1})\beta_m)(g_1-1)\dots(g_{d+1}-1)\\
&=\sum_{g_1\in\Gamma_1}\cdots\sum_{g_{d+1}\in\Gamma_{d+1}} \phi^1((g_1^{-1}\dots g_{d+1}^{-1})\beta_m)\sum_{I\subseteq B}(-1)^{d+1-\#I}\prod_{i\in I} g_i\\
&=\sum_{I\subseteq B}(-1)^{d+1-\#I}\sum_{g\in\Gamma_I}\phi^1(g^{-1}\prod_{j\notin I} N_{p_j^{a_j}}\beta_m)g \qquad (11)
\end{aligned}$$

We recall (Lemma 5.2) that, if $I\neq\emptyset$, we have $\prod_{j\notin I} N_{p_j^{a_j}}\beta_m = \prod_{j\notin I}(1-\sigma_{p_j,m(I)}^{-1})\beta_{m(I)}$, and we note that $\sigma_{p_j,m(I)}\in\Gamma_I$. For $I=\emptyset$, $\prod_j N_{p_j^{a_j}}\beta_m = 1$ by Lemma 5.2. So (11) is equal to

$$\sum_{\emptyset\neq I\subseteq B}(-1)^{d+1-\#I}\phi_{(I)}(\beta_{m(I)})\prod_{j\notin I}(1-\sigma_{p_j,m(I)}^{-1}),$$

where $\phi_{(I)}$ means the $\mathbb{Z}[\Gamma_I]$-homomorphism $U_{\mathbb{Q}(\zeta_{m(I)}),S}\longrightarrow\mathbb{Z}[\Gamma_I]$ associated to the restriction of ϕ^1 to $U_{\mathbb{Q}(\zeta_{m(I)}),S}$.

Lemma 5.4 shows that $\sigma_{p,m(I)} = \prod_{i\in I} f_{p_i}(p)^{-1}$ for $p\nmid m(I)$, and our induction hypothesis gives $\phi_{(I)}(\beta_{m(I)})\in I_\Gamma^{\#I-1}$ for $I\neq B$. So if we reduce our equality modulo I_Γ^{d+1}, we obtain

$$\begin{aligned}
0&\equiv\sum_{I\subseteq B}(-1)^{d+1-\#I}\phi_{(I)}(\beta_{m(I)})\prod_{j\notin I}\big(-\sum_{k\in I}(f_{p_k}(p_j)-1)\big)\\
&\equiv\sum_{I\subseteq B}\phi_{(I)}(\beta_{m(I)})\prod_{j\notin I}\big(\sum_{k\in I}a_{jk}\big)\pmod{I_\Gamma^{d+1}}.
\end{aligned}$$

By the induction hypothesis, $\phi_{(I)}(\beta_{m(I)})\equiv\sum_{i\in I}\phi^1(p_i)A_i^I \pmod{I_\Gamma^{\#I}}$ if $I\neq B$. Therefore

$$0\equiv\phi(\beta_m)+\sum_{i=1}^{d+1}\phi^1(p_i)\sum_{\{i\}\subseteq I\subsetneq B}A_i^I\prod_{j\in B-I}\big(\sum_{k\in I}a_{jk}\big).$$

Now Proposition 5.3 shows that the i-th term of the sum is $-\phi^1(p_i)A_i^B$, as required. □

5.4 The congruence statement for a real abelian extension of $\mathbb{Q}$

Let $F/\mathbb{Q}$ be a finite, real, abelian extension. Let G be the Galois group and $m = p_1^{a_1}\dots p_{d+1}^{a_{d+1}}$ the conductor of this extension. Recall from §5.1 that

$$2\eta_{F/\mathbb{Q}} = \delta_{T_\mathbb{Q}} N_{\mathbb{Q}(\zeta_m)/F}\beta_m,$$

where $\beta_m = (1-\zeta_m)$. We set $S_\mathbb{Q} = S = \{\infty\}\cup\{p\mid m\}$, $S_1=\{\infty\}$, $r=1$, and T to be as required by Hypothesis 2.1.

Set $B=\{1,\dots,d+1\}$, and define $a_{ij}\in\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]$ as in §5.3. Then $a_{ij}\mapsto f_{p_j}(p_i)-1\in\mathbb{Z}[G]$ under the natural projection to $\mathbb{Z}[G]$. We relate the projections $\bar{A}_i^I$ of the A_i^I determinants from Proposition 5.5 to our group ring-valued regulators. We must choose an ordered set of d places with respect to which to calculate, and we choose $p_1,\dots,p_d$.

Proposition 5.6. *Let $\sigma\in S_{d+1}$ (the symmetric group on B) such that $\sigma(1)<\ldots<\sigma(d)$. Then*

$$\mathrm{Reg}_G(p_{\sigma(1)}\wedge\cdots\wedge p_{\sigma(d)}) = \mathrm{sign}(\sigma)\bar{A}_{\sigma(d+1)}^B.$$

Proof. If $d+1=1$ then both sides are the determinants of 0×0 matrices and so are 1. So we suppose $d+1>1$. First assume $\sigma=id$. The regulator is, by the product rule, the determinant of

(the projection of)

$$\begin{pmatrix} c_1 & a_{12} & \cdots & a_{1d} \\ a_{21} & c_2 & \cdots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \cdots & c_d \end{pmatrix},$$

with $c_j = -\sum_k a_{jk}$, where k runs over $\{1,\dots,d+1\}$-$\{j\}$. This is what we need.

Now assume $\sigma \neq id$. Let $b = \sigma(d+1)$. In the matrix defining the regulator, add the other columns to the column for the place p_b. This gives, by the product rule,

$$-\mathrm{Reg}_G^{(1,2,\dots,b-1,n+1,b+1,\dots,d)}(p_{\sigma(1)} \wedge \cdots \wedge p_{\sigma(d)}),$$

where the upper d-tuple is the ordered set of places with respect to which the regulator is calculated. This ordered set differs from the ordered set $(\sigma(1),\dots,\sigma(d))$ by the permutation $\sigma \circ (n+1 \quad b)$. So

$$\mathrm{Reg}_G^{(1,\dots,d)}(p_{\sigma(1)} \wedge \cdots \wedge p_{\sigma(d)}) = \mathrm{sign}(\sigma)\mathrm{Reg}_G^{(\sigma(1),\dots,\sigma(d))}(p_{\sigma(1)} \wedge \cdots \wedge p_{\sigma(d)}).$$

Now the result follows from the case $\sigma = id$ by renaming the primes $p_{\sigma(i)}$ to p_i. □

Now Proposition 5.5 and a "lowering the top field" argument entirely analogous to the proof of Proposition 3.2 together prove the following:

PROPOSITION 5.7. *For all $\phi : U_{F,S} \longrightarrow \mathbb{Z}[G]$, we have*

$$\phi(N_{\mathbb{Q}(\zeta_m)/F}\beta_m) \equiv (-1)^d \sum_{i=1}^{d+1} (-1)^{i+1}\phi^1(p_i)\mathrm{Reg}_G(p_1 \wedge \cdots \wedge \hat{p}_i \wedge \cdots \wedge p_{d+1}) \pmod{I_G^{d+1}},$$

where "$\hat{}$" means "omit". □

Remark 5.8. If $F/\mathbb{Q}$ is cyclic of prime-power degree, and all the p_i are totally tamely ramified in F, then this result can be deduced from Theorem 1 of [GK03].

Finally, we deduce a T-modified version. Let $\delta_T = \prod_{v\in T}(1 - N_v\mathrm{Frob}_v^{-1})$.

LEMMA 5.9. *Let $u_1,\dots,u_{d+1}$ be a $\mathbb{Z}$-basis for $U_{\mathbb{Q},S,T}$. Then we have the following equality in $\wedge_{\mathbb{Z}}^{d+1}U_{\mathbb{Q},S}$:*

$$2u_1 \wedge \cdots \wedge u_{d+1} = (U_{\mathbb{Q},S} : U_{\mathbb{Q},S,T})p_1 \wedge \cdots \wedge p_{d+1}.$$

Proof. Write $u_j = \pm\prod_{i=1}^{d+1} p_i^{c_{ji}}$ for $1 \leqslant j \leqslant d+1$ and a $d+1$ square matrix $C = (c_{ji})$ over $\mathbb{Z}$. The $d+2$ square matrix of relations between the $d+2$ generators $-1, p_1,\dots,p_{d+1}$ of $U_{\mathbb{Q},S}/U_{\mathbb{Q},S,T}$ is of the form

$$\left(\begin{array}{c|c} 2 & 0 \\ \hline ? & C \end{array}\right)$$

where 2 is the top-left entry. Hence the index $(U_{\mathbb{Q},S} : U_{\mathbb{Q},S,T}) = 2\det C$. On the other hand we have

$$u_1 \wedge \cdots \wedge u_{d+1} = (\det C)p_1 \wedge \cdots \wedge p_{d+1} + X$$

where X is a sum of terms of the form $(-1) \wedge x$, so that $2X = 0$. Multiplying this by 2 gives the stated result. □

Now we can show:

THEOREM 5.10. *Let $F/\mathbb{Q}$ be a real abelian extension with Galois group G and conductor m, $r = 1$, $S_1 = \{\infty\}$, $S = \{\infty\} \cup \{p \mid m\}$, $T \nsubseteq \{2\}$ a finite nonempty set of primes of $\mathbb{Q}$ disjoint from S. Then*

$F/\mathbb{Q}, S \supseteq S_1, T, r$ satisfies Hypothesis 2.1. In this case the congruence of Conjecture 2.6 is satisfied up to a factor of 2. That is, for all $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(U_{F,S,T}, \mathbb{Z}[G])$ we have

$$2\phi(\eta) \equiv 2(\pm h_{\mathbb{Q},S,T}\mathrm{Reg}_G^{\phi}) \pmod{I_G^{d+1}}.$$

Proof. Let $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(U_{F,S,T}, \mathbb{Z}[G])$. Applying Proposition 5.7 to the map $x \mapsto \phi(x^{\delta_T})$ and using Lemma 5.9 shows

$$2\phi(\delta_T N_{\mathbb{Q}(\zeta_m)/F}\beta_m) \equiv 2\frac{\mathrm{aug}(\delta_T)}{(U_{\mathbb{Q},S} : U_{\mathbb{Q},S,T})}\mathrm{Reg}_G(\tilde{\phi}(u_1 \wedge \cdots \wedge u_{d+1})) \pmod{I_G^{d+1}}.$$

Finally, equation (3) for $\mathbb{Q}$ shows that $h_{\mathbb{Q},S,T} = (-1)^{\#T}\mathrm{aug}(\delta_T)/(U_{\mathbb{Q},S} : U_{\mathbb{Q},S,T})$. $\square$

Remark 5.11. In particular, if G is of odd order then Conjecture 2.6 holds.

6. Base change via a conjecture of Darmon

We now move on to studying what happens when we make a quadratic extension of the base field k in Conjecture 2.6.

6.1 Quadratic extension of the base field for L-functions

Let K/k and $\tilde{L}/k$ be linearly disjoint finite abelian extensions of global fields. Assume $[K : k] = 2$. Write $L = \tilde{L}K$. Hence we have the following diagram of fields:

$$\begin{tikzcd} & L \arrow[dl, dash] \arrow[dr, dash] & \\ K \arrow[dr, dash] & & \tilde{L} \arrow[dl, dash] \\ & k & \end{tikzcd}$$

L/k is Galois with group $\mathrm{Gal}(L/k) = \mathrm{Gal}(\tilde{L}/k) \times \mathrm{Gal}(K/k)$. Let ω be the nontrivial character of $\mathrm{Gal}(K/k)$, and let $G := \mathrm{Gal}(L/K) = \mathrm{Gal}(\tilde{L}/k)$. Let $S = S_k$ and $T = T_k$ be disjoint finite sets of places of k with S nonempty and containing all infinite places. The Euler factors defining L functions in the extensions $\tilde{L}/k$ and L/K are related as follows. Let v be a finite prime of k, then for each place w of K lying over v we have a Frobenius element $\mathrm{Frob}_w \in \mathrm{Gal}(L/K) \hookrightarrow \mathrm{Gal}(L/k)$. We compare these with $\mathrm{Frob}_v \in \mathrm{Gal}(L/k)$ to obtain:

$$\prod_{\substack{w \text{ place of } K \\ w|v}} (1 - N_w^s\mathrm{Frob}_w^{-1}) = (1 - N_v^s\mathrm{Frob}_v^{-1})(1 - \omega(v)N_v^s\mathrm{Frob}_v^{-1}). \tag{12}$$

This follows by considering each Euler factor in the three cases $\omega(v) = 1$ (v splits in K/k), $\omega(v) = -1$ (v is inert) and $\omega(v) = 0$ (v ramifies). We see that the L-functions satisfy the following base-change factorisation when passing from L/K to L/k:

$$\Theta_{L/K,S_K,T_K}(s) = \Theta_{L/K/k,S_k,T_k}(s,\omega)\Theta_{\tilde{L}/k,S_k,T_k}(s), \tag{13}$$

where $\Theta_{L/K/k,S,T}(s,\omega)$ is the twisted Stickelberger function defined as

$$\begin{aligned} &\Big(\prod_{t\in T_k}(1 - \omega(t)N_t^{1-s}\mathrm{Frob}_t^{-1})\Big)\sum_{\chi\in\hat{G}} L_{L/k,S_k}(s,\omega\chi^{-1})e_{\omega\chi}(L/k). \\ &= \Big(\prod_{t\in T_k}(1 - \omega(t)N_t^{1-s}\mathrm{Frob}_t^{-1})\Big)\Big(\prod_{v\notin S_k}(1 - \omega(v)N_v^{-s}\mathrm{Frob}_v^{-1})\Big)^{-1}. \end{aligned}$$

The validity of equation (13) follows from the lemma and the definition (4) in the region of convergence $\operatorname{Re} s > 1$ and then everywhere by meromorphic continuation.

We will also use the notation

$$\delta_T^\omega := \prod_{v \in T_k} (1 - \omega(v) N_v \mathrm{Frob}_v^{-1})$$

for the relative T-modification factor at $s = 0$.

6.2 The circular unit

Here we show how the "circular unit" defined in [Da95] corresponds to the change in L-functions which results from raising the base field from $\mathbb{Q}$ to a linearly disjoint real quadratic field.

For comparison with [Da95], we assume the following hypothesis for the rest of §6.

HYPOTHESIS 6.1 (DARMON'S SETUP). Let N and S be coprime integers with $N > 1$ and $S > 1$. Let ω be a primitive, quadratic, even Dirichlet character defined modulo N. Set $K = \mathbb{Q}(\zeta_N)^{\ker \omega}$, a real quadratic field, and call its nontrivial automorphism τ. Let $\tilde{L}$ be a real subfield of $\mathbb{Q}(\zeta_S)$, normal over $\mathbb{Q}$. Write $L = \tilde{L}K$.

Hence we are in the situation of §6.1 with the further assumptions that $k = \mathbb{Q}$ and that $\tilde{L}$ and K are totally real and have coprime conductors. We define the set $S_{\mathbb{Q}}$ from the integer S in the obvious way: $S_{\mathbb{Q}} = \{p \mid S, \infty\}$.

All the characters of these extensions come from even Dirichlet characters because the fields are totally real. Since, by (10), L-functions of even characters vanish at $s = 0$, differentiating (13) twice shows that we have the following equality in $\mathbb{C}[G]$

$$\Theta^2_{L/K,S_K,T_K}(0) = \frac{1}{2}\Theta''_{L/K,S_K,T_K}(0) = \Theta'_{L/K/\mathbb{Q},S_{\mathbb{Q}},T_{\mathbb{Q}}}(0,\omega)\Theta'_{\tilde{L}/\mathbb{Q},S_{\mathbb{Q}},T_{\mathbb{Q}}}(0). \tag{14}$$

We now relate the base-change factor $\Theta'_{L/K/\mathbb{Q},S_{\mathbb{Q}},T_{\mathbb{Q}}}(0,\omega)$ to the circular unit defined in [Da95], §4. This is the following element of $K_S := K(\zeta_S)$.

$$\alpha_S := \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{NS})/\mathbb{Q}(\zeta_S))} \sigma(\zeta_{NS} - 1)^{\omega(\sigma)} \in U_{K_S}.$$

Write ∞_L for the place of L corresponding to the embedding of L into $\mathbb{R}$ given by $\zeta_{NS} \mapsto e^{2\pi i/NS}$, and $\overline{\infty_L}$ its conjugate by τ.

LEMMA 6.2. $\Theta'_{L/K/\mathbb{Q},S_{\mathbb{Q}},\emptyset}(0,\omega) \cdot (\infty_L - \overline{\infty_L}) = \frac{1}{2}\lambda_{S_L}(N_{K_S/L}\alpha_S)$.

Proof. Because NS is not a prime power, $(\zeta_{NS} - 1)$ is a global unit in $\mathbb{Q}(\zeta_{NS})$. Hence $\lambda_{S_L}(N_{K_S/L}\alpha_S)$ is zero outside the archimedean places. Now

$$\lambda_{S_L}(N_{K_S/L}\alpha_S) = -\sum_{\gamma \in \mathrm{Gal}(L/\mathbb{Q})} \ln|\gamma^{-1}N_{K_S/L}\alpha_S|\gamma\infty_L = -\sum_{\gamma \in G} \ln|\gamma^{-1}N_{K_S/L}\alpha_S|\gamma(\infty_L - \overline{\infty_L}),$$

since the generator of $\mathrm{Gal}(K/\mathbb{Q})$ inverts α_S. For χ a character of L/K, it suffices to prove

$$2\Theta'_{L/K/\mathbb{Q},S_{\mathbb{Q}},\emptyset}(0,\omega)e_{\omega\chi}(L/\mathbb{Q}) = -\sum_{\gamma \in G} \ln|\gamma^{-1}N_{K_S/L}\alpha_S|\omega\chi(\gamma)e_{\omega\chi}(L/\mathbb{Q}).$$

This is easy to show using the formulas (10) for values of Dirichlet L-series at $s = 0$, as in Lemma 5.1. □

DEFINITION 6.3. We set $\eta_\omega := \delta_T^\omega N_{K_S/L}\alpha_S$.

Then $\Theta'_{L/K/\mathbb{Q},S_{\mathbb{Q}},T_{\mathbb{Q}}}(0,\omega) \cdot (\infty_L - \overline{\infty_L}) = \frac{1}{2}\lambda_{S_L}(\eta_\omega)$.

6.3 Calculation of η

By Hypothesis 6.1, $\mathrm{Gal}(L/\mathbb{Q}) = \mathrm{Gal}(\tilde{L}/\mathbb{Q}) \times \mathrm{Gal}(K/\mathbb{Q})$ and the restriction map $\mathrm{Gal}(L/K) = G \longrightarrow \mathrm{Gal}(\tilde{L}/\mathbb{Q})$ is an isomorphism. Let $\#S_s = \#S_{split}$ be the number of primes p dividing S with $\omega(p) = 1$, and $\#S_i = \#S_{inert}$ the number of p with $\omega(p) = -1$.

We consider Conjecture 2.6 for the extensions L/K and $\tilde{L}/\mathbb{Q}$ in turn. We then show that the results and conjecture of [Da95] relate them.

- L/K: Since K is a real quadratic field, there are two infinite places of K, which we call ∞_K and $\overline{\infty_K}$. Because L is also real, these split completely in L/K. To avoid confusion, we write r' and d' for the "r" and "d" of §2 for the extension L/K. We take $r' = 2$ and $S_{1,K} = \{\infty_K, \overline{\infty_K}\}$. The element $\eta_{L/K} \in \mathbb{C} \wedge_G^2 U_{K,S_K,T_K} e_2$ is defined by:
$$\Theta^2_{L/K,S_K,T_K}(0)(\infty_L - w_0) \wedge (\overline{\infty_L} - w_0) = \lambda^{(2)}_{S_L}(\eta_{L/K})$$
for some finite place w_0 of S_L.
- $\tilde{L}/\mathbb{Q}$: In the notation of §2 we take $r = 1$ and $S_{1,\mathbb{Q}} = \{\infty\}$. Then $d = \#S_s + \#S_i$. The element $\eta_{\tilde{L}/\mathbb{Q}} \in \mathbb{C}U_{\mathbb{Q},S_\mathbb{Q},T_\mathbb{Q}} e_1(\tilde{L}/\mathbb{Q})$ is defined by
$$\Theta^1_{\tilde{L}/\mathbb{Q},S_\mathbb{Q},T_\mathbb{Q}}(0)(\infty_{\tilde{L}} - v_0) = \lambda_{S_{\tilde{L}}}(\eta_{\tilde{L}/\mathbb{Q}}),$$
for some finite place $v_0 \in S_{\tilde{L}}$.

The results of §6.2 allow us to express $\eta_{L/K}$ in terms of η_ω and $\eta_{\tilde{L}/\mathbb{Q}}$. We have:

$$\begin{aligned}&2\Theta^2_{L/K,S_K,T_K}(0)(\infty_L - w_0) \wedge (\overline{\infty_L} - w_0) = \\ &\qquad \Theta^1_{L/K/\mathbb{Q},S_\mathbb{Q},T_\mathbb{Q}}(0,\omega)(\infty_L - \overline{\infty_L}) \wedge \Theta'_{\tilde{L}/\mathbb{Q},S_\mathbb{Q},T_\mathbb{Q}}(0)(\infty_L + \overline{\infty_L} - 2w_0). \qquad (15)\end{aligned}$$

We now calculate $\lambda_{S_L}(\eta_{\tilde{L}/\mathbb{Q}})$. Let us review the various identifications and inclusions. Along with the canonical identification of Galois groups $G = \mathrm{Gal}(\tilde{L}/\mathbb{Q})$, we define homomorphisms $U_{\tilde{L}} \hookrightarrow U_L$ (inclusion) and

$$\begin{array}{rccc} i_{L/\tilde{L}}: & Y_{S_{\tilde{L}}} & \longrightarrow & Y_{S_L} \\ & v & \longmapsto & w + \bar{w}, \end{array}$$

where w is a place of L chosen arbitrarily above the place v of $\tilde{L}$, and $\bar{w} = w^\tau$. With these maps, the following diagrams commute:

$$\begin{array}{cccccccc} \mathrm{Gal}(\tilde{L}/\mathbb{Q}) \times Y_{S_{\tilde{L}}} & \longrightarrow & Y_{S_{\tilde{L}}} & \qquad & U_{\tilde{L}} & \xrightarrow{\lambda_{S_{\tilde{L}}}} & \mathbb{R}X_{S_{\tilde{L}}} \\ \downarrow & \square & \downarrow i_{L/\tilde{L}} & & \downarrow & \square & \downarrow i_{L/\tilde{L}} \\ G \times Y_{S_L} & \longrightarrow & Y_{S_L}, & & U_L & \xrightarrow{\lambda_{S_L}} & \mathbb{R}X_{S_L}. \end{array} \qquad (16)$$

Hence $\lambda_{S_L}(\eta_{\tilde{L}/\mathbb{Q}}) = \Theta'_{\tilde{L}/\mathbb{Q},S_\mathbb{Q},T_\mathbb{Q}}(0)(\infty_L + \overline{\infty_L} - w_1 - \overline{w_1})$, where w_1 is a place of L chosen above v_0. If we assume $\#S_i \neq 0$, we can choose v_0 such that $w_1 = \overline{w_1}$. Then setting $w_0 = w_1$ in (15) shows that
$$\lambda^{(2)}_{S_L}(\eta_\omega \wedge \eta_{\tilde{L}/\mathbb{Q}}) = 4\Theta^2_{L/K,S_K,T_K}(0)(\infty_L - w_0) \wedge (\overline{\infty_L} - w_0),$$
and hence $4\eta_{L/K} = \eta_\omega \wedge \eta_{\tilde{L}/\mathbb{Q}}$.

In [Da95], Darmon makes a congruence conjecture for his circular unit. We propose to interpret this as a base-change statement for Conjecture 2.6, under the following assumptions:

Hypothesis 6.4. i) $\#S_i \neq 0$
ii) For every place p in $T_\mathbb{Q}$, we have $\omega(p) = 1$.

For a group U on which τ acts, we define $U^- = \{u \in U : u^\tau = u^{-1}\}$.

PROPOSITION 6.5. *Assuming Hypothesis 6.4,* $\delta_T^\omega(U_{L,S}^-) \subseteq U_{L,S,T}^-$. *Therefore* $\eta_\omega \in U_{L,S,T}^-$.

Proof. Let $x \in U_{L,S}^-$ and $y = x^{\delta_T^\omega}$. Let $v \in T$ split into $w, \bar{w}$ in K. Then δ_T^ω contains a factor $(1 - N_w \mathrm{Frob}_w^{-1})$ by (12). Hence $w(y-1) > 0$. But we also have $\bar{w}(y-1) = w(y^{-1}-1) = w((1-y)/y) = w(1-y) > 0$. Therefore $y \equiv 1 \pmod{t}$ for all $t \in T_K$ as required. Setting $x = N_{K_S/L}\alpha_S$ proves the second assertion. ☐

6.4 Indices of minus units

Let K/k be a quadratic Galois extension of global fields with Galois group generated by τ. For this section we only need to assume that S is a finite, nonempty set of places of k containing all infinite places, and that T is any finite disjoint set of places of k.

LEMMA 6.6. $(U_{K,S} : U_{K,S,T}) = (U_{k,S} : U_{k,S,T})(U_{K,S}^{1-\tau} : U_{K,S,T}^{1-\tau})$

Proof. Consider the following commutative diagram, in which the rows are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_{k,S,T} & \longrightarrow & U_{K,S,T} & \xrightarrow{1-\tau} & U_{K,S,T}^{1-\tau} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_{k,S} & \longrightarrow & U_{K,S} & \xrightarrow{1-\tau} & U_{K,S}^{1-\tau} & \longrightarrow & 0, \end{array}$$

The vertical arrows are inclusions. Applying the snake lemma, we obtain the exact sequence

$$0 \longrightarrow U_{k,S}/U_{k,S,T} \longrightarrow U_{K,S}/U_{K,S,T} \xrightarrow{1-\tau} U_{K,S}^{1-\tau}/U_{K,S,T}^{1-\tau} \longrightarrow 0.$$

This shows the result. ☐

We consider the subgroup $U_{K,S,T}^- := \{u \in U_{K,S,T} : u^\tau = u^{-1}\}$ of "minus S-units" in $U_{K,S,T}$. This contains $U_{K,S,T}^{1-\tau}$. The quotient is, by Tate's finite group cohomology ([Se79], Chapter VIII),

$$U_{K,S,T}^-/U_{K,S,T}^{1-\tau} = H^1(\langle\tau\rangle, U_{K,S,T}).$$

LEMMA 6.7. *Suppose* $h_{k,S} = 1$. *Then:*

i) $H^1(\langle\tau\rangle, U_{K,S,\emptyset}) = 0$.
ii) $(U_{K,S}^{1-\tau} : U_{K,S,T}^{1-\tau}) = (U_{K,S}^- : U_{K,S,T}^-)\#H^1(\langle\tau\rangle, U_{K,S,T})$.

Proof. Corollary 2 of [Ri65] shows that $H^1(\langle\tau\rangle, U_{K,S,\emptyset})$ embeds into the S-class group of k, which is trivial. This shows (i), and (ii) follows immediately. ☐

Finally we adapt the method of [Ta84] (§II.2 and Theorem IV.5.4) to show the following. Let n denote the number of places of S which split in K/k.

LEMMA 6.8.

$$(U_{K,S,T} : U_{k,S,T}U_{K,S,T}^-) = \frac{2^n\#(U_{K,S,T}^- \cap \{\pm 1\})}{\#H^1(\langle\tau\rangle, U_{K,S,T})}$$

Proof. The sequence

$$0 \longrightarrow U_{k,S,T}U_{K,S,T}^- \longrightarrow U_{K,S,T} \xrightarrow{1-\tau} \frac{U_{K,S,T}^{1-\tau}}{(U_{K,S,T}^-)^2} \longrightarrow 0$$

is exact. This shows that $(U_{K,S,T} : U_{k,S,T}U_{K,S,T}^-) = (U_{K,S,T}^{1-\tau} : (U_{K,S,T}^-)^2)$. We also have:

$$(U_{K,S,T}^- : (U_{K,S,T}^-)^2) = (U_{K,S,T}^- : U_{K,S,T}^{1-\tau})(U_{K,S,T}^{1-\tau} : (U_{K,S,T}^-)^2).$$

The first factor on the right is $\#H^1(\langle\tau\rangle, U_{K,S,T})$. The factor on the left can be calculated from the standard decomposition of the finitely generated abelian group $U_{K,S,T}^-$. The free rank of this group is n, and the cokernel of squaring on the torsion part has the same size as the kernel, which is $U^-_{K,S,T} \cap \{\pm 1\}$. The second factor on the right is what we want to calculate. Hence

$$2^n \#(U^-_{K,S,T} \cap \{\pm 1\}) = \# H^1(\langle \tau \rangle, U_{K,S,T})(U_{K,S,T} : U_{k,S,T} U^-_{K,S,T}),$$

as required. □

6.5 Darmon's conjecture

We return to the situation of Hypothesis 6.1. We first state Darmon's conjecture in our notation.

Write $\Gamma := \mathrm{Gal}(K(\zeta_S)/K)$. For each prime $l_i | S$ such that $\omega(l_i) = 1$, l_i splits into two distinct places λ_i and $\bar{\lambda}_i$ in K. Darmon claims that $U^-_{K,S,\emptyset}$ is a free $\mathbb{Z}$-module ([Da95], §4), but this is not in fact the case since it contains -1, so actually $U^-_{K,S,\emptyset} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{\# S_s + 1}$. Taking either $T = \emptyset$ or T such that $U_{K,S,T}$ is torsion-free, choose a basis $u_1, \ldots, u_{\# S_s + 1}$ for a maximal free subgroup of $U^-_{K,S,T}$, which will have index 2 if $T = \emptyset$. Following Darmon, we define a regulator

$$R_{S,T} := \sum_{i=1}^{\# S_s + 1} (-1)^{i+1} u_i \otimes \det(f_{\lambda_k}(u_j) - 1)_{k,j} \in U_{K,S,T} \otimes \frac{I_\Gamma^{\# S_s}}{I_\Gamma^{\# S_s + 1}}, \tag{17}$$

where, in the matrix, k runs from 1 to $\# S_s$ and j runs from 1 to $\# S_s + 1$, omitting i. Note that $R_{S,\emptyset}$ might depend upon the choice of maximal free subgroup if the torsion element -1 is not in the kernel of the local Artin maps.

We state Darmon's conjecture [Da95], Conjecture 4.3, under the ring automorphism involution of $\mathbb{Z}[\Gamma]$ given by $g \mapsto g^{-1}$, which amounts to a sign change in the statement, and then ignoring all issues of sign.

Conjecture 6.9 (Darmon). We have the following equality in $U_{K(\zeta_S),S} \otimes I_\Gamma^{\# S_s} / I_\Gamma^{\# S_s + 1}$:

$$\sum_{\sigma \in \Gamma} \sigma^{-1} \alpha_S \otimes \sigma = \pm 2^{\# S_i + 1} h_{K,S} R_{S,\emptyset}.$$

We consider a T-modified version. This will fit with our general framework, and avoids the problem of torsion in the unit group. We assume Hypothesis 6.4 (ii) which implies that each v in T splits into w and $\bar{w}$ in K, with $N_w = N_v$. Then by (3), we have the following:

$$h_{\mathbb{Q},S,T} = h_{\mathbb{Q},S} \cdot \frac{\prod_{v \in T}(N_v - 1)}{(U_{\mathbb{Q},S} : U_{\mathbb{Q},S,T})}, \qquad h_{K,S,T} = h_{K,S} \cdot \frac{\prod_{v \in T}(N_v - 1)^2}{(U_{K,S} : U_{K,S,T})},$$

where we note $h_{\mathbb{Q},S} = 1$. The quotient is

$$\frac{h_{K,S,T}}{h_{\mathbb{Q},S,T}} = h_{K,S} \cdot \frac{\prod_{v \in T}(N_v - 1)}{(U_{K,S}^{1-\tau} : U_{K,S,T}^{1-\tau})} = h_{K,S} \cdot \frac{\prod_{v \in T}(N_v - 1)}{(U^-_{K,S} : U^-_{K,S,T}) \# H^1(\langle \tau \rangle, U_{K,S,T})}, \tag{18}$$

using Lemmas 6.6 and 6.7 (ii).

Lemma 6.10. *Under hypothesis 6.4,*

$$\frac{h_{K,S,T}}{h_{\mathbb{Q},S,T}} \# H^1(\langle \tau \rangle, U_{K,S,T})$$

is an integer.

Proof. For T empty, this is clear. Now let $T = \{v_1, \ldots, v_n\}$ and choose a place w_i of K above each v_i. Let $K(w_i)$ be the residue field of K at w_i. Then the natural sequence

$$0 \longrightarrow U^-_{K,S,T} \longrightarrow U^-_{K,S} \longrightarrow \bigoplus_{i=1}^{n} K(w_i)^\times$$

is exact. For if $u \in U^-_{K,S}$ reduces to 1 modulo each w_i, then $\overline{w_i}(u-1) = w_i(\bar{u}-1) = w_i((1-u)/u) = w_i(1-u) > 0$, as in Proposition 6.5. Hence $u \in U_{K,S,T} \cap U^-_{K,S} = U^-_{K,S,T}$.

This shows that $(U^-_{K,S} : U^-_{K,S,T}) \mid \prod_{v\in T}(N_v - 1)$, and by (18) this gives the result. □

We propose the following slight modification of Darmon's conjecture:

Conjecture 6.11. Assume T satisfies Hypothesis 6.4 (ii). Then we have the following equality in $U_{K(\zeta_S),S,T} \otimes I_\Gamma^{\#S_s}/I_\Gamma^{\#S_s+1}$:

$$\sum_{\sigma\in\Gamma} \sigma^{-1}\alpha_S^{\delta_T^\omega} \otimes \sigma = \pm 2^{\#S_i} \frac{h_{K,S,T}}{h_{\mathbb{Q},S,T}} \#H^1(\langle\tau\rangle, U_{K,S,T})\#(U_{K,S,T})_{tors} R_{S,T}. \tag{19}$$

If we put $T = \emptyset$ in this statement, then $h_{\mathbb{Q},S,T} = 1, \#H^1(\langle\tau\rangle, U_{K,S,T}) = 1$ (by Lemma 6.7 (i)), and $\#(U_{K,S,T})_{tors} = 2$. Hence we recover Conjecture 6.9. On the other hand, let us look at how Conjecture 6.11 varies when we replace T by $T \cup \{v\}$. If T is empty, then for the comparison statement we will have to assume that the regulator in Conjecture 6.9 is calculated with respect to a maximal free subgroup of $U_{K,S}$ which contains $U_{K,S,\{v\}}$. Examining how the various factors change on increase of T shows that Conjecture 6.11 behaves well, and that it follows from Conjecture 6.9 when $U^-_{K,S,T}$ can be embedded in a maximal free submodule of $U^-_{K,S}$.

The consequence of Darmon's conjecture that we wish to use is the following.

Proposition 6.12. *Assume the setup of Hypothesis 6.1 and let T satisfy Hypothesis 6.4 (ii) with $U_{L,S,T}$ torsion-free. Then Conjecture 6.11 implies that for each $\phi \in \mathrm{Hom}_G(U_{L,S,T}, \mathbb{Z}[G])$, we have*

$$\phi(\eta_\omega) \equiv \pm 2^{\#S_i} \frac{h_{K,S,T}}{h_{\mathbb{Q},S,T}} \#H^1(\langle\tau\rangle, U_{K,S,T})\phi^1(R_{S,T}) \pmod{I_G^{\#S_s+1}}.$$

Proof. Denote as usual $G = \mathrm{Gal}(L/K)$. We apply the natural projection $U_{K(\zeta_S),S} \otimes \mathbb{Z}[\Gamma] \longrightarrow U_{K(\zeta_S),S} \otimes \mathbb{Z}[G]$, which maps the left-hand side of (19) to $\sum_{\sigma\in G} \sigma^{-1}(\delta_T^\omega N_{K(\zeta_S)/L}\alpha_S) \otimes \sigma$ in $U_{L,S} \otimes \mathbb{Z}[G]$. Then Conjecture 6.11 implies the following equality in $U_{L,S} \otimes I_G^{\#S_s}/I_G^{\#S_s+1}$:

$$\sum_{\sigma\in G} \sigma^{-1}\eta_\omega \otimes \sigma = \pm 2^{\#S_i} \frac{h_{K,S,T}}{h_{\mathbb{Q},S,T}} \#H^1(\langle\tau\rangle, U_{K,S,T}) R_{S,T},$$

with η_ω from Definition 6.3.

Recall the isomorphism (5). Applying the homomorphism

$$\phi^1 \otimes id : U_{L,S,T} \otimes I_G^{\#S_s}/I_G^{\#S_s+1} \longrightarrow I_G^{\#S_s}/I_G^{\#S_s+1}$$

gives the stated result. □

6.6 Factorisation of the regulator

We assume T is such that $U_{L,S,T}$ is torsion-free, and that Hypothesis 6.4 is satisfied. We let $u_1, \dots, u_{\#S_s+1}$ be a basis for $U^-_{K,S,T}$ and $u_{\#S_s+2}, \dots, u_{2+d'}$ be a basis for $U_{\mathbb{Q},S,T}$. Then these u_i form a basis for $U_{\mathbb{Q},S,T}U^-_{K,S,T}$. The index of this group in $U_{K,S,T}$ was calculated in Lemma 6.8. We calculate the regulator from Conjecture 2.6 for these u_i. Let $\Phi = \phi_1 \wedge \phi_2 \in \wedge^2_G \mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S,T}, \mathbb{Z}[G])$. Let $R_{S,T}$ be the regulator defined in §6.5 in terms of the u_i. Write $\mathbf{u}_\mathbb{Q} = u_{\#S_s+2} \wedge \cdots \wedge u_{2+d'}$.

Recall that for each prime $l_i|S$ such that $\omega(l_i) = 1$, l_i splits into distinct places $\lambda_i, \bar{\lambda}_i$ in K. The other $\#S_i$ primes dividing S are inert in $K/\mathbb{Q}$, and will be denoted $q_1, \dots, q_{\#S_i}$. For reference, we summarise $S_{\mathbb{Q}}$ and S_K:

$$\begin{gathered}S_{\mathbb{Q}} = \{\infty, l_1, \ldots, l_{\#S_s}, q_1, \ldots, q_{\#S_i}\},\\ S_{1,\mathbb{Q}} = \{\infty\}, \quad r=1, \quad \#S_{\mathbb{Q}} = r+d+1, \quad \text{so } d = \#S_s + \#S_i;\\ S_K = \{\infty_L, \overline{\infty_L}, \lambda_1, \ldots, \lambda_{\#S_s}, \bar{\lambda}_1, \ldots, \bar{\lambda}_{\#S_s}, q_1, \ldots, q_{\#S_i}\},\\ S_{1,K} = \{\infty_L, \overline{\infty_L}\}, \quad r'=2, \quad \#S_K = r'+d'+1, \quad \text{so } d' = 2\#S_s + \#S_i.\end{gathered}$$

Proposition 6.13. *We have the following equality in $\mathbb{Z}[G]/I_G^{d'+1}$:*

$$\mathrm{Reg}_{L/K}(\tilde{\Phi}(u_1 \wedge \cdots \wedge u_{2+d'})) = \pm 2^{\#S_s} \begin{vmatrix} 2^{\#S_i-1}\phi_1^1(R_{S,T}) & 2^{\#S_i-1}\phi_2^1(R_{S,T}) \\ \mathrm{Reg}_{\tilde{L}/\mathbb{Q}}(\tilde{\phi_1}(\mathbf{u}_{\mathbb{Q}})) & \mathrm{Reg}_{\tilde{L}/\mathbb{Q}}(\tilde{\phi_2}(\mathbf{u}_{\mathbb{Q}}))\end{vmatrix}.$$

Proof. We saw in Lemma 2.4 that

$$\tilde{\Phi}(u_1 \wedge \cdots \wedge u_{2+d'}) = \sum_{\sigma \in \left[\frac{2+d'}{2}\right]} \mathrm{sign}(\sigma) \begin{vmatrix} \phi_1^1(u_{\sigma(1)}) & \phi_2^1(u_{\sigma(1)}) \\ \phi_1^1(u_{\sigma(2)}) & \phi_2^1(u_{\sigma(2)}) \end{vmatrix} u_{\sigma(3)} \wedge \cdots \wedge u_{\sigma(2+d')}. \tag{20}$$

The terms $u_{\sigma(3)} \wedge \cdots \wedge u_{\sigma(2+d')}$ are made by choosing two of the u_i for the integer determinant. So each σ excludes 0, 1 or 2 units of the $U^-_{K,S,T}$ basis from the wedge of units. Let $m_\sigma = \#(\sigma(\{3,4,\ldots,2+d'\}) \cap \{1,\ldots,\#S_s+1\})$ be the number of minus-units included in $u_{\sigma(3)} \wedge \cdots \wedge u_{\sigma(2+d')}$ in the term corresponding to σ, so $m_\sigma = \#S_s - 1, \#S_s$ or $\#S_s + 1$.

We calculate our matrix with respect to the following places of K, using Hypothesis 6.4 to exclude $q_{\#S_i}$:

$$\lambda_1, \ldots, \lambda_{\#S_s}, \bar{\lambda}_1, \ldots, \bar{\lambda}_{\#S_s}, q_1, \ldots, q_{\#S_i - 1}.$$

This means we have the determinant of the following $d' \times d'$ matrix to calculate for $\mathrm{Reg}_{L/K}(u_{\sigma(3)} \wedge \cdots \wedge u_{\sigma(2+d')})$:

$$\begin{array}{c||c|c|c} & \lambda_1, \ldots, \lambda_{\#S_s} & \bar{\lambda}_1, \ldots, \bar{\lambda}_{\#S_s} & q_1, \ldots, q_{\#S_i-1} \\ \hline\hline \begin{matrix} u_{\sigma(3)} \\ \vdots \\ u_{\sigma(m_\sigma+2)} \end{matrix} & f_{\lambda_j}(u_i) - 1 & f_{\bar{\lambda}_j}(u_i) - 1 & f_{q_j}(u_i) - 1 \\ \hline \begin{matrix} u_{\sigma(m_\sigma+3)} \\ \vdots \\ u_{\sigma(2+d')} \end{matrix} & f_{\lambda_j}(u_i) - 1 & f_{\bar{\lambda}_j}(u_i) - 1 & f_{q_j}(u_i) - 1 \end{array} \tag{21}$$

where the units at the top are in $U^-_{K,S,T}$ and the units at the bottom are in $U_{\mathbb{Q},S,T}$. We will distinguish between the cases where m_σ takes the different values.

First consider the case $m_\sigma = \#S_s + 1$. We may add the column for λ_j to the column for $\overline{\lambda_j}$ for $j = 1, \ldots, \#S_s$ without altering the value of the determinant. The (i,j)-th entry in the top-centre $(\#S_s + 1) \times (\#S_s)$ block is then congruent mod I_G^2 to $f_{\lambda_j}(\bar{u_i} u_i) - 1 = 0$. On the other hand, for each q_j, the local extension $K_{q_j}/\mathbb{Q}_{q_j}$ has degree 2, and $f_{q_j}(u_i)$ only depends on the norm of u_i in this local extension. If $u_i \in U^-_{K,S,T}$ then $u_i^{1+\tau} = 1$. Hence $f_{q_j}(u_i) - 1 = 0$ for these u_i. Therefore the entire top-right $(\#S_s + 1) \times (\#S_s + \#S_i - 1)$ block is zero. Hence there are at most $\#S$ columns which are nonzero in their first $\#S + 1$ rows. Therefore the determinant is zero.

Now in (21) we subtract the column for $\bar{\lambda}_j$ from the column for λ_j for $j = 1, \ldots, \#S_s$ to show the determinant is the same as the determinant of the following matrix.

$$\begin{array}{c||c|c|c} & \lambda_1,\dots,\lambda_{\#S_s} & \bar{\lambda}_1,\dots,\bar{\lambda}_{\#S_s} & q_1,\dots,q_{\#S_i-1} \\ \hline\hline u_{\sigma(3)} & & & \\ \vdots & f_{\lambda_j}(u_i)-f_{\bar{\lambda}_j}(u_i) & f_{\lambda_j}(u_i)-1 & f_{q_j}(u_i)-1 \\ u_{\sigma(m_\sigma+2)} & & & \\ \hline u_{\sigma(m_\sigma+3)} & & & \\ \vdots & 0 & f_{\bar{\lambda}_j}(u_i)-1 & f_{q_j}(u_i)-1 \\ u_{\sigma(2+d')} & & & \end{array} \tag{22}$$

If $m_\sigma = \#S_s - 1$, the first $\#S_s$ columns have all zeros except perhaps in the first $\#S_s - 1$ rows. Therefore the determinant is again zero.

We are left with the case $m_\sigma = \#S_s$. In this case the matrix (22) is block-upper-triangular. Let us consider the top-left $(\#S_s)\times(\#S_s)$ block first. We note that $f_{\lambda_j}(u_{\sigma(i)})-f_{\bar{\lambda}_j}(u_{\sigma(i)}) \equiv 2(f_{\lambda_j}(u_{\sigma(i)})-1) \pmod{I_G^2}$. So the top-left block has determinant $2^{\#S_s}\det(f_{\lambda_j}(u_{\sigma(i)})-1)_{i,j}$. Note the relation to the regulator $R_{S,T}$ of (17).

Now we calculate the determinant of the bottom-right block. We have $f_{\bar{\lambda}_j}(u) = f'_{l_j}(u)$, $f_{q_j}(u) = f'_{q_j}(u)^2$ for each j and each u appearing, where the f' denote the local symbols coming from the extension $\tilde{L}/\mathbb{Q}$. So the bottom-right block is $2^{\#S_i-1}\mathrm{Reg}_{\tilde{L}/\mathbb{Q}}(u_{\sigma(m_\sigma+3)}\wedge\cdots\wedge u_{\sigma(2+d')})$.

Referring back to (20), the only terms which appear in the sum after applying $\mathrm{Reg}_{L/K}$ are those for σ such that $\sigma(1)\leqslant \#S_s+1$ and $\sigma(2) > \#S_s+1$. We put this in correspondence with a pair $(i,j), 1\leqslant i\leqslant \#S_s+1, 1\leqslant j\leqslant d+1$ such that $\sigma(1)=i, \sigma(2)=\#S_s+1+j$. Then one may check that $\mathrm{sign}(\sigma) = (-1)^{\#S_s}(-1)^{i+1}(-1)^{j+1}$.

Putting all this together with (20) gives the stated result, with sign $(-1)^{\#S_s}$ on the right. □

6.7 Base change for the congruence

We are now ready to show the base change statement for Conjecture 2.6.

Theorem 6.14. *We use the setup of Hypothesis 6.1, assume Hypothesis 6.4 and use the definition of u_i from §6.6.*

Assume Conjecture 2.6 holds for the extension $\tilde{L}/\mathbb{Q}$, ie. that

$$\phi(\eta_{\tilde{L}/\mathbb{Q}}) \equiv \pm h_{\mathbb{Q},S,T}\mathrm{Reg}_{\tilde{L}/K}(\tilde{\phi}(u_{\#S_s+2}\wedge\cdots\wedge u_{2+d'})) \pmod{I_G^{d+1}},$$

for all $\phi\in\mathrm{Hom}_G(U_{\tilde{L},S,T},\mathbb{Z}[G])$. Assume also that the modified Darmon conjecture 6.11 holds. Then Conjecture 2.6 holds for the extension L/K up to a power of 2. Explicitly, for all $\Phi\in\wedge^2_G\mathrm{Hom}_G(U_{L,S,T},\mathbb{Z}[G])$, we have

$$4\cdot 2^{\#S_s}\Phi(\eta_{L/K}) \equiv \pm 4\cdot 2^{\#S_s}h_{K,S,T}\mathrm{Reg}_{L/K}(\tilde{\Phi}(\epsilon_1\wedge\cdots\wedge\epsilon_{2+d'})) \pmod{I_G^{d'+1}},$$

where the ϵ_i form a $\mathbb{Z}$-basis for $U_{K,S,T}$.

Proof. Write $\mathbf{u} = u_1\wedge\cdots\wedge u_{2+d'}$, $\epsilon = \epsilon_1\wedge\cdots\wedge\epsilon_{2+d'}$. Set $\Phi = \phi_1\wedge\phi_2$, and recall from §6.3 that $4\eta_{L/K} = \eta_\omega\wedge\eta_{\tilde{L}/\mathbb{Q}}$. The conjectures tell us, using Proposition 6.12,

$$4\Phi(\eta_{L/K}) = \begin{vmatrix} \phi_1(\eta_\omega) & \phi_2(\eta_\omega) \\ \phi_1(\eta_{\tilde{L}/\mathbb{Q}}) & \phi_2(\eta_{\tilde{L}/\mathbb{Q}}) \end{vmatrix} \equiv \pm \begin{vmatrix} 2^{\#S_i}\dfrac{h_{K,S,T}}{h_{\mathbb{Q},S,T}}\#H^1\phi_1^1(R_{S,T}) & 2^{\#S_i}\dfrac{h_{K,S,T}}{h_{\mathbb{Q},S,T}}\#H^1\phi_2^1(R_{S,T}) \\ h_{\mathbb{Q},S,T}\mathrm{Reg}_{\tilde{L}/\mathbb{Q}}(\tilde{\phi_1}(\mathbf{u}_\mathbb{Q})) & h_{\mathbb{Q},S,T}\mathrm{Reg}_{\tilde{L}/\mathbb{Q}}(\tilde{\phi_2}(\mathbf{u}_\mathbb{Q})) \end{vmatrix}$$

modulo $I_G^{d+1+\#S_s+1}$, where $\#H^1 = \#H^1(\langle\tau\rangle, U_{K,S,T})$ and $R_{S,T}$ is calculated with respect to the

$\mathbb{Z}$-basis $u_1, \ldots, u_{\#S_s+1}$ of $U^-_{K,S,T}$. Noting that $d' = d + \#S_s$, this is

$$\equiv \pm 2^{\#S_i} h_{K,S,T} \#H^1 \begin{vmatrix} \phi_1^1(R_{S,T}) & \phi_2^1(R_{S,T}) \\ \mathrm{Reg}_{\tilde{L}/\mathbb{Q}}(\tilde{\phi_1}(\mathbf{u}_\mathbb{Q})) & \mathrm{Reg}_{\tilde{L}/\mathbb{Q}}(\tilde{\phi_2}(\mathbf{u}_\mathbb{Q})) \end{vmatrix} \pmod{I_G^{d'+2}}.$$

Hence by Proposition 6.13, we have $4 \cdot 2^{\#S_s} \Phi(\eta_{L/K}) \equiv \pm h_{K,S,T} \cdot 2\#H^1 \mathrm{Reg}(\tilde{\Phi}(\mathbf{u})) \pmod{I_G^{d'+1}}$. Now we know from Lemma 6.8 that $(U_{K,S,T} : U_{\mathbb{Q},S,T} U^-_{K,S,T}) = 2^{\#S_s+1}/\#H^1$, so $\mathbf{u} = (2^{\#S_s+1}/\#H^1)\epsilon$. This gives the result. □

Note that if $\#G$ is odd, this last congruence is the full statement of Conjecture 2.6.

Remark 6.15. Using the method of [Da95], Lemma 8.1, it is possible to prove that $\phi(N_{K(\zeta_S)/L}\alpha_S) \in I_G^{\#S_s}$ for all $\phi \in \mathrm{Hom}_G(U^-_{L,S}, \mathbb{Z}[G])$, without assuming the validity of Darmon's conjecture. It then follows that $\phi'(\eta_\omega) \in I_G^{\#S_s}$ for all $\phi' \in \mathrm{Hom}_G(U_{L,S,T}, \mathbb{Z}[G])$. Thus if Conjecture 2.6 holds for $\tilde{L}/\mathbb{Q}$, then for L/K we have $4\Phi(\eta_{L/K}) \in I_G^{d'}$, for all $\Phi \in \wedge^2_G \mathrm{Hom}_G(U_{L,S,T}, \mathbb{Z}[G])$, consistent with the "order of vanishing" implied by Conjecture 2.6.

7. Base change via Gross's conjecture on the L-functions of tori

In §8 of [Gr88], Gross makes a conjecture motivated by considering algebraic tori. Similarly to Darmon's later conjecture, this involves a quadratic extension of the base field and consideration of the "minus-units" in this extension. It also involves a "Θ" element which is twisted by the nontrivial character of the extension. In the previous section we saw that Darmon's conjecture, which was related to the first derivative of the relative factor in (14), gave us a base change property for Conjecture 2.6 where the order of vanishing, r, increased by 1. Gross's conjecture, by contrast, concerns the value (zero-th derivative) of the relative factor and correspondingly it gives us a base change property where r doesn't change.

7.1 Setup and calculation of η

Our setup is as follows. Let k be a global field and $\tilde{L}/k$ and K/k linearly disjoint abelian extensions, with $[K:k] = 2$. Let $\omega : \mathrm{Gal}(K/k) \longrightarrow \{\pm 1\}$ be the nontrivial character of K/k. Setting $L = \tilde{L}K$, we are in the situation of §6.1. Write $G = \mathrm{Gal}(L/K) = \mathrm{Gal}(\tilde{L}/k)$. Let $S = S_k$ be a set of places of k containing all infinite places and all places ramifying in L/k. That is, both $\tilde{L}/k$ *and* K/k are unramified outside S. Take $T = T_k$ such that $U_{L,S,T}$ is torsion-free. We define n to be the number of places in S splitting in the quadratic extension K/k, and refer to the other $\#S - n$ inert or ramified places as nonsplit. We write τ for the nontrivial automorphism of this extension. This is the situation of §8 of [Gr88] except we have an unfortunate clash of notation, summarised in the following table.

Gross's notation	L	K	χ	σ
Our notation	K	$\tilde{L}$	ω	τ

Assume there are r places $S_{1,k}$ in S_k splitting completely in $\tilde{L}/k$. Then all the places above these in K split completely in L/K, and there are at least r of them, so the two sets of data $L/K, S_K \supseteq S_{1,K}, T_K, r$ and $\tilde{L}/k, S_k \supseteq S_{1,k}, T_k, r$ both satisfy Hypothesis 2.1. Differentiating the base-change factorisation of the L-functions (13) r times and evaluating at $s = 0$ gives

$$\Theta^r_{L/K,S_K,T_K}(0) = \Theta_{L/K/k,S_k,T_k}(0,\omega)\Theta^r_{\tilde{L}/k,S_k,T_k}(0). \tag{23}$$

The base-change factor $\Theta_{L/K/k,S_k,T_k}(0,\omega)$ lies in $\mathbb{Z}[G]$ by the argument following [Gr88] (8.7), where the corresponding element is denoted $\theta_G(\chi)$. Gross's tori conjecture concerns this element, and we will show that its validity would imply that Conjecture 2.6 for $L/K, S_K \supseteq S_{1,K}, T_K, r$ (weakened by powers of 2, similarly to the case of Darmon's conjecture) follows from the conjecture for $\tilde{L}/k, S_k \supseteq S_{1,k}, T_k, r$.

First note that we may assume that the r places in $S_{1,k}$ are nonsplit in K/k, since otherwise more than r places in S_K split in L/K and Conjecture 2.6 already holds for $L/K, S_K, T_K, r$ by Proposition 3.10. We also impose the following assumption, which is the same as Hypothesis 6.4 (i).

HYPOTHESIS 7.1. There is a place in $S_k - S_{1,k}$ which is nonsplit in K/k. That is, $d \geqslant n$.

Let v_0 be such a place. Write $S_{1,k} = \{v_1, \ldots, v_r\}$. Choose w_i a place of L above v_i for $i = 0, 1, \ldots, r$. Set $\mathbf{b}_L = (w_1 - w_0) \wedge \cdots \wedge (w_r - w_0)$. Write $\tilde{w}_i$ for the place of $\tilde{L}$ induced by w_i for $i = 0, \ldots, r$, and set $\mathbf{b}_{\tilde{L}} = (\tilde{w}_1 - \tilde{w}_0) \wedge \cdots \wedge (\tilde{w}_r - \tilde{w}_0)$. Then with these choices of the W in the definition of η, we have that $\eta_{\tilde{L}/k} \in \mathbb{C} \wedge^r_{\mathbb{Z}[G]} U_{\tilde{L},S,T}$ is defined by the equation

$$\lambda_{\tilde{L}}(\eta_{\tilde{L}/k}) = \Theta^r_{\tilde{L}/k,S_k,T_k}(0)\mathbf{b}_{\tilde{L}}.$$

The commutative diagrams (16) hold here (with k instead of $\mathbb{Q}$). Since $v_0, \ldots, v_r$ are all nonsplit in K/k, there is a unique w_i over $\tilde{w}_i$ for $i = 0, \ldots, r$. Hence

$$\lambda_L(\eta_{\tilde{L}/k}) = \Theta^r_{\tilde{L}/k,S_k,T_k}(0)\big((2w_1 - 2w_0) \wedge \cdots \wedge (2w_r - 2w_0)\big) = 2^r \Theta^r_{\tilde{L}/k,S_k,T_k}(0)\mathbf{b}_L.$$

Therefore we have, using (23), $\lambda_L(\eta_{\tilde{L}/k}^{\Theta_{L/K/k,S,T}(0,\omega)}) = 2^r \Theta^r_{L/K,S_K,T_K}(0)\mathbf{b}_L$. It follows that

$$\eta_{L/K} = \frac{1}{2^r}\eta_{\tilde{L}/k}^{\Theta_{L/K/k,S,T}(0,\omega)}. \tag{24}$$

7.2 Regulator calculations

Keeping the assumptions of the previous section, we now go on to study the regulators involved in the various conjectures. Break up S_k as follows:

$$S_k = \{ \overbrace{v_1, \ldots, v_r,}^{S_{1,k},\text{ nonsplit in } K} \quad \overbrace{v_{r+1}, \ldots, v_{r+n},}^{n \text{ split in } K} \quad \overbrace{v_{r+n+1}, \ldots, v_{r+d+1}}^{d-n+1 \text{ nonsplit in } K} \quad \}.$$

Note that $d-n+1 > 0$ by Hypothesis 7.1. Choose v'_i to be place of K above v_i for $i = 1, \ldots, r+d+1$. Choose a $\mathbb{Z}$-basis $\mu_1, \ldots, \mu_n$ of $U^-_{K,S,T}$. Then we can define a minus-unit regulator $R^-_G \in \mathbb{Z}[G]/I_G^{n+1}$ by the determinant of the $n \times n$ matrix with (i,j)-th entry $f_{v'_{r+j}}(\mu_i) - 1$ for $1 \leqslant i,j \leqslant n$. This is denoted $\det_G(\lambda_\tau)$ in [Gr88].

We choose a $\mathbb{Z}$-basis $u_1, \ldots, u_{r+d+n}$ for $U_{K,S,T}$ such that $u_{1+n}, \ldots, u_{r+d+n}$ is a basis for $U_{k,S,T}$, which is possible by Lemma 3.1.

The analogue of Proposition 6.13 in this situation is the following.

PROPOSITION 7.2. *We have the following in* $\mathbb{Z}[G]/I_G^{d+n+1}$*:*

$$\mathrm{Reg}_{L/K}(\tilde{\Phi}(u_1 \wedge \cdots \wedge u_{r+d+n})) = \pm 2^{d-n} R^-_G \mathrm{Reg}_{\tilde{L}/k}(\tilde{\Phi}(u_{1+n} \wedge \cdots \wedge u_{r+d+n}))(U^-_{K,S,T} : U^{1-\tau}_{K,S,T}).$$

Proof. The regulator on the left is

$$\sum_{\sigma \in \left[{r+d+n \atop r}\right]} \mathrm{sign}(\sigma) \det\left(\phi^1_j(u_{\sigma(i)})\right)_{1 \leqslant i,j \leqslant r} \mathrm{Reg}_{L/K}(u_{\sigma(r+1)} \wedge \cdots \wedge u_{\sigma(r+d+n)}), \tag{25}$$

where, after manipulations as in the proof of Proposition 6.13, $\mathrm{Reg}_{L/K}(u_{\sigma(r+1)} \wedge \cdots \wedge u_{\sigma(r+d+n)})$ is seen to be the determinant in $\mathbb{Z}[G]/I_G^{d+n+1}$ of the matrix

	$v'_{r+1},\ldots,v'_{r+n}$	$\bar{v}'_{r+1},\ldots,\bar{v}'_{r+n}$	$v'_{r+n+1},\ldots,v'_{r+n+d}$
$u_{\sigma(r+1)}$ $\vdots$ $u_{\sigma(r+m_\sigma)}$	$f_{v'_j}(u_i^{1-\tau})-1$	$f_{v'_j}(u_i^{\tau})-1$	$f_{v'_j}(u_i)-1$
$u_{\sigma(r+m_\sigma+1)}$ $\vdots$ $u_{\sigma(r+d+n)}$	0	$f_{v_j}(u_i)-1$	$2(f_{v_j}(u_i)-1)$

in which $m_\sigma = \#\sigma(\{r+1,\ldots,r+d+n\})\cap\{1,\ldots,n\}$. Now if $m_\sigma < n$ then this determinant is clearly 0. So for nonzero terms in the sum (25) we must have $m_\sigma = n$, ie. $\sigma(r+1)=1,\ldots,\sigma(r+n)=n$. Then $u_{\sigma(r+1)}^{1-\tau},\ldots,u_{\sigma(r+n)}^{1-\tau}$ is a $\mathbb{Z}$-basis for $U_{K,S,T}^{1-\tau}$ and so the determinant of the top left $n\times n$ block is $(U_{K,S,T}^- : U_{K,S,T}^{1-\tau})R_G^-$. The determinant of the bottom-right $d\times d$ block is $2^{d-n}\mathrm{Reg}_{\tilde{L}/k}(u_{\sigma(r+n+1)}\wedge\cdots\wedge u_{\sigma(r+d+n)})$.

Note that for such σ, the map $\sigma\circ(1\quad 2\quad \ldots\quad r+n)^r$ is a permutation of $\{n+1,\ldots,r+d+n\}$ of the form $n+k\mapsto n+\sigma'(k)$ for $\sigma'\in\left[{r+d \atop r}\right]$. We have

$$\mathrm{Reg}_{L/K}(\tilde{\Phi}(u_1\wedge\cdots\wedge u_{r+d+n})) = (U_{K,S,T}^- : U_{K,S,T}^{1-\tau})R_G^-\sum_{\sigma'}\Bigg((-1)^{r(r+n+1)}\mathrm{sign}(\sigma') \det\left(\phi_j^1(u_{n+\sigma'(i)})\right)_{1\leqslant i,j\leqslant r}\mathrm{Reg}_{\tilde{L}/k}(u_{n+\sigma'(r+1)}\wedge\cdots\wedge u_{n+\sigma'(r+d)})\Bigg),$$

which gives the result. ☐

7.3 Gross's conjecture on the L-functions of tori

We will now state Conjecture 8.8 of [Gr88]. The analytic class number formula makes it possible to calculate the coefficient of the leading term of the Taylor expansion of $L_{K/k,S,T}(s,\omega)$ at $s=0$, as in [Ta84], Chapter II, §2. It is $m_\omega R^-$, where R^- is a logarithmic regulator calculated with respect to bases of the minus-parts of $U_{K,S,T}$ and X_{S_K}, and

$$m_\omega = \pm\frac{h_{K,S,T}}{h_{k,S,T}}2^{\#S-n-1}(U_{K,S,T}^- : U_{K,S,T}^{1-\tau}).$$

The reader is warned that the factor $2^{\#S-n-1}$ is missing in equation (8.5) of [Gr88].

LEMMA 7.3. *Assuming Hypothesis 7.1, m_ω is an integer.*

Proof. The hypothesis shows that $\#S-n-1\geqslant 0$. Also K/k is a quadratic extension unramified outside S such that at least one place in S is inert. Therefore Lemma 4.6 (i) shows that $h_{k,S,T}$ divides $h_{K,S,T}$. This gives the result. ☐

We can now state Gross's tori conjecture, which in our setup, assuming Hypothesis 7.1 in order to have the conclusion of Lemma 7.3, states the following

CONJECTURE 7.4 (GROSS). We have

$$\Theta_{L/K/k,S,T}(0,\omega)\equiv m_\omega R_G^- \pmod{I_G^{n+1}}.$$

7.4 Base change

THEOREM 7.5. *Let $\tilde{L}/k, K/k$ be finite linearly disjoint abelian extensions of a global field k, with $[K:k]=2$. Set $L=\tilde{L}K$. Assume $S=S_k$, $T=T_k$ are such that L/k is unramified outside S_k and*

$U_{L,S,T}$ is torsion-free. Let $S_1 \subseteq S$ be a set of r places which split in $\tilde{L}/k$ but not in K/k. Assume Hypothesis 7.1 for these data.

Assume that Conjecture 2.6 holds for $\tilde{L}/k, S, T, r$ and that Conjecture 7.4 holds. Then for all $\Phi \in \wedge^r_{\mathbb{Z}[G]}\mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S,T}, \mathbb{Z}[G])$ we have

$$2^r\Phi(\eta_{L/K}) \equiv \pm 2^r h_{K,S,T}\mathrm{Reg}_G^{\Phi} \pmod{I_G^{d+n+1}}.$$

That is, the conclusion of Conjecture 2.6 for $L/K, S_K, T_K, r$ holds with a factor of 2^r on each side.

Proof. By (24) we have $2^r\Phi(\eta_{L/K}) = \Theta_{L/K/k,S,T}(0,\omega)\Phi(\eta_{\tilde{L}/k})$. Multiplying the congruences of Conjecture 7.4 and Conjecture 2.6 gives

$$\begin{aligned}2^r\Phi(\eta_{L/K}) &\equiv \pm\frac{h_{K,S,T}}{h_{k,S,T}}2^{r+d-n}(U^-_{K,S,T} : U^{1-\tau}_{K,S,T})R^-_G \cdot \\ &\qquad h_{k,S,T}\mathrm{Reg}_{\tilde{L}/k}(\tilde{\Phi}(u_{1+n}\wedge\cdots\wedge u_{r+d+n})) \pmod{I_G^{d+n+2}} \\ &\equiv \pm 2^r h_{K,S,T}\mathrm{Reg}_{L/K}(\tilde{\Phi}(u_1\wedge\cdots\wedge u_{r+d+n})) \pmod{I_G^{d+n+1}},\end{aligned}$$

by the regulator calculation in Proposition 7.2. □

Note that if $r = 0$ then this shows that, under Hypothesis 7.1, Gross's conjecture on tori actually gives a base-change property with no weakening factor for Conjecture 4.1 in [Gr88].

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